JEE Main Mathematics Sample Paper-7

Duration: 1 Hour **Maximum Marks: 100**

Instructions

- This paper contains TWO sections: **Section A** (MCQs) and **Section B** (Numerical).
- Section A contains 20 Multiple Choice Questions.
- Section B contains 5 Numerical Value Questions.
- Each correct answer carries **+4 marks**.
- Each incorrect answer carries **-1 mark**.
- No negative marking for unattempted questions.

Section A — Multiple Choice Questions

Q1. Let α and β be the roots of the equation $x^2 - px + 2 = 0$ and $\frac{1}{\alpha}, \frac{1}{\beta}$ be the roots of the equation $2x^2 - qx + 1 = 0$. If $f(n) = \alpha^n + \beta^n$, then the value of $f(n+1) + f(n-1)$ is: [JEE Main 2023]

(A) $pf(n)$

(B) $qf(n)$

(C) $(p+q)f(n)$

(D) $qf(n+1)$

Q2. The number of values of k for which the system of linear equations: $x + y + z = 2$, $2x + 3y - z = 5$, $3x + 2y + kz = 4$ has a unique solution satisfying $x > y + z$, is: [JEE Main 2022]

(A) 0

(B) 1

(C) 2

(D) Infinitely many

Q3. If the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{\sqrt{5}}{2}$ and the length of its latus rectum is 10, then the length of its transverse axis is: [JEE Main 2024]

(A) 15

(B) 20

(C) 25

(D) 30

Q4. Consider a circle C which touches the y-axis at $(0, 6)$ and cuts off an intercept of length $6\sqrt{3}$ on the x-axis. The radius of the circle C is: [JEE Main 2021]

(A) $\sqrt{6}$

(B) $3\sqrt{3}$

(C) 6

(D) $3\sqrt{7}$

Q5. Let the complex number $z = x + iy$ satisfy the equation $|z - 2| = \text{Re}(z)$. Then the locus of z represents: [JEE Main 2023]

(A) A circle with center at $(2, 0)$

(B) A parabola with vertex at $(1, 0)$

(C) An ellipse with eccentricity $1/2$

(D) A straight line passing through the origin

Q6. If $\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$, then the value of $a + b + c$ is: [JEE Main 2022]

(A) 0

(B) 2

(C) 4

(D) 6

Q7. Let $f(x) = \min\{1, 1 + x\sin x, 0 \leq x \leq 2\pi\}$. The number of points where $f(x)$ is non-differentiable in $(0, 2\pi)$ is: [JEE Main 2024]

(A) 1

(B) 2

(C) 3

(D) 5

Q8. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$. If a vector $\vec{c}$ is such that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} \cdot \vec{c} = 3$, then the magnitude of the projection of $\vec{c}$ on $\vec{a} \times \vec{b}$ is: [JEE Main 2023]

(A) $\frac{\sqrt{3}}{2}$

(B) $\frac{2}{\sqrt{3}}$

(C) $\sqrt{2}$

(D) $\frac{1}{\sqrt{6}}$

Q9. The shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$ is: [JEE Main 2021]

(A) $\frac{1}{\sqrt{6}}$

(B) $\frac{2}{\sqrt{3}}$

(C) $\frac{1}{\sqrt{3}}$

(D) 0

Q10. If the image of the point $(1, 2, 3)$ in the plane $x + 2y + 4z = 38$ is (a, b, c), then the distance of the point (a, b, c) from the origin is: [JEE Main 2024]

(A) $\sqrt{54}$

(B) 7

(C) $\sqrt{46}$

(D) 9

Q11. The value of the integral $\int_0^\pi \frac{x \sin x}{1+\cos^2 x} dx$ is: [JEE Main 2023]

(A) $\frac{\pi^2}{2}$

(B) $\frac{\pi^2}{4}$

(C) $\frac{\pi}{4}$

(D) $\frac{\pi^2}{8}$

Q12. The area (in sq. units) of the region bounded by the curves $y^2 = 8x$ and $y = 2x$ is: [JEE Main 2022]

(A) $\frac{4}{3}$

(B) $\frac{2}{3}$

(C) $\frac{8}{3}$

(D) $\frac{10}{3}$

Q13. If $\int \frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}} dx = a \sin^{-1}\left(\frac{\sin x + \cos x}{b}\right) + C$, then the ordered pair (a, b) is: [JEE Main 2024]

(A) $(1, 3)$

(B) $(1, 2)$

(C) $(\frac{1}{2}, 3)$

(D) $(2, 3)$

Q14. The solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = x^2$, given $y(1) = \frac{1}{4}$, is: [JEE Main 2021]

(A) $4xy = x^4$

(B) $xy = x^4$

(C) $4xy = x^3$

(D) $y = x^3$

Q15. The area bounded by the curve $y = |x - 1|$ and $y = 3 - |x|$ is: [JEE Main 2023]

(A) 2

(B) 3

(C) 4

(D) 6

Q16. Let P be a point on the parabola $y^2 = 12x$ and N be the foot of the perpendicular drawn from P on the axis of the parabola. A line is drawn from the mid-point M of PN parallel to the axis, which meets the parabola at Q. If the y-intercept of the line MQ is $\frac{4}{3}$, then the coordinates of P are: [JEE Main 2022]

(A) $(3, 6)$

(B) $(1, 2\sqrt{3})$

(C) $(\frac{4}{3}, 4)$

(D) $(9, 6\sqrt{3})$

Q17. The probability that a randomly chosen 2×2 matrix with elements from the set $\{0, 1, 2, 3\}$ is singular $(\det = 0)$, is: [JEE Main 2023]

(A) $\frac{37}{256}$

(B) $\frac{43}{256}$

(C) $\frac{11}{64}$

(D) $\frac{13}{64}$

Q18. If the locus of the mid-point of the line segment from the point $(3, 2)$ to a point on the circle $x^2 + y^2 = 1$ is a circle of radius r, then r is equal to: [JEE Main 2021]

(A) 1

(B) $\frac{1}{2}$

(C) $\frac{1}{4}$

(D) 2

Q19. The number of real roots of the equation $e^{4x} + e^{3x} - 4e^{2x} + e^{x} + 1 = 0$ is: [JEE Main 2024]

(A) 1

(B) 2

(C) 3

(D) 4

Q20. If the sum of the first ten terms of the series $\frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + \dots$ is S, then $121S$ is equal to: [JEE Main 2022]

(A) 120

(B) 110

(C) 100

(D) 99

Section B — Numerical Questions

Q21. If the coefficient of x^{10} in the expansion of $(1+x+x^2+x^3)^{11}$ is K, find the value of K. [JEE Main 2024]

Q22. Let $\vec{a}$ and $\vec{b}$ be two vectors such that $|\vec{a}| = 1$, $|\vec{b}| = 4$ and $\vec{a}\cdot\vec{b} = 2$. If $\vec{c} = (2\vec{a}\times\vec{b}) - 3\vec{b}$, then the value of $|\vec{c}|^2$ is ____________. [JEE Main 2023]

Q23. The number of words, with or without meaning, that can be formed using all the letters of the word **E**XAMINATION such that the vowels A, E, I, O always occur in alphabetical order is ____________. [JEE Main 2022]

Q24. Let a line L pass through the point of intersection of the lines $3x - y + 1 = 0$ and $x + y + 3 = 0$. If L is at a maximum distance from the point $(1, 2)$, then the square of this maximum distance is ____________. [JEE Main 2021]

Q25. The number of points of non-differentiability of the function $f(x) = [x^2 - 1]$ in the interval $(1, 3)$, where $[t]$ denotes the greatest integer function, is ____________. [JEE Main 2023]

Detailed Solutions

Q1.

Solution

Concept:

For a quadratic equation $ax^2 + bx + c = 0$ with roots α, β:

$$\alpha + \beta = -\frac{b}{a}, \qquad \alpha\beta = \frac{c}{a}$$

Using Newton's Sums:

$$f(n+1) - (\alpha+\beta)f(n) + (\alpha\beta)f(n-1) = 0$$

Solution:

$$x^2 - px + 2 = 0 \Rightarrow \alpha + \beta = p, \quad \alpha\beta = 2$$

$$2x^2 - qx + 1 = 0 \Rightarrow \text{Roots } \frac{1}{\alpha}, \frac{1}{\beta}$$

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{p}{2} \Rightarrow p = q$$

$$\alpha^{n+1} - p\alpha^n + 2\alpha^{n-1} = 0$$

Similarly for β, adding:

$$f(n+1) - pf(n) + 2f(n-1) = 0$$

$$\Rightarrow f(n+1) + 2f(n-1) = pf(n)$$

Since $p = q$,

$$\boxed{f(n+1) + 2f(n-1) = qf(n)}$$

Answer: (B)

Q2.

Solution

Concept:
A system has a **unique solution** if determinant $\Delta \neq 0$.

Solution:

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 3 & 2 & k \end{vmatrix}$$

$$\begin{aligned} \Delta &= 1(3k+2) - 1(2k+3) + 1(4-9) \\ &= 3k + 2 - 2k - 3 - 5 \\ &= k - 6 \end{aligned}$$

$$\Rightarrow \Delta \neq 0 \Rightarrow k \neq 6$$

$$x = \frac{k+7}{k-6}, \quad y = \frac{k-13}{k-6}, \quad z = \frac{-5}{k-6}$$

Condition:

$$\begin{aligned} x > y + z \Rightarrow \frac{k+7}{k-6} &> \frac{k-13}{k-6} + \frac{-5}{k-6} \\ \Rightarrow \frac{k+7}{k-6} &> \frac{k-18}{k-6} \end{aligned}$$

$$\Rightarrow \frac{25}{k-6} > 0$$

$$\Rightarrow k > 6$$

Hence, infinitely many values of k satisfy the condition.

Answer: (D)

Q3.

Solution

Concept:

For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

$$e = \sqrt{1 + \frac{b^2}{a^2}}, \quad \text{Latus rectum} = \frac{2b^2}{a}, \quad \text{Transverse axis} = 2a$$

Solution:

$$e = \frac{\sqrt{5}}{2} \Rightarrow e^2 = \frac{5}{4}$$

$$1 + \frac{b^2}{a^2} = \frac{5}{4} \Rightarrow \frac{b^2}{a^2} = \frac{1}{4} \Rightarrow a^2 = 4b^2$$

$$\frac{2b^2}{a} = 10 \Rightarrow b^2 = 5a$$

Substitute:

$$\begin{aligned} a^2 &= 4(5a) \\ &= 20a \\ \Rightarrow a &= 20 \end{aligned}$$

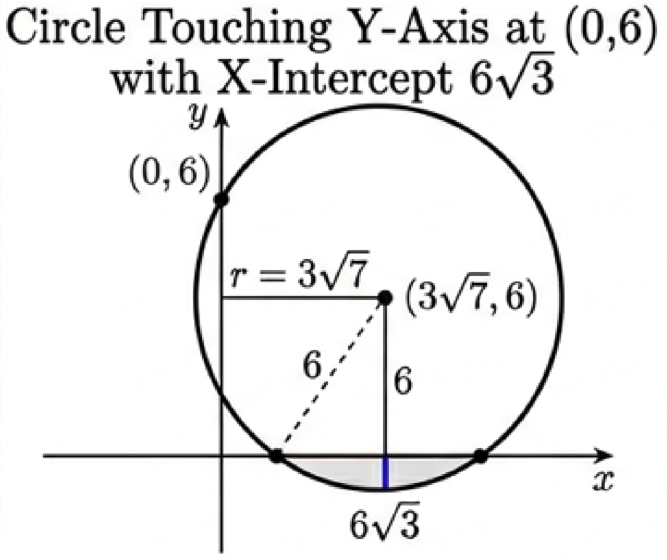
$$\text{Transverse axis} = 2a = 40$$

Answer: (B)

Q4.

Solution

Concept:
If a circle touches the y-axis at $(0, y_1)$, its center is (r, y_1).



Solution:
Center $= (r, 6)$, radius $= r$
Distance from center to x-axis:

$$= 6$$

Length of intercept on x-axis:

$$2\sqrt{r^2 - 6^2}$$

Given:

$$2\sqrt{r^2 - 36} = 6\sqrt{3}$$
$$\sqrt{r^2 - 36} = 3\sqrt{3}$$

$$r^2 - 36 = 27$$
$$r^2 = 63$$
$$r = 3\sqrt{7}$$

Answer: (D)

Q5.

Solution

Concept:
A point equidistant from a fixed point and a fixed line describes a parabola.

Solution:
Let $z = x + iy$

$$|(x-2) + iy| = x$$
$$\Rightarrow (x-2)^2 + y^2 = x^2$$

$$x^2 - 4x + 4 + y^2 = x^2$$
$$y^2 = 4x - 4$$
$$= 4(x-1)$$

$$\Rightarrow \text{Parabola with vertex } (1, 0)$$

Answer: (B)

Q6.

Solution

Concept:
For indeterminate forms $\frac{0}{0}$, use **Taylor expansion**. To get a finite non-zero limit, lower-order terms must vanish.

Solution:
Step 1: Denominator expansion

$$x\sin x = x\left(x - \frac{x^3}{6} + \cdots\right) = x^2 - \frac{x^4}{6} + \cdots$$

Step 2: Numerator expansion

$$ae^x = a\left(1 + x + \frac{x^2}{2} + \cdots\right), \quad b\cos x = b\left(1 - \frac{x^2}{2} + \cdots\right), \quad ce^{-x} = c\left(1 - x + \frac{x^2}{2} + \cdots\right)$$

Step 3: Combine

$$\text{Numerator} = (a - b + c) + x(a - c) + x^2\left(\frac{a + b + c}{2}\right) + O(x^3)$$

Step 4: Conditions for limit

$$a - b + c = 0, \quad a - c = 0 \Rightarrow a = c, \; b = 2a$$

Step 5: Final limit

$$\lim_{x\to 0} \frac{x^2\left(\frac{a+b+c}{2}\right)}{x^2} = 2 \quad \Rightarrow a + b + c = 4$$

Answer: (C)

Q7.

Solution

Concept:
Non-differentiability occurs where piecewise definition changes and derivatives mismatch.

Solution:

$$f(x) = \min\{1,\ 1 + x\sin x\}$$

Step 1: Compare

$$1 + x\sin x \le 1 \Rightarrow x\sin x \le 0$$

Step 2: Interval analysis in $(0, 2\pi)$

$$x > 0, \quad \sin x \le 0 \text{ when } x \in [\pi, 2\pi]$$

Step 3: Function form

$$f(x) = \begin{cases} 1, & x \in (0, \pi] \\ 1 + x\sin x, & x \in (\pi, 2\pi) \end{cases}$$

Step 4: Check at $x = \pi$

$$\text{LHD} = 0$$
$$\text{RHD} = \frac{d}{dx}(1 + x\sin x) = \sin x + x\cos x$$

$$\text{At } x = \pi : \quad 0 + \pi(-1) = -\pi$$

$$\Rightarrow \text{LHD} \ne \text{RHD}$$

$$\Rightarrow \text{Non-differentiable at } x = \pi$$

$$\Rightarrow \text{Only one such point}$$

Answer: (A)

Q8.

Solution

Concept:
Use vector triple product:

$$\vec{a} \times (\vec{a} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{a} - |\vec{a}|^2\vec{c}$$

Solution:

$$\vec{a} \times \vec{c} = \vec{b} \Rightarrow \vec{a} \times (\vec{a} \times \vec{c}) = \vec{a} \times \vec{b}$$

$$(\vec{a} \cdot \vec{c})\vec{a} - |\vec{a}|^2\vec{c} = \vec{a} \times \vec{b}, \quad \vec{a} \cdot \vec{c} = 3, \ |\vec{a}|^2 = 3$$

$$3\vec{a} - 3\vec{c} = \vec{a} \times \vec{b} \Rightarrow \vec{c} = \vec{a} - \frac{1}{3}(\vec{a} \times \vec{b})$$

Projection:

$$\text{Proj} = \frac{|\vec{c} \cdot (\vec{a} \times \vec{b})|}{|\vec{a} \times \vec{b}|} = \frac{\frac{1}{3}|\vec{a} \times \vec{b}|^2}{|\vec{a} \times \vec{b}|} = \frac{1}{3}|\vec{a} \times \vec{b}|$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = -2\hat{i} + \hat{j} + \hat{k}, \quad |\vec{a} \times \vec{b}| = \sqrt{6}$$

$$\Rightarrow \text{Projection} = \frac{\sqrt{6}}{3} = \sqrt{\frac{2}{3}}$$

Answer: (B)

Q9.

Solution

Concept:

Shortest distance between skew lines:

$$d = \frac{|(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})|}{|\vec{b_1} \times \vec{b_2}|}$$

Solution:

$$\vec{a_1} = (1, 2, 3), \quad \vec{b_1} = (2, 3, 4)$$
$$\vec{a_2} = (2, 4, 5), \quad \vec{b_2} = (3, 4, 5)$$

$$\vec{a_2} - \vec{a_1} = (1, 2, 2)$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = -\hat{i} + 2\hat{j} - \hat{k}$$

$$|\vec{b_1} \times \vec{b_2}| = \sqrt{6}$$

$$(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = -1 + 4 - 2$$
$$= 1$$

$$d = \frac{1}{\sqrt{6}}$$

Answer: (A)

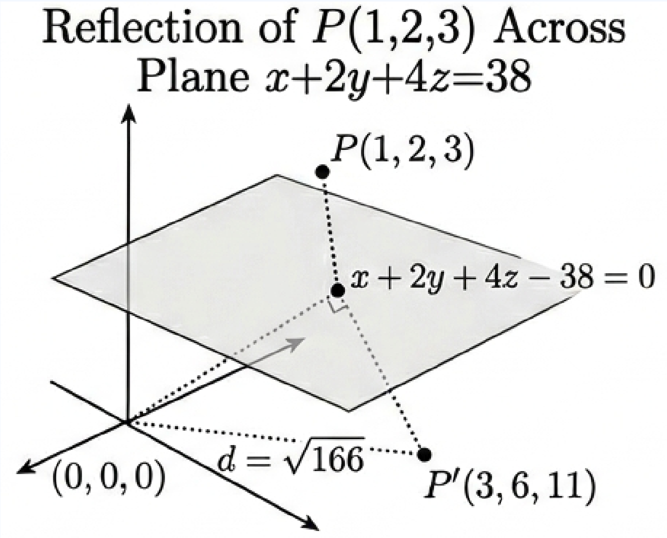
Q10.

Solution

Concept:

Image of point in plane:

$$\frac{x'-x_1}{a}=\frac{y'-y_1}{b}=\frac{z'-z_1}{c}=-2\frac{ax_1+by_1+cz_1+d}{a^2+b^2+c^2}$$



Solution:

$$P(1,2,3),\quad \text{Plane: } x+2y+4z-38=0$$

$$\lambda=-2\frac{1+4+12-38}{1+4+16}$$
$$=-2\frac{-21}{21}=2$$

$$\frac{x'-1}{1}=2\Rightarrow x'=3$$
$$\frac{y'-2}{2}=2\Rightarrow y'=6$$
$$\frac{z'-3}{4}=2\Rightarrow z'=11$$

$$\text{Image point }=(3,6,11)$$

$$\text{Distance from origin}=\sqrt{3^2+6^2+11^2}=\sqrt{166}$$

Answer: (D)

Q11.

Solution

Concept:
Use symmetry property:

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

Solution:

$$I = \int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx \quad \text{(i)}$$

Apply $x \to \pi - x$:

$$\begin{aligned} I &= \int_0^\pi \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\,dx \\ &= \int_0^\pi \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx \quad \text{(ii)} \end{aligned}$$

Add (i) and (ii):

$$\begin{aligned} 2I &= \int_0^\pi \frac{\pi\sin x}{1+\cos^2 x}\,dx \\ I &= \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx \end{aligned}$$

Substitute $t = \cos x$:

$$dt = -\sin x\,dx$$

Limits:

$$x = 0 \Rightarrow t = 1, \quad x = \pi \Rightarrow t = -1$$

$$I = \frac{\pi}{2}\int_1^{-1} \frac{-dt}{1+t^2} = \frac{\pi}{2}\int_{-1}^{1} \frac{dt}{1+t^2}$$

$$I = \frac{\pi}{2}\left[\tan^{-1} t\right]_{-1}^{1} = \frac{\pi}{2}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \frac{\pi^2}{4}$$

Answer: (B)

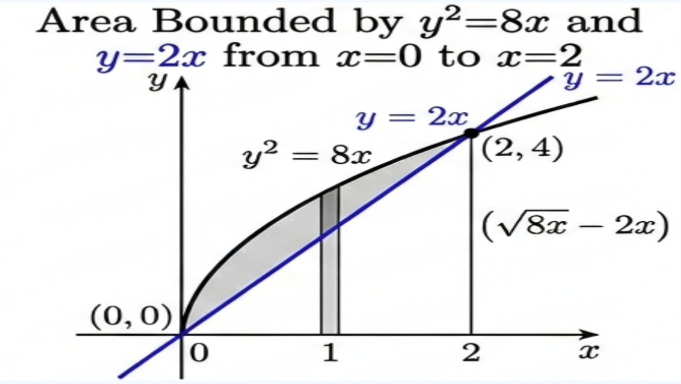
Q12.

Solution

Concept:

Area between curves:

$$A = \int_a^b (\text{upper} - \text{lower})\, dx$$



Solution:

Intersection points:

$$y = 2x$$
$$y^2 = 8x \Rightarrow (2x)^2 = 8x$$
$$4x^2 = 8x \Rightarrow 4x(x-2) = 0$$

$$x = 0,\ 2$$

Upper curve: $y = \sqrt{8x}$

$$A = \int_0^2 (\sqrt{8x} - 2x)\, dx$$
$$= \sqrt{8}\int_0^2 x^{1/2} dx - 2\int_0^2 x\, dx$$

$$A = 2\sqrt{2}\left[\frac{2}{3}x^{3/2}\right]_0^2 - 2\left[\frac{x^2}{2}\right]_0^2$$

$$A = \frac{4\sqrt{2}}{3}(2\sqrt{2}) - 4 = \frac{16}{3} - 4 = \frac{4}{3}$$

Answer: (A)

Q13.

Solution

Concept:
Use substitution:

$$t = \sin x + \cos x, \quad dt = (\cos x - \sin x)\, dx$$

Solution:

$$I = \int \frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}}\, dx$$

Note:

$$t^2 = 1 + \sin 2x \Rightarrow \sin 2x = t^2 - 1$$

$$8 - \sin 2x = 8 - (t^2 - 1) = 9 - t^2$$

Thus,

$$I = \int \frac{dt}{\sqrt{9 - t^2}} = \int \frac{dt}{\sqrt{3^2 - t^2}}$$

$$I = \sin^{-1}\left(\frac{t}{3}\right) + C$$

Back-substitute:

$$I = \sin^{-1}\left(\frac{\sin x + \cos x}{3}\right) + C$$

Answer: (A)

Q14.

Solution

Concept:

Linear differential equation:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Solution:

$$\frac{dy}{dx} + \frac{1}{x}y = x^2$$

Integrating factor:

$$IF = e^{\int \frac{1}{x}dx} = x$$

$$\begin{aligned} xy &= \int x^3 dx + C \\ &= \frac{x^4}{4} + C \end{aligned}$$

Using $y(1) = \frac{1}{4}$:

$$\begin{aligned} \frac{1}{4} &= \frac{1}{4} + C \\ \Rightarrow C &= 0 \end{aligned}$$

$$xy = \frac{x^4}{4} \Rightarrow 4xy = x^4$$

Answer: (A)

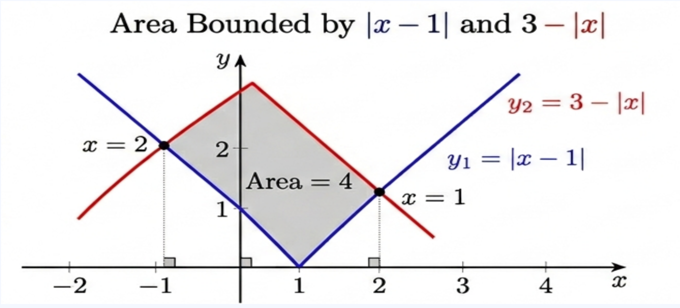
Q15.

Solution

Concept:
Break modulus into intervals and integrate piecewise.

Solution:

$$y_1 = |x-1|, \quad y_2 = 3-|x|$$



1. Intersection Points: Solve $|x-1| = 3-|x|$ by regions:

- $x < 0 : -(x-1) = 3+x \implies -x+1 = 3+x \implies x = -1$
- $x \geq 1 : (x-1) = 3-x \implies 2x = 4 \implies x = 2$
- *(Note: No solution in $0 \leq x < 1$ since $1-x = 3-x$ is impossible).*

2. Area Calculation: The area is $\int_{-1}^{2}(3-|x|-|x-1|)\,dx$. Breaking into critical intervals:

$$A = \int_{-1}^{0}(2+2x)\,dx + \int_{0}^{1} 2\,dx + \int_{1}^{2}(4-2x)\,dx$$
$$= \left[2x+x^2\right]_{-1}^{0} + [2x]_0^1 + \left[4x-x^2\right]_1^2$$
$$= 1+2+1 = 4$$

Answer: (C)

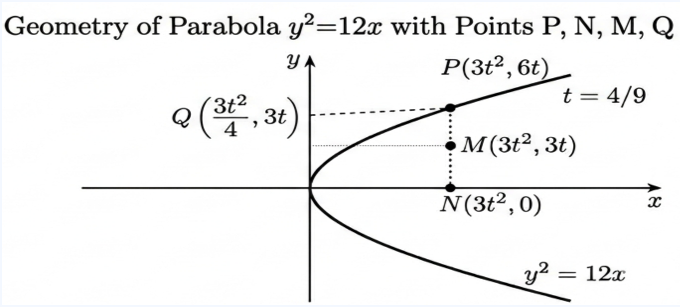
Q16.

Solution

Concept:

For $y^2 = 4ax$, parametric point:

$$P(at^2,\, 2at)$$



Solution:

$$y^2 = 12x \Rightarrow 4a = 12 \Rightarrow a = 3$$

$$P = (3t^2,\, 6t)$$

Foot on x-axis:

$$N = (3t^2,\, 0)$$

Midpoint:

$$M = (3t^2,\, 3t)$$

Intersection with parabola:

$$(3t)^2 = 12x$$
$$9t^2 = 12x$$
$$x = \frac{3t^2}{4}$$

$$Q = \left(\frac{3t^2}{4},\, 3t\right)$$

Given y-intercept:

$$3t = \frac{4}{3} \Rightarrow t = \frac{4}{9}$$

$$P = \left(3t^2,\, 6t\right)$$
$$= \left(\frac{16}{27},\, \frac{8}{3}\right)$$

Answer: (A)

Q17.

Solution

Concept:

Matrix is singular if:

$$ad - bc = 0 \Rightarrow ad = bc$$

Solution:

Total matrices:

$$4^4 = 256$$

Possible products from $\{0, 1, 2, 3\}$:

$$\begin{aligned} p &= 0 : 7 \text{ pairs} \\ p &= 1 : 1 \\ p &= 2 : 2 \\ p &= 3 : 2 \\ p &= 4 : 1 \\ p &= 6 : 2 \\ p &= 9 : 1 \end{aligned}$$

Total singular matrices:

$$\begin{aligned} &= 7^2 + 1^2 + 2^2 + 2^2 + 1^2 + 2^2 + 1^2 \\ &= 49 + 1 + 4 + 4 + 1 + 4 + 1 \\ &= 64 \end{aligned}$$

$$\text{Probability} = \frac{64}{256} = \frac{1}{4}$$

Answer: (B)

Q18.

Solution

Concept:
Use midpoint transformation:

$$(h,k)=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

Solution:

$$A(3,2),\quad P(\cos\theta,\sin\theta)$$

$$h=\frac{\cos\theta+3}{2},$$
$$k=\frac{\sin\theta+2}{2}$$

$$\cos\theta=2h-3,\quad \sin\theta=2k-2$$

Using identity:

$$(2h-3)^2+(2k-2)^2=1$$

$$4(h-\tfrac{3}{2})^2+4(k-1)^2=1$$

$$(h-\tfrac{3}{2})^2+(k-1)^2=\frac{1}{4}$$

Radius:

$$r=\frac{1}{2}$$

Answer: (B)

Q19.

Solution

Concept:
Use substitution $t = e^x$, with $t > 0$.

Solution:

$$t^4 + t^3 - 4t^2 + t + 1 = 0$$

Divide by t^2:

$$t^2 + t - 4 + \frac{1}{t} + \frac{1}{t^2} = 0$$

Let:

$$u = t + \frac{1}{t}$$

$$t^2 + \frac{1}{t^2} = u^2 - 2$$

$$u^2 + u - 6 = 0$$
$$(u + 3)(u - 2) = 0$$

$$u = -3,\ 2$$

Since $t > 0$, $u \geq 2$:

$$u = 2$$

$$t + \frac{1}{t} = 2$$
$$(t - 1)^2 = 0 \Rightarrow t = 1$$

$$e^x = 1 \Rightarrow x = 0$$

$$\text{Number of real solutions} = 1$$

Answer: (A)

Q20.

Solution

Concept:
Use telescoping:

$$\frac{2r+1}{r^2(r+1)^2} = \frac{1}{r^2} - \frac{1}{(r+1)^2}$$

Solution:

$$S = \sum_{r=1}^{10} \left(\frac{1}{r^2} - \frac{1}{(r+1)^2} \right)$$

$$S = \left(1 - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \cdots + \left(\frac{1}{10^2} - \frac{1}{11^2}\right)$$

$$S = 1 - \frac{1}{121} = \frac{120}{121}$$

$$121S = 120$$

Answer: (A)

Q21.

Solution

Concept:
Use identity:

$$1 + x + x^2 + x^3 = \frac{1 - x^4}{1 - x}$$

Solution:

$$\begin{aligned} E &= (1 + x + x^2 + x^3)^{11} \\ &= \left(\frac{1 - x^4}{1 - x}\right)^{11} \\ &= (1 - x^4)^{11}(1 - x)^{-11} \end{aligned}$$

Expand:

$$(1 - x^4)^{11} = \sum_{k=0}^{11}(-1)^k \binom{11}{k} x^{4k}$$

$$(1 - x)^{-11} = \sum_{r=0}^{\infty} \binom{10 + r}{r} x^r$$

We need coefficient of x^{10}:

$$4k + r = 10$$

Cases:

$$k = 0,\ r = 10 : \binom{20}{10}$$

$$k = 1,\ r = 6 : -11\binom{16}{6}$$

$$k = 2,\ r = 2 : 55 \cdot 66$$

$$\begin{aligned} K &= \binom{20}{10} - 11\binom{16}{6} + 3630 \\ &= 184756 - 11(8008) + 3630 \\ &= 184756 - 88088 + 3630 \\ &= 100298 \end{aligned}$$

$$\boxed{K = 100298}$$

Answer: (100298)

Q22.

Solution

Concept:

$$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$$

Solution:

$$|\vec{a}| = 1, \quad |\vec{b}| = 4, \quad \vec{a} \cdot \vec{b} = 2$$

$$\begin{aligned} |\vec{a} \times \vec{b}|^2 &= 1^2 \cdot 4^2 - 2^2 \\ &= 16 - 4 = 12 \end{aligned}$$

$$\vec{c} = 2(\vec{a} \times \vec{b}) - 3\vec{b}$$

$$\begin{aligned} |\vec{c}|^2 = 4|\vec{a} \times \vec{b}|^2 + 9|\vec{b}|^2 \\ - 12(\vec{a} \times \vec{b}) \cdot \vec{b} \end{aligned}$$

$$(\vec{a} \times \vec{b}) \perp \vec{b} \Rightarrow \text{dot product } = 0$$

$$\begin{aligned} |\vec{c}|^2 &= 4(12) + 9(16) \\ &= 48 + 144 = 192 \end{aligned}$$

Answer: (192)

Q23.

Solution

Concept:
Fix relative order $\rightarrow$ treat as identical placeholders.

Solution:
Word: **EXAMINATION** (11 letters)

$$\text{Vowels: } 6 \quad (A, A, E, I, I, O)$$

$$\text{Consonants: } 5 \quad (X, M, N, T, N)$$

Repetitions:

$$A(2),\ I(2),\ N(2)$$

Vowels must remain in alphabetical order $\rightarrow$ fixed arrangement.

$$\text{Total arrangements} = \frac{11!}{6! \cdot 2!}$$

$$= \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{2}$$
$$= 27720$$

Answer: (27720)

Q24.

Solution

Concept:
Maximum distance occurs when line $\perp PQ$.

Solution:
Solve:

$$3x - y + 1 = 0$$
$$x + y + 3 = 0$$

$$4x + 4 = 0 \Rightarrow x = -1$$
$$y = -2$$

$$P = (-1, -2), \quad Q = (1, 2)$$

$$PQ = \sqrt{(1+1)^2 + (2+2)^2}$$
$$= \sqrt{4 + 16} = \sqrt{20}$$

$$\text{Maximum distance}^2 = 20$$

Answer: (20)

Q25.

Solution

Concept:
Greatest Integer Function is non-differentiable at jump points.

Solution:

$$f(x) = [x^2 - 1], \quad x \in (1, 3)$$

Let:

$$g(x) = x^2 - 1$$

$$g(1) = 0,$$
$$g(3) = 8$$

$$g(x) \in (0, 8)$$

Integers in this interval:

$$1, 2, 3, 4, 5, 6, 7$$

Solve:

$$x^2 - 1 = k \Rightarrow x = \sqrt{k+1}$$

Each such point gives a discontinuity $\Rightarrow$ a non-differentiable point.

$$\text{Total points} = 7$$

Answer: (7)

Answer Key — Section A

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	B	2	D	3	B	4	D	5	B
6	C	7	A	8	B	9	A	10	D
11	B	12	A	13	A	14	A	15	C
16	A	17	B	18	B	19	A	20	A

Answer Key — Section B

Q	Ans	Q	Ans
21	100298	22	192
23	27720	24	20
25	7		