MHT-CET Mathematics Sample Paper-10

Duration: 90 Minutes **Maximum Marks: 100**

Instructions

- This paper contains a total of **50** Multiple Choice Questions.
- Each correct answer carries **+2 marks**.
- No negative marking for incorrect questions.
- Use of mobile phones, smartwatches, or any electronic gadgets is strictly prohibited.
- No marks will be deducted for questions that are left unattempted.

Q1. Let $f(x) = |x-1| + |x-2|$. If the derivative of the function at $x = 1.5$ is k, then the value of $k^2 + 1$ is:

(A) 1

(B) 2

(C) 5

(D) 0

Q2. A stone is dropped into a quiet lake and waves move in circles at a speed of 5 cm/s. At the instant when the radius of the circular wave is 12 cm, the rate at which the enclosed area is increasing is:

(A) 120π cm^2/s

(B) 60π cm^2/s

(C) 100π cm^2/s

(D) 144π cm^2/s

Q3. The maximum value of the function $f(x) = x^2e^{-2x}$ for $x > 0$ occurs at x equal to:

(A) 1

(B) $1/e$

(C) 2

(D) $1/2$

Q4. Using differentials, the approximate value of $\sqrt{101}$ is:

(A) 10.050

(B) 10.010

(C) 10.100

(D) 10.005

Q5. The function $f(x) = x^3 - 6x^2 + 15x + 10$ is strictly increasing in the interval:

(A) $(-\infty, \infty)$

(B) $(2, 3)$

(C) $(0, \infty)$

(D) $(-\infty, 0)$

Q6. The integral $\int \frac{1}{x(x^4+1)} dx$ is equal to:

(A) $\frac{1}{4} \ln |\frac{x^4}{x^4+1}| + C$

(B) $\ln |\frac{x^4}{x^4+1}| + C$

(C) $\frac{1}{4} \ln |\frac{x^4+1}{x^4}| + C$

(D) $4 \ln |\frac{x^4}{x^4+1}| + C$

Q7. The value of the definite integral $\int_0^{\pi/2} \frac{\sin^{10} x}{\sin^{10} x + \cos^{10} x} dx$ is:

(A) π

(B) $\pi/2$

(C) $\pi/4$

(D) 0

Q8. The integral $\int e^x (\cot x + \ln \sin x) dx$ evaluates to:

(A) $e^x \cot x + C$

(B) $e^x \ln \sin x + C$

(C) $e^x(\cot x + \ln \sin x) + C$

(D) $-e^x \cot x + C$

Q9. The area bounded by the parabola $y^2 = 8x$ and its latus rectum is:

(A) 16/3 sq. units

(B) 32/3 sq. units

(C) 8/3 sq. units

(D) 64/3 sq. units

Q10. The value of the integral $\int_{-2}^{2} |x + 1| dx$ is:

(A) 4

(B) 5

(C) 2

(D) 1

Q11. The area of the region bounded by the curve $y = \cos x$ between $x = 0$ and $x = \pi/2$ is:

(A) 1 sq. unit

(B) 2 sq. units

(C) 0.5 sq. unit

(D) π sq. units

Q12. The area enclosed between the two parabolas $y^2 = 4x$ and $x^2 = 4y$ is:

(A) 16/3 sq. units

(B) 8/3 sq. units

(C) 4/3 sq. units

(D) 32/3 sq. units

Q13. The order and degree of the differential equation $\left[1+\left(\frac{dy}{dx}\right)^3\right]^{5/3}=\frac{d^2y}{dx^2}$ are respectively:

(A) 2, 3

(B) 2, 5

(C) 3, 2

(D) 2, 2

Q14. The general solution of the differential equation $\frac{dy}{dx}+y\cot x=2\cos x$ is:

(A) $y\sin x=\sin^2 x+C$

(B) $y\cos x=\sin^2 x+C$

(C) $y\sin x=-\cos 2x+C$

(D) $y=\sin x+C\cos x$

Q15. The solution of the differential equation $\frac{dy}{dx}=\frac{y}{x}+\tan\left(\frac{y}{x}\right)$ is:

(A) $\sin(y/x)=Cx$

(B) $\cos(y/x)=Cx$

(C) $\tan(y/x)=Cx$

(D) $\sin(y/x)=C/x$

Q16. If the complex number z is given by $z=\frac{2+i}{3-2i}$, then the modulus of z, denoted by $|z|$, is equal to:

(A) $\sqrt{\frac{5}{13}}$

(B) $\frac{5}{13}$

(C) 1

(D) $\sqrt{\frac{13}{5}}$

Q17. The value of the expression $(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})^{12}$ is:

(A) 1

(B) -1

(C) i

(D) $-i$

Q18. The principal argument of the complex number $z = -1 - i\sqrt{3}$ is:

(A) $2\pi/3$

(B) $-2\pi/3$

(C) $4\pi/3$

(D) $-\pi/3$

Q19. If the roots of the quadratic equation $ax^2 + bx + c = 0$ are in the ratio $m : n$, then which of the following relations holds true?

(A) $mb^2 = ac(m + n)^2$

(B) $mnb^2 = ac(m + n)^2$

(C) $mnc^2 = ab(m + n)^2$

(D) $(m + n)b^2 = mnac$

Q20. The value of k for which the quadratic equation $x^2 - (k - 3)x + k = 0$ has equal roots is:

(A) 1 or 9

(B) 3 or 5

(C) 2 or 6

(D) 4 or 8

Q21. If one root of the quadratic equation $ax^2 + bx + c = 0$ is the square of the other root, then the condition is:

(A) $b^3 + ac^2 + a^2c = 3abc$

(B) $b^3 + a^2c + ac^2 = 0$

(C) $a^2c + ac^2 - b^3 = 3abc$

(D) $b^3 + a^2c + ac^2 = 3abc$

Q22. The sum to infinity of the arithmetic-geometric series $1 + \frac{2}{5} + \frac{3}{25} + \frac{4}{125} + \dots$ is:

(A) $25/16$

(B) $5/4$

(C) $25/4$

(D) $16/25$

Q23. The n-th term of the sequence $2, 5, 10, 17, 26, \dots$ is given by:

(A) $n^2 + 1$

(B) $2n + 1$

(C) $n^2 - 1$

(D) $n^2 + n$

Q24. If the Arithmetic Mean (A.M.) and Geometric Mean (G.M.) of two positive numbers are 25 and 20 respectively, then the numbers are:

(A) $40, 10$

(B) $30, 20$

(C) $45, 5$

(D) $25, 25$

Q25. In the binomial expansion of $(2x - \frac{1}{3x^2})^9$, the term that is independent of x is:

(A) T_3

(B) T_4

(C) T_5

(D) T_6

Q26. The coefficient of x^7 in the expansion of $(1 + x)^{11}$ is:

(A) 330

(B) 462

(C) 792

(D) 210

Q27. The number of distinct ways in which 6 people can be seated around a circular table is:

(A) 720

(B) 120

(C) 240

(D) 60

Q28. A bag contains 5 red and 4 black balls. If 3 balls are drawn at random, the probability that at least one of the balls is red is:

(A) $20/21$

(B) $1/21$

(C) $5/21$

(D) $19/21$

Q29. Two unbiased dice are thrown simultaneously. The probability that the sum of the numbers appearing on the dice is a prime number is:

(A) $5/12$

(B) $7/18$

(C) $1/2$

(D) $13/36$

Q30. The equation of a straight line passing through the point $(1, -2)$ and perpendicular to the line $3x + 4y + 5 = 0$ is:

(A) $4x - 3y - 10 = 0$

(B) $4x + 3y + 2 = 0$

(C) $3x - 4y - 11 = 0$

(D) $4x - 3y + 10 = 0$

Q31. The distance between the parallel lines $5x + 12y - 7 = 0$ and $5x + 12y + 19 = 0$ is:

(A) 2 units

(B) $26/13$ units

(C) 1 unit

(D) $12/13$ units

Q32. The center and the radius of the circle $x^2 + y^2 - 8x + 10y - 12 = 0$ are respectively:

(A) $(4, -5)$ and $\sqrt{53}$

(B) $(-4, 5)$ and $\sqrt{53}$

(C) $(4, -5)$ and $\sqrt{43}$

(D) $(8, -10)$ and 12

Q33. The length of the tangent drawn from the point $(5, 4)$ to the circle $x^2 + y^2 = 25$ is:

(A) 3

(B) 4

(C) 5

(D) 16

Q34. The eccentricity of the hyperbola $16x^2 - 9y^2 = 144$ is:

(A) $4/3$

(B) $5/3$

(C) $5/4$

(D) $\sqrt{7}/3$

Q35. The length of the latus rectum of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ is:

(A) $9/2$

(B) $3/2$

(C) $18/4$

(D) 4.5

Q36. The coordinates of the focus of the parabola $y^2 = -20x$ are:

(A) $(5, 0)$

(B) $(0, -5)$

(C) $(-5, 0)$

(D) $(0, 5)$

Q37. If $\vec{a} = 3\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} + \hat{k}$, then the vector $\vec{a} \times \vec{b}$ is:

(A) $-7\hat{i} + \hat{j} + 11\hat{k}$

(B) $7\hat{i} - \hat{j} - 11\hat{k}$

(C) $-7\hat{i} + \hat{j} - 11\hat{k}$

(D) $7\hat{i} + \hat{j} + 11\hat{k}$

Q38. The projection of the vector $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $\vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}$ is:

(A) $60/\sqrt{114}$

(B) $50/\sqrt{114}$

(C) $\sqrt{114}/60$

(D) $60/114$

Q39. The direction cosines of the line joining the points $(1, 0, 0)$ and $(0, 1, 1)$ are:

(A) $-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$

(B) $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$

(C) $1, 1, 1$

(D) $-1, 1, 1$

Q40. The distance of the point $(2, 3, 4)$ from the plane $3x - 6y + 2z + 11 = 0$ is:

(A) 1 unit

(B) 2 units

(C) 3 units

(D) 0 units

Q41. The angle θ between the planes $x + y + 2z = 9$ and $2x - y + z = 15$ is:

(A) $\pi/3$

(B) $\pi/4$

(C) $\pi/6$

(D) $\pi/2$

Q42. If $\sin\alpha + \sin\beta = a$ and $\cos\alpha + \cos\beta = b$, then the value of $\tan\left(\frac{\alpha-\beta}{2}\right)$ in terms of a and b is:

(A) $\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$

(B) $\frac{a}{b}$

(C) $\frac{b}{a}$

(D) $\sqrt{\frac{a^2+b^2}{4-a^2-b^2}}$

Q43. The value of $\tan 15° + \cot 15°$ is equal to:

(A) 2

(B) 4

(C) $\sqrt{3}$

(D) $2\sqrt{3}$

Q44. The value of the product $\cos 20° \cos 40° \cos 80°$ is:

(A) $1/2$

(B) $1/4$

(C) $1/8$

(D) $1/16$

Q45. The number of solutions for x in the interval $[0, 2\pi]$ for the equation $2\sin^2 x + \sin^2 2x = 2$ is:

(A) 2

(B) 4

(C) 6

(D) 8

Q46. The derivative of $e^{\sin x}$ with respect to $\cos x$ is:

(A) $-e^{\sin x}\cot x$

(B) $e^{\sin x}\tan x$

(C) $-e^{\sin x}\tan x$

(D) $e^{\sin x}\cot x$

Q47. If $y = \sqrt{\sin x + \sqrt{\sin x + \ldots\infty}}$, then the value of $\frac{dy}{dx}$ is:

(A) $\frac{\cos x}{2y-1}$

(B) $\frac{\sin x}{2y-1}$

(C) $\frac{\cos x}{2y+1}$

(D) $\frac{-\cos x}{2y-1}$

Q48. The value of the limit $\lim_{x\to 0}\frac{1-\cos 4x}{x^2}$ is:

(A) 4

(B) 8

(C) 16

(D) 2

Q49. The point on the curve $y = x^2$ where the tangent is parallel to the line $y = 4x - 5$ is:

(A) $(2, 4)$

(B) $(1, 1)$

(C) $(0, 0)$

(D) $(-2, 4)$

Q50. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} = 0$ and the angle between $\vec{b}$ and $\vec{c}$ is $\pi/6$, then $\vec{a}$ is equal to:

(A) $\pm 2(\vec{b} \times \vec{c})$

(B) $\pm(\vec{b} \times \vec{c})$

(C) $\pm\frac{1}{2}(\vec{b} \times \vec{c})$

(D) $\vec{b} + \vec{c}$

Detailed Solutions

Q1.

Solution

Concept:
For a function involving absolute values like $f(x) = |x - 1| + |x - 2|$, the derivative at a point is found by determining the sign of the expressions inside the absolute value within the local neighborhood of that point. Once the signs are fixed, the function becomes a simple polynomial, allowing direct differentiation.

Solution:

(a) We are given $f(x) = |x - 1| + |x - 2|$ and we need the derivative at $x = 1.5$.

(b) At $x = 1.5$, the expression $(x - 1)$ is $1.5 - 1 = 0.5$, which is positive. Thus, $|x - 1| = (x - 1)$.

(c) At $x = 1.5$, the expression $(x - 2)$ is $1.5 - 2 = -0.5$, which is negative. Thus, $|x - 2| = -(x - 2) = 2 - x$.

(d) Substituting these into the function for the interval $1 < x < 2$: $f(x) = (x - 1) + (2 - x) = 1$.

(e) The derivative of a constant function $f(x) = 1$ is $f'(x) = 0$.

(f) Therefore, $k = 0$.

(g) The question asks for $k^2 + 1$, which is $0^2 + 1 = 1$.

Final Answer: The value is 1.

Answer: (A)

Go Back to Question 1

Q2.

Solution

Concept:
In related rates problems, the rate of change of one quantity is found using the rate of change of another related quantity via differentiation. For a circle, the area A is related to the radius r by $A = \pi r^2$. Differentiating with respect to time t gives $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$.
Solution:

(a) We are given the rate of increase of the radius: $\frac{dr}{dt} = 5$ cm/s.

(b) We need to find the rate of increase of the area $\frac{dA}{dt}$ when $r = 12$ cm.

(c) The formula for the area of a circle is $A = \pi r^2$.

(d) Differentiating both sides with respect to time t: $\frac{dA}{dt} = \frac{d}{dt}(\pi r^2) = 2\pi r \frac{dr}{dt}$.

(e) Substitute the known values $r = 12$ and $\frac{dr}{dt} = 5$: $\frac{dA}{dt} = 2\pi(12)(5)$.

(f) $\frac{dA}{dt} = 120\pi$ cm^2/s.

Final Answer: The rate of increase is 120π cm^2/s.

Answer: (A)

Go Back to Question 2

Q3.

Solution

Concept:
To find the maximum value of a function $f(x)$, we calculate its first derivative $f'(x)$ and set it to zero to find critical points. We then use the second derivative test or observe the sign change of $f'(x)$ to confirm the maximum.
Solution:

(a) Given $f(x) = x^2e^{-2x}$.

(b) Using the product rule $\frac{d}{dx}(uv) = u'v + uv'$, we find $f'(x)$: $f'(x) = (2x)e^{-2x} + x^2(e^{-2x} \cdot (-2))$.

(c) Factor out $2xe^{-2x}$: $f'(x) = 2xe^{-2x}(1 - x)$.

(d) For critical points, set $f'(x) = 0$. Since $e^{-2x} \neq 0$ and $x > 0$, we have $1 - x = 0$, so $x = 1$.

(e) For $x < 1$, $f'(x) > 0$ (increasing); for $x > 1$, $f'(x) < 0$ (decreasing).

(f) Thus, a local maximum occurs at $x = 1$.

Final Answer: The maximum occurs at $x = 1$.

Answer: (A)

Go Back to Question 3

Q4.

Solution

Concept:
Differentials allow us to approximate the change in a function $y = f(x)$ using $\Delta y \approx f'(x)\Delta x$. For square roots, we choose a perfect square x close to the value and let Δx be the small difference.
Solution:

(a) Let $f(x) = \sqrt{x}$. We want to approximate $\sqrt{101}$.

(b) Choose $x = 100$ (a perfect square) and $\Delta x = 1$.

(c) The derivative is $f'(x) = \frac{1}{2\sqrt{x}}$.

(d) At $x = 100$, $f(100) = 10$ and $f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20} = 0.05$.

(e) Using the approximation formula: $f(x + \Delta x) \approx f(x) + f'(x)\Delta x$.

(f) $\sqrt{101} \approx 10 + (0.05)(1) = 10.05$.

Final Answer: The approximate value is 10.050.

Answer: (A)

Go Back to Question 4

Q5.

Solution

Concept:
A function is strictly increasing in an interval if its first derivative $f'(x)$ is strictly greater than zero for all x in that interval. For quadratic derivatives, we analyze the discriminant to determine if the function is always increasing.

Solution:

(a) Given $f(x) = x^3 - 6x^2 + 15x + 10$.

(b) Differentiating with respect to x: $f'(x) = 3x^2 - 12x + 15$.

(c) To check if $f'(x) > 0$, we look at the discriminant $D = b^2 - 4ac$ of the quadratic $3x^2 - 12x + 15$.

(d) $D = (-12)^2 - 4(3)(15) = 144 - 180 = -36$.

(e) Since $D < 0$ and the leading coefficient $a = 3$ is positive, the quadratic $f'(x)$ is always positive for all real x.

(f) Thus, the function is strictly increasing on the entire real line $(-\infty, \infty)$.

Final Answer: The interval is $(-\infty, \infty)$.

Answer: (A)

Go Back to Question 5

Q6.

Solution

Concept:

To evaluate the integral $\int \frac{1}{x(x^4+1)}dx$, we use a substitution method to simplify the denominator. By multiplying the numerator and denominator by x^3, we can set $t = x^4$, which transforms the integrand into a form suitable for partial fraction decomposition or standard logarithmic integration.

Solution:

(a) Let $I = \int \frac{1}{x(x^4+1)}dx$.

(b) Multiply the numerator and denominator by x^3: $I = \int \frac{x^3}{x^4(x^4+1)}dx$.

(c) Let $x^4 = t$. Then $4x^3dx = dt$, which means $x^3dx = \frac{1}{4}dt$.

(d) Substitute these into the integral: $I = \int \frac{1/4}{t(t+1)}dt = \frac{1}{4}\int \left(\frac{1}{t} - \frac{1}{t+1}\right) dt$.

(e) Integrating term by term: $I = \frac{1}{4}[\ln|t| - \ln|t+1|] + C$.

(f) Using logarithmic properties: $I = \frac{1}{4}\ln|\frac{t}{t+1}| + C$.

(g) Substituting back $t = x^4$: $I = \frac{1}{4}\ln|\frac{x^4}{x^4+1}| + C$.

Final Answer: The integral is $\frac{1}{4}\ln|\frac{x^4}{x^4+1}| + C$.

Answer: (A)

Go Back to Question 6

Q7.

Solution

Concept:
This integral is a classic application of the property $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$. This property is especially powerful for symmetric denominators in trigonometric integrals, as adding the original and the transformed integral often leads to a constant integrand.

Solution:

(a) Let $I = \int_0^{\pi/2} \frac{\sin^{10} x}{\sin^{10} x + \cos^{10} x} dx$.

(b) Using the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$: $I = \int_0^{\pi/2} \frac{\sin^{10}(\pi/2 - x)}{\sin^{10}(\pi/2 - x) + \cos^{10}(\pi/2 - x)} dx$.

(c) Since $\sin(\pi/2 - x) = \cos x$ and $\cos(\pi/2 - x) = \sin x$: $I = \int_0^{\pi/2} \frac{\cos^{10} x}{\cos^{10} x + \sin^{10} x} dx$.

(d) Adding the two expressions for I: $2I = \int_0^{\pi/2} \frac{\sin^{10} x + \cos^{10} x}{\sin^{10} x + \cos^{10} x} dx = \int_0^{\pi/2} 1 dx$.

(e) $2I = [x]_0^{\pi/2} = \pi/2$.

(f) Therefore, $I = \pi/4$.

Final Answer: The value is $\pi/4$.

Answer: (C)

Go Back to Question 7

Q8.

Solution

Concept:
This problem utilizes the standard integral form $\int e^x[f(x)+f'(x)]dx = e^x f(x)+C$. We identify a function $f(x)$ such that its derivative $f'(x)$ is also present within the parentheses multiplied by e^x.

Solution:

(a) The given integral is $\int e^x(\cot x + \ln \sin x)dx$.

(b) Let $f(x) = \ln \sin x$.

(c) Differentiating $f(x)$ with respect to x: $f'(x) = \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) = \frac{\cos x}{\sin x} = \cot x$.

(d) Now the integral is in the form $\int e^x[f'(x) + f(x)]dx$.

(e) According to the theorem, $\int e^x[f(x) + f'(x)]dx = e^x f(x) + C$.

(f) Substituting $f(x) = \ln \sin x$, we get $e^x \ln \sin x + C$.

Final Answer: The integral is $e^x \ln \sin x + C$.

Answer: (B)

Go Back to Question 8

Q9.

Solution

Concept:
The area of a region bounded by a parabola $y^2 = 4ax$ and its latus rectum $(x = a)$ is calculated by integrating the function $y = \sqrt{4ax}$ from $x = 0$ to $x = a$ and doubling the result to account for the symmetry across the x-axis.

Solution:

(a) Given parabola: $y^2 = 8x$. Comparing with $y^2 = 4ax$, we get $4a = 8$, so $a = 2$.

(b) The latus rectum is the line $x = a = 2$.

(c) The area is given by $A = 2\int_0^2 y dx$.

(d) $A = 2\int_0^2 \sqrt{8x}dx = 2\sqrt{8}\int_0^2 x^{1/2}dx$.

(e) $A = 2(2\sqrt{2})[\frac{x^{3/2}}{3/2}]_0^2 = 4\sqrt{2} \cdot \frac{2}{3}[2^{3/2} - 0]$.

(f) $A = \frac{8\sqrt{2}}{3} \cdot (2\sqrt{2}) = \frac{8 \cdot 2 \cdot 2}{3} = 32/3$ sq. units.

Final Answer: The area is $32/3$ sq. units.

Answer: (B)

Go Back to Question 9

Q10.

Solution

Concept:
To solve an integral involving an absolute value $|g(x)|$, we must split the integration interval at the points where $g(x) = 0$. This allows us to remove the absolute value signs by applying the correct sign (positive or negative) based on the sub-interval.
Solution:

(a) The integral is $\int_{-2}^{2} |x + 1| dx$.

(b) The expression inside the absolute value, $x + 1$, is zero at $x = -1$.

(c) For $x \in [-2, -1]$, $(x + 1) \leq 0$, so $|x + 1| = -(x + 1)$.

(d) For $x \in [-1, 2]$, $(x + 1) \geq 0$, so $|x + 1| = (x + 1)$.

(e) Split the integral: $\int_{-2}^{-1} -(x + 1)dx + \int_{-1}^{2} (x + 1)dx$.

(f) First part: $[-\frac{x^2}{2} - x]_{-2}^{-1} = [(-1/2 + 1) - (-2 + 2)] = 1/2$.

(g) Second part: $[\frac{x^2}{2} + x]_{-1}^{2} = [(2 + 2) - (1/2 - 1)] = 4 + 1/2 = 9/2$.

(h) Total integral: $1/2 + 9/2 = 10/2 = 5$.

Final Answer: The value is 5.

Answer: (B)

Go Back to Question 10

Q11.

Solution

Concept:
The area under a curve is a fundamental application of definite integration. For a function $y = f(x)$, the area bounded by the curve, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by the integral of $f(x)$ from a to b. In the case of the cosine function, we must be mindful of the interval provided; between 0 and $\pi/2$, the cosine function remains non-negative, meaning the integral directly represents the physical area without requiring absolute value splits.

Solution:

(a) We are tasked with finding the area of the region bounded by the trigonometric curve $y = \cos x$.

(b) The boundaries for the independent variable x are specified as $x = 0$ (the y-axis) and $x = \pi/2$.

(c) The area A is defined by the definite integral: $A = \int_0^{\pi/2} \cos x dx$.

(d) To evaluate this, we find the antiderivative of $\cos x$, which is $\sin x$.

(e) Applying the Fundamental Theorem of Calculus, we evaluate the antiderivative at the upper and lower limits: $A = [\sin x]_0^{\pi/2}$.

(f) Substituting the upper limit: $\sin(\pi/2) = 1$.

(g) Substituting the lower limit: $\sin(0) = 0$.

(h) Subtracting the lower limit value from the upper limit value: $A = 1 - 0 = 1$.

(i) Since area is a physical quantity, we express it in square units.

(j) The calculation shows that exactly one unit of area is contained under the first quarter-period of the cosine wave.

Final Answer: The area is 1 sq. unit.

Answer: (A)

Go Back to Question 11

Q12.

Solution

Concept:
When calculating the area enclosed between two intersecting curves, we first determine their points of intersection by solving their equations simultaneously. Once the intersection points (x_1, y_1) and (x_2, y_2) are found, the area is calculated by integrating the difference between the upper curve and the lower curve. For two parabolas of the form $y^2 = 4ax$ and $x^2 = 4by$, there is a standard shortcut formula, but the rigorous approach involves setting up the integral of the square root function minus the quadratic function.

Solution:

(a) The two given parabolas are $y^2 = 4x$ and $x^2 = 4y$.

(b) To find the intersection points, substitute $y = x^2/4$ from the second equation into the first: $(x^2/4)^2 = 4x \implies x^4/16 = 4x \implies x^4 = 64x$.

(c) Rearranging gives $x(x^3 - 64) = 0$, so the intersection points occur at $x = 0$ and $x = 4$.

(d) In the interval $[0, 4]$, the curve $y = \sqrt{4x} = 2\sqrt{x}$ lies above the curve $y = x^2/4$.

(e) The area A is given by the integral: $A = \int_0^4 (2\sqrt{x} - x^2/4)dx$.

(f) Integrate each term separately: $\int 2x^{1/2}dx = 2 \cdot (x^{3/2}/(3/2)) = \frac{4}{3}x^{3/2}$. $\int (x^2/4)dx = \frac{1}{4} \cdot (x^3/3) = \frac{x^3}{12}$.

(g) Evaluate from 0 to 4: $A = [\frac{4}{3}(4)^{3/2} - \frac{4^3}{12}] - [0]$.

(h) Since $4^{3/2} = 8$, we have $A = \frac{4\cdot 8}{3} - \frac{64}{12} = \frac{32}{3} - \frac{16}{3}$.

(i) $A = 16/3$ sq. units.

Final Answer: The area is $16/3$ sq. units.

Answer: (A)

Go Back to Question 12

Q13.

Solution

Concept:
The order of a differential equation is defined as the order of the highest derivative present in the equation. The degree of a differential equation is the power to which the highest order derivative is raised, provided the equation is expressed as a polynomial in its derivatives. This requires removing any fractional exponents or radical signs involving the derivatives. In this problem, we must rationalize the equation by raising both sides to a suitable power to eliminate the fractional exponent of $5/3$.

Solution:

(a) We are given the equation: $\left[1+\left(\frac{dy}{dx}\right)^3\right]^{5/3}=\frac{d^2y}{dx^2}$.

(b) First, identify the derivatives: the first derivative is dy/dx and the highest derivative is the second derivative d^2y/dx^2.

(c) The order is determined by the highest derivative, so the Order is 2.

(d) To find the degree, we must eliminate the fraction $5/3$ in the exponent.

(e) Cube both sides of the equation to remove the denominator: $\left(\left[1+\left(\frac{dy}{dx}\right)^3\right]^{5/3}\right)^3=\left(\frac{d^2y}{dx^2}\right)^3$.

(f) This simplifies to: $\left[1+\left(\frac{dy}{dx}\right)^3\right]^5=\left(\frac{d^2y}{dx^2}\right)^3$.

(g) Now the equation is a polynomial in terms of its derivatives.

(h) The highest order derivative is d^2y/dx^2, and its power is 3.

(i) Therefore, the Degree is 3.

(j) The order and degree are 2 and 3 respectively.

Final Answer: The order and degree are 2 and 3.

Answer: (A)

Go Back to Question 13

Q14.

Solution

Concept:
A first-order linear differential equation has the standard form $dy/dx + P(x)y = Q(x)$. The solution process involves finding an Integrating Factor (I.F.), defined as $e^{\int P(x)dx}$. Multiplying the entire equation by this factor transforms the left side into the derivative of the product of the dependent variable and the I.F., which can then be integrated directly. This method ensures that the differential equation is reduced to a simple integration problem.

Solution:

(a) The given equation is $\frac{dy}{dx} + y\cot x = 2\cos x$.

(b) Comparing this with $\frac{dy}{dx} + Py = Q$, we identify $P(x) = \cot x$ and $Q(x) = 2\cos x$.

(c) Calculate the Integrating Factor (I.F.): $I.F. = e^{\int \cot x dx} = e^{\ln(\sin x)}$.

(d) Since $e^{\ln u} = u$, the I.F. is $\sin x$.

(e) The general solution is given by: $y \cdot (I.F.) = \int Q(x) \cdot (I.F.)dx + C$.

(f) Substitute the values: $y\sin x = \int (2\cos x)(\sin x)dx + C$.

(g) Use the trigonometric identity $2\sin x\cos x = \sin 2x$: $y\sin x = \int \sin 2x dx + C$.

(h) Integrate the right side: $y\sin x = \frac{-\cos 2x}{2} + C$.

(i) Alternatively, $\int 2\sin x\cos x dx$ can be written as $\sin^2 x + C'$.

(j) Thus, $y\sin x = \sin^2 x + C$ is a valid representation of the general solution.

Final Answer: The solution is $y\sin x = \sin^2 x + C$.

Answer: (A)

Go Back to Question 14

Q15.

Solution

Concept:
A differential equation is called homogeneous if it can be written in the form $dy/dx = f(y/x)$. Such equations are solved using the substitution $y = vx$, which implies $dy/dx = v + x(dv/dx)$. This substitution transforms the original equation into a variable-separable form in terms of v and x. Once separated, both sides are integrated, and v is replaced back with y/x to obtain the final solution in terms of the original variables.

Solution:

(a) The given equation is $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$.

(b) This is a homogeneous equation. Let $y = vx$.

(c) Differentiating y with respect to x: $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

(d) Substitute these into the original equation: $v + x\frac{dv}{dx} = v + \tan v$.

(e) Subtract v from both sides: $x\frac{dv}{dx} = \tan v$.

(f) Separate the variables v and x: $\frac{dv}{\tan v} = \frac{dx}{x} \implies \cot v\,dv = \frac{dx}{x}$.

(g) Integrate both sides: $\int \cot v\,dv = \int \frac{1}{x}dx$.

(h) $\ln(\sin v) = \ln x + \ln C$.

(i) Using log properties: $\ln(\sin v) = \ln(Cx)$.

(j) Exponentiating both sides: $\sin v = Cx$.

(k) Substitute $v = y/x$ back into the equation: $\sin(y/x) = Cx$.

Final Answer: The solution is $\sin(y/x) = Cx$.

Answer: (A)

Go Back to Question 15

Q16.

Solution

Concept:
In complex algebra, the modulus of a complex number $z = a + bi$ is defined as $|z| = \sqrt{a^2 + b^2}$. A critical property that simplifies calculations is that the modulus of a quotient of two complex numbers is equal to the quotient of their individual moduli. This means for $z = z_1/z_2$, the modulus is $|z| = |z_1|/|z_2|$. This property is highly efficient as it removes the need to rationalize the denominator by multiplying with the conjugate before finding the modulus.
Solution:

(a) We are given the complex number $z = \frac{2+i}{3-2i}$.

(b) Let the numerator be $z_1 = 2 + i$.

(c) Let the denominator be $z_2 = 3 - 2i$.

(d) Calculate the modulus of the numerator z_1: $|z_1| = \sqrt{2^2 + 1^2} = \sqrt{4+1} = \sqrt{5}$.

(e) Calculate the modulus of the denominator z_2: $|z_2| = \sqrt{3^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13}$.

(f) Using the modulus property $|z| = \frac{|z_1|}{|z_2|}$: $|z| = \frac{\sqrt{5}}{\sqrt{13}} = \sqrt{\frac{5}{13}}$.

(g) This result provides the magnitude of the vector representing the complex number in the Argand plane.

(h) Note that if we had rationalized z first, the real and imaginary parts would have been more complex, but the modulus would remain the same.

(i) Squaring the modulus would give $5/13$, but the question asks for the modulus itself.

Final Answer: The modulus is $\sqrt{\frac{5}{13}}$.

Answer: (A)

Go Back to Question 16

Q17.

Solution

Concept:
De Moivre's Theorem is a powerful tool in complex numbers which states that for any real number n and any angle θ, the expression $(\cos\theta + i\sin\theta)^n$ is equivalent to $\cos(n\theta) + i\sin(n\theta)$. This theorem allows us to raise a complex number in polar form to any power efficiently. When the power is an integer, it essentially rotates the complex number around the origin in the Argand plane and scales its magnitude, though in the case of unit vectors (where the magnitude is 1), only the rotation occurs.

Solution:

(a) The given expression is $(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})^{12}$.

(b) This expression is already in the standard polar form $\cos\theta + i\sin\theta$, where $\theta = \pi/6$.

(c) Applying De Moivre's Theorem with $n = 12$: $(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})^{12} = \cos(12\cdot\frac{\pi}{6}) + i\sin(12\cdot\frac{\pi}{6})$.

(d) Simplify the argument: $12\cdot\frac{\pi}{6} = 2\pi$.

(e) Substitute the simplified argument back into the trigonometric functions: Result $= \cos(2\pi) + i\sin(2\pi)$.

(f) We know from the unit circle that $\cos(2\pi) = 1$.

(g) We also know that $\sin(2\pi) = 0$.

(h) Therefore, the value becomes $1 + i(0) = 1$.

(i) The complex number has completed one full rotation of 360 degrees or 2π radians, returning to the positive real axis.

Final Answer: The value is 1.

Answer: (A)

Go Back to Question 17

Q18.

Solution

Concept:
The principal argument of a complex number $z = x + iy$ is the angle θ that the vector makes with the positive real axis, such that $-\pi < \theta \leq \pi$. To find this, we first determine the basic acute angle $\alpha = \tan^{-1}(|y/x|)$. The actual value of the principal argument then depends on the quadrant in which the complex number lies. If the point (x, y) is in the third quadrant (where both x and y are negative), the principal argument is calculated as $\theta = -(\pi - \alpha)$ or $\theta = \alpha - \pi$.
Solution:

(a) We are given $z = -1 - i\sqrt{3}$. Here, $x = -1$ and $y = -\sqrt{3}$.

(b) Since both x and y are negative, the complex number z lies in the third quadrant.

(c) First, find the basic angle α: $\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-\sqrt{3}}{-1}\right| = \tan^{-1}(\sqrt{3})$.

(d) The value of $\tan^{-1}(\sqrt{3})$ is $\pi/3$ or 60 degrees.

(e) In the third quadrant, the principal argument θ is given by $\alpha - \pi$.

(f) $\theta = \frac{\pi}{3} - \pi$.

(g) Calculating the difference: $\theta = \frac{\pi - 3\pi}{3} = -2\pi/3$.

(h) This angle measures 120 degrees in the clockwise direction from the positive real axis.

(i) Note that while $4\pi/3$ represents the same direction, it is not the principal argument because it exceeds the standard range of $(-\pi, \pi]$.

Final Answer: The principal argument is $-2\pi/3$.

Answer: (B)

Go Back to Question 18

Q19.

Solution

Concept:

In quadratic theory, the relationship between the roots and the coefficients is dictated by Vieta's formulas: the sum of roots $\alpha + \beta = -b/a$ and the product of roots $\alpha\beta = c/a$. If the roots are given in a specific ratio $m : n$, we can represent them as $\alpha = mk$ and $\beta = nk$ for some constant k. By substituting these representations into Vieta's formulas, we can eliminate the constant k to derive a direct relationship between the ratio m, n and the coefficients a, b, c. This process involves squaring the sum of roots to create a term comparable to the product of roots.

Solution:

(a) Let the roots of $ax^2 + bx + c = 0$ be α and β.

(b) Given $\alpha : \beta = m : n$. Let $\alpha = mk$ and $\beta = nk$.

(c) From Vieta's formulas: $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$.

(d) Substitute mk and nk into the sum: $k(m+n) = -b/a \implies k = \frac{-b}{a(m+n)}$.

(e) Substitute into the product: $(mk)(nk) = c/a \implies k^2mn = c/a$.

(f) Substitute the expression for k from the sum into the product equation: $\left(\frac{-b}{a(m+n)}\right)^2 mn = \frac{c}{a}$.

(g) Expanding the square: $\frac{b^2}{a^2(m+n)^2}mn = \frac{c}{a}$.

(h) Multiply both sides by $a^2(m+n)^2$: $b^2mn = \frac{c}{a} \cdot a^2(m+n)^2$.

(i) Simplify the right side: $b^2mn = ac(m+n)^2$.

(j) This identifies choice B as the correct algebraic relation.

Final Answer: The relation is $mnb^2 = ac(m+n)^2$.

Answer: (B)

Go Back to Question 19

Q20.

Solution

Concept:

For a quadratic equation $ax^2 + bx + c = 0$, the roots are equal if and only if the discriminant $D = b^2 - 4ac$ is equal to zero. This condition indicates that the quadratic is a perfect square and touches the x-axis at exactly one point. Setting the discriminant to zero for an equation where coefficients are functions of a parameter k results in a new equation (often quadratic) in terms of k. Solving this "discriminant equation" provides the specific values of the parameter that satisfy the condition of equal roots.

Solution:

(a) The given quadratic equation is $x^2 - (k-3)x + k = 0$.

(b) Here, the coefficients are $a = 1$, $b = -(k-3)$, and $c = k$.

(c) For the roots to be equal, we must have $D = b^2 - 4ac = 0$.

(d) Substituting the coefficients: $[-(k-3)]^2 - 4(1)(k) = 0$.

(e) Expanding the square: $(k-3)^2 - 4k = 0 \implies k^2 - 6k + 9 - 4k = 0$.

(f) Simplify the equation: $k^2 - 10k + 9 = 0$.

(g) This is a quadratic equation in k. Factorize it: $k^2-9k-k+9 = 0 \implies k(k-9)-1(k-9) = 0$.

(h) Thus, $(k-1)(k-9) = 0$.

(i) The possible values for k are $k = 1$ and $k = 9$.

(j) Both values will make the original equation a perfect square $(x+1)^2 = 0$ or $(x-3)^2 = 0$.

Final Answer: The values of k are 1 or 9.

Answer: (A)

Go Back to Question 20

Q21.

Solution

Concept:
In the study of quadratic equations, the relationship between the coefficients and the roots is governed by Vieta's formulas. For the standard quadratic equation $ax^2 + bx + c = 0$, the sum of the roots is given by $-b/a$ and the product of the roots is given by c/a. When a specific geometric or algebraic condition is imposed on the roots, such as one root being the square of the other, we can express the roots as α and α^2. By substituting these into the sum and product formulas, we can eliminate the variable α to find a unique relation involving only the coefficients a, b, and c. This usually involves algebraic manipulation such as cubing the sum of the roots to match the power of the product.

Solution:

(a) Let the two roots of the quadratic equation be α and β.

(b) According to the given condition, one root is the square of the other, so we set $\beta = \alpha^2$.

(c) From the properties of roots, the sum of the roots is: $\alpha + \alpha^2 = -b/a$.

(d) The product of the roots is: $\alpha \cdot \alpha^2 = \alpha^3 = c/a$.

(e) To eliminate α, we take the cube of the sum equation: $(\alpha + \alpha^2)^3 = (-b/a)^3$.

(f) Expanding the left side using $(x+y)^3 = x^3+y^3+3xy(x+y)$: $\alpha^3+(\alpha^2)^3+3\alpha\cdot\alpha^2(\alpha+\alpha^2) = -b^3/a^3$.

(g) Simplify the powers: $\alpha^3 + \alpha^6 + 3\alpha^3(\alpha + \alpha^2) = -b^3/a^3$.

(h) Now, substitute the known values $\alpha^3 = c/a$ and $(\alpha + \alpha^2) = -b/a$ into the equation: $(c/a) + (c/a)^2 + 3(c/a)(-b/a) = -b^3/a^3$.

(i) This gives: $c/a + c^2/a^2 - 3bc/a^2 = -b^3/a^3$.

(j) Multiply the entire equation by a^3 to clear the denominators: $a^2c + ac^2 - 3abc = -b^3$.

(k) Rearranging the terms to match the standard identity format: $b^3 + a^2c + ac^2 = 3abc$.

Final Answer: The condition is $b^3 + a^2c + ac^2 = 3abc$.

Answer: (D)

Go Back to Question 21

Q22.

Solution

Concept:
An arithmetic-geometric series (AGS) is a sequence where each term is the product of the corresponding terms of an arithmetic progression (AP) and a geometric progression (GP). The general form is $\sum(a + (n-1)d)r^{n-1}$. When the common ratio $|r| < 1$, the sum to infinity S_∞ can be found using a specific derivation method. By writing the sum S, multiplying it by the common ratio r, and subtracting the two series (the "shift and subtract" method), we reduce the AGS to a standard infinite geometric series. This process effectively isolates the geometric components, allowing for the application of the formula $S_\infty = a/(1-r) + dr/(1-r)^2$.
Solution:

(a) The given series is $S = 1 + 2/5 + 3/25 + 4/125 + \ldots$.

(b) We identify the AP part: $1, 2, 3, 4, \ldots$ with first term $a = 1$ and common difference $d = 1$.

(c) We identify the GP part: $1, 1/5, 1/25, 1/125, \ldots$ with common ratio $r = 1/5$.

(d) Since $|r| < 1$, the sum to infinity exists.

(e) Write the sum: $S = 1 + 2(1/5) + 3(1/5)^2 + 4(1/5)^3 + \ldots$ (Equation 1).

(f) Multiply S by $r = 1/5$: $(1/5)S = 1/5 + 2(1/5)^2 + 3(1/5)^3 + \ldots$ (Equation 2).

(g) Subtract Equation 2 from Equation 1: $S - (1/5)S = 1 + (2/5 - 1/5) + (3/25 - 2/25) + (4/125 - 3/125) + \ldots$.

(h) This simplifies to: $(4/5)S = 1 + 1/5 + 1/25 + 1/125 + \ldots$.

(i) The right side is an infinite GP with $a = 1$ and $r = 1/5$. The sum is $1/(1 - 1/5) = 1/(4/5) = 5/4$.

(j) So, $(4/5)S = 5/4$.

(k) Solving for S: $S = (5/4) \cdot (5/4) = 25/16$.

Final Answer: The sum to infinity is 25/16.

Answer: (A)

Go Back to Question 22

Q23.

Solution

Concept:
Determining the n-th term of a sequence involves identifying the underlying pattern or rule that generates each term from its position n. Sequences often follow polynomial patterns, where the general term T_n can be expressed as $an^2 + bn + c$ or similar forms. One effective method for identifying such patterns is the method of differences. If the first differences between consecutive terms are not constant but form an arithmetic progression, the n-th term is likely a quadratic function of n. By observing the squares of natural numbers $(1, 4, 9, 16, \dots)$ and comparing them to the given sequence, one can often find a constant shift that defines the entire series.

Solution:

(a) The provided sequence is $2, 5, 10, 17, 26, \dots$.

(b) Let us look at the first term $T_1 = 2$.

(c) Let us look at the second term $T_2 = 5$.

(d) Let us look at the third term $T_3 = 10$.

(e) Let us look at the fourth term $T_4 = 17$.

(f) Now, we calculate the differences between consecutive terms: $5 - 2 = 3$. $10 - 5 = 5$. $17 - 10 = 7$. $26 - 17 = 9$.

(g) The differences $(3, 5, 7, 9, \dots)$ form an arithmetic progression, suggesting a quadratic pattern n^2.

(h) Compare the sequence to perfect squares (n^2): For $n = 1$, $n^2 = 1$. The sequence has 2. Difference is $+1$. For $n = 2$, $n^2 = 4$. The sequence has 5. Difference is $+1$. For $n = 3$, $n^2 = 9$. The sequence has 10. Difference is $+1$. For $n = 4$, $n^2 = 16$. The sequence has 17. Difference is $+1$.

(i) In every case, the term is exactly one more than the square of its position.

(j) Therefore, the general term is $T_n = n^2 + 1$.

Final Answer: The n-th term is $n^2 + 1$.

Answer: (A)

Go Back to Question 23

Q24.

Solution

Concept:
For any two positive numbers x and y, the Arithmetic Mean (A.M.) is defined as $(x+y)/2$ and the Geometric Mean (G.M.) is defined as $\sqrt{xy}$. These two values can be used to reconstruct the quadratic equation whose roots are the original numbers. Specifically, any two numbers are the roots of the equation $t^2-(x+y)t+xy=0$. Since $x+y=2\cdot A.M.$ and $xy=(G.M.)^2$, the quadratic equation becomes $t^2-(2\cdot A.M.)t+(G.M.)^2=0$. Solving this quadratic equation using the standard quadratic formula allows us to find the individual values of the two numbers.

Solution:

(a) Let the two positive numbers be x and y.

(b) Given A.M. $=(x+y)/2=25$. This implies the sum of the numbers $x+y=50$.

(c) Given G.M. $=\sqrt{xy}=20$. Squaring both sides gives the product of the numbers $xy=400$.

(d) The two numbers x and y are the roots of the quadratic equation: $t^2-(\text{sum})t+(\text{product})=0$.

(e) Substitute the values: $t^2-50t+400=0$.

(f) We solve this quadratic equation by factoring. We look for two numbers that multiply to 400 and add to -50.

(g) The numbers are -40 and -10.

(h) So, $(t-40)(t-10)=0$.

(i) The solutions for t are 40 and 10.

(j) Thus, the two original numbers are 40 and 10.

(k) Verification: A.M. $=(40+10)/2=25$ and G.M. $=\sqrt{40\cdot 10}=\sqrt{400}=20$. The values match the given conditions perfectly.

Final Answer: The numbers are 40 and 10.

Answer: (A)

Go Back to Question 24

Q25.

Solution

Concept:
The General Term of a binomial expansion $(a+b)^n$ is given by $T_{r+1} = \binom{n}{r} a^{n-r} b^r$. A term is considered "independent of x" if the net exponent of the variable x in that term is zero. To find such a term, we write the general term in its most expanded form, collect all powers of x into a single exponent using laws of indices, and set that resultant exponent equal to zero. Solving this linear equation for r gives us the position of the term in the expansion. Note that the $(r+1)$-th term corresponds to the index r used in the selection of the binomial coefficient.

Solution:

(a) We have the expansion $(2x - \frac{1}{3x^2})^9$. Here $a = 2x$, $b = -1/(3x^2)$, and $n = 9$.

(b) Write the general term T_{r+1}: $T_{r+1} = \binom{9}{r}(2x)^{9-r}(-\frac{1}{3x^2})^r$.

(c) Separate the constants from the powers of x: $T_{r+1} = \binom{9}{r} 2^{9-r}(-1/3)^r \cdot x^{9-r} \cdot (x^{-2})^r$.

(d) Combine the powers of x: $x^{9-r} \cdot x^{-2r} = x^{9-r-2r} = x^{9-3r}$.

(e) For the term to be independent of x, the power $9 - 3r$ must be 0.

(f) $9 - 3r = 0 \implies 3r = 9 \implies r = 3$.

(g) Since $r = 3$, the term is the $(r+1)$-th term, which is $T_{3+1} = T_4$.

(h) The fourth term in the expansion contains no x variable.

(i) Calculation of the coefficient (optional): $T_4 = \binom{9}{3} 2^6 (-1/3)^3$.

Final Answer: The term independent of x is T_4.

Answer: (B)

Go Back to Question 25

Q26.

Solution

Concept:

In binomial theorem, the expansion of $(1+x)^n$ is a fundamental series where the general term is expressed using the combination formula. The coefficient of any specific power of x, say x^k, in the expansion of $(1+x)^n$ is given by the binomial coefficient $\binom{n}{k}$, which represents the number of ways to choose k items from a set of n distinct items. This is calculated using the formula $n!/(k!(n-k)!)$. For large values of n, these coefficients follow a symmetric distribution, increasing until they reach the middle term and then decreasing. Understanding how to compute these factorials efficiently is key to solving MHT-CET problems within the time limit.

Solution:

(a) We are given the binomial expression $(1+x)^{11}$ and we need to determine the coefficient of the x^7 term.

(b) The general term T_{r+1} in the expansion of $(1+x)^n$ is given by the formula $\binom{n}{r}x^r$.

(c) Comparing x^r with x^7, we identify that $r = 7$ and the total power $n = 11$.

(d) Therefore, the required coefficient is $\binom{11}{7}$.

(e) Using the property of combinations where $\binom{n}{r} = \binom{n}{n-r}$, we can simplify the calculation: $\binom{11}{7} = \binom{11}{11-7} = \binom{11}{4}$.

(f) Now we expand $\binom{11}{4}$ using the factorial formula: $\binom{11}{4} = \frac{11\cdot10\cdot9\cdot8}{4\cdot3\cdot2\cdot1}$.

(g) Performing the division: $4 \cdot 2 = 8$, so the 8 in the numerator and denominator cancel out.

(h) This leaves us with $11 \cdot 10 \cdot 9/3$.

(i) Further simplifying, $9/3 = 3$.

(j) The final product is $11 \cdot 10 \cdot 3 = 11 \cdot 30 = 330$.

Final Answer: The coefficient is 330.

Answer: (A)

Go Back to Question 26

Q27.

Solution

Concept:
Circular permutation is a branch of combinatorics dealing with the arrangement of objects in a closed loop. Unlike linear permutations where the starting and ending positions are distinct, a circular arrangement remains the same under rotation. To account for this rotational symmetry, we fix one person's position to serve as a reference point, which effectively turns the remaining arrangement into a linear one for the other individuals. Consequently, if there are n distinct objects to be arranged in a circle, the number of unique arrangements is $(n-1)!$. This principle is widely used in seating problems and necklace or garland configurations, though the latter may require further division by two if the orientation (clockwise vs anti-clockwise) is indistinguishable.
Solution:

(a) The problem asks for the number of ways to seat 6 people around a circular table.

(b) First, let us consider the case of linear seating. If these 6 people were sitting in a straight row, the number of arrangements would be $6!$, which equals 720.

(c) However, in a circular seating arrangement, there is no designated "head" of the table. Shifting every person one seat to the left results in the same relative arrangement.

(d) For 6 people, there are 6 such rotations that are considered identical in a circular context.

(e) To find the unique arrangements, we divide the linear total by the number of people: $6!/6$.

(f) This simplifies to $(6-1)!$, which is $5!$.

(g) Calculating the value of $5!$: $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

(h) $5 \cdot 4 = 20$.

(i) $20 \cdot 3 = 60$.

(j) $60 \cdot 2 = 120$.

(k) Therefore, there are exactly 120 distinct ways to arrange the 6 individuals.

Final Answer: The number of ways is 120.

Answer: (B)
Go Back to Question 27

Q28.

Solution

Concept:

In probability theory, calculating the likelihood of "at least one" event occurring is often most efficiently done using the complement rule. Instead of summing the probabilities of exactly one, exactly two, or exactly three successful outcomes, we subtract the probability of the "none" case from the total probability of one. This approach relies on the fact that the sum of the probabilities of all mutually exclusive and exhaustive events is exactly one. By focusing on the scenario where no red balls are drawn (meaning only black balls are selected), we simplify the combinatorial calculation significantly and avoid the repetition or omission of cases.

Solution:

(a) We have a bag containing a total of $5 + 4 = 9$ balls.

(b) We are drawing 3 balls at random. The total number of ways to choose 3 balls from 9 is given by $\binom{9}{3}$.

(c) $\binom{9}{3} = \frac{9\cdot8\cdot7}{3\cdot2\cdot1} = 3 \cdot 4 \cdot 7 = 84$.

(d) We want the probability of getting "at least one red ball". The complement is getting "no red balls", which means all 3 balls must be black.

(e) The number of ways to choose 3 black balls from the 4 available black balls is $\binom{4}{3}$.

(f) $\binom{4}{3} = \binom{4}{1} = 4$.

(g) The probability of drawing only black balls is $P(\text{No Red}) = 4/84 = 1/21$.

(h) Using the complement rule: $P(\text{At least one Red}) = 1 - P(\text{No Red})$.

(i) $P(\text{At least one Red}) = 1 - 1/21$.

(j) $P(\text{At least one Red}) = (21 - 1)/21 = 20/21$.

(k) Thus, there is a very high probability that at least one red ball will be present in a sample of three.

Final Answer: The probability is 20/21.

Answer: (A)

Go Back to Question 28

Q29.

Solution

Concept:
When two dice are thrown, the sample space consists of $6 \cdot 6 = 36$ equally likely outcomes. To find the probability of a specific sum, we must identify all ordered pairs $(d1, d2)$ whose sum satisfies the condition. A prime number is a natural number greater than 1 that has no positive divisors other than 1 and itself. For two dice, the possible sums range from $1 + 1 = 2$ to $6 + 6 = 12$. The prime numbers within this range are $2, 3, 5, 7$, and 11. We systematically list the favorable outcomes for each of these prime sums to determine the total count of favorable outcomes and then divide by the total sample size.

Solution:

(a) Total number of outcomes when two dice are thrown = $6 \times 6 = 36$.

(b) We need the sum of the dice to be a prime number. The possible prime sums are $\{2, 3, 5, 7, 11\}$.

(c) Favorable cases for Sum = 2: $(1, 1)$ [1 case].

(d) Favorable cases for Sum = 3: $(1, 2), (2, 1)$ [2 cases].

(e) Favorable cases for Sum = 5: $(1, 4), (4, 1), (2, 3), (3, 2)$ [4 cases].

(f) Favorable cases for Sum = 7: $(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)$ [6 cases].

(g) Favorable cases for Sum = 11: $(5, 6), (6, 5)$ [2 cases].

(h) Total number of favorable outcomes = $1 + 2 + 4 + 6 + 2 = 15$.

(i) The probability is given by (Favorable outcomes) / (Total outcomes).

(j) $P(\text{Sum is prime}) = 15/36$.

(k) Simplifying the fraction by dividing both numerator and denominator by 3: $15/3 = 5$ and $36/3 = 12$.

(l) The simplified probability is $5/12$.

Final Answer: The probability is 5/12.

Answer: (A)

Go Back to Question 29

Q30.

Solution

Concept:
In coordinate geometry, the relationship between the slopes of two perpendicular lines is central to finding their equations. If a line has a slope m_1, any line perpendicular to it must have a slope m_2 such that $m_1 \cdot m_2 = -1$. For a line given in the general form $Ax + By + C = 0$, the slope is $-A/B$. A line perpendicular to this will have the form $Bx - Ay + \lambda = 0$. This general perpendicular form allows us to find the specific equation by substituting the coordinates of a known point through which the line passes to solve for the constant λ. This method is more direct than finding the slope and using the point-slope formula.

Solution:

(a) Given the line equation: $3x + 4y + 5 = 0$.

(b) Any line perpendicular to $Ax + By + C = 0$ is of the form $Bx - Ay + \lambda = 0$.

(c) Here $A = 3$ and $B = 4$. So, the perpendicular line is of the form: $4x - 3y + \lambda = 0$.

(d) We are given that this line passes through the point $(1, -2)$.

(e) Substitute $x = 1$ and $y = -2$ into the perpendicular line equation to find λ: $4(1)-3(-2)+\lambda = 0$.

(f) $4 + 6 + \lambda = 0$.

(g) $10 + \lambda = 0 \implies \lambda = -10$.

(h) Now, substitute $\lambda = -10$ back into the perpendicular line equation: $4x - 3y - 10 = 0$.

(i) This is the final equation of the straight line.

(j) Verification: The slope of the first line is $-3/4$ and the slope of the second line is $4/3$. Their product is $(-3/4)(4/3) = -1$, confirming they are perpendicular.

Final Answer: The equation is $4x - 3y - 10 = 0$.

Answer: (A)

Go Back to Question 30

Q31.

Solution

Concept:
In two-dimensional coordinate geometry, the perpendicular distance between two parallel lines is a constant value regardless of where it is measured along the lines. For two lines represented in the general form $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$, the distance d is determined by the absolute difference of their constant terms divided by the square root of the sum of the squares of the coefficients of x and y. This formula arises from finding the distance of a point on one line to the other line using the point-to-line distance formula. It is essential to ensure that the coefficients A and B are identical in both equations before applying the formula.
Solution:

(a) We are given two parallel lines: $L_1 : 5x + 12y - 7 = 0$ and $L_2 : 5x + 12y + 19 = 0$.

(b) Identifying the coefficients, we see $A = 5$ and $B = 12$ for both lines, confirming they are parallel.

(c) The constant terms are $C_1 = -7$ and $C_2 = 19$.

(d) The formula for the distance d between parallel lines is: $d = \frac{|C_2 - C_1|}{\sqrt{A^2+B^2}}$.

(e) Substitute the values into the formula: $d = \frac{|19-(-7)|}{\sqrt{5^2+12^2}}$.

(f) Simplify the numerator: $|19 + 7| = |26| = 26$.

(g) Simplify the denominator: $\sqrt{25 + 144} = \sqrt{169} = 13$.

(h) Perform the final division: $d = 26/13 = 2$.

(i) Therefore, the perpendicular distance between these two linear paths is exactly 2 units.

Final Answer: The distance is 2 units.

Answer: (A)

Go Back to Question 31

Q32.

Solution

Concept:
The general equation of a circle is expressed as $x^2 + y^2 + 2gx + 2fy + c = 0$. From this form, the center of the circle is located at the coordinates $(-g, -f)$ and the radius r is calculated using the relation $r = \sqrt{g^2 + f^2 - c}$. To find these parameters from a given equation, one must divide the coefficients of x and y by 2 and negate them to find the center. The constant term c is then used in the square root formula. This process essentially completes the square for the x and y terms to transform the general equation into the standard central form $(x - h)^2 + (y - k)^2 = r^2$.

Solution:

(a) The given equation of the circle is $x^2 + y^2 - 8x + 10y - 12 = 0$.

(b) Comparing this with the general form $x^2 + y^2 + 2gx + 2fy + c = 0$: $2g = -8 \implies g = -4$. $2f = 10 \implies f = 5$. $c = -12$.

(c) The center of the circle is defined as $(-g, -f)$.

(d) Center $= (-(-4), -(5)) = (4, -5)$.

(e) The radius r is calculated as $r = \sqrt{g^2 + f^2 - c}$.

(f) Substitute the values: $r = \sqrt{(-4)^2 + 5^2 - (-12)}$.

(g) $r = \sqrt{16 + 25 + 12}$.

(h) $r = \sqrt{53}$.

(i) The geometric analysis reveals a circle centered in the fourth quadrant with a radius slightly greater than 7 units.

Final Answer: The center is (4, -5) and radius is $\sqrt{53}$.

Answer: (A)

Go Back to Question 32

Q33.

Solution

Concept:

The length of a tangent drawn from an external point $P(x_1, y_1)$ to a circle $S = x^2+y^2+2gx+2fy+c = 0$ is given by the formula $L = \sqrt{S_1}$, where S_1 is the value obtained by substituting the coordinates of point P into the circle's equation. If the circle is centered at the origin, $x^2 + y^2 - r^2 = 0$, the length simplifies to $\sqrt{x_1^2 + y_1^2 - r^2}$. Geometrically, this represents the side of a right-angled triangle where the hypotenuse is the distance from the point to the center and the other leg is the radius of the circle, according to the Pythagorean theorem.

Solution:

(a) The circle is given by $x^2 + y^2 = 25$. This is a circle centered at $(0, 0)$ with radius $r = 5$.

(b) The external point is $P(5, 4)$.

(c) To find the length of the tangent, we first evaluate the expression $x^2 + y^2 - 25$ at the point $(5, 4)$.

(d) $S_1 = (5)^2 + (4)^2 - 25$.

(e) $S_1 = 25 + 16 - 25$.

(f) $S_1 = 16$.

(g) The length of the tangent L is the square root of S_1.

(h) $L = \sqrt{16} = 4$.

(i) Alternatively, using the distance formula: distance from $(0, 0)$ to $(5, 4)$ is $\sqrt{5^2 + 4^2} = \sqrt{41}$.

(j) Using Pythagoras: $L^2 + r^2 = \text{distance}^2 \implies L^2 + 5^2 = (\sqrt{41})^2 \implies L^2 + 25 = 41 \implies L^2 = 16 \implies L = 4$.

Final Answer: The length of the tangent is 4.

Answer: (B)

Go Back to Question 33

Q34.

Solution

Concept:
Eccentricity is a measure of how much a conic section deviates from being a circle. For a hyperbola in the standard form $x^2/a^2 - y^2/b^2 = 1$, the eccentricity e is always greater than 1 and is calculated using the formula $e = \sqrt{1 + b^2/a^2}$. The values a and b represent the lengths of the semi-transverse axis and semi-conjugate axis, respectively. To determine these values, the given equation must first be converted into the standard form by dividing the entire equation by the constant on the right side. Eccentricity is fundamentally related to the distance from the center to the foci.
Solution:

(a) The given equation is $16x^2 - 9y^2 = 144$.

(b) Convert to standard form by dividing both sides by 144: $\frac{16x^2}{144} - \frac{9y^2}{144} = \frac{144}{144}$.

(c) Simplify the fractions: $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

(d) Comparing with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we find $a^2 = 9$ and $b^2 = 16$.

(e) The formula for eccentricity e of a hyperbola is $e = \sqrt{1 + \frac{b^2}{a^2}}$.

(f) Substitute the values: $e = \sqrt{1 + \frac{16}{9}}$.

(g) $e = \sqrt{\frac{9+16}{9}} = \sqrt{\frac{25}{9}}$.

(h) $e = 5/3$.

(i) Since $5/3 > 1$, this confirms the eccentricity of a hyperbola.

Final Answer: The eccentricity is 5/3.

Answer: (B)

Go Back to Question 34

Q35.

Solution

Concept:

The latus rectum of a conic section is a chord passing through the focus and perpendicular to the major axis. For an ellipse with the standard equation $x^2/a^2 + y^2/b^2 = 1$, where $a > b$, the length of the latus rectum is given by the formula $2b^2/a$. Here, a is the semi-major axis and b is the semi-minor axis. This length represents the "width" of the ellipse at the focal points. If $b > a$, the formula would adjust to $2a^2/b$ as the major axis would then lie along the y-axis. Accurate identification of which denominator is larger is critical for applying the correct formula.

Solution:

(a) The given equation of the ellipse is $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

(b) Comparing with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we identify $a^2 = 16$ and $b^2 = 9$.

(c) Taking the square root, $a = 4$ and $b = 3$.

(d) Since $a > b$, the major axis is along the x-axis.

(e) The formula for the length of the latus rectum is $L = \frac{2b^2}{a}$.

(f) Substitute the values into the formula: $L = \frac{2 \cdot 9}{4}$.

(g) $L = \frac{18}{4}$.

(h) Simplify the fraction: $L = 9/2$.

(i) In decimal form, this is 4.5 units. The option $9/2$ is the standard fractional representation.

Final Answer: The length of the latus rectum is 9/2.

Answer: (A)

Go Back to Question 35

Q36.

Solution

Concept:
In coordinate geometry, a parabola is defined as the locus of a point that is equidistant from a fixed point called the focus and a fixed line called the directrix. For a parabola opening towards the left, the standard equation is $y^2 = -4ax$, where a is a positive constant representing the distance from the vertex at $(0, 0)$ to the focus. Because the parabola opens in the negative x-direction, the focus must lie on the negative x-axis. The focus is a critical geometric parameter as it determines the curvature and the position of the latus rectum, which is a chord passing through this focus perpendicular to the axis of symmetry.
Solution:

(a) We are given the equation of the parabola as $y^2 = -20x$.

(b) We compare this equation with the standard form for a parabola opening to the left: $y^2 = -4ax$.

(c) By comparing the coefficients of x, we set up the equality: $4a = 20$.

(d) Dividing both sides by 4 gives $a = 5$.

(e) The vertex of this parabola is at the origin $(0, 0)$ because there are no horizontal or vertical shifts in the variables.

(f) For the form $y^2 = -4ax$, the focus is located at the coordinates $(-a, 0)$.

(g) Substituting the value of a we found: Focus $= (-5, 0)$.

(h) Geometrically, this means the focus is 5 units to the left of the origin on the x-axis.

(i) Correspondingly, the equation of the directrix would be $x = 5$.

(j) Any point (x, y) on this curve satisfies the property that its distance to $(-5, 0)$ is equal to its perpendicular distance to the line $x = 5$.

Final Answer: The coordinates of the focus are (-5, 0).

Answer: (C)

Go Back to Question 36

Q37.

Solution

Concept:
The vector cross product, also known as the Gibbs vector product, is a binary operation on two vectors in three-dimensional space. It results in a vector that is perpendicular to both of the original vectors, and its magnitude is proportional to the area of the parallelogram that the vectors span. The direction of the resulting vector is determined by the right-hand rule. Algebraically, the cross product is calculated using the determinant of a 3×3 matrix where the first row consists of the unit basis vectors $\hat{i}, \hat{j},$ and $\hat{k}$, the second row contains the components of the first vector, and the third row contains the components of the second vector.
Solution:

(a) Given vectors: $\vec{a} = 3\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} + \hat{k}$.

(b) To find $\vec{a} \times \vec{b}$, we set up the determinant:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

(c) Expand the determinant along the first row: $\vec{a} \times \vec{b} = \hat{i}[(-1)(1) - (2)(3)] - \hat{j}[(3)(1) - (2)(2)] + \hat{k}[(3)(3) - (-1)(2)]$.

(d) Calculate the components inside the brackets: $\hat{i}$ component: $(-1 - 6) = -7$. $\hat{j}$ component: $-(3 - 4) = -(-1) = 1$. $\hat{k}$ component: $(9 + 2) = 11$.

(e) Combine the components to write the final vector: $\vec{a} \times \vec{b} = -7\hat{i} + 1\hat{j} + 11\hat{k}$.

(f) This resulting vector is orthogonal to both $\vec{a}$ and $\vec{b}$. We can verify this by checking that the dot product of the result with either $\vec{a}$ or $\vec{b}$ equals zero.

Final Answer: The vector is $-7\hat{i} + \hat{j} + 11\hat{k}$.

Answer: (A)

Go Back to Question 37

Q38.

Solution

Concept:
The projection of a vector $\vec{a}$ onto a vector $\vec{b}$ is a scalar value that represents the "shadow" or the component of $\vec{a}$ that lies in the direction of $\vec{b}$. It is mathematically defined as the dot product of vector $\vec{a}$ and the unit vector of $\vec{b}$. The formula is $Proj_{\vec{b}}\vec{a} = (\vec{a}\cdot\vec{b})/|\vec{b}|$. This operation is fundamental in physics and engineering for decomposing forces or velocities into specific directions. While the scalar projection gives the magnitude and sense, the vector projection would involve multiplying this scalar by the unit vector $\vec{b}/|\vec{b}|$. In MHT-CET, the term "projection" usually refers to the scalar projection unless specified otherwise.

Solution:

(a) We have vectors $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ and $\vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}$.

(b) First, we calculate the dot product $\vec{a}\cdot\vec{b}$: $\vec{a}\cdot\vec{b} = (1)(7) + (3)(-1) + (7)(8)$.

(c) $\vec{a}\cdot\vec{b} = 7 - 3 + 56 = 60$.

(d) Next, we calculate the magnitude of vector $\vec{b}$, denoted by $|\vec{b}|$: $|\vec{b}| = \sqrt{7^2 + (-1)^2 + 8^2}$.

(e) $|\vec{b}| = \sqrt{49 + 1 + 64} = \sqrt{114}$.

(f) The scalar projection is given by the formula $(\vec{a}\cdot\vec{b})/|\vec{b}|$.

(g) Substituting the values: Projection $= 60/\sqrt{114}$.

(h) This value represents the length of the segment on the line of $\vec{b}$ cut off by perpendiculars from the ends of $\vec{a}$.

(i) Since the dot product is positive, the component of $\vec{a}$ is in the same direction as $\vec{b}$.

Final Answer: The projection is $60/\sqrt{114}$.

Answer: (A)

Go Back to Question 38

Q39.

Solution

Concept:
Direction cosines of a line in 3D space are the cosines of the angles that the line makes with the positive $x, y,$ and z axes. If a line passes through two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, we first find the direction ratios by subtracting the coordinates: $a = x_2 - x_1$, $b = y_2 - y_1$, and $c = z_2 - z_1$. These direction ratios are components of a vector parallel to the line. To convert these into direction cosines (l, m, n), we normalize the direction ratios by dividing each by the magnitude of the vector, which is $\sqrt{a^2 + b^2 + c^2}$. A key property of direction cosines is that the sum of their squares is always equal to unity ($l^2 + m^2 + n^2 = 1$).

Solution:

(a) The line joins points $A(1, 0, 0)$ and $B(0, 1, 1)$.

(b) First, calculate the direction ratios (a, b, c) of the line segment AB: $a = 0 - 1 = -1$. $b = 1 - 0 = 1$. $c = 1 - 0 = 1$.

(c) Calculate the distance d between the two points, which is the magnitude of the direction vector: $d = \sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{1 + 1 + 1} = \sqrt{3}$.

(d) The direction cosines (l, m, n) are obtained by dividing the direction ratios by d: $l = a/d = -1/\sqrt{3}$. $m = b/d = 1/\sqrt{3}$. $n = c/d = 1/\sqrt{3}$.

(e) Thus, the direction cosines are $(-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

(f) Verification: $(-1/\sqrt{3})^2 + (1/\sqrt{3})^2 + (1/\sqrt{3})^2 = 1/3 + 1/3 + 1/3 = 1$. This confirms the result is correct.

Final Answer: The direction cosines are $-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$.

Answer: (A)

Go Back to Question 39

Q40.

Solution

Concept:

The perpendicular distance from a point $P(x_1, y_1, z_1)$ to a plane $Ax + By + Cz + D = 0$ is a fundamental calculation in 3D geometry. The distance d is given by the formula $d = |Ax_1 + By_1 + Cz_1 + D|/\sqrt{A^2 + B^2 + C^2}$. This formula represents the length of the normal segment dropped from the point to the plane. If the result of the numerator $Ax_1 + By_1 + Cz_1 + D$ is zero, it implies that the point lies exactly on the plane, resulting in a distance of zero. The denominator represents the magnitude of the normal vector (A, B, C) to the plane, ensuring the distance is scaled correctly according to the units of the coordinate system.

Solution:

(a) We are given the point $(2, 3, 4)$ and the plane $3x - 6y + 2z + 11 = 0$.

(b) Identify the coefficients from the plane equation: $A = 3$, $B = -6$, $C = 2$, and $D = 11$.

(c) Identify the coordinates of the point: $x_1 = 2$, $y_1 = 3$, $z_1 = 4$.

(d) Use the distance formula: $d = \frac{|3(2)+(-6)(3)+2(4)+11|}{\sqrt{3^2+(-6)^2+2^2}}$.

(e) Simplify the numerator: $|6 - 18 + 8 + 11| = |17 - 18 + 8| = |7|$.

(f) Simplify the denominator: $\sqrt{9 + 36 + 4} = \sqrt{49} = 7$.

(g) Calculate the final distance: $d = 7/7 = 1$.

(h) Therefore, the point $(2, 3, 4)$ is exactly 1 unit away from the specified plane.

Final Answer: The distance is 1 unit.

Answer: (A)

Go Back to Question 40

Q41.

Solution

Concept:
The angle between two planes is defined as the angle between their normal vectors. In three dimensional geometry, a plane represented by the equation $Ax+By+Cz+D=0$ has a normal vector given by the coefficients (A, B, C). If we have two planes with normal vectors n_1 and n_2, the cosine of the angle θ between them is found using the dot product formula: $\cos\theta = |(n_1 \cdot n_2)/(|n_1| \cdot |n_2|)|$. This calculation effectively reduces a 3D orientation problem into a simple vector algebra exercise. It is important to use the absolute value in the numerator to ensure that the angle found is the acute angle between the planes, which is the standard convention in geometry.

Solution:

(a) We are given two planes: $P_1 : x + y + 2z = 9$ and $P_2 : 2x - y + z = 15$.

(b) Extract the normal vector to the first plane: $\vec{n}_1 = (1, 1, 2)$.

(c) Extract the normal vector to the second plane: $\vec{n}_2 = (2, -1, 1)$.

(d) Calculate the dot product $\vec{n}_1 \cdot \vec{n}_2$: $(1)(2) + (1)(-1) + (2)(1) = 2 - 1 + 2 = 3$.

(e) Calculate the magnitude of $\vec{n}_1$: $|\vec{n}_1| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{1 + 1 + 4} = \sqrt{6}$.

(f) Calculate the magnitude of $\vec{n}_2$: $|\vec{n}_2| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4 + 1 + 1} = \sqrt{6}$.

(g) Use the angle formula: $\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1| \cdot |\vec{n}_2|} = \frac{3}{\sqrt{6} \cdot \sqrt{6}}$.

(h) Simplify the expression: $\cos\theta = 3/6 = 1/2$.

(i) We know that $\cos(60°)$ or $\cos(\pi/3)$ is equal to $1/2$.

(j) Therefore, the angle between the two planes is $\pi/3$.

Final Answer: The angle is $\pi/3$.

Answer: (A)

Go Back to Question 41

Q42.

Solution

Concept:

This problem involves trigonometric sum to product identities. When given equations involving the sum of sines and cosines, the objective is often to find a specific trigonometric ratio of the half difference or half sum of the angles. The identities $\sin\alpha + \sin\beta = 2\sin((\alpha+\beta)/2)\cos((\alpha-\beta)/2)$ and $\cos\alpha + \cos\beta = 2\cos((\alpha+\beta)/2)\cos((\alpha-\beta)/2)$ are the primary tools used here. By squaring and adding these transformed equations, one can eliminate the half sum terms due to the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$. This manipulation reveals a relationship between the given constants a and b and the cosine of the difference of the angles, which can then be converted to the tangent of the half difference.

Solution:

(a) We are given two equations: (1) $\sin\alpha + \sin\beta = a$ and (2) $\cos\alpha + \cos\beta = b$.

(b) Using the sum to product formulas: $2\sin((\alpha+\beta)/2)\cos((\alpha-\beta)/2) = a$ $2\cos((\alpha+\beta)/2)\cos((\alpha-\beta)/2) = b$

(c) Square both of these resulting equations: $4\sin^2((\alpha+\beta)/2)\cos^2((\alpha-\beta)/2) = a^2$ $4\cos^2((\alpha+\beta)/2)\cos^2((\alpha-\beta)/2) = b^2$

(d) Adding the two squared equations together: $4\cos^2((\alpha-\beta)/2)[\sin^2((\alpha+\beta)/2) + \cos^2((\alpha+\beta)/2)] = a^2 + b^2$.

(e) Since the expression in the brackets is equal to 1, we get: $4\cos^2((\alpha-\beta)/2) = a^2 + b^2$.

(f) This implies $\cos^2((\alpha-\beta)/2) = (a^2 + b^2)/4$.

(g) We know the identity $\sec^2\theta = 1 + \tan^2\theta$. Therefore, $1/\cos^2\theta = 1 + \tan^2\theta$.

(h) $\frac{4}{a^2+b^2} = 1 + \tan^2((\alpha-\beta)/2)$.

(i) $\tan^2((\alpha-\beta)/2) = \frac{4}{a^2+b^2} - 1 = \frac{4-a^2-b^2}{a^2+b^2}$.

(j) Taking the square root gives the final value.

Final Answer: The value is $\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$.

Answer: (A)

Go Back to Question 42

Q43.

Solution

Concept:
Evaluating trigonometric expressions for non standard angles like 15 degrees usually involves using either the half angle formulas or the sum and difference identities. For the expression $\tan\theta + \cot\theta$, a useful simplification is to convert everything to sine and cosine. Using the identity $\tan\theta + \cot\theta = (\sin^2\theta + \cos^2\theta)/(\sin\theta\cos\theta) = 1/(\sin\theta\cos\theta)$, we see that the expression is equivalent to $2/\sin 2\theta$. This reduction is highly effective because it transforms a sum of two difficult ratios into a single ratio of a well known standard angle. In this specific case, 2θ becomes 30 degrees, for which the sine value is easily accessible.

Solution:

(a) The expression given is $E = \tan 15^\circ + \cot 15^\circ$.

(b) Convert the terms to sine and cosine: $E = \frac{\sin 15^\circ}{\cos 15^\circ} + \frac{\cos 15^\circ}{\sin 15^\circ}$.

(c) Combine the fractions using a common denominator: $E = \frac{\sin^2 15^\circ + \cos^2 15^\circ}{\sin 15^\circ \cos 15^\circ}$.

(d) Use the fundamental identity $\sin^2\theta + \cos^2\theta = 1$: $E = \frac{1}{\sin 15^\circ \cos 15^\circ}$.

(e) Multiply the numerator and denominator by 2 to use the double angle formula for sine: $E = \frac{2}{2\sin 15^\circ \cos 15^\circ}$.

(f) Recall that $2\sin\theta\cos\theta = \sin 2\theta$: $E = \frac{2}{\sin(2\cdot 15^\circ)} = \frac{2}{\sin 30^\circ}$.

(g) We know that $\sin 30^\circ = 1/2$.

(h) Substitute this value back into the expression: $E = \frac{2}{1/2} = 4$.

(i) This elegant method avoids the need to calculate the specific radical values of $\tan 15^\circ$ (which is $2 - \sqrt{3}$).

Final Answer: The value is 4.

Answer: (B)

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Q44.

Solution

Concept:
This problem showcases a standard product involving the cosine of angles that are in a geometric progression with a common ratio of 2. For such products of the form $\cos\theta \cos 2\theta \cos 4\theta \ldots \cos 2^{n-1}\theta$, there is a well known identity: the product equals $\sin(2^n\theta)/(2^n \sin\theta)$. This identity is derived by repeatedly applying the double angle formula for sine. Specifically, multiplying the expression by $\sin\theta$ and dividing by it allows for a "telescoping" effect where each cosine term merges with a preceding sine term to form a sine of double the angle. This process continues until the entire product is condensed into a single sine term.

Solution:

(a) We need to evaluate $P = \cos 20^\circ \cos 40^\circ \cos 80^\circ$.

(b) Notice the angles are $20^\circ, 2 \cdot 20^\circ,$ and $4 \cdot 20^\circ$.

(c) Let $\theta = 20^\circ$. The number of terms n is 3.

(d) Apply the identity $\prod_{k=0}^{n-1} \cos(2^k\theta) = \frac{\sin(2^n\theta)}{2^n \sin\theta}$.

(e) For our specific values: $P = \frac{\sin(2^3 \cdot 20^\circ)}{2^3 \sin 20^\circ}$.

(f) $P = \frac{\sin(8 \cdot 20^\circ)}{8 \sin 20^\circ} = \frac{\sin 160^\circ}{8 \sin 20^\circ}$.

(g) Use the supplementary angle identity $\sin(180^\circ - \phi) = \sin\phi$: $\sin 160^\circ = \sin(180^\circ - 20^\circ) = \sin 20^\circ$.

(h) Substitute this back into the fraction: $P = \frac{\sin 20^\circ}{8 \sin 20^\circ}$.

(i) Since $\sin 20^\circ$ is not zero, they cancel out.

(j) The final result is $1/8$.

Final Answer: The product is 1/8.

Answer: (C)

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Q45.

Solution

Concept:
Solving trigonometric equations within a specific interval requires transforming the equation into a polynomial form of a single trigonometric ratio. In this problem, we have a mix of $\sin^2 x$ and $\sin^2 2x$. The double angle formula $\sin 2x = 2\sin x \cos x$ is utilized to expand the term, and then the Pythagorean identity is used to express the entire equation in terms of $\cos^2 x$ or $\sin^2 x$. After simplification, the equation typically factors into basic trigonometric statements like $\cos^2 x = k$. It is essential to consider the full range of the interval $[0, 2\pi]$ and count each distinct solution that satisfies the equation to find the total count.

Solution:

(a) The equation is $2\sin^2 x + \sin^2 2x = 2$.

(b) Use the identity $\sin 2x = 2\sin x \cos x$: $2\sin^2 x + (2\sin x \cos x)^2 = 2$.

(c) $2\sin^2 x + 4\sin^2 x \cos^2 x = 2$.

(d) Divide the entire equation by 2: $\sin^2 x + 2\sin^2 x \cos^2 x = 1$.

(e) Rearrange the equation: $2\sin^2 x \cos^2 x = 1 - \sin^2 x$.

(f) Use the identity $1 - \sin^2 x = \cos^2 x$: $2\sin^2 x \cos^2 x = \cos^2 x$.

(g) Move all terms to one side: $\cos^2 x(2\sin^2 x - 1) = 0$.

(h) Case 1: $\cos^2 x = 0 \implies \cos x = 0$. In $[0, 2\pi]$, the solutions are $x = \pi/2$ and $x = 3\pi/2$.

(i) Case 2: $2\sin^2 x - 1 = 0 \implies \sin^2 x = 1/2 \implies \sin x = \pm 1/\sqrt{2}$.

(j) In $[0, 2\pi]$, these solutions are $x = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.

(k) Total number of distinct solutions: $2 + 4 = 6$.

Final Answer: There are 6 solutions.

Answer: (C)

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Q46.

Solution

Concept:
The derivative of one function with respect to another function is a variation of the chain rule. If we are asked to find the derivative of $u(x)$ with respect to $v(x)$, we calculate it as the ratio of their individual derivatives with respect to x: $du/dv = (du/dx)/(dv/dx)$. This technique is essential for comparing rates of change between two mathematical models. In this specific problem, we differentiate an exponential function involving a trigonometric exponent and a standard trigonometric function. The chain rule is applied to the exponential term, where the derivative of $e^{g(x)}$ is $e^{g(x)} \cdot g'(x)$. The final result is simplified by expressing the ratio of cosine and sine as the cotangent function.
Solution:

(a) Let $u = e^{\sin x}$ and $v = \cos x$.

(b) We need to find $\frac{du}{dv}$, which is given by $\frac{du/dx}{dv/dx}$.

(c) First, calculate the derivative of u with respect to x using the chain rule: $\frac{du}{dx} = \frac{d}{dx}(e^{\sin x}) = e^{\sin x} \cdot \frac{d}{dx}(\sin x) = e^{\sin x} \cos x$.

(d) Next, calculate the derivative of v with respect to x: $\frac{dv}{dx} = \frac{d}{dx}(\cos x) = -\sin x$.

(e) Now, substitute these derivatives into the ratio formula: $\frac{du}{dv} = \frac{e^{\sin x} \cos x}{-\sin x}$.

(f) Separate the exponential term and the trigonometric fraction: $\frac{du}{dv} = -e^{\sin x} \cdot \left(\frac{\cos x}{\sin x}\right)$.

(g) Use the trigonometric identity $\cot x = \frac{\cos x}{\sin x}$: $\frac{du}{dv} = -e^{\sin x} \cot x$.

(h) This result represents how the value of $e^{\sin x}$ changes relative to the change in $\cos x$ at any point x.

Final Answer: The derivative is $-e^{\sin x} \cot x$.

Answer: (A)

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Q47.

Solution

Concept:
Infinite nested radicals represent a recursive relationship where the function is defined in terms of itself. For a function $y = \sqrt{f(x) + \sqrt{f(x) + \dots}}$, we can observe that the entire expression under the first square root, starting from the second $f(x)$, is identical to the original function y. This allows us to replace the infinite tail with y, resulting in the algebraic equation $y = \sqrt{f(x) + y}$. By squaring both sides and using implicit differentiation, we can find dy/dx. This method effectively turns a complex limit problem into a simple differential equation. The resulting formula $dy/dx = f'(x)/(2y - 1)$ is a standard result for this specific class of infinite series problems.

Solution:

(a) The given function is $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \dots \infty}}}$.

(b) Because the sequence is infinite, we can rewrite it as: $y = \sqrt{\sin x + y}$.

(c) To remove the square root, square both sides of the equation: $y^2 = \sin x + y$.

(d) Now, differentiate both sides with respect to x using implicit differentiation.

(e) The derivative of y^2 is $2y\frac{dy}{dx}$ and the derivative of y is $\frac{dy}{dx}$.

(f) So, $2y\frac{dy}{dx} = \cos x + \frac{dy}{dx}$.

(g) Rearrange the equation to group all terms containing $\frac{dy}{dx}$ on one side: $2y\frac{dy}{dx} - \frac{dy}{dx} = \cos x$.

(h) Factor out $\frac{dy}{dx}$: $\frac{dy}{dx}(2y - 1) = \cos x$.

(i) Solve for the derivative: $\frac{dy}{dx} = \frac{\cos x}{2y-1}$.

(j) This elegant result shows that the slope of this infinite curve is directly proportional to the cosine of x and inversely proportional to the value of y.

Final Answer: The derivative is $\frac{\cos x}{2y-1}$.

Answer: (A)

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Q48.

Solution

Concept:

Evaluating limits involving trigonometric functions often requires the use of standard limits, specifically $\lim_{\theta\to 0}(\sin\theta/\theta) = 1$. When dealing with $1 - \cos(kx)$, the identity $1 - \cos\theta = 2\sin^2(\theta/2)$ is extremely powerful as it converts a subtraction problem into a product of sines. This transformation allows us to align the terms with the denominator to use the standard limit. Alternatively, L'Hopital's Rule can be applied because the limit initially presents an indeterminate form of $0/0$. By differentiating the numerator and denominator twice, we can resolve the indeterminacy and find the specific numerical value of the limit.

Solution:

(a) We need to evaluate $L = \lim_{x\to 0} \frac{1-\cos 4x}{x^2}$.

(b) Substituting $x = 0$ gives $(1 - \cos 0)/0 = (1 - 1)/0 = 0/0$, which is indeterminate.

(c) Method 1: Using Trigonometric Identity.

(d) Recall that $1 - \cos\theta = 2\sin^2(\theta/2)$.

(e) Here $\theta = 4x$, so $1 - \cos 4x = 2\sin^2(2x)$.

(f) Substitute this into the limit: $L = \lim_{x\to 0} \frac{2\sin^2 2x}{x^2}$.

(g) Rewrite the expression to use the standard limit $(\sin\theta/\theta)$: $L = 2 \cdot \lim_{x\to 0} \left(\frac{\sin 2x}{x}\right)^2$.

(h) Multiply and divide by 2 inside the parenthesis: $L = 2 \cdot \lim_{x\to 0} \left(2 \cdot \frac{\sin 2x}{2x}\right)^2$.

(i) Since $\lim_{x\to 0} \frac{\sin 2x}{2x} = 1$, we have: $L = 2 \cdot (2 \cdot 1)^2 = 2 \cdot 4 = 8$.

(j) Method 2: L'Hopital's Rule. Differentiating numerator and denominator: $\lim \frac{4\sin 4x}{2x}$. Differentiating again: $\lim \frac{16\cos 4x}{2} = 16(1)/2 = 8$.

Final Answer: The limit is 8.

Answer: (B)

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Q49.

Solution

Concept:
The slope of the tangent to a curve $y = f(x)$ at any point is given by the derivative dy/dx. If a tangent is parallel to a given line, their slopes must be equal. This geometric condition allows us to set the derivative of the curve equal to the slope of the line, creating an equation that can be solved for the x-coordinate of the point of tangency. Once the x-coordinate is found, the corresponding y-coordinate is determined by substituting x back into the original curve equation. This process is a classic application of derivatives in determining the local linear properties of functions and identifying specific points on a graph that satisfy global geometric constraints.
Solution:

(a) The given curve is $y = x^2$.

(b) The slope of the tangent at any point (x, y) is $dy/dx = 2x$.

(c) The tangent is parallel to the line $y = 4x - 5$.

(d) The slope of the line $y = mx + c$ is m. Here, the slope is 4.

(e) Since parallel lines have equal slopes, we set the derivative equal to the slope of the line: $2x = 4$.

(f) Solving for x: $x = 2$.

(g) Now, find the corresponding y-coordinate by substituting $x = 2$ into the equation of the curve $y = x^2$.

(h) $y = (2)^2 = 4$.

(i) The point of tangency is $(2, 4)$.

(j) Verification: The tangent at $(2, 4)$ has an equation $y - 4 = 4(x - 2)$, which simplifies to $y = 4x - 4$. This line is indeed parallel to $y = 4x - 5$ as they share the same slope but have different intercepts.

Final Answer: The point is (2, 4).

Answer: (A)

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Q50.

Solution

Concept:
In vector geometry, if a vector $\vec{a}$ is perpendicular to both vectors $\vec{b}$ and $\vec{c}$, then $\vec{a}$ must be parallel to the cross product $\vec{b} \times \vec{c}$. This means $\vec{a}$ can be expressed as $k(\vec{b} \times \vec{c})$ for some scalar k. The magnitude of the cross product is given by $|\vec{b}||\vec{c}| \sin\theta$, where θ is the angle between them. Since $\vec{a}, \vec{b},$ and $\vec{c}$ are unit vectors, their magnitudes are all 1. By setting the magnitude of $\vec{a}$ equal to 1, we can solve for the scalar k. This identifies the unit vector $\vec{a}$ in terms of the cross product, which is a fundamental way to define a normal vector to a plane spanned by two other vectors.
Solution:

(a) We are given that $\vec{a}, \vec{b}, \vec{c}$ are unit vectors, so $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$.

(b) Given $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \cdot \vec{c} = 0$, which means $\vec{a}$ is perpendicular to both $\vec{b}$ and $\vec{c}$.

(c) Therefore, $\vec{a}$ is parallel to $\vec{b} \times \vec{c}$. We can write $\vec{a} = \lambda(\vec{b} \times \vec{c})$.

(d) Take the magnitude of both sides: $|\vec{a}| = |\lambda||\vec{b} \times \vec{c}|$.

(e) We know $|\vec{a}| = 1$. Also, $|\vec{b} \times \vec{c}| = |\vec{b}||\vec{c}| \sin(\pi/6)$.

(f) Substitute the known magnitudes and the angle: $1 = |\lambda| \cdot (1) \cdot (1) \cdot \sin(30°)$.

(g) $1 = |\lambda| \cdot (1/2)$.

(h) This gives $|\lambda| = 2$, so $\lambda = \pm 2$.

(i) Substituting the value of λ back into the vector equation: $\vec{a} = \pm 2(\vec{b} \times \vec{c})$.

(j) This shows that $\vec{a}$ is a unit vector that is orthogonal to the plane of $\vec{b}$ and $\vec{c}$, scaled by the reciprocal of the sine of the angle between them.

Final Answer: The vector $\vec{a} = \pm 2(\vec{b} \times \vec{c})$.

Answer: (A)

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Answer Key

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	A	2	A	3	A	4	A	5	A
6	A	7	C	8	B	9	B	10	B
11	A	12	A	13	A	14	A	15	A
16	A	17	A	18	B	19	B	20	A
21	D	22	A	23	A	24	A	25	B
26	A	27	B	28	A	29	A	30	A
31	A	32	A	33	B	34	B	35	A
36	C	37	A	38	A	39	A	40	A
41	A	42	A	43	B	44	C	45	C
46	A	47	A	48	B	49	A	50	A