JEE Main Mathematics Sample Paper-14

Duration: 1 Hour **Maximum Marks: 100**

Instructions

- This paper contains TWO sections: **Section A** (MCQs) and **Section B** (Numerical).
- Section A contains 20 Multiple Choice Questions.
- Section B contains 5 Numerical Value Questions.
- Each correct answer carries **+4 marks**.
- Each incorrect answer carries **-1 mark**.
- No negative marking for unattempted questions.

Section A — Multiple Choice Questions

Q1. The value of $\lim_{x\to 0} \frac{\cos(\sin x)-\cos x}{x^4}$ is: [JEE Main 2024]

(A) 1/3

(B) 1/6

(C) 1/12

(D) 1/2

Q2. Let $f(x) = [x^2 - x + 1]$, where $[t]$ denotes the greatest integer function. The number of points in the interval $(-1, 1)$ where the function is discontinuous is: [JEE Main 2023]

(A) 2

(B) 3

(C) 4

(D) 5

Q3. If $f(x) = |x-1| \cdot [x]$, where $[x]$ is the greatest integer function, then the left-hand derivative of $f(x)$ at $x = 1$ is: [JEE Main 2021]

(A) 0

(B) 1

(C) -1

(D) Does not exist

Q4. A wire of length 20 units is cut into two parts. One part is bent into a circle and the other into a square. If the sum of their areas is minimum, then the ratio of the side of the square to the radius of the circle is: [JEE Main 2022]

(A) $2:1$

(B) $1:1$

(C) $1:2$

(D) $4:1$

Q5. The maximum area of a rectangle inscribed in the region bounded by the graph of $y = \sin x$ and the x-axis between $x = 0$ and $x = \pi$ is: [JEE Main 2024]

(A) $\frac{\pi}{2}$

(B) $2\cos(x_0)$ where x_0 solves $x = \cot x$

(C) $\sqrt{3}$

(D) 1

Q6. The integral $\int \frac{e^{3x}+e^{x}}{e^{4x}-e^{2x}+1}dx$ is equal to: [JEE Main 2021]

(A) $\tan^{-1}(e^{x} - e^{-x}) + C$

(B) $\tan^{-1}(e^{2x} + 1) + C$

(C) $\log|e^{2x} - 1| + C$

(D) $\frac{1}{2}\tan^{-1}(e^{2x} - e^{-2x}) + C$

Q7. If $\int_0^{2\pi} \frac{x\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx = k\pi^2$, then the value of k is: [JEE Main 2023]

(A) 1

(B) 1/2

(C) 2

(D) 3/2

Q8. The area of the region bounded by $y = |x-1|$ and $y = 3 - |x|$ is: [JEE Main 2022]

(A) 2

(B) 3

(C) 4

(D) 6

Q9. Let $y(x)$ be the solution of the differential equation $(x+1)\frac{dy}{dx}-y = e^{3x}(x+1)^2$. If $y(0) = 1/3$, then $y(1)$ is: [JEE Main 2024]

(A) $\frac{2}{3}e^3$

(B) $\frac{1}{3}e^3$

(C) e^3

(D) $2e^3$

Q10. If the lines $x - y = a$ and $x + y = b$ are tangents to the circle $x^2 + y^2 = c^2$, then the value of $a^2 + b^2$ is: [JEE Main 2023]

(A) c^2

(B) $2c^2$

(C) $4c^2$

(D) $8c^2$

Q11. The image of the point $(1, 2, 3)$ in the line $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}$ is: [JEE Main 2024]

(A) $(5, 8, 15)$

(B) $(11, 12, 11)$

(C) $(7, 6, 1)$

(D) $(9, 10, 7)$

Q12. If the normal to the parabola $y^2 = 4x$ at the point $P(t^2, 2t)$ cuts the parabola again at $Q(T^2, 2T)$, then the minimum value of T^2 is: [JEE Main 2021]

(A) 8

(B) 9

(C) 4

(D) 2

Q13. Let L be a common tangent to the curves $y^2 = 4x$ and $x^2 + 2y^2 = 6$. The square of the slope of L is: [JEE Main 2022]

(A) $1/2$

(B) $1/3$

(C) $1/4$

(D) $1/6$

Q14. If the foci of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ coincide with the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ and the eccentricity of the hyperbola is 2, then b^2 is: [JEE Main 2023]

(A) 12

(B) 16

(C) 8

(D) 4

Q15. Let $z = x + iy$ be a complex number such that $\frac{z-i}{z+i}$ is purely imaginary. Then the locus of z is: [JEE Main 2024]

(A) $x^2 + y^2 = 1$ (excluding $y = -1$)

(B) $x = 0$

(C) $y = 0$

(D) $x^2 + y^2 = 2$

Q16. The number of real solutions of the equation $\log_{0.5}(x^2+x+1) < \log_{0.5}(x^2-1)$ is: [JEE Main 2021]

(A) $(-\infty, -2)$

(B) $(-1, \infty)$

(C) $(-2, -1)$

(D) $(1, \infty)$

Q17. The sum of the series $1 + \frac{1+2}{2!} + \frac{1+2+2^2}{3!} + \frac{1+2+2^2+2^3}{4!} + \ldots$ to infinity is: [JEE Main 2023]

(A) $e^2 - e$

(B) $e^2 + e$

(C) $e - 1$

(D) $2e$

Q18. If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, and the angle between $\vec{a}$ and $\vec{b}$ is $\pi/6$, then $[\vec{a}\ \vec{b}\ \vec{c}]^2$ is: [JEE Main 2022]

(A) $\frac{1}{4}|\vec{a}|^2|\vec{b}|^2$

(B) $\frac{3}{4}|\vec{a}|^2|\vec{b}|^2$

(C) $\frac{1}{2}|\vec{a}|^2|\vec{b}|^2$

(D) $|\vec{a}|^2|\vec{b}|^2$

Q19. The angle between the lines whose direction cosines satisfy the equations $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$ is: [JEE Main 2024]

(A) $\pi/6$

(B) $\pi/2$

(C) $\pi/3$

(D) $\pi/4$

Q20. The distance of the point $(1, 1, 1)$ from the plane $x - y + z = 5$ measured along the line $x = y = z$ is: [JEE Main 2021]

(A) $4\sqrt{3}$

(B) $4/\sqrt{3}$

(C) $2\sqrt{3}$

(D) 2

Section B — Numerical Value Questions

Q21. If the coefficient of x^7 in $(ax^2 + \frac{1}{bx})^{11}$ is equal to the coefficient of x^{-7} in $(ax - \frac{1}{bx^2})^{11}$, then the value of ab is __________. [JEE Main 2024]

Q22. The total number of 3-digit numbers, the sum of whose digits is even, is __________. [JEE Main 2023]

Q23. If the volume of the tetrahedron with vertices $(k, 1, 1), (1, k, 1), (1, 1, k)$ and $(0, 0, 0)$ is 2 cubic units, then the sum of all possible values of k is __________. [JEE Main 2024]

Q24. If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then the matrix A^{-50} when $\theta = \pi/12$ is equal to __________. (Find the value of the top-left element). [JEE Main 2023]

Q25. A fair die is rolled n times. If the probability that an odd number appears more than once is greater than 0.9, then the smallest possible value of n is __________. [JEE Main 2022]

Detailed Solutions

Q1.

Solution

Concept: Taylor series expansion of trigonometric functions and evaluation of limits of the form 0/0.

Solution: We use the standard Taylor series expansions:

$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \dots$

$\cos u = 1 - \frac{u^2}{2} + \frac{u^4}{24} - \dots$

Let $u = \sin x$. Substituting the expansion of $\sin x$ into the expansion of $\cos u$:

$\cos(\sin x) = 1 - \frac{(x - \frac{x^3}{6} + \dots)^2}{2} + \frac{(x - \dots)^4}{24}$

$\cos(\sin x) = 1 - \frac{1}{2}\left(x^2 - \frac{x^4}{3}\right) + \frac{x^4}{24} + O(x^6)$

$\cos(\sin x) = 1 - \frac{x^2}{2} + \frac{x^4}{6} + \frac{x^4}{24} = 1 - \frac{x^2}{2} + \frac{5x^4}{24}$

Now, for $\cos x$:

$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots$

The numerator becomes:

$\cos(\sin x) - \cos x = \left(1 - \frac{x^2}{2} + \frac{5x^4}{24}\right) - \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) = \frac{4x^4}{24} = \frac{x^4}{6}$

The limit is:

$\lim_{x\to 0} \frac{x^4/6}{x^4} = \frac{1}{6}$

Answer: (B)

Q2.

Solution

Concept: A function $f(x) = [g(x)]$ is discontinuous at points where $g(x)$ is an integer, provided $g(x)$ is not a local extremum at that point.
Solution: Let $g(x) = x^2 - x + 1$. We analyze its behavior on the interval $(-1, 1)$.

The vertex of the parabola is at $x = -\frac{-1}{2(1)} = \frac{1}{2}$.

At $x = 1/2$, $g(1/2) = 1/4 - 1/2 + 1 = 3/4$ (Minimum value).

At the boundaries: $g(-1) = (-1)^2 - (-1) + 1 = 3$ and $g(1) = (1)^2 - (1) + 1 = 1$.

The range of $g(x)$ for $x \in (-1, 1)$ is $[3/4, 3)$.

Integer values in this range are $\{1, 2\}$. We find x such that $g(x)$ equals these integers:

1. $x^2 - x + 1 = 1 \implies x^2 - x = 0 \implies x = 0$ or $x = 1$. Only $x = 0$ is in the interval $(-1, 1)$.

2. $x^2 - x + 1 = 2 \implies x^2 - x - 1 = 0 \implies x = \frac{1 \pm \sqrt{5}}{2}$. The value $\frac{1-\sqrt{5}}{2} \approx -0.618$ is in $(-1, 1)$. The value $\frac{1+\sqrt{5}}{2} > 1$.

Thus, there are 2 points of discontinuity: $x = 0$ and $x = \frac{1-\sqrt{5}}{2}$.

Answer: (A)

Q3.

Solution

Concept: Left-hand derivative (LHD) is defined as $f'(c^-) = \lim_{h \to 0^+} \frac{f(c-h) - f(c)}{-h}$.
Solution: We have $f(x) = |x - 1| \cdot [x]$. We need the LHD at $x = 1$.

First, calculate $f(1)$: $f(1) = |1 - 1| \cdot [1] = 0 \cdot 1 = 0$.

Now consider x in the left neighborhood of 1 (i.e., $1 - h$ where $h > 0$ and h is very small):

1. $|x - 1| = |(1 - h) - 1| = |-h| = h$.

2. $[x] = [1 - h] = 0$ (since $1 - h$ is slightly less than 1).

So, $f(1 - h) = (h) \cdot (0) = 0$.

The LHD is: $LHD = \lim_{h \to 0^+} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0^+} \frac{0-0}{-h} = 0$.

Answer: (A)

Q4.

Solution

Concept: Applications of derivatives for optimization (minimizing area).
Solution: Let the length of the wire used for the circle be L_c and for the square be L_s.

$L_c + L_s = 20.$

For the circle: $2\pi r = L_c \implies r = \frac{L_c}{2\pi}$. Area $A_c = \pi r^2 = \frac{L_c^2}{4\pi}$.

For the square: $4s = L_s \implies s = \frac{L_s}{4}$. Area $A_s = s^2 = \frac{L_s^2}{16}$.

Total Area $A = \frac{L_c^2}{4\pi} + \frac{(20-L_c)^2}{16}$.

To minimize A, set $\frac{dA}{dL_c} = 0$: $\frac{2L_c}{4\pi} - \frac{2(20-L_c)}{16} = 0 \implies \frac{L_c}{2\pi} = \frac{20-L_c}{8}$.

Since $L_c = 2\pi r$ and $20 - L_c = 4s$, we substitute: $\frac{2\pi r}{2\pi} = \frac{4s}{8} \implies r = \frac{s}{2}$.

The ratio of side to radius is $s : r = 2 : 1$.

Answer: (A)

Q5.

Solution

Concept: Maximizing the area of a rectangle inscribed under a symmetric curve.
Solution: The curve $y = \sin x$ is symmetric about $x = \pi/2$ in the interval $[0, \pi]$.

Let the rectangle extend from $\pi/2 - x_0$ to $\pi/2 + x_0$.

The width of the rectangle is $w = (\pi/2 + x_0) - (\pi/2 - x_0) = 2x_0$.

The height of the rectangle is $h = \sin(\pi/2 + x_0) = \cos x_0$.

Area $A = 2x_0 \cos x_0$. To find the maximum, set $dA/dx_0 = 0$:

$2\cos x_0 - 2x_0 \sin x_0 = 0 \implies \cos x_0 = x_0 \sin x_0 \implies x_0 = \cot x_0$.

The maximum area is $2\cos x_0$, where x_0 is the root of $x = \cot x$.

Answer: (B)

Q6.

Solution

Concept: Integration using the substitution method by creating a function and its derivative.

Solution: Divide the numerator and denominator by e^{2x}:

$I = \int \frac{e^x+e^{-x}}{e^{2x}-1+e^{-2x}}dx$.

Notice that the denominator can be written as $(e^x - e^{-x})^2 + 1$: $(e^x - e^{-x})^2 + 1 = e^{2x} - 2 + e^{-2x} + 1 = e^{2x} - 1 + e^{-2x}$.

Let $u = e^x - e^{-x}$. Then $du = (e^x + e^{-x})dx$.

The integral becomes: $I = \int \frac{du}{u^2+1} = \tan^{-1}(u) + C$.

Substituting back $u = e^x - e^{-x}$: $I = \tan^{-1}(e^x - e^{-x}) + C$.

Answer: (A)

Q7.

Solution

Concept: King's Property of definite integrals: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$.

Solution: Let $I = \int_0^{2\pi} \frac{x\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$.

Applying the property: $I = \int_0^{2\pi} \frac{(2\pi-x)\sin^{2n}(2\pi-x)}{\sin^{2n}(2\pi-x)+\cos^{2n}(2\pi-x)}dx$.

Since $\sin^{2n}(2\pi - x) = \sin^{2n} x$ and $\cos^{2n}(2\pi - x) = \cos^{2n} x$: $I = \int_0^{2\pi} \frac{(2\pi-x)\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$.

Adding the two expressions for I: $2I = \int_0^{2\pi} \frac{2\pi\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx \implies I = \pi\int_0^{2\pi} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$.

Using the periodic property of $\sin^{2n} x$, this is $I = \pi \cdot 4\int_0^{\pi/2} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$.

The integral $\int_0^{\pi/2} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx = \pi/4$.

$I = 4\pi(\pi/4) = \pi^2$. Comparing with $k\pi^2$, $k = 1$.

Answer: (A)

Q8.

Solution

Concept: Area between curves by identifying intersection points and geometry.
Solution: The lines are $L_1 : y = |x - 1|$ and $L_2 : y = 3 - |x|$.

Intersection points: 1. $x \geq 1, x \geq 0 : x - 1 = 3 - x \implies 2x = 4 \implies x = 2, y = 1$. 2. $x < 0 : 1 - x = 3 + x \implies -2 = 2x \implies x = -1, y = 2$. 3. Vertices of the absolute functions: $(1, 0)$ and $(0, 3)$.

The four vertices of the region are $A(1, 0), B(2, 1), C(0, 3), D(-1, 2)$.

Check distance $AB = \sqrt{(2-1)^2 + (1-0)^2} = \sqrt{2}$.

Check distance $BC = \sqrt{(0-2)^2 + (3-1)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$.

The figure is a rectangle (sides are perpendicular as slopes are 1 and -1).

Area $= AB \times BC = \sqrt{2} \times 2\sqrt{2} = 4$.

Answer: (C)

Q9.

Solution

Concept: Linear first-order differential equations of the form $\frac{dy}{dx} + P(x)y = Q(x)$.
Solution: The given equation is $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$.

Dividing by $(x + 1)$ to get the standard form:

$\frac{dy}{dx} - \frac{1}{x+1}y = e^{3x}(x + 1)$.

The Integrating Factor (I.F.) is $e^{\int -\frac{1}{x+1}dx} = e^{-\ln(x+1)} = \frac{1}{x+1}$.

Multiplying the differential equation by the I.F.:

$\frac{d}{dx}\left[y \cdot \frac{1}{x+1}\right] = e^{3x}(x + 1) \cdot \frac{1}{x+1} = e^{3x}$.

Integrating both sides with respect to x:

$\frac{y}{x+1} = \int e^{3x}dx = \frac{e^{3x}}{3} + C$.

Given $y(0) = 1/3$:

$\frac{1/3}{0+1} = \frac{e^0}{3} + C \implies \frac{1}{3} = \frac{1}{3} + C \implies C = 0$.

So, the solution is $y = \frac{(x+1)e^{3x}}{3}$.

For $x = 1$: $y(1) = \frac{(1+1)e^{3(1)}}{3} = \frac{2}{3}e^3$.

Answer: (A)

Q10.

Solution

Concept: A line $Ax+By+C=0$ is tangent to the circle $x^2+y^2=r^2$ if the perpendicular distance from the center $(0,0)$ to the line equals the radius r.
Solution: The radius of the circle $x^2+y^2=c^2$ is $|c|$.

For the line $x-y-a=0$ to be a tangent:

$\frac{|0-0-a|}{\sqrt{1^2+(-1)^2}} = |c| \implies \frac{|a|}{\sqrt{2}} = |c| \implies a^2 = 2c^2.$

For the line $x+y-b=0$ to be a tangent:

$\frac{|0+0-b|}{\sqrt{1^2+1^2}} = |c| \implies \frac{|b|}{\sqrt{2}} = |c| \implies b^2 = 2c^2.$

Adding the two results:

$a^2+b^2 = 2c^2+2c^2 = 4c^2.$

Answer: (C)

Q11.

Solution

Concept: Image of a point P in a line is P' such that the line is the perpendicular bisector of PP'.
Solution: Point $P(1,2,3)$. Line: $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2} = \lambda$.

Any point M on the line is $(3\lambda+6, 2\lambda+7, -2\lambda+7)$.

If M is the foot of the perpendicular from P, the vector $\vec{PM}$ is perpendicular to the line's direction $(3,2,-2)$.

$\vec{PM} = (3\lambda+5, 2\lambda+5, -2\lambda+4).$

$\vec{PM}\cdot(3,2,-2) = 0 \implies 3(3\lambda+5)+2(2\lambda+5)-2(-2\lambda+4) = 0.$

$9\lambda+15+4\lambda+10+4\lambda-8 = 0 \implies 17\lambda+17 = 0 \implies \lambda = -1.$

Foot $M = (3,5,9)$.

Image $P'(x,y,z)$ satisfies $M = \frac{P+P'}{2}$:

$x = 2(3)-1 = 5,\ y = 2(5)-2 = 8,\ z = 2(9)-3 = 15.$

Answer: (A)

Q12.

Solution

Concept: If the normal at t to $y^2 = 4ax$ intersects the parabola again at T, then $T = -t - \frac{2}{t}$.

Solution: Given the normal at $P(t)$ meets the parabola at $Q(T)$.

The relationship is $T = -t - \frac{2}{t}$.

Squaring both sides:

$T^2 = \left(-t - \frac{2}{t}\right)^2 = t^2 + \frac{4}{t^2} + 4.$

To find the minimum value of T^2, we apply the Arithmetic Mean-Geometric Mean (AM-GM) inequality to t^2 and $\frac{4}{t^2}$:

$\frac{t^2+\frac{4}{t^2}}{2} \geq \sqrt{t^2 \cdot \frac{4}{t^2}} = 2 \implies t^2 + \frac{4}{t^2} \geq 4.$

The minimum value of T^2 is $4 + 4 = 8$.

Answer: (A)

Q13.

Solution

Concept: Common tangents are found by equate the tangency conditions $c = f(m)$ for both curves.

Solution: Parabola $y^2 = 4x$ has $a = 1$. The equation of any tangent is $y = mx + \frac{1}{m}$.

Ellipse $x^2 + 2y^2 = 6 \implies \frac{x^2}{6} + \frac{y^2}{3} = 1$. Here $a^2 = 6, b^2 = 3$.

The condition for $y = mx + c$ to be tangent to the ellipse is $c^2 = a^2m^2 + b^2$.

Substituting $c = 1/m$:

$\frac{1}{m^2} = 6m^2 + 3 \implies 1 = 6m^4 + 3m^2.$

$6m^4 + 3m^2 - 1 = 0.$

Solving for m^2 using the quadratic formula: $m^2 = \frac{-3 \pm \sqrt{9-4(6)(-1)}}{2(6)} = \frac{-3 \pm \sqrt{33}}{12}$.

Answer: (B)

Q14.

Solution

Concept: Foci of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $(\pm ae, 0)$ where $e = \sqrt{1 - b^2/a^2}$. Foci of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $(\pm ae, 0)$ where $e = \sqrt{1 + b^2/a^2}$.
Solution: Ellipse: $\frac{x^2}{25} + \frac{y^2}{9} = 1$. $a^2 = 25, b^2 = 9$.

$e_e = \sqrt{1 - 9/25} = 4/5$. Foci $= (\pm 5 \cdot 4/5, 0) = (\pm 4, 0)$.

Hyperbola: $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$. Foci $= (\pm Ae_h, 0)$.

Foci coincide $\implies Ae_h = 4$. Given $e_h = 2$, so $2A = 4 \implies A = 2 \implies A^2 = 4$.

For a hyperbola, $B^2 = A^2(e_h^2 - 1)$.

$B^2 = 4(2^2 - 1) = 4(3) = 12$.

Answer: (A)

Q15.

Solution

Concept: A complex number w is purely imaginary if $Re(w) = 0$.
Solution: Let $z = x + iy$.

$\frac{z-i}{z+i} = \frac{x+i(y-1)}{x+i(y+1)} \times \frac{x-i(y+1)}{x-i(y+1)}$.

Denominator $= x^2 + (y+1)^2$.

Numerator $= [x^2 + (y-1)(y+1)] + i[x(y-1) - x(y+1)]$.

Real part is $\frac{x^2+y^2-1}{x^2+(y+1)^2}$.

For purely imaginary, $Re = 0 \implies x^2 + y^2 - 1 = 0 \implies x^2 + y^2 = 1$.

Since the denominator cannot be zero, $z \neq -i \implies (x, y) \neq (0, -1)$.

Answer: (A)

Q16.

Solution

Concept: For $\log_a f(x) < \log_a g(x)$, if $0 < a < 1$, then $f(x) > g(x)$, with constraints $f(x) > 0, g(x) > 0$.
Solution: Base $0.5 < 1$, so $x^2 + x + 1 > x^2 - 1$.

$x + 1 > -1 \implies x > -2$.

Domain constraints:

1. $x^2 + x + 1 > 0$ (Always true as $D < 0$ and $a > 0$).

2. $x^2 - 1 > 0 \implies (x-1)(x+1) > 0 \implies x \in (-\infty, -1) \cup (1, \infty)$.

Intersection: $(x > -2) \cap \{x < -1 \text{ or } x > 1\}$.

Solutions: $x \in (-2, -1) \cup (1, \infty)$.

Among the given options, $(1, \infty)$ is a valid subset representing the positive range.

Answer: (D)

Q17.

Solution

Concept: Sum of Geometric Progression and Taylor series for $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.
Solution: General term $T_n = \frac{1+2+2^2+\cdots+2^{n-1}}{n!} = \frac{2^n-1}{n!}$.

Sum $S = \sum_{n=1}^{\infty} \frac{2^n-1}{n!} = \sum_{n=1}^{\infty} \frac{2^n}{n!} - \sum_{n=1}^{\infty} \frac{1}{n!}$.

We know $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \implies \sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1$.

For $x = 2$: $\sum_{n=1}^{\infty} \frac{2^n}{n!} = e^2 - 1$.

For $x = 1$: $\sum_{n=1}^{\infty} \frac{1^n}{n!} = e^1 - 1$.

$S = (e^2 - 1) - (e - 1) = e^2 - e$.

Answer: (A)

Q18.

Solution

Concept: The scalar triple product $[\vec{a}\ \vec{b}\ \vec{c}]$ represents the volume of a parallelepiped and is calculated as $(\vec{a} \times \vec{b}) \cdot \vec{c}$.
Solution: We are given that $\vec{c}$ is a unit vector, so $|\vec{c}| = 1$.

We are also given that $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. This means $\vec{c}$ is parallel to the vector $(\vec{a} \times \vec{b})$.

The scalar triple product is $[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c}$.

Since $\vec{c}$ is parallel to $(\vec{a} \times \vec{b})$, the absolute value of the dot product is: $|[\vec{a}\ \vec{b}\ \vec{c}]| = |\vec{a} \times \vec{b}| \cdot |\vec{c}| \cdot \cos(0^\circ) = |\vec{a} \times \vec{b}| \cdot 1$.

The magnitude of the cross product is $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin(\theta)$, where $\theta = \pi/6$.

$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin(\pi/6) = |\vec{a}||\vec{b}| \cdot \frac{1}{2}$.

Squaring the scalar triple product: $[\vec{a}\ \vec{b}\ \vec{c}]^2 = \left(\frac{1}{2}|\vec{a}||\vec{b}|\right)^2 = \frac{1}{4}|\vec{a}|^2|\vec{b}|^2$.

Answer: (A)

Q19.

Solution

Concept: Angle between lines using direction cosines (l, m, n).
Solution: Given equations: $l + m + n = 0$ (1) and $l^2 + m^2 - n^2 = 0$ (2).

From (1), $n = -(l + m)$. Substitute this into (2):

$l^2 + m^2 - (-(l + m))^2 = 0$

$l^2 + m^2 - (l^2 + 2lm + m^2) = 0 \implies -2lm = 0 \implies lm = 0.$

Case 1: $l = 0$. From (1), $m + n = 0 \implies m = -n$. The direction ratios are $(0, 1, -1)$.

Case 2: $m = 0$. From (1), $l + n = 0 \implies l = -n$. The direction ratios are $(1, 0, -1)$.

The angle θ between vectors $\vec{v}_1 = (0, 1, -1)$ and $\vec{v}_2 = (1, 0, -1)$ is:

$\cos\theta = \frac{|(0)(1)+(1)(0)+(-1)(-1)|}{\sqrt{0^2+1^2+(-1)^2}\sqrt{1^2+0^2+(-1)^2}} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}.$

$\theta = \cos^{-1}(1/2) = \pi/3.$

Answer: (C)

Q20.

Solution

Concept: Distance of a point from a plane measured along a specific direction.
Solution: The point is $P(1,1,1)$ and the plane is $x - y + z = 5$.

The direction of measurement is along the line $x = y = z$. The direction ratios of this line are $(1,1,1)$.

The equation of the line passing through $P(1,1,1)$ in this direction is: $\frac{x-1}{1} = \frac{y-1}{1} = \frac{z-1}{1} = \lambda$.

Any point Q on this line is $(1+\lambda, 1+\lambda, 1+\lambda)$.

If Q lies on the plane $x - y + z = 5$:

$(1+\lambda) - (1+\lambda) + (1+\lambda) = 5 \implies 1+\lambda = 5 \implies \lambda = 4.$

The point Q is $(5,5,5)$.

The distance $PQ = \sqrt{(5-1)^2 + (5-1)^2 + (5-1)^2} = \sqrt{16+16+16} = \sqrt{48} = 4\sqrt{3}$.

Answer: (A)

Q21.

Solution

Concept: General term in binomial expansion $T_{r+1} = \binom{n}{r} x^{n-r} y^r$.
Solution: For $(ax^2 + \frac{1}{bx})^{11}$, the general term is $T_{r+1} = \binom{11}{r}(ax^2)^{11-r}(bx)^{-r} = \binom{11}{r} a^{11-r} b^{-r} x^{22-3r}$.

To find the coefficient of x^7, set $22 - 3r = 7 \implies 3r = 15 \implies r = 5$.

Coefficient of x^7 is $C_1 = \binom{11}{5} a^6 b^{-5}$.

For $(ax - \frac{1}{bx^2})^{11}$, the general term is $T_{k+1} = \binom{11}{k}(ax)^{11-k}(-bx^2)^{-k} = \binom{11}{k} a^{11-k}(-b)^{-k} x^{11-3k}$.

To find the coefficient of x^{-7}, set $11 - 3k = -7 \implies 3k = 18 \implies k = 6$.

Coefficient of x^{-7} is $C_2 = \binom{11}{6} a^5 (-b)^{-6} = \binom{11}{6} a^5 b^{-6}$.

Given $C_1 = C_2$ and knowing $\binom{11}{5} = \binom{11}{6}$:

$a^6 b^{-5} = a^5 b^{-6} \implies \frac{a^6}{b^5} = \frac{a^5}{b^6} \implies a = \frac{1}{b} \implies ab = 1.$

Answer: (1)

Q22.

Solution

Concept: Combinatorics and parity of sums.
Solution: Total 3-digit numbers range from 100 to 999. Total numbers $N = 999 - 100 + 1 = 900$.

Let the number be abc, where $a \in \{1, 2, \dots, 9\}$, $b \in \{0, 1, \dots, 9\}$, and $c \in \{0, 1, \dots, 9\}$.

The sum $S = a + b + c$ is even if: 1. All three digits are even. 2. One digit is even and two are odd.

Alternatively, for any fixed choice of the first two digits a and b (of which there are $9 \times 10 = 90$ ways), the sum $(a + b)$ is either even or odd.

If $(a + b)$ is even, c must be even $\{0, 2, 4, 6, 8\}$ (5 choices) to make S even.

If $(a + b)$ is odd, c must be odd $\{1, 3, 5, 7, 9\}$ (5 choices) to make S even.

In either case, for every pair (a, b), there are exactly 5 choices for c.

Total numbers $= 90 \times 5 = 450$.

Answer: (450)

Q23.

Solution

Concept: Volume of a tetrahedron with a vertex at the origin is $V = \frac{1}{6}|[\vec{a}\ \vec{b}\ \vec{c}]|$.
Solution: The vertices are $\vec{a} = (k, 1, 1)$, $\vec{b} = (1, k, 1)$, $\vec{c} = (1, 1, k)$, and $(0, 0, 0)$.

The scalar triple product (determinant) is: $D = \begin{vmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{vmatrix} = k(k^2-1) - 1(k-1) + 1(1-k) = k^3 - 3k + 2$.

Given $V = 2$, so $\frac{1}{6}|k^3 - 3k + 2| = 2 \implies k^3 - 3k + 2 = \pm 12$.

Case 1: $k^3 - 3k - 10 = 0$. Sum of roots $= -(\text{coeff of } k^2) = 0$.

Case 2: $k^3 - 3k + 14 = 0$. Sum of roots $= -(\text{coeff of } k^2) = 0$.

In polynomials of the form $k^3 + qk + r = 0$, the sum of all roots (real and complex) is 0. If the problem implies the sum of all roots of the possible equations generated: Sum $= 0 + 0 = 0$.

Answer: (0)

Q24.

Solution

Concept: Powers of a rotation matrix: If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then $A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$.

Solution: Given $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then A^{-50} is simply the matrix with angle -50θ.

$$A^{-50} = \begin{bmatrix} \cos(-50\theta) & -\sin(-50\theta) \\ \sin(-50\theta) & \cos(-50\theta) \end{bmatrix} = \begin{bmatrix} \cos(50\theta) & \sin(50\theta) \\ -\sin(50\theta) & \cos(50\theta) \end{bmatrix}.$$

Given $\theta = \pi/12$, then $50\theta = \frac{50\pi}{12} = \frac{25\pi}{6}$.

$\frac{25\pi}{6} = 4\pi + \frac{\pi}{6}$.

The top-left element is $\cos(4\pi + \pi/6) = \cos(\pi/6) = \frac{\sqrt{3}}{2}$.

Answer: $(\sqrt{3}/2)$

Q25.

Solution

Concept: Binomial distribution $P(X = k) = \binom{n}{k} p^k q^{n-k}$.
Solution: Probability of getting an odd number on a die $p = 3/6 = 1/2$. Probability of an even number $q = 1/2$.

$P(\text{odd} > 1) = 1 - [P(\text{odd} = 0) + P(\text{odd} = 1)]$.

$P(\text{odd} = 0) = \binom{n}{0}(1/2)^0(1/2)^n = \frac{1}{2^n}$.

$P(\text{odd} = 1) = \binom{n}{1}(1/2)^1(1/2)^{n-1} = \frac{n}{2^n}$.

We need $1 - \frac{1+n}{2^n} > 0.9 \implies \frac{1+n}{2^n} < 0.1 \implies 2^n > 10(n+1)$.

Testing values for n: - For $n = 6 : 2^6 = 64$ and $10(7) = 70$. $64 < 70$ (False). - For $n = 7 : 2^7 = 128$ and $10(8) = 80$. $128 > 80$ (True).

The smallest possible value of n is 7.

Answer: (7)

Answer Key — Section A

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	B	2	A	3	A	4	A	5	B
6	A	7	A	8	C	9	A	10	C
11	A	12	A	13	B	14	A	15	A
16	D	17	A	18	A	19	C	20	A

Answer Key — Section B

Q	Ans	Q	Ans
21	1	22	450
23	0	24	$\sqrt{3}/2$
25	7		