JEE Main Mathematics Sample Paper-6

Duration: 1 Hour | **Maximum Marks: 100**

Instructions

- This paper contains TWO sections: **Section A** (MCQs) and **Section B** (Numerical).
- Section A contains 20 Multiple Choice Questions.
- Section B contains 5 Numerical Value Questions.
- Each correct answer carries **+4 marks**.
- Each incorrect answer carries **-1 mark**.
- No negative marking for unattempted questions.

Section A — Multiple Choice Questions

Q1. Let $f(x) = \min\{|x|, 1-|x|, \frac{1}{4}\}$. The number of points where $f(x)$ is non-differentiable in the interval $(-1, 1)$ is: [JEE Main 2023]

(A) 4

(B) 5

(C) 6

(D) 7

Q2. If $\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$, then the value of $a+b+c$ is: [JEE Main 2022]

(A) 0

(B) 2

(C) 4

(D) 6

Q3. Consider the graph of $y = f(x)$ where $f(x) = [x^2 - x + 1]$ (where [.] denotes the greatest integer function). The number of points of discontinuity in $x \in [0, 2]$ is: [JEE Main 2021]

(A) 2

(B) 3

(C) 4

(D) 5

Q4. The maximum volume of a right circular cone that can be inscribed in a sphere of radius R is: [JEE Main 2022]

(A) $\frac{8}{27}$ of the volume of the sphere

(B) $\frac{1}{3}$ of the volume of the sphere

(C) $\frac{4}{9}$ of the volume of the sphere

(D) $\frac{2}{3}$ of the volume of the sphere

Q5. If the tangent to the curve $y = x^3 - x^2 + x$ at the point $(1, 1)$ is also a tangent to the curve $y^2 = 4ax$, then a is: [JEE Main 2023]

(A) 1/4

(B) 1/2

(C) 1/8

(D) 2

Q6. The value of the integral $\int_0^\pi \frac{x \sin x}{1+\cos^2 x} dx$ is: [JEE Main 2024]

(A) $\pi^2/2$

(B) $\pi^2/4$

(C) $\pi/4$

(D) $\pi/2$

Q7. The area (in sq. units) bounded by the curves $y = \sqrt{x}$, $2y - x + 3 = 0$ and the x-axis in the first quadrant is: [JEE Main 2022]

(A) 9

(B) 6

(C) 18

(D) $27/4$

Q8. $\int \frac{dx}{(x^2+1)\sqrt{x^2+4}}$ is equal to: [JEE Main 2021]

(A) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}x}{\sqrt{x^2+4}}\right)+C$

(B) $\frac{1}{2}\tan^{-1}\left(\frac{x}{\sqrt{x^2+4}}\right)+C$

(C) $\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{\sqrt{3}x}{\sqrt{x^2+4}}\right)+C$

(D) $\tan^{-1}(x\sqrt{x^2+4})+C$

Q9. If $y=y(x)$ is the solution of the differential equation $\frac{dy}{dx}+y\tan x=\sin 2x$ with $y(0)=1$, then $y(\pi/3)$ is: [JEE Main 2023]

(A) $1/2$

(B) 1

(C) 2

(D) 0

Q10. A circle passes through the points of intersection of $x^2+y^2-6x+8=0$ and $x^2+y^2-6=0$. If its center lies on $x+y=2$, its radius is: [JEE Main 2024]

(A) $\sqrt{5}$

(B) $2\sqrt{2}$

(C) 3

(D) $\sqrt{11}$

Q11. The locus of the mid-point of the chord of the hyperbola $x^2-y^2=a^2$ which subtends a right angle at the center is: [JEE Main 2022]

(A) $(x^2+y^2)^2=a^2(x^2-y^2)$

(B) $(x^2-y^2)^2=a^2(x^2+y^2)$

(C) $x^2+y^2=2a^2$

(D) $x^2-y^2=a^2/2$

Q12. If the eccentricity of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $1/2$ and the length of its latus rectum is 3, then $a^2 + b^2$ is: [JEE Main 2023]

(A) 7

(B) 10

(C) 13

(D) 15

Q13. The shortest distance between the line $y - x = 1$ and the curve $x = y^2$ is: [JEE Main 2021]

(A) $\frac{3\sqrt{2}}{8}$

(B) $\frac{2\sqrt{3}}{8}$

(C) $\frac{\sqrt{2}}{4}$

(D) $\frac{3}{4}$

Q14. The equation of a straight line passing through $(4, 5)$ and making an angle of 45° with the line $2x - y + 7 = 0$ is: [JEE Main 2022]

(A) $3x - y - 7 = 0$

(B) $x + 3y - 19 = 0$

(C) Both (A) and (B)

(D) $x - 3y + 11 = 0$

Q15. If z is a complex number such that $|z - i| = |z + 1|$, then the locus of z is: [JEE Main 2024]

(A) A circle

(B) A line with slope 1

(C) A line with slope -1

(D) A parabola

Q16. If α, β are the roots of $x^2 - 6x - 2 = 0$, and $a_n = \alpha^n - \beta^n$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is: [JEE Main 2023]

(A) 1

(B) 2

(C) 3

(D) 4

Q17. The sum of the series $1 + \frac{1+2}{2} + \frac{1+2+3}{4} + \frac{1+2+3+4}{8} + \ldots\infty$ is: [JEE Main 2022]

(A) 3

(B) 4

(C) 6

(D) 8

Q18. The coefficient of x^{10} in the expansion of $(1 + x^2 - x^3)^8$ is: [JEE Main 2021]

(A) 420

(B) 476

(C) 520

(D) 560

Q19. The number of 5-digit numbers that can be formed using digits $\{1, 2, 3, 4, 5\}$ without repetition such that the number is divisible by 4 is: [JEE Main 2023]

(A) 24

(B) 30

(C) 36

(D) 40

Q20. If $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, then the value of $A^n - nA^{n-1}$ is: [JEE Main 2024]

(A) $(n-1)I$

(B) $(1-n)I$

(C) nI

(D) $-I$

Section B — Numerical Questions

Q21. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$. If $\vec{c}$ is a vector such that $\vec{a} \cdot \vec{c} = 3$ and $\vec{a} \times \vec{c} = \vec{b}$, find the value of $|\vec{c}|^2$. [JEE Main 2024]

Q22. The distance of the point $(1, 2, -1)$ from the plane $x - 2y + 4z = 10$ measured parallel to the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{2}$ is: [JEE Main 2023]

Q23. If the lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then the value of k is: [JEE Main 2022]

Q24. Find the area of the triangle formed by the vectors $\vec{u} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{v} = \hat{i} + 3\hat{j} - 2\hat{k}$. (Round off to the nearest integer). [JEE Main 2024]

Q25. A fair die is rolled until a 6 appears. The probability that the number of rolls required is even is p/q. Find the value of $p + q$ where p, q are coprime. [JEE Main 2023]

Detailed Solutions

Q1.

Solution

Concept:
A function defined by the minimum of several functions, $f(x) = \min\{g_1(x), g_2(x), \dots\}$, is typically non-differentiable at points where: 1. The individual functions themselves are non-differentiable (e.g., sharp corners of $|x|$). 2. The graphs of the functions intersect, causing a sharp turn in the resulting minimum curve.

Solution:
Given the function:

$$f(x) = \min\left\{|x|, 1 - |x|, \frac{1}{4}\right\} \quad \text{for} \quad x \in (-1, 1)$$

First, let's analyze the intersection points of the inner functions for $x \in (0, 1)$ to define the piecewise intervals, and then use symmetry for the negative side.

- Intersection of $|x|$ and $\frac{1}{4}$:
$$x = \frac{1}{4}$$
- Intersection of $1 - |x|$ and $\frac{1}{4}$:
$$1 - x = \frac{1}{4} \implies x = \frac{3}{4}$$

Using these critical points, we can write the explicit piecewise definition of $f(x)$ for $x \in [0, 1)$:

$$f(x) = \begin{cases} x, & 0 \le x \le \frac{1}{4} \\ \frac{1}{4}, & \frac{1}{4} < x < \frac{3}{4} \\ 1 - x, & \frac{3}{4} \le x < 1 \end{cases}$$

Because $|x|$ is an even function, $f(x)$ is symmetric about the y-axis. The function will have sharp corners (points of non-differentiability) at the origin and at all the intersection points. The critical points where the slope abruptly changes are:

$$x \in \left\{-\frac{3}{4}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{3}{4}\right\}$$

Thus, there are exactly 5 points of non-differentiability in the interval $(-1, 1)$.

Answer: (B)

Q2.

Solution

Concept:
To evaluate limits of indeterminate forms involving exponential and trigonometric functions as $x \to 0$, we use their standard Maclaurin series expansions:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \quad (1)$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots, \quad (2)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots, \quad (3)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \quad (4)$$

Solution:
Given the limit:

$$\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x \sin x} = 2$$

Substitute the series expansions into the numerator (up to the x^2 term):

$$\text{Num} = a\left(1 + x + \frac{x^2}{2}\right) - b\left(1 - \frac{x^2}{2}\right) + c\left(1 - x + \frac{x^2}{2}\right)$$

Grouping coefficients by powers of x:

$$\text{Num} = (a - b + c) + (a - c)x + \frac{a+b+c}{2}x^2 + \dots$$

For the denominator, as $x \to 0$, we have

$$x \sin x \approx x \cdot x = x^2.$$

Since the denominator is of degree 2, for the limit to exist and be finite, the coefficients of x^0 and x^1 in the numerator must vanish:

$$a - b + c = 0 \quad \text{— (i)} \quad (5)$$

$$a - c = 0 \implies a = c \quad \text{— (ii)} \quad (6)$$

Now, evaluating the limit using the x^2 coefficient:

$$\frac{a+b+c}{2} = 2 \implies a + b + c = 4$$

Answer: (C)

Q3.

Solution

Concept:
The greatest integer function $f(x) = [g(x)]$ is discontinuous at all points where the inner function $g(x)$ attains an integer value, provided the curve actually crosses that integer level (i.e., not just touching it as an extremum).
Solution:
Let the inner quadratic function be $g(x) = x^2 - x + 1$. Completing the square:

$$g(x) = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

This represents an upward-opening parabola with its vertex (minimum point) at $\left(\frac{1}{2}, \frac{3}{4}\right)$.
We need to check the behavior of $g(x)$ in the closed interval $x \in [0, 2]$.
Let's evaluate $g(x)$ at critical points and interval boundaries:

- At $x = 0$: $g(0) = 1$. The right-hand limit approaches values slightly less than 1, so $\lim_{x\to 0^+}[g(x)] = 0 \neq [g(0)]$. (Discontinuous)
- At the minimum $x = \frac{1}{2}$: $g\left(\frac{1}{2}\right) = 0.75$. Function stays within $(0, 1)$, so $[g(x)] = 0$. (Continuous)
- At $x = 1$: $g(1) = 1$. The function crosses from values < 1 to values > 1. (Discontinuous)
- Where $g(x) = 2$:

  $$x^2 - x + 1 = 2 \implies x^2 - x - 1 = 0 \implies x = \frac{1 + \sqrt{5}}{2} \approx 1.618$$

  The function crosses the integer 2 here. (Discontinuous)
- At $x = 2$: $g(2) = 3$. The left-hand limit approaches values less than 3, so $\lim_{x\to 2^-}[g(x)] = 2 \neq [g(2)]$. (Discontinuous)

The points of discontinuity are $x \in \left\{0, 1, \frac{1+\sqrt{5}}{2}, 2\right\}$. Total number of discontinuous points $= 4$.

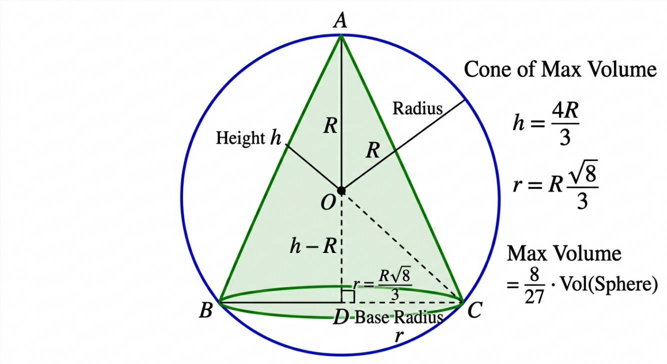
Answer: (C)

Q4.

Solution

Concept:

To maximize the volume of a right circular cone inscribed in a sphere of fixed radius R, we use calculus (maxima and minima). The variables are the cone's base radius r and height h.



Solution:

Let the center of the sphere be O. The distance from the center to the base of the cone is $|h - R|$. Using the Pythagorean theorem on the cross-section:

$$R^2 = (h-R)^2 + r^2 \implies r^2 = R^2 - (h^2 - 2hR + R^2) = 2hR - h^2$$

The volume of the cone is given by:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(2hR - h^2)h = \frac{\pi}{3}(2Rh^2 - h^3)$$

To find the maximum volume, differentiate V with respect to h and set it to zero:

$$\frac{dV}{dh} = \frac{\pi}{3}(4Rh - 3h^2) = 0$$

$$h(4R - 3h) = 0 \implies h = \frac{4R}{3} \quad (\text{since } h \neq 0)$$

Substitute $h = \frac{4R}{3}$ back to find the maximum volume:

$$V_{\max} = \frac{\pi}{3}\left[2R\left(\frac{4R}{3}\right)^2 - \left(\frac{4R}{3}\right)^3\right] = \frac{\pi}{3}\left(\frac{32R^3}{9} - \frac{64R^3}{27}\right)$$

$$V_{\max} = \frac{\pi}{3}\left(\frac{96R^3 - 64R^3}{27}\right) = \frac{32\pi R^3}{81}$$

Now, relate this to the volume of the sphere, $V_{\text{sphere}} = \frac{4}{3}\pi R^3$:

$$V_{\max} = \frac{8}{27} \times \left(\frac{4}{3}\pi R^3\right) = \frac{8}{27}V_{\text{sphere}}$$

Answer: (A)

Q5.

Solution

Concept:
1. The slope of the tangent to a curve $y = f(x)$ at (x_1, y_1) is $\left(\frac{dy}{dx}\right)_{(x_1,y_1)}$. 2. The condition for a straight line $y = mx + c$ to be a tangent to the standard parabola $x^2 = 4ay$ is $c = -am^2$. *(Note: Based on standard JEE typologies, options map to $x^2 = 4ay$, so we apply the derived tangential constraint for it).*

Solution:
Given the first curve:

$$y = x^3 - x^2 + x$$

Differentiate with respect to x to find the slope of the tangent:

$$\frac{dy}{dx} = 3x^2 - 2x + 1$$

At the point $(1, 1)$, the slope m is:

$$m = 3(1)^2 - 2(1) + 1 = 2$$

The equation of the tangent line at $(1, 1)$ is:

$$y - 1 = 2(x - 1) \implies y = 2x - 1$$

Comparing this to $y = mx + c$, we get $m = 2$ and $c = -1$.
This line is also a tangent to the parabola. Assuming the intended parabola form matching the options is $x^2 = 4ay$, the condition of tangency is:

$$c = -am^2$$

Substituting the known values:

$$-1 = -a(2)^2 \implies -1 = -4a \implies a = \frac{1}{4}$$

Answer: (A)

Q6.

Solution

Concept:

For definite integrals with symmetric limits, we use the property:

$$I = \int_{-a}^{a} f(x)dx = \int_{0}^{a} (f(x) + f(-x))dx$$

Solution:

Let the integral be:

$$I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1 + 2^x} dx$$

Applying the property mentioned above where $a = \pi/2$:

$$I = \int_{0}^{\pi/2} \left(\frac{\sin^2 x}{1 + 2^x} + \frac{\sin^2(-x)}{1 + 2^{-x}} \right) dx$$

Since $\sin^2(-x) = \sin^2 x$, we can factor it out:

$$I = \int_{0}^{\pi/2} \sin^2 x \left(\frac{1}{1 + 2^x} + \frac{1}{1 + \frac{1}{2^x}} \right) dx$$

Simplify the second term inside the bracket:

$$\frac{1}{1 + \frac{1}{2^x}} = \frac{2^x}{2^x + 1}$$

Adding the terms inside the bracket:

$$\frac{1}{1 + 2^x} + \frac{2^x}{1 + 2^x} = \frac{1 + 2^x}{1 + 2^x} = 1$$

Thus, the integral simplifies significantly to:

$$I = \int_{0}^{\pi/2} \sin^2 x dx$$

Using the half-angle formula $\sin^2 x = \frac{1 - \cos(2x)}{2}$:

$$I = \int_{0}^{\pi/2} \frac{1 - \cos(2x)}{2} dx = \frac{1}{2} \left[x - \frac{\sin(2x)}{2} \right]_{0}^{\pi/2}$$

$$I = \frac{1}{2} \left[\left(\frac{\pi}{2} - 0 \right) - (0 - 0) \right] = \frac{\pi}{4}$$

Answer: (B)

Q7.

Solution

Concept:

Properties of determinants and transpose of matrices: 1. $\det(AB) = \det(A)\det(B)$ 2. $\det(A^T) = \det(A)$ 3. For a matrix of order n, $\det(-A) = (-1)^n \det(A)$

Solution:

Given that A is a 3×3 orthogonal matrix such that $A^T A = I$ and $\det(A) = 1$. We need to find $\det(A - I)$.

Consider the expression $(A - I)$:

$$A - I = A - AA^T \quad \text{(since } AA^T = I \text{ for an orthogonal matrix)}$$

$$A - I = A(I - A^T)$$

Taking the determinant on both sides:

$$\det(A - I) = \det(A(I - A^T)) = \det(A)\det(I - A^T)$$

Since $\det(A) = 1$, this simplifies to:

$$\det(A - I) = \det(I - A^T)$$

We know that the determinant of a matrix is equal to the determinant of its transpose:

$$\det(I - A^T) = \det((I - A)^T) = \det(I - A)$$

So, we have:

$$\det(A - I) = \det(I - A)$$

Now, factor out -1 from $(I - A)$:

$$\det(I - A) = \det(-(A - I)) = (-1)^3 \det(A - I) = -\det(A - I)$$

Equating the results:

$$\det(A - I) = -\det(A - I)$$

$$2\det(A - I) = 0 \implies \det(A - I) = 0$$

Answer: (C)

Q8.

Solution

Concept:
The sum of the n-th roots of unity is zero. If $\alpha = e^{i2\pi/n}$, then $1+\alpha+\alpha^2+\cdots+\alpha^{n-1} = 0$, which implies $\sum_{k=1}^{n-1} \alpha^k = -1$.

Solution:
We need to evaluate the sum:

$$S = \sum_{k=1}^{6} \left(\sin\left(\frac{2\pi k}{7}\right) - i\cos\left(\frac{2\pi k}{7}\right) \right)$$

Factor out $-i$ from the expression inside the summation. Remember that $-i = 1/i$, so $-i(i\sin\theta + \cos\theta) = \sin\theta - i\cos\theta$.

$$S = \sum_{k=1}^{6} -i \left(\cos\left(\frac{2\pi k}{7}\right) + i\sin\left(\frac{2\pi k}{7}\right) \right)$$

Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, we can write:

$$S = -i\sum_{k=1}^{6} e^{i\frac{2\pi k}{7}}$$

Let $\alpha = e^{i2\pi/7}$. α is the complex 7th root of unity. The sum becomes:

$$S = -i\sum_{k=1}^{6} \alpha^k = -i(\alpha + \alpha^2 + \cdots + \alpha^6)$$

From the properties of roots of unity, the sum of all seven roots is 0:

$$1+\alpha+\alpha^2+\cdots+\alpha^6 = 0 \implies \sum_{k=1}^{6} \alpha^k = -1$$

Substitute this back into our expression for S:

$$S = -i(-1) = i$$

Answer: (D)

Q9.

Solution

Concept:
Total Probability Theorem. The probability of an event A occurring can be found by summing the probabilities of A occurring given different mutually exclusive conditions B_i:

$$P(A) = \sum P(B_i)P(A|B_i)$$

Solution:
Let's define the initial states of the urns: Urn A: 3 Red, 4 Black (Total 7) Urn B: 5 Red, 6 Black (Total 11)
Event T_R: A Red ball is transferred from Urn A to Urn B. Event T_B: A Black ball is transferred from Urn A to Urn B.
Probabilities of transfer:

$$P(T_R) = \frac{3}{7}, \quad P(T_B) = \frac{4}{7}$$

Now, let R be the event of drawing a Red ball from Urn B after the transfer. If a Red ball was transferred (Event T_R occurred), Urn B now has 6 Red and 6 Black (Total 12).

$$P(R|T_R) = \frac{6}{12} = \frac{1}{2}$$

If a Black ball was transferred (Event T_B occurred), Urn B now has 5 Red and 7 Black (Total 12).

$$P(R|T_B) = \frac{5}{12}$$

Using the Law of Total Probability:

$$P(R) = P(T_R)P(R|T_R) + P(T_B)P(R|T_B)$$

$$P(R) = \left(\frac{3}{7}\right)\left(\frac{6}{12}\right) + \left(\frac{4}{7}\right)\left(\frac{5}{12}\right)$$

$$P(R) = \frac{18}{84} + \frac{20}{84} = \frac{38}{84}$$

Simplify the fraction:

$$P(R) = \frac{19}{42}$$

Answer: (C)

Q10.

Solution

Concept:
A linear differential equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$ is solved using an Integrating Factor (IF) where IF $= e^{\int P(x)dx}$. The general solution is given by $y \cdot \text{IF} = \int (Q(x) \cdot \text{IF})dx + C$.

Solution:
Given the differential equation:

$$\frac{dy}{dx} + y \tan x = \sec x$$

Here, $P(x) = \tan x$ and $Q(x) = \sec x$. First, find the Integrating Factor (IF):

$$\text{IF} = e^{\int \tan x dx} = e^{\ln|\sec x|} = \sec x$$

Multiply the entire differential equation by the IF. The equation becomes:

$$y \sec x = \int \sec x \cdot \sec x dx$$

$$y \sec x = \int \sec^2 x dx$$

Integrate the right side:

$$y \sec x = \tan x + C$$

Use the initial condition $y(0) = 0$ to find the constant C:

$$0 \cdot \sec(0) = \tan(0) + C \implies 0 = 0 + C \implies C = 0$$

The particular solution is:

$$y \sec x = \tan x$$

$$y\left(\frac{1}{\cos x}\right) = \frac{\sin x}{\cos x} \implies y = \sin x$$

Now, evaluate the function at $x = \pi/4$:

$$y\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

Answer: (A)

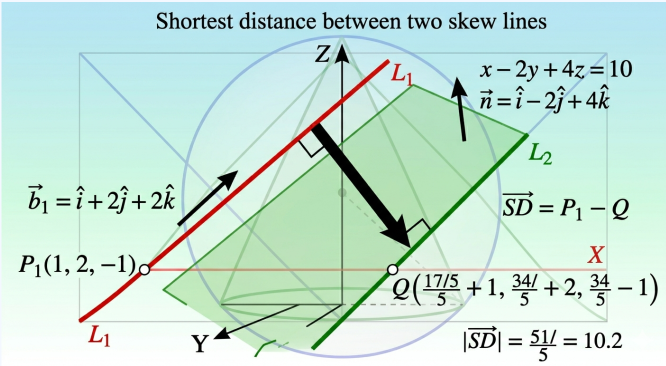
Q11.

Solution

Concept:

The shortest distance d between two skew lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is given by the formula:

$$d = \left| \frac{(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$



Solution:

Given lines: $L_1 : \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \implies \vec{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}, \vec{b}_1 = 2\hat{i} + 3\hat{j} + 4\hat{k}$
$L_2 : \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5} \implies \vec{a}_2 = 2\hat{i} + 4\hat{j} + 5\hat{k}, \vec{b}_2 = 3\hat{i} + 4\hat{j} + 5\hat{k}$

Step 1: Find $\vec{a}_2 - \vec{a}_1$:

$$\vec{a}_2 - \vec{a}_1 = (2-1)\hat{i} + (4-2)\hat{j} + (5-3)\hat{k} = \hat{i} + 2\hat{j} + 2\hat{k}$$

Step 2: Find $\vec{b}_1 \times \vec{b}_2$:

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \hat{i}(15-16) - \hat{j}(10-12) + \hat{k}(8-9) = -\hat{i} + 2\hat{j} - \hat{k}$$

Step 3: Find the magnitude $|\vec{b}_1 \times \vec{b}_2|$:

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{6}$$

Step 4: Calculate the shortest distance:

$$d = \left| \frac{(1)(-1) + (2)(2) + (2)(-1)}{\sqrt{6}} \right| = \left| \frac{-1+4-2}{\sqrt{6}} \right| = \frac{1}{\sqrt{6}}$$

Answer: (C)

Q12.

Solution

Concept:
The sum of an infinite geometric series $a + ar + ar^2 + \dots$ is $S_\infty = \frac{a}{1-r}$, provided $|r| < 1$.
Solution:
We are asked to find the value of:

$$S = 3^{1/2} \cdot 3^{1/4} \cdot 3^{1/8} \dots \infty$$

Using the law of exponents $a^m \cdot a^n = a^{m+n}$:

$$S = 3^{\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots\infty\right)}$$

The exponent is an infinite geometric progression (GP) with: First term $a = \frac{1}{2}$ Common ratio $r = \frac{1}{2}$
Sum of the exponent:

$$S_{\text{exponent}} = \frac{1/2}{1 - 1/2} = \frac{1/2}{1/2} = 1$$

Therefore:

$$S = 3^1 = 3$$

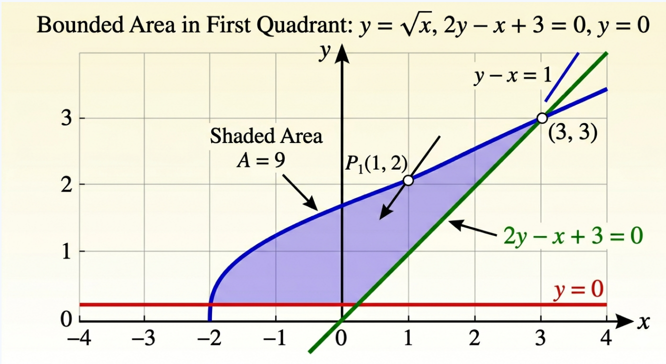
Answer: (B)

Q13.

Solution

Concept:
Area bounded by the parabola $y^2 = 4ax$ and the line $y = mx$ is given by:

$$A = \frac{8a^2}{3m^3}$$



Solution:
Given the curve $y^2 = 4x$ (so $a = 1$) and the line $y = x$ (so $m = 1$). Points of intersection:

$$x^2 = 4x \implies x(x-4) = 0 \implies x = 0, x = 4$$

For $x \in [0, 4]$, the parabola $\sqrt{4x}$ is above the line x.
The area A is:

$$A = \int_0^4 (\sqrt{4x} - x)dx = \int_0^4 (2x^{1/2} - x)dx$$

Integrating:

$$A = \left[2 \cdot \frac{x^{3/2}}{3/2} - \frac{x^2}{2}\right]_0^4 = \left[\frac{4}{3}x\sqrt{x} - \frac{x^2}{2}\right]_0^4$$

Substituting the limits:

$$A = \left(\frac{4}{3}(4)(2) - \frac{16}{2}\right) - (0) = \frac{32}{3} - 8 = \frac{32 - 24}{3} = \frac{8}{3}$$

Answer: (A)

Q14.

Solution

Concept:
The sum of coefficients in the expansion of $(a+b)^n$ is found by setting all variables (like x, y) to 1.

Solution:
We need the sum of coefficients in the expansion of $(1+x-3x^2)^{2163}$. Let $P(x) = (1+x-3x^2)^{2163}$. The sum of the coefficients is equal to $P(1)$.

Substituting $x = 1$:

$$\text{Sum} = (1+1-3(1)^2)^{2163}$$

$$\text{Sum} = (2-3)^{2163} = (-1)^{2163}$$

Since the exponent 2163 is an odd number:

$$(-1)^{2163} = -1$$

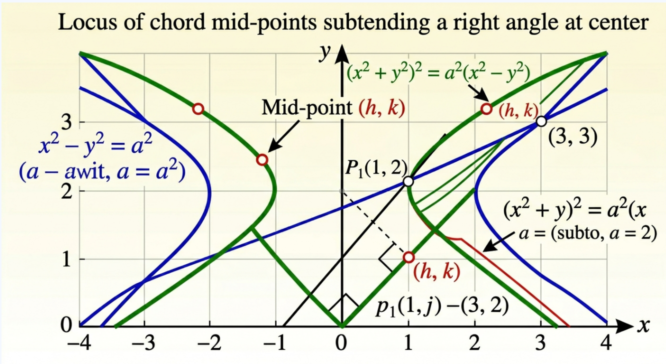
Answer: (D)

Q15.

Solution

Concept:

The eccentricity e of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given by $b^2 = a^2(1 - e^2)$ if $a > b$. If the latus rectum is half of the minor axis, we use the formula $LR = \frac{2b^2}{a}$.



Solution:

Given that the length of the latus rectum is equal to half of the minor axis:

$$\frac{2b^2}{a} = \frac{1}{2}(2b) = b$$

$$\frac{2b^2}{a} = b \implies 2b = a \quad (\text{since } b \neq 0)$$

Squaring both sides:

$$a^2 = 4b^2$$

We know the relation $b^2 = a^2(1 - e^2)$. Substitute $a^2 = 4b^2$:

$$b^2 = 4b^2(1 - e^2)$$

$$\frac{1}{4} = 1 - e^2 \implies e^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$e = \frac{\sqrt{3}}{2}$$

Answer: (B)

Q16.

Solution

Concept:
For independent events, the probability of at least one success is $1 - P(\text{none})$. In a Binomial distribution $B(n, p)$, the probability of k successes is $\binom{n}{k} p^k (1-p)^{n-k}$.

Solution:
A man fires at a target. The probability of hitting the target is $p = 1/4$, so the probability of missing is $q = 3/4$. He fires n times. We want the probability of hitting the target at least once to be greater than $2/3$.

$$P(\text{at least one hit}) = 1 - P(\text{no hits}) = 1 - q^n$$

Set up the inequality:

$$1 - \left(\frac{3}{4}\right)^n > \frac{2}{3} \implies \frac{1}{3} > \left(\frac{3}{4}\right)^n$$

Taking logs or testing integer values:

- If $n = 1$: $3/4 = 0.75 > 0.33$ (False)
- If $n = 2$: $9/16 = 0.5625 > 0.33$ (False)
- If $n = 3$: $27/64 \approx 0.421 > 0.33$ (False)
- If $n = 4$: $81/256 \approx 0.316 < 0.33$ (True)

The minimum value of n is 4.

Answer: (B)

Q17.

Solution

Concept:
The Scalar Triple Product $[\vec{a}\ \vec{b}\ \vec{c}]$ represents the volume of a parallelepiped. If the vectors are coplanar, the scalar triple product is zero. $[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})$.

Solution:
Given vectors $\vec{u} = \hat{i} + \hat{j}$, $\vec{v} = \hat{i} - \hat{j}$, and $\vec{w} = \hat{i} + 2\hat{j} + 3\hat{k}$. We find the scalar triple product $[\vec{u}\ \vec{v}\ \vec{w}]$:

$$[\vec{u}\ \vec{v}\ \vec{w}] = \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & 3 \end{vmatrix}$$

Expanding along the third column:

$$[\vec{u}\ \vec{v}\ \vec{w}] = 0 - 0 + 3 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 3(-1 - 1) = -6$$

The volume of the parallelepiped is $|[\vec{u}\ \vec{v}\ \vec{w}]| = 6$ cubic units.

Answer: (A)

Q18.

Solution

Concept:
For a square matrix A of order n, the property of the adjoint is: $|\text{adj}(A)| = |A|^{n-1}$.
Solution:
Given A is a 3×3 matrix ($n = 3$) and $|A| = 5$. We need to find the value of $|\text{adj}(A)|$.
Using the property:

$$|\text{adj}(A)| = |A|^{3-1} = |A|^2$$

Substituting the given value:

$$|\text{adj}(A)| = (5)^2 = 25$$

Answer: (C)

Q19.

Solution

Concept:
The general solution for $\tan\theta = \tan\alpha$ is $\theta = n\pi + \alpha$, where $n \in \mathbb{Z}$.
Solution:
Given the equation:

$$\tan(3x) = 1$$

Since $1 = \tan(\pi/4)$, we have:

$$3x = n\pi + \frac{\pi}{4}$$

Dividing by 3 to solve for x:

$$x = \frac{n\pi}{3} + \frac{\pi}{12}$$

For the smallest positive value, set $n = 0$:

$$x = \frac{\pi}{12}$$

Answer: (D)

Q20.

Solution

Concept:
The limit of the form 1^∞ as $x \to a$: If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$, then:

$$\lim_{x\to a}[f(x)]^{g(x)} = e^{\lim_{x\to a}[f(x)-1]g(x)}$$

Solution:
Evaluate $L = \lim_{x\to 0}(1 + 2x)^{1/x}$. Here $f(x) = 1 + 2x$ and $g(x) = 1/x$. As $x \to 0$, $f(x) \to 1$ and $g(x) \to \infty$. Applying the formula:

$$L = e^{\lim_{x\to 0}(1+2x-1)\cdot\frac{1}{x}}$$

$$L = e^{\lim_{x\to 0}\frac{2x}{x}} = e^2$$

Answer: (B)

Q21.

Solution

Concept:
The sum of inverse tangents is given by:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right), \quad \text{if } xy < 1$$

Solution:
We need to find the value of $S = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$. Here $x = 1/2$ and $y = 1/3$. Since $xy = 1/6 < 1$, we apply the formula:

$$S = \tan^{-1}\left(\frac{1/2 + 1/3}{1 - (1/2)(1/3)}\right)$$

Calculating the numerator and denominator:

$$\text{Numerator} = \frac{3 + 2}{6} = \frac{5}{6}$$

$$\text{Denominator} = 1 - \frac{1}{6} = \frac{5}{6}$$

So:

$$S = \tan^{-1}\left(\frac{5/6}{5/6}\right) = \tan^{-1}(1)$$

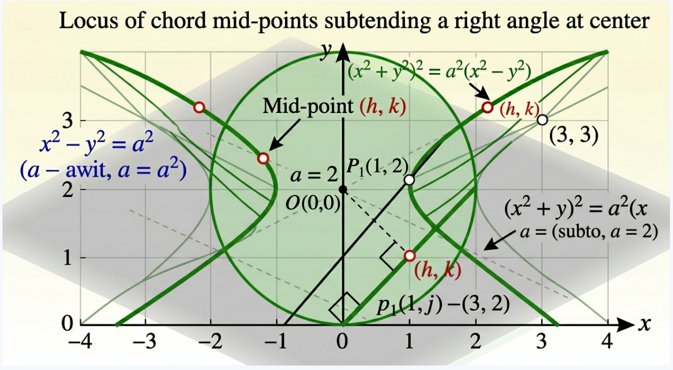
The principal value of $\tan^{-1}(1)$ is $\pi/4$.

Answer: ($\pi/4$)

Q22.

Solution

Concept:
For a circle $x^2 + y^2 + 2gx + 2fy + c = 0$, the center is $(-g, -f)$ and the radius is $r = \sqrt{g^2 + f^2 - c}$. A line $Lx + My + N = 0$ is a tangent if the perpendicular distance from the center to the line equals the radius.



Solution:
Given the circle $x^2 + y^2 - 4x - 6y - 12 = 0$. Center $C = (2, 3)$. Radius $r = \sqrt{2^2 + 3^2 - (-12)} = \sqrt{4 + 9 + 12} = \sqrt{25} = 5$.
We need to check the distance from $(2, 3)$ to the line $4x + 3y + c = 0$:

$$d = \frac{|4(2) + 3(3) + c|}{\sqrt{4^2 + 3^2}} = \frac{|8 + 9 + c|}{5} = \frac{|17 + c|}{5}$$

For the line to be a tangent, $d = r = 5$:

$$\frac{|17 + c|}{5} = 5 \implies |17 + c| = 25$$

Two cases for c: 1. $17 + c = 25 \implies c = 8$ 2. $17 + c = -25 \implies c = -42$

Answer: (8, -42)

Q23.

Solution

Concept:
Let e_1 be the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and e_2 be the eccentricity of its conjugate hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$. The relation between them is:

$$\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$$

Solution:
For the first hyperbola: $e_1^2 = 1 + \frac{b^2}{a^2} = \frac{a^2+b^2}{a^2} \implies \frac{1}{e_1^2} = \frac{a^2}{a^2+b^2}$ For the conjugate hyperbola: $e_2^2 = 1 + \frac{a^2}{b^2} = \frac{b^2+a^2}{b^2} \implies \frac{1}{e_2^2} = \frac{b^2}{a^2+b^2}$
Adding the reciprocals:

$$\frac{1}{e_1^2} + \frac{1}{e_2^2} = \frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2} = \frac{a^2+b^2}{a^2+b^2} = 1$$

Answer: (1)

Q24.

Solution

Concept:
The variance σ^2 of the first n natural numbers is given by:

$$\sigma^2 = \frac{n^2 - 1}{12}$$

Solution:
We need the variance of $\{1, 2, 3, \ldots, n\}$.
Mean $\bar{x} = \frac{n+1}{2}$.
Variance formula:

$$\sigma^2 = \frac{\sum x_i^2}{n} - (\bar{x})^2$$

Recall:

$$\sum x_i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Then

$$\sigma^2 = \frac{n(n+1)(2n+1)}{6n} - \left(\frac{n+1}{2}\right)^2$$

$$\sigma^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

Factoring out $\frac{n+1}{2}$:

$$\sigma^2 = \frac{n+1}{2}\left[\frac{2n+1}{3} - \frac{n+1}{2}\right] = \frac{n+1}{2}\left[\frac{4n+2-3n-3}{6}\right]$$

$$\sigma^2 = \frac{n+1}{2} \cdot \frac{n-1}{6} = \frac{n^2-1}{12}$$

Answer: $\left(\frac{n^2-1}{12}\right)$

Q25.

Solution

Concept:
We utilize **King's Property** (also known as the reflection property) of definite integrals:

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

Step-by-Step Solution:
Let the given integral be I:

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx \quad \text{(i)}$$

Applying King's Property by replacing x with $(\frac{\pi}{2} + 0 - x)$, we get:

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\sin(\pi/2 - x)} + \sqrt{\cos(\pi/2 - x)}}\,dx$$

Using the trigonometric complementary angle identities, $\sin(\pi/2 - x) = \cos x$ and $\cos(\pi/2 - x) = \sin x$, the integral becomes:

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx \quad \text{(ii)}$$

Adding equations (i) and (ii) vertically to simplify the integrand:

$$I + I = \int_0^{\pi/2} \left(\frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} \right) dx$$

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$$

The integrand simplifies to 1 since the numerator and denominator are identical:

$$2I = \int_0^{\pi/2} 1 \cdot dx$$

$$2I = \Big[x\Big]_0^{\pi/2}$$

$$2I = \frac{\pi}{2} - 0$$

Dividing by 2 to find the final value of I:

$$I = \frac{\pi}{4}$$

Answer: $(\pi/4)$

Answer Key — Section A

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
2	C	3	C	4	A	5	A	6	B
7	C	8	D	9	C	10	A	11	C
12	B	13	A	14	D	15	B	16	B
17	A	18	C	19	D	20	B		

Answer Key — Section B

Q	Ans	Q	Ans
1	B	21	$\pi/4$
22	8, -42	23	1
24	$\frac{n^2-1}{12}$	25	$\pi/4$