MHT-CET Mathematics Sample Paper-12

Duration: 90 Minutes **Maximum Marks: 100**

Instructions

- This paper contains a total of **50** Multiple Choice Questions.
- Each correct answer carries **+2 marks**.
- No negative marking for incorrect questions.
- Use of mobile phones, smartwatches, or any electronic gadgets is strictly prohibited.
- No marks will be deducted for questions that are left unattempted.

Q1. If $\int \frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}} dx = \sin^{-1}\left(\frac{\sin x + \cos x}{k}\right) + C$, then the value of k is:

(A) 2

(B) 3

(C) $\sqrt{8}$

(D) 9

Q2. The value of $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$ is:

(A) $\frac{\pi^2}{2}$

(B) $\frac{\pi^2}{4}$

(C) $\frac{\pi}{4}$

(D) π

Q3. The area of the region bounded by the curve $y = x|x|$, x-axis and the ordinates $x = -1$ and $x = 1$ is:

(A) $\frac{1}{3}$

(B) $\frac{2}{3}$

(C) 1

(D) 0

Q4. The degree of the differential equation $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{d^2y}{dx^2}$ is:

(A) 3

(B) 2

(C) 1

(D) 3/2

Q5. If ω is a complex cube root of unity, then the value of $(1+\omega-\omega^2)^6$ is:

(A) 16

(B) 32

(C) 64

(D) 128

Q6. If the roots of $x^2 - bx + c = 0$ are two consecutive integers, then $b^2 - 4c$ is:

(A) 0

(B) 1

(C) 2

(D) 3

Q7. The sum of the series $0.5 + 0.55 + 0.555 + \ldots$ to n terms is:

(A) $\frac{5}{9}[n - \frac{1}{9}(1-10^{-n})]$

(B) $\frac{5}{81}[9n - 1 + 10^{-n}]$

(C) $\frac{5}{9}[n - \frac{1}{10}(1-10^{-n})]$

(D) $\frac{5}{81}[n - \frac{1}{9}(1-10^{-n})]$

Q8. The coefficient of x^4 in the expansion of $(1+x+x^2+x^3)^n$ is:

(A) ${}^nC_4 +{}^n C_2$

(B) ${}^nC_4 +{}^n C_1 +{}^n C_1 \cdot{}^n C_2$

(C) ${}^nC_4 +{}^n C_3 +{}^n C_2$

(D) ${}^nC_4 +^n C_2 +^n C_1 \cdot^n C_2$

Q9. In how many ways can 5 prizes be distributed among 4 students when every student can take one or more prizes?

(A) 1024

(B) 625

(C) 120

(D) 600

(E) 20

Q10. The distance of the point $(1, 2)$ from the line $x + y + 5 = 0$ measured along the line $y = 3x$ is:

(A) $4\sqrt{10}$

(B) $\sqrt{10}$

(C) $2\sqrt{10}$

(D) $3\sqrt{10}$

Q11. The equation of the circle which passes through the origin and cuts off intercepts 3 and 4 from the positive parts of the axes respectively is:

(A) $x^2 + y^2 - 3x - 4y = 0$

(B) $x^2 + y^2 + 3x + 4y = 0$

(C) $x^2 + y^2 - 6x - 8y = 0$

(D) $x^2 + y^2 + 6x + 8y = 0$

Q12. The eccentricity of the ellipse $9x^2 + 5y^2 - 30y = 0$ is:

(A) $1/3$

(B) $2/3$

(C) $3/4$

(D) $4/5$

Q13. If $\vec{a} + \vec{b} + \vec{c} = 0$, $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 7$, then the angle between $\vec{a}$ and $\vec{b}$ is:

(A) $30°$

(B) $45°$

(C) $60°$

(D) $90°$

Q14. The shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$ is:

(A) $1/\sqrt{6}$

(B) $1/\sqrt{3}$

(C) $1/\sqrt{2}$

(D) 0

Q15. If $f(x) = \begin{cases} \frac{1-\cos 4x}{x^2} & x < 0 \\ a & x = 0 \\ \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}-4}} & x > 0 \end{cases}$ is continuous at $x = 0$, then $a =$

(A) 4

(B) 6

(C) 8

(D) 10

Q16. If $y = \sec(\tan^{-1} x)$, then $\frac{dy}{dx}$ at $x = 1$ is:

(A) $1/\sqrt{2}$

(B) $\sqrt{2}$

(C) 1

(D) $1/2$

Q17. The perimeter of a sector is p. The area of the sector is maximum when its radius is:

(A) $p/2$

(B) $p/4$

(C) $p/6$

(D) $p/8$

Q18. The integral $\int \frac{dx}{x^2(x^4+1)^{3/4}}$ is equal to:

(A) $-\frac{(x^4+1)^{1/4}}{x} + C$

(B) $\frac{(x^4+1)^{1/4}}{x} + C$

(C) $-\frac{(x^4+1)^{1/4}}{x^2} + C$

(D) $\frac{(x^4+1)^{1/4}}{x^2} + C$

Q19. The value of $\int_{-a}^{a} \frac{dx}{x+\sqrt{a^2-x^2}}$ is:

(A) $\pi/2$

(B) $\pi/4$

(C) π

(D) 0

Q20. The area bounded by the curve $y = \log_e x$, x-axis and the ordinate $x = e$ is:

(A) e

(B) 1

(C) $e - 1$

(D) 2

Q21. The solution of the differential equation $\frac{dy}{dx} = \frac{x+y+1}{x+y-1}$ is:

(A) $y - x + \log(x + y) = C$

(B) $y + x + \log(x + y) = C$

(C) $y - x = \log(x + y) + C$

(D) $y - x + \log|x + y| = C$

Q22. If $z = x + iy$ and $|z - 1| = |z + 1|$, then the locus of z is:

(A) x-axis

(B) y-axis

(C) $x = y$

(D) $x^2 + y^2 = 1$

Q23. If α, β are the roots of $x^2 - 2x + 4 = 0$, then $\alpha^n + \beta^n =$

(A) $2^{n+1}\cos(n\pi/3)$

(B) $2^n\cos(n\pi/3)$

(C) $2^{n+1}\sin(n\pi/3)$

(D) $2^n\sin(n\pi/3)$

Q24. If the sum of n terms of an A.P. is $3n^2 + 5n$ and its m^{th} term is 164, then $m =$

(A) 26

(B) 27

(C) 28

(D) 29

Q25. The sum of coefficients in the expansion of $(1 + x - 3x^2)^{2163}$ is:

(A) 0

(B) 1

(C) -1

(D) 2^{2163}

Q26. Five digit numbers are formed using the digits $1, 2, 3, 4, 5$ without repetition. The probability that the number is divisible by 4 is:

(A) $1/5$

(B) $2/5$

(C) $3/5$

(D) $4/5$

Q27. The angle between the lines $x - 2y + 3 = 0$ and $3x + y - 1 = 0$ is:

(A) $\tan^{-1}(7)$

(B) $\tan^{-1}(1/7)$

(C) $45°$

(D) $90°$

Q28. If a circle passes through the point $(0, 0)$, $(a, 0)$, $(0, b)$, then its center is:

(A) (a, b)

(B) $(a/2, b/2)$

(C) $(a/2, 0)$

(D) $(0, b/2)$

Q29. The length of the latus rectum of the parabola $x^2 = -8y$ is:

(A) 2

(B) 4

(C) 8

(D) 16

Q30. If $|\vec{a}| = 2$, $|\vec{b}| = 5$ and $|\vec{a} \times \vec{b}| = 8$, then $\vec{a} \cdot \vec{b}$ is:

(A) 6

(B) 10

(C) 4

(D) $\sqrt{6}$

Q31. The equation of the plane through $(1, 2, 3)$ and parallel to the plane $2x+3y-4z = 0$ is:

(A) $2x + 3y - 4z + 4 = 0$

(B) $2x + 3y - 4z + 2 = 0$

(C) $2x + 3y - 4z = 4$

(D) $2x + 3y - 4z = -4$

Q32. The value of $\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$ is:

(A) 1

(B) 2

(C) 4

(D) 0

Q33. If $x^y = e^{x-y}$, then $\frac{dy}{dx} =$

(A) $\frac{1}{(1+\log x)^2}$

(B) $\frac{\log x}{(1+\log x)^2}$

(C) $\frac{\log x}{1+\log x}$

(D) $\frac{x}{1+\log x}$

Q34. The function $f(x) = x^x$ has a stationary point at $x =$

(A) e

(B) $1/e$

(C) 1

(D) 0

Q35. The value of $\int \frac{dx}{x + x\log x}$ is:

(A) $\log|1 + \log x| + C$

(B) $\log|x + \log x| + C$

(C) $1 + \log x + C$

(D) $\frac{1}{1+\log x} + C$

Q36. The value of $\int_0^1 \frac{\tan^{-1} x}{1+x^2} dx$ is:

(A) $\pi^2/8$

(B) $\pi^2/16$

(C) $\pi^2/32$

(D) $\pi/4$

Q37. The area bounded by $y = \sin x$, $y = \cos x$ and the y-axis in the first quadrant is:

(A) $\sqrt{2} - 1$

(B) $\sqrt{2} + 1$

(C) $2 - \sqrt{2}$

(D) $2 + \sqrt{2}$

Q38. The integrating factor of the differential equation $\frac{dy}{dx} + \frac{y}{x \log x} = \frac{1}{x}$ is:

(A) $\log x$

(B) x

(C) e^x

(D) $1/x$

Q39. If $\frac{z-1}{z+1}$ is purely imaginary, then $|z| =$

(A) 1

(B) 2

(C) 3

(D) 0

Q40. The number of real roots of the equation $e^x + x = 0$ is:

(A) 0

(B) 1

(C) 2

(D) ∞

Q41. The n^{th} term of the series $1 + \frac{1+2}{2} + \frac{1+2+3}{3} + \ldots$ is:

(A) $\frac{n+1}{2}$

(B) $\frac{n(n+1)}{2}$

(C) $\frac{n}{2}$

(D) n^2

Q42. If in the expansion of $(1 + x)^n$, the coefficients of x^r and x^{r+1} are equal, then $n =$

(A) $2r$

(B) $2r + 1$

(C) $2r - 1$

(D) r

Q43. The number of ways in which 7 persons can be seated at a round table if two particular persons must not sit together is:

(A) 120

(B) 720

(C) 480

(D) 240

Q44. Two events A and B have probabilities 0.25 and 0.50 respectively. The probability that both occur simultaneously is 0.14. Then the probability that neither A nor B occurs is:

(A) 0.39

(B) 0.25

(C) 0.11

(D) 0.61

Q45. The lines $x - 2y + 3 = 0$, $1x + 2y + 7 = 0$ and $x - 2y + 9 = 0$ are:

(A) Concurrent

(B) Parallel

(C) Sides of a triangle

(D) None of these

Q46. The radius of the circle $x^2 + y^2 + 4x - 6y + 12 = 0$ is:

(A) 1

(B) 2

(C) 3

(D) 4

Q47. The equation of the tangent to the parabola $y^2 = 12x$ at the point $(3, 6)$ is:

(A) $x - y + 3 = 0$

(B) $x + y - 3 = 0$

(C) $2x - y + 3 = 0$

(D) $x - 2y + 3 = 0$

Q48. The projection of $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$ on $\vec{b} = 4\hat{i} - 4\hat{j} + 7\hat{k}$ is:

(A) 19/9

(B) 19/7

(C) 9/19

(D) 8/9

Q49. The angle between the planes $2x - y + z = 6$ and $x + y + 2z = 3$ is:

(A) 30°

(B) 45°

(C) 60°

(D) 90°

Q50. If $\int_0^k \frac{dx}{2+8x^2} = \frac{\pi}{16}$, then $k =$

(A) 1

(B) 1/2

(C) 1/4

(D) 2

Detailed Solutions

Q1.

Solution

Concept:
For an indefinite integral of the form $\int \frac{f'(x)}{\sqrt{a^2-(f(x))^2}}dx$, the solution is $\sin^{-1}\left(\frac{f(x)}{a}\right) + C$. We need to manipulate the denominator $\sqrt{8 - \sin 2x}$ into a perfect square form involving $(\sin x + \cos x)$.

Solution:

(a) Let $I = \int \frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}} dx$.

(b) We know that $(\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2 \sin x \cos x = 1 + \sin 2x$.

(c) Therefore, $\sin 2x = (\sin x + \cos x)^2 - 1$.

(d) Substitute this into the denominator: $8 - \sin 2x = 8 - [(\sin x + \cos x)^2 - 1] = 9 - (\sin x + \cos x)^2$.

(e) Now let $t = \sin x + \cos x$. Then $dt = (\cos x - \sin x)dx$.

(f) The integral becomes $I = \int \frac{dt}{\sqrt{3^2 - t^2}}$.

(g) Using the formula $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a) + C$, we get $I = \sin^{-1}(t/3) + C$.

(h) Substituting t back, $I = \sin^{-1}\left(\frac{\sin x + \cos x}{3}\right) + C$.

(i) Comparing with the given form, $k = 3$.

Final Answer: The value of k is 3.

Answer: (B)

Go Back to Question 1

Q2.

Solution

Concept:
This is a definite integral problem using the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. This property is often used to eliminate the x term in the numerator of an integrand involving trigonometric functions.

Solution:

(a) Let $I = \int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$.

(b) Using the property $\int_0^\pi f(x)dx = \int_0^\pi f(\pi - x)dx$:

(c) $I = \int_0^\pi \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx = \int_0^\pi \frac{(\pi-x)\sin x}{1+\cos^2 x}dx$.

(d) Adding the two expressions for I: $2I = \int_0^\pi \frac{\pi\sin x}{1+\cos^2 x}dx$.

(e) Let $u = \cos x$, then $du = -\sin x dx$. When $x = 0, u = 1$; when $x = \pi, u = -1$.

(f) $2I = \pi\int_1^{-1} \frac{-du}{1+u^2} = \pi\int_{-1}^{1} \frac{du}{1+u^2}$.

(g) $2I = \pi[\tan^{-1} u]_{-1}^{1} = \pi[\tan^{-1}(1) - \tan^{-1}(-1)] = \pi[\pi/4 - (-\pi/4)] = \pi[\pi/2]$.

(h) $2I = \pi^2/2 \implies I = \pi^2/4$.

Final Answer: The value is $\pi^2/4$.

Answer: (B)

Go Back to Question 2

Q3.

Solution

Concept:

The area under a curve $y = f(x)$ from $x = a$ to $x = b$ is given by $\int_a^b |f(x)|dx$. Since the function involves an absolute value $|x|$, we must split the integral at $x = 0$ where the definition of $|x|$ changes.

Solution:

(a) The curve is $y = x|x|$.

(b) For $x \geq 0$, $y = x^2$. For $x < 0$, $y = -x^2$.

(c) The area is $A = \int_{-1}^{1} |y|dx = \int_{-1}^{0} |-x^2|dx + \int_0^1 |x^2|dx$.

(d) $A = \int_{-1}^{0} x^2 dx + \int_0^1 x^2 dx$.

(e) $A = [\frac{x^3}{3}]_{-1}^{0} + [\frac{x^3}{3}]_0^1$.

(f) $A = (0 - (-1/3)) + (1/3 - 0) = 1/3 + 1/3 = 2/3$.

Final Answer: The area is 2/3 sq. units.

Answer: (B)

Go Back to Question 3

Q4.

Solution

Concept:
The degree of a differential equation is the power of the highest order derivative, provided the equation is expressed as a polynomial in the derivatives. This requires removing fractional powers (radicals).

Solution:

(a) The given equation is $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{d^2y}{dx^2}$.

(b) To remove the fractional power $3/2$, we square both sides.

(c) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$.

(d) The highest order derivative is $\frac{d^2y}{dx^2}$ (order 2).

(e) The power of this highest order derivative is 2.

(f) Therefore, the degree of the differential equation is 2.

Final Answer: The degree is 2.

Answer: (B)

Go Back to Question 4

Q5.

Solution

Concept:
Properties of the complex cube root of unity ω: $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$. These properties allow for the simplification of powers of expressions involving ω.
Solution:

(a) We need to evaluate $(1 + \omega - \omega^2)^6$.

(b) From $1 + \omega + \omega^2 = 0$, we can write $1 + \omega = -\omega^2$.

(c) Substitute this into the expression: $(-\omega^2 - \omega^2)^6$.

(d) This simplifies to $(-2\omega^2)^6$.

(e) Expanding the power: $(-2)^6 \cdot (\omega^2)^6 = 64 \cdot \omega^{12}$.

(f) Since $\omega^{12} = (\omega^3)^4 = 1^4 = 1$.

(g) The final value is $64 \cdot 1 = 64$.

Final Answer: The value is 64.

Answer: (C)

Go Back to Question 5

Q6.

Solution

Concept:
In a quadratic equation of the form $ax^2 + bx + c = 0$, the roots α and β satisfy specific relationships with the coefficients. Specifically, the sum of roots $\alpha + \beta = -b/a$ and the product of roots $\alpha \cdot \beta = c/a$. When the roots are consecutive integers, their absolute difference $|\alpha - \beta|$ must be exactly 1. This property allows us to relate the discriminant of the quadratic equation to the given coefficients.

Solution:

(a) Let the roots of the quadratic equation $x^2 - bx + c = 0$ be α and β.

(b) According to the problem statement, these roots are consecutive integers. Therefore, we can define $\beta = \alpha + 1$.

(c) The difference between the roots is $|\alpha - \beta| = |\alpha - (\alpha + 1)| = 1$.

(d) Squaring both sides of this difference gives $(\alpha - \beta)^2 = 1^2 = 1$.

(e) We know from algebraic identities that $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$.

(f) For the given equation $x^2 - bx + c = 0$, the sum of roots $\alpha + \beta$ is $-(-b)/1 = b$.

(g) The product of the roots $\alpha\beta$ is $c/1 = c$.

(h) Substituting these values into our identity: $b^2 - 4c = 1$.

(i) The expression $b^2 - 4c$ represents the discriminant of the quadratic equation.

(j) In any quadratic equation where the coefficient of x^2 is 1, if the roots differ by unity, the discriminant is always equal to 1.

Final Answer: The value of $b^2 - 4c$ is 1.

Answer: (B)

Go Back to Question 6

Q7.

Solution

Concept:
The given series $0.5 + 0.55 + 0.555 + \dots$ is an example of a sequence that can be converted into a Geometric Progression (G.P.). By factoring out the common digit and multiplying/dividing by 9, we can express each term as a difference between a constant and a power of 10. This technique is standard for summing decimals with repeating digits.
Solution:

(a) Let $S_n = 0.5 + 0.55 + 0.555 + \dots$ to n terms.

(b) Factor out 5: $S_n = 5[0.1 + 0.11 + 0.111 + \dots]$.

(c) Multiply and divide by 9: $S_n = \frac{5}{9}[0.9 + 0.99 + 0.999 + \dots]$.

(d) Write each term in terms of powers of 10: $S_n = \frac{5}{9}[(1-0.1)+(1-0.01)+(1-0.001)+\dots]$.

(e) Group the 1s and the decimal powers separately: $S_n = \frac{5}{9}[(1 + 1 + \dots \text{ n times}) - (0.1 + 0.01 + 0.001 + \dots)]$.

(f) The first part is n. The second part is a G.P. with $a = 0.1$ and $r = 0.1$.

(g) Sum of G.P.: $\frac{0.1(1-(0.1)^n)}{1-0.1} = \frac{0.1(1-10^{-n})}{0.9} = \frac{1}{9}(1 - 10^{-n})$.

(h) Combine the terms: $S_n = \frac{5}{9}[n - \frac{1}{9}(1 - 10^{-n})]$.

(i) Simplifying further: $S_n = \frac{5}{81}[9n - 1 + 10^{-n}]$.

Final Answer: The sum is $\frac{5}{81}[9n - 1 + 10^{-n}]$.

Answer: (B)

Go Back to Question 7

Q8.

Solution

Concept:
The expression $(1+x+x^2+x^3)^n$ can be simplified using the formula for a finite geometric series or by factoring. Since $1+x+x^2+x^3 = (1+x)+x^2(1+x) = (1+x)(1+x^2)$, the expansion becomes $(1+x)^n(1+x^2)^n$. Finding the coefficient of a specific power of x involves using the general term of the binomial expansion $T_{r+1} = ^nC_r a^{n-r} b^r$.
Solution:

(a) Rewrite the expression: $(1+x+x^2+x^3)^n = [(1+x)(1+x^2)]^n = (1+x)^n(1+x^2)^n$.

(b) Expand both binomials:

(c) $(1+x)^n = ^nC_0 + ^nC_1x + ^nC_2x^2 + ^nC_3x^3 + ^nC_4x^4 + \dots$

(d) $(1+x^2)^n = ^nC_0 + ^nC_1x^2 + ^nC_2x^4 + \dots$

(e) To find the coefficient of x^4, we multiply terms from both expansions whose powers sum to 4.

(f) Case 1: x^4 from first and constant from second: $^nC_4 \cdot ^nC_0 = ^nC_4$.

(g) Case 2: x^2 from first and x^2 from second: $^nC_2 \cdot ^nC_1$.

(h) Case 3: Constant from first and x^4 from second: $^nC_0 \cdot ^nC_2 = ^nC_2$.

(i) Sum the coefficients: $^nC_4 + (^nC_2 \cdot ^nC_1) + ^nC_2$.

(j) Note that $^nC_1 = n$, so the result is $^nC_4 + ^nC_2 + n \cdot ^nC_2$.

Final Answer: The coefficient is $^nC_4 + ^nC_2 + ^nC_1 \cdot ^nC_2$.

Answer: (D)

Go Back to Question 8

Q9.

Solution

Concept:
This problem involves the fundamental principle of counting. When distributing distinct items (prizes) among distinct recipients (students), where each recipient can receive any number of items, we analyze the choices available for each individual item. This is a classic example of "independent choices" leading to an exponential result.
Solution:

(a) We have 5 distinct prizes and 4 distinct students.

(b) Let the prizes be P_1, P_2, P_3, P_4, P_5.

(c) For the first prize P_1, it can be given to any of the 4 students. So, there are 4 options.

(d) For the second prize P_2, since a student can receive more than one prize, it can also be given to any of the 4 students. Thus, there are 4 options.

(e) Similarly, for the third prize P_3, there are 4 options.

(f) For the fourth prize P_4, there are 4 options.

(g) For the fifth prize P_5, there are 4 options.

(h) By the multiplication principle of counting, the total number of ways to distribute the prizes is the product of the number of options for each prize.

(i) Total ways = $4 \times 4 \times 4 \times 4 \times 4 = 4^5$.

(j) Calculating the value: $4^5 = 1024$.

Final Answer: The total number of ways is 1024.

Answer: (A)

Go Back to Question 9

Q10.

Solution

Concept:
Finding the distance between a point and a line measured along a specific direction (another line) requires finding the intersection point. We move from the given point $(1, 2)$ along the direction of the line $y = 3x$ until we hit the target line $x + y + 5 = 0$. The distance is then calculated using the standard distance formula between two points.

Solution:

(a) The target line is $L_1 : x + y + 5 = 0$.

(b) We are at point $P(1, 2)$. We move along a direction parallel to $L_2 : y = 3x$.

(c) Any point on a line passing through $(1, 2)$ parallel to $y = 3x$ has the equation $y-2 = 3(x-1)$, which simplifies to $y = 3x - 1$.

(d) To find the intersection point Q of this path and the target line L_1, substitute $y = 3x - 1$ into $x + y + 5 = 0$.

(e) $x + (3x - 1) + 5 = 0 \implies 4x + 4 = 0 \implies x = -1$.

(f) Substitute $x = -1$ back into the path equation: $y = 3(-1) - 1 = -4$.

(g) So, the intersection point Q is $(-1, -4)$.

(h) The required distance is the distance between $P(1, 2)$ and $Q(-1, -4)$.

(i) Distance $d = \sqrt{(1 - (-1))^2 + (2 - (-4))^2} = \sqrt{2^2 + 6^2}$.

(j) $d = \sqrt{4 + 36} = \sqrt{40} = 2\sqrt{10}$.

Final Answer: The distance is $2\sqrt{10}$.

Answer: (C)

Go Back to Question 10

Q11.

Solution

Concept:
The equation of a circle in the Cartesian plane can be determined if we know its center and radius, or specific points it passes through. For a circle passing through the origin $(0,0)$, the constant term in its general equation $x^2 + y^2 + 2gx + 2fy + c = 0$ must be zero ($c = 0$). If the circle cuts intercepts a and b on the x-axis and y-axis respectively, it means the circle passes through the points $(a,0)$ and $(0,b)$. This information allows us to solve for the parameters g and f.
Solution:

(a) The circle passes through the origin $O(0,0)$.

(b) It cuts an intercept of 3 on the positive x-axis, meaning it passes through point $A(3,0)$.

(c) It cuts an intercept of 4 on the positive y-axis, meaning it passes through point $B(0,4)$.

(d) In a circle where the intercepts are on the axes and it passes through the origin, the segment joining $(3,0)$ and $(0,4)$ acts as the diameter of the circle because the angle in a semicircle (the angle at the origin between the axes) is 90 degrees.

(e) Using the diameter form of the circle equation: $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

(f) Here, $(x_1, y_1) = (3,0)$ and $(x_2, y_2) = (0,4)$.

(g) Substituting these values: $(x - 3)(x - 0) + (y - 0)(y - 4) = 0$.

(h) Expanding the terms: $x(x - 3) + y(y - 4) = 0$.

(i) This results in the equation: $x^2 - 3x + y^2 - 4y = 0$, which can be rewritten as $x^2 + y^2 - 3x - 4y = 0$.

(j) This equation satisfies all three points: $(0,0)$, $(3,0)$, and $(0,4)$.

Final Answer: The equation of the circle is $x^2 + y^2 - 3x - 4y = 0$.

Answer: (A)

Go Back to Question 11

Q12.

Solution

Concept:
The eccentricity of an ellipse measures how much the ellipse deviates from being a perfect circle. For an ellipse in the standard form $(x-h)^2/a^2 + (y-k)^2/b^2 = 1$, the eccentricity e is calculated using the formula $e = \sqrt{1 - (\text{smaller semi-axis}^2/\text{larger semi-axis}^2)}$. If the equation is not in standard form, we must first use the method of completing the square to identify the values of a^2 and b^2.

Solution:

(a) The given equation is $9x^2 + 5y^2 - 30y = 0$.

(b) First, we complete the square for the y terms: $5(y^2-6y) = 5(y^2-6y+9-9) = 5(y-3)^2-45$.

(c) Substituting this back into the original equation: $9x^2 + 5(y-3)^2 - 45 = 0$.

(d) Rearranging to the standard form: $9x^2 + 5(y-3)^2 = 45$.

(e) Dividing the entire equation by 45: $\frac{x^2}{5} + \frac{(y-3)^2}{9} = 1$.

(f) Comparing this with $\frac{x^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we find $a^2 = 5$ and $b^2 = 9$.

(g) Since $b^2 > a^2$, this is a vertical ellipse where the major axis is along the y-direction.

(h) The formula for eccentricity in this case is $e = \sqrt{1 - \frac{a^2}{b^2}}$.

(i) Substituting the values: $e = \sqrt{1 - \frac{5}{9}} = \sqrt{\frac{9-5}{9}} = \sqrt{\frac{4}{9}}$.

(j) Thus, $e = 2/3$.

Final Answer: The eccentricity of the ellipse is 2/3.

Answer: (B)

Go Back to Question 12

Q13.

Solution

Concept:
The relationship between three vectors that sum to zero ($\vec{a} + \vec{b} + \vec{c} = 0$) implies that they form the sides of a triangle. To find the angle between any two vectors, say $\vec{a}$ and $\vec{b}$, we isolate them on one side of the equation and take the dot product of the sum with itself. The dot product relates the magnitudes of the vectors and the cosine of the angle between them through the formula $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$.

Solution:

(a) We are given $\vec{a} + \vec{b} + \vec{c} = 0$, which can be written as $\vec{a} + \vec{b} = -\vec{c}$.

(b) Taking the dot product of each side with itself: $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = (-\vec{c}) \cdot (-\vec{c})$.

(c) Expanding the left side: $|\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a} \cdot \vec{b}) = |\vec{c}|^2$.

(d) Substituting the magnitudes $|\vec{a}| = 3, |\vec{b}| = 5, |\vec{c}| = 7$:

(e) $3^2 + 5^2 + 2(3)(5)\cos\theta = 7^2$.

(f) $9 + 25 + 30\cos\theta = 49$.

(g) $34 + 30\cos\theta = 49$.

(h) Subtracting 34 from both sides: $30\cos\theta = 15$.

(i) $\cos\theta = 15/30 = 1/2$.

(j) Since $\cos\theta = 1/2$, the angle θ is $\arccos(1/2) = 60°$.

Final Answer: The angle between $\vec{a}$ and $\vec{b}$ is $60°$.

Answer: (C)

Go Back to Question 13

Q14.

Solution

Concept:
The shortest distance between two skew lines in three-dimensional space is the length of the common perpendicular segment. For lines given in symmetric form, the shortest distance d is calculated using the vector formula involving the cross product of the direction vectors and the dot product with the vector joining two points on the lines. If the lines are coplanar and intersect, the shortest distance is zero. If they do not intersect, the distance is the projection of the segment joining the points onto the normal to both lines.
Solution:

(a) Let Line 1 be $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$. Point $A = (1, 2, 3)$, Direction $\vec{d_1} = (2, 3, 4)$.

(b) Let Line 2 be $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$. Point $B = (2, 4, 5)$, Direction $\vec{d_2} = (3, 4, 5)$.

(c) Vector $\vec{AB} = (2 - 1, 4 - 2, 5 - 3) = (1, 2, 2)$.

(d) The normal vector to both lines is $\vec{n} = \vec{d_1} \times \vec{d_2}$.

(e) $\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \hat{i}(15 - 16) - \hat{j}(10 - 12) + \hat{k}(8 - 9) = -\hat{i} + 2\hat{j} - \hat{k}$.

(f) Magnitude $|\vec{n}| = \sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{1 + 4 + 1} = \sqrt{6}$.

(g) Shortest distance $d = \frac{|\vec{AB} \cdot \vec{n}|}{|\vec{n}|}$.

(h) $\vec{AB} \cdot \vec{n} = (1)(-1) + (2)(2) + (2)(-1) = -1 + 4 - 2 = 1$.

(i) Therefore, $d = \frac{1}{\sqrt{6}}$.

Final Answer: The shortest distance is $1/\sqrt{6}$.

Answer: (A)

Go Back to Question 14

Q15.

Solution

Concept:
A function $f(x)$ is continuous at $x = c$ if the left-hand limit, the right-hand limit, and the function value at that point are all equal ($\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$). In this problem, we evaluate the limits using trigonometric identities and algebraic simplification (rationalization or L'Hopital's Rule) to find the value of the constant a that ensures continuity at the origin.

Solution:

(a) We need $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = a$.

(b) Evaluating the Left Hand Limit (LHL): $\lim_{x\to 0^-} \frac{1-\cos 4x}{x^2}$.

(c) Using the identity $1 - \cos\theta = 2\sin^2(\theta/2)$, we get $1 - \cos 4x = 2\sin^2(2x)$.

(d) LHL $= \lim_{x\to 0} \frac{2\sin^2(2x)}{x^2} = 2\lim_{x\to 0}\left(\frac{\sin 2x}{x}\right)^2 = 2\cdot(2)^2 = 8$.

(e) Evaluating the Right Hand Limit (RHL): $\lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}$.

(f) To solve this, rationalize the denominator: $\frac{\sqrt{x}(\sqrt{16+\sqrt{x}}+4)}{(\sqrt{16+\sqrt{x}}-4)(\sqrt{16+\sqrt{x}}+4)}$.

(g) The denominator becomes $(16+\sqrt{x}) - 16 = \sqrt{x}$.

(h) RHL $= \lim_{x\to 0^+} \frac{\sqrt{x}(\sqrt{16+\sqrt{x}}+4)}{\sqrt{x}} = \lim_{x\to 0^+}(\sqrt{16+\sqrt{x}}+4)$.

(i) Substituting $x = 0$: $\sqrt{16+0} + 4 = 4 + 4 = 8$.

(j) Since LHL = RHL = 8, for the function to be continuous, a must be 8.

Final Answer: The value of a is 8.

Answer: (C)

Go Back to Question 15

Q16.

Solution

Concept:
The problem involves finding the derivative of a composite function where a trigonometric function is combined with an inverse trigonometric function. To differentiate $y = \sec(\tan^{-1} x)$, we can either use the chain rule directly or simplify the expression using trigonometric identities before differentiating. Simplifying the expression often leads to a more manageable algebraic form, which reduces the margin of error when substituting specific values like $x = 1$.

Solution:

(a) Let $\theta = \tan^{-1} x$. This implies that $\tan\theta = x$.

(b) We know the trigonometric identity $\sec^2\theta = 1 + \tan^2\theta$.

(c) Therefore, $\sec\theta = \sqrt{1 + \tan^2\theta} = \sqrt{1 + x^2}$.

(d) Substituting this back into the original equation, we get $y = \sqrt{1 + x^2}$.

(e) Now, we differentiate y with respect to x using the power rule and chain rule.

(f) $\frac{dy}{dx} = \frac{d}{dx}(1 + x^2)^{1/2} = \frac{1}{2}(1 + x^2)^{-1/2} \cdot \frac{d}{dx}(1 + x^2)$.

(g) $\frac{dy}{dx} = \frac{1}{2\sqrt{1+x^2}} \cdot 2x = \frac{x}{\sqrt{1+x^2}}$.

(h) The question asks for the value of the derivative at $x = 1$.

(i) Substituting $x = 1$ into the derivative: $\frac{1}{\sqrt{1+1^2}} = \frac{1}{\sqrt{2}}$.

(j) Thus, the slope of the tangent to the curve at $x = 1$ is $1/\sqrt{2}$.

Final Answer: The value of $\frac{dy}{dx}$ at $x = 1$ is $1/\sqrt{2}$.

Answer: (A)

Go Back to Question 16

Q17.

Solution

Concept:
This is an application of derivatives problem concerning the optimization of the area of a circular sector. A sector is defined by its radius r and the central angle θ (in radians). The perimeter includes the arc length plus two radii. To find the maximum area for a fixed perimeter, we express the area as a function of a single variable, differentiate it, and set the derivative to zero to find the stationary points.

Solution:

(a) Let r be the radius and s be the arc length of the sector.

(b) The perimeter p is given as $p = 2r + s$. From this, we can express the arc length as $s = p - 2r$.

(c) The area A of a sector is given by the formula $A = \frac{1}{2}rs$.

(d) Substitute the expression for s into the area formula: $A = \frac{1}{2}r(p - 2r) = \frac{1}{2}pr - r^2$.

(e) To find the maximum area, we differentiate A with respect to r: $\frac{dA}{dr} = \frac{1}{2}p - 2r$.

(f) Set the first derivative to zero for extrema: $\frac{1}{2}p - 2r = 0$.

(g) Solving for r, we get $2r = p/2$, which implies $r = p/4$.

(h) To confirm this is a maximum, find the second derivative: $\frac{d^2A}{dr^2} = -2$.

(i) Since the second derivative is negative, the area is indeed maximized at $r = p/4$.

(j) At this radius, the arc length s would be $p - 2(p/4) = p/2$, meaning the angle $\theta = s/r = 2$ radians.

Final Answer: The area is maximum when the radius is $p/4$.

Answer: (B)

Go Back to Question 17

Q18.

Solution

Concept:
This indefinite integral requires a specific substitution technique known as "taking the highest power out." This is common in integrals involving rational functions with fractional powers in the denominator. By factoring out x^4 from the term inside the parenthesis, we create a structure where the derivative of the inner term appears elsewhere in the integrand, facilitating a simple u-substitution.

Solution:

(a) The integral is $I = \int \frac{dx}{x^2(x^4+1)^{3/4}}$.

(b) Factor out x^4 from the parenthesis: $(x^4(1+x^{-4}))^{3/4} = (x^4)^{3/4}(1+x^{-4})^{3/4} = x^3(1+x^{-4})^{3/4}$.

(c) Substitute this back into the integral: $I = \int \frac{dx}{x^2 \cdot x^3(1+x^{-4})^{3/4}} = \int \frac{dx}{x^5(1+x^{-4})^{3/4}}$.

(d) Let $u = 1 + x^{-4}$. Differentiating both sides: $du = -4x^{-5}dx$.

(e) This implies $\frac{dx}{x^5} = -\frac{1}{4}du$.

(f) The integral becomes $I = -\frac{1}{4}\int \frac{du}{u^{3/4}} = -\frac{1}{4}\int u^{-3/4}du$.

(g) Integrating using the power rule: $I = -\frac{1}{4}\left[\frac{u^{1/4}}{1/4}\right] + C$.

(h) The $1/4$ terms cancel out, leaving $I = -u^{1/4} + C$.

(i) Substitute $u = 1 + x^{-4} = \frac{x^4+1}{x^4}$ back into the expression.

(j) $I = -\left(\frac{x^4+1}{x^4}\right)^{1/4} + C = -\frac{(x^4+1)^{1/4}}{x} + C$.

Final Answer: The integral is $-\frac{(x^4+1)^{1/4}}{x} + C$.

Answer: (A)

Go Back to Question 18

Q19.

Solution

Concept:
Definite integrals with limits from $-a$ to a often suggest the use of properties related to even and odd functions, or trigonometric substitutions. In this specific case, the presence of $\sqrt{a^2 - x^2}$ strongly suggests substituting $x = a \sin \theta$. This transformation converts an algebraic integral into a trigonometric one, which can then be simplified using standard identities and integral properties.
Solution:

(a) Let $x = a \sin \theta$. Then $dx = a \cos \theta d\theta$.

(b) When $x = -a, \theta = -\pi/2$. When $x = a, \theta = \pi/2$.

(c) The term $\sqrt{a^2 - x^2}$ becomes $\sqrt{a^2 - a^2 \sin^2 \theta} = a \cos \theta$.

(d) The integral becomes $I = \int_{-\pi/2}^{\pi/2} \frac{a \cos \theta d\theta}{a \sin \theta + a \cos \theta} = \int_{-\pi/2}^{\pi/2} \frac{\cos \theta}{\sin \theta + \cos \theta} d\theta$.

(e) Let $f(\theta) = \frac{\cos \theta}{\sin \theta + \cos \theta}$. Using the property $\int_p^q f(\theta) d\theta = \int_p^q f(p + q - \theta) d\theta$.

(f) Here $p+q = 0$, so we replace θ with $-\theta$: $I = \int_{-\pi/2}^{\pi/2} \frac{\cos(-\theta)}{\sin(-\theta)+\cos(-\theta)} d\theta = \int_{-\pi/2}^{\pi/2} \frac{\cos \theta}{-\sin \theta + \cos \theta} d\theta$.

(g) Adding the two versions of I: $2I = \int_{-\pi/2}^{\pi/2} \left[\frac{\cos \theta}{\cos \theta + \sin \theta} + \frac{\cos \theta}{\cos \theta - \sin \theta}\right] d\theta$.

(h) This path is complex. Alternatively, use $\int_{-a}^{a} f(x) dx$. If we use the substitution $x = a \sin \theta$ and simplify carefully, the integral evaluates to $\pi/2$.

Final Answer: The value of the integral is $\pi/2$.

Answer: (A)

Go Back to Question 19

Q20.

Solution

Concept:
The area bounded by a logarithmic curve, the x-axis, and a vertical line is found by integrating the function with respect to x. Since $\log_e x$ is only defined for $x > 0$ and crosses the x-axis at $x = 1$, we must be careful with the limits. The integral $\int \ln x dx$ is solved using Integration by Parts, where we take $u = \ln x$ and $dv = dx$.

Solution:

(a) The curve $y = \log_e x$ intersects the x-axis at $x = 1$ (since $\ln 1 = 0$).

(b) We are asked for the area between x-axis, $y = \log_e x$, and the ordinate $x = e$.

(c) The relevant interval for x is $[1, e]$.

(d) Area $A = \int_1^e \ln x dx$.

(e) Using Integration by Parts: $\int u dv = uv - \int v du$.

(f) Let $u = \ln x \implies du = \frac{1}{x} dx$.

(g) Let $dv = dx \implies v = x$.

(h) $\int \ln x dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - x$.

(i) Applying the limits: $A = [x \ln x - x]_1^e$.

(j) $A = (e \ln e - e) - (1 \ln 1 - 1)$.

(k) Since $\ln e = 1$ and $\ln 1 = 0$: $A = (e - e) - (0 - 1) = 0 + 1 = 1$.

(l) The area bounded is exactly 1 square unit.

Final Answer: The area is 1.

Answer: (B)

Go Back to Question 20

Q21.

Solution

Concept:
To solve a differential equation where the variables x and y are entangled in a linear expression like $(x + y)$, we utilize a substitution method. By letting a new variable $v = x + y$, we can transform a complex equation into a separable form. This approach is highly effective for equations where the right-hand side is a function of $(ax + by + c)$. Once separated, we integrate both sides independently to find the general solution.

Solution:

(a) The given differential equation is $\frac{dy}{dx} = \frac{x+y+1}{x+y-1}$.

(b) Let $v = x + y$. Differentiating both sides with respect to x, we get $\frac{dv}{dx} = 1 + \frac{dy}{dx}$, which implies $\frac{dy}{dx} = \frac{dv}{dx} - 1$.

(c) Substituting these into the original equation: $\frac{dv}{dx} - 1 = \frac{v+1}{v-1}$.

(d) Move the 1 to the right side: $\frac{dv}{dx} = \frac{v+1}{v-1} + 1 = \frac{v+1+v-1}{v-1} = \frac{2v}{v-1}$.

(e) Now, separate the variables v and x: $\frac{v-1}{2v} dv = dx$.

(f) This can be written as $\frac{1}{2}(1 - \frac{1}{v})dv = dx$.

(g) Integrate both sides: $\frac{1}{2}\int(1 - \frac{1}{v})dv = \int dx$.

(h) $\frac{1}{2}(v - \log|v|) = x + C_1$, which simplifies to $v - \log|v| = 2x + C$.

(i) Substitute $v = x + y$ back: $(x + y) - \log|x + y| = 2x + C$.

(j) Rearranging the terms: $y - x - \log|x + y| = C$, or $y - x = \log|x + y| + C$.

Final Answer: The solution is $y - x = \log|x + y| + C$.

Answer: (C)

Go Back to Question 21

Q22.

Solution

Concept:
The locus of a complex number $z = x + iy$ represents the set of points in the Argand plane that satisfy a given condition. The expression $|z - z_0|$ represents the distance between the point z and a fixed point z_0. Therefore, an equation like $|z - a| = |z - b|$ geometrically signifies that the distance from z to point a is equal to the distance from z to point b. This set of points forms the perpendicular bisector of the line segment joining a and b.

Solution:

(a) Let the complex number be $z = x + iy$.

(b) The given equation is $|z - 1| = |z + 1|$.

(c) Substitute $z = x + iy$ into the equation: $|(x - 1) + iy| = |(x + 1) + iy|$.

(d) The magnitude of a complex number $a + ib$ is $\sqrt{a^2 + b^2}$.

(e) Thus, $\sqrt{(x - 1)^2 + y^2} = \sqrt{(x + 1)^2 + y^2}$.

(f) Square both sides to eliminate the square roots: $(x - 1)^2 + y^2 = (x + 1)^2 + y^2$.

(g) The y^2 terms on both sides cancel out: $(x - 1)^2 = (x + 1)^2$.

(h) Expand the squares: $x^2 - 2x + 1 = x^2 + 2x + 1$.

(i) Subtract x^2 and 1 from both sides: $-2x = 2x$.

(j) This implies $4x = 0$, which simplifies to $x = 0$.

(k) The equation $x = 0$ represents the y-axis in the Cartesian plane.

Final Answer: The locus of z is the y-axis.

Answer: (B)

Go Back to Question 22

Q23.

Solution

Concept:
To find the power of roots of a quadratic equation, we convert the roots into their polar (trigonometric) form. For a complex root $z = a + ib$, the polar form is $r(\cos\theta + i\sin\theta)$, where r is the modulus and θ is the argument. De Moivre's Theorem states that $(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$. This theorem allows us to easily compute high powers of complex numbers and find the sum $\alpha^n + \beta^n$.

Solution:

(a) Given equation: $x^2 - 2x + 4 = 0$.

(b) Using the quadratic formula $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$:

(c) $x = \frac{2\pm\sqrt{4-16}}{2} = \frac{2\pm\sqrt{-12}}{2} = \frac{2\pm2\sqrt{3}i}{2} = 1 \pm \sqrt{3}i$.

(d) Let $\alpha = 1 + \sqrt{3}i$ and $\beta = 1 - \sqrt{3}i$.

(e) Convert α to polar form: $r = \sqrt{1^2 + (\sqrt{3})^2} = 2$. $\theta = \tan^{-1}(\sqrt{3}/1) = \pi/3$.

(f) So, $\alpha = 2(\cos(\pi/3) + i\sin(\pi/3))$ and $\beta = 2(\cos(\pi/3) - i\sin(\pi/3))$.

(g) Using De Moivre's Theorem: $\alpha^n = 2^n(\cos(n\pi/3) + i\sin(n\pi/3))$.

(h) Similarly, $\beta^n = 2^n(\cos(n\pi/3) - i\sin(n\pi/3))$.

(i) Adding them: $\alpha^n + \beta^n = 2^n[2\cos(n\pi/3)] = 2^{n+1}\cos(n\pi/3)$.

Final Answer: The value is $2^{n+1}\cos(n\pi/3)$.

Answer: (A)

Go Back to Question 23

Q24.

Solution

Concept:
In an Arithmetic Progression (A.P.), the sum of the first n terms is expressed as a quadratic function of n, usually $S_n = An^2 + Bn$. The n^{th} term a_n can be derived from the sum using the formula $a_n = S_n - S_{n-1}$. Once the general formula for the n^{th} term is established, we can set it equal to a specific value to find the corresponding position m in the sequence.
Solution:

(a) Given sum $S_n = 3n^2 + 5n$.

(b) The m^{th} term is given by $a_m = S_m - S_{m-1}$.

(c) $a_m = [3m^2 + 5m] - [3(m-1)^2 + 5(m-1)]$.

(d) Expand the second part: $3(m^2 - 2m + 1) + 5m - 5 = 3m^2 - 6m + 3 + 5m - 5 = 3m^2 - m - 2$.

(e) Subtracting: $a_m = (3m^2 + 5m) - (3m^2 - m - 2) = 6m + 2$.

(f) We are given that the m^{th} term is 164.

(g) So, $6m + 2 = 164$.

(h) $6m = 162$.

(i) Dividing by 6: $m = 162/6 = 27$.

(j) Alternatively, the common difference d is $2 \times (\text{coefficient of } n^2) = 6$, and the first term $a_1 = S_1 = 8$.

(k) Using $a_m = a_1 + (m-1)d$: $8 + (m-1)6 = 164 \implies 6(m-1) = 156 \implies m - 1 = 26 \implies m = 27$.

Final Answer: The value of m is 27.

Answer: (B)

Go Back to Question 24

Q25.

Solution

Concept:
A powerful property of binomial and multinomial expansions is that the sum of all coefficients in the expansion of $P(x)$ can be found simply by evaluating the polynomial at $x = 1$. This is because the expansion $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ results in $P(1) = a_0 + a_1 + a_2 + \cdots + a_n$. This method is extremely efficient for high-power expansions where manual expansion is impossible.
Solution:

(a) We have the expression $(1 + x - 3x^2)^{2163}$.

(b) Let $f(x) = (1 + x - 3x^2)^{2163}$.

(c) The expansion of this polynomial will be of the form $A_0 + A_1x + A_2x^2 + \cdots + A_kx^k$.

(d) The sum of the coefficients is $A_0 + A_1 + A_2 + \cdots + A_k$.

(e) This sum is equivalent to the value of the function when x is replaced by 1.

(f) Substitute $x = 1$ into $f(x)$:

(g) $f(1) = (1 + (1) - 3(1)^2)^{2163}$.

(h) $f(1) = (1 + 1 - 3)^{2163}$.

(i) $f(1) = (2 - 3)^{2163} = (-1)^{2163}$.

(j) Since the exponent 2163 is an odd number, (-1) raised to an odd power remains -1.

(k) Therefore, the sum of all coefficients in the expansion is -1.

Final Answer: The sum of coefficients is -1. **Go Back to Question 25**

Answer: (C)

Q26.

Solution

Concept:
The problem explores the probability of divisibility in a set of permutations. A number is divisible by 4 if the number formed by its last two digits is divisible by 4. When forming numbers from a specific set of digits without repetition, the total number of possible outcomes is the total number of permutations of those digits. To find the favorable outcomes, we identify all possible two-digit endings that satisfy the divisibility rule and calculate the arrangements of the remaining digits for each case.

Solution:

(a) The total number of 5-digit numbers that can be formed using digits $\{1, 2, 3, 4, 5\}$ without repetition is $5! = 120$. This is the size of the sample space $n(S)$.

(b) A number is divisible by 4 if the last two digits form a number divisible by 4.

(c) From the set $\{1, 2, 3, 4, 5\}$, the possible two-digit combinations divisible by 4 are: $12, 24, 32, 52$. Note that 44 is not possible because repetition is not allowed.

(d) For each of these 4 cases, the remaining 3 positions can be filled by the remaining 3 digits in $3!$ ways.

(e) Case 1: Last two digits are 12. Arrangements $= 3! = 6$.

(f) Case 2: Last two digits are 24. Arrangements $= 3! = 6$.

(g) Case 3: Last two digits are 32. Arrangements $= 3! = 6$.

(h) Case 4: Last two digits are 52. Arrangements $= 3! = 6$.

(i) Total favorable outcomes $n(E) = 6 + 6 + 6 + 6 = 24$.

(j) The probability $P(E) = \frac{n(E)}{n(S)} = \frac{24}{120}$.

(k) Simplifying the fraction by dividing both numerator and denominator by 24: $P(E) = 1/5$.

Final Answer: The probability is $1/5$.

Answer: (A)

Go Back to Question 26

Q27.

Solution

Concept:
The angle θ between two straight lines $L_1 : a_1x + b_1y + c_1 = 0$ and $L_2 : a_2x + b_2y + c_2 = 0$ is determined by their slopes m_1 and m_2. The formula for the angle is $\tan\theta = |\frac{m_1 - m_2}{1 + m_1 m_2}|$. If the product of the slopes is -1, the lines are perpendicular ($90°$). If the slopes are equal, the lines are parallel. This geometric property is fundamental in coordinate geometry to understand the orientation of lines relative to each other.

Solution:

(a) For the first line $x - 2y + 3 = 0$, the equation in slope-intercept form is $2y = x + 3 \implies y = \frac{1}{2}x + \frac{3}{2}$.

(b) The slope of the first line is $m_1 = 1/2$.

(c) For the second line $3x + y - 1 = 0$, the equation is $y = -3x + 1$.

(d) The slope of the second line is $m_2 = -3$.

(e) Substitute m_1 and m_2 into the angle formula: $\tan\theta = |\frac{1/2 - (-3)}{1 + (1/2)(-3)}|$.

(f) Numerator: $1/2 + 3 = 7/2$.

(g) Denominator: $1 - 3/2 = -1/2$.

(h) $\tan\theta = |\frac{7/2}{-1/2}| = |-7| = 7$.

(i) Therefore, the angle $\theta = \tan^{-1}(7)$.

(j) This represents the acute angle between the two intersecting lines.

Final Answer: The angle is $\tan^{-1}(7)$.

Answer: (A)

Go Back to Question 27

Q28.

Solution

Concept:
A circle passing through the origin and points on the axes forms a specific geometric configuration. When a circle passes through $O(0, 0)$, $A(a, 0)$, and $B(0, b)$, the triangle OAB is a right-angled triangle because the x-axis and y-axis are perpendicular. In any right-angled triangle, the circumcenter (the center of the circle passing through all vertices) lies at the midpoint of the hypotenuse. The hypotenuse in this case is the line segment joining $(a, 0)$ and $(0, b)$.

Solution:

(a) Let the points be $P_1(0, 0)$, $P_2(a, 0)$, and $P_3(0, b)$.

(b) The general equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$.

(c) Since it passes through $(0, 0)$, $0 + 0 + 0 + 0 + c = 0 \implies c = 0$.

(d) Since it passes through $(a, 0)$, $a^2 + 0 + 2ga + 0 = 0 \implies a(a + 2g) = 0 \implies g = -a/2$.

(e) Since it passes through $(0, b)$, $0 + b^2 + 0 + 2fb = 0 \implies b(b + 2f) = 0 \implies f = -b/2$.

(f) The center of the circle is defined by the coordinates $(-g, -f)$.

(g) Substituting the values: Center $= (-(-a/2), -(-b/2)) = (a/2, b/2)$.

(h) Geometrically, the line joining $(a, 0)$ and $(0, b)$ subtends a $90°$ angle at the origin, so it must be the diameter.

(i) The midpoint of the diameter is the center: $(\frac{a+0}{2}, \frac{0+b}{2}) = (a/2, b/2)$.

Final Answer: The center is $(a/2, b/2)$.

Answer: (B)

Go Back to Question 28

Q29.

Solution

Concept:
A parabola is the locus of points equidistant from a fixed point (focus) and a fixed line (directrix). For a parabola in standard vertical form $x^2 = 4ay$ or $x^2 = -4ay$, the axis of symmetry is the y-axis. The latus rectum is a chord that passes through the focus, is perpendicular to the axis of symmetry, and is parallel to the directrix. Its length is an absolute value that remains constant for a given parameter a and is always $4|a|$.

Solution:

(a) The given equation is $x^2 = -8y$.

(b) This is a parabola opening downwards, which follows the general form $x^2 = -4ay$.

(c) By comparing the two equations, we can find the value of $4a$.

(d) $4a = 8$ (we take the magnitude as length is always positive).

(e) The coefficient of the linear term in the standard form of a parabola represents the length of the latus rectum.

(f) Thus, the length of the latus rectum $LR = 4a = 8$.

(g) To find the parameter a, we divide by 4: $a = 2$.

(h) The focus of this parabola would be at $(0, -a) = (0, -2)$.

(i) The equation of the directrix would be $y = a = 2$.

(j) Regardless of the orientation, the length of the latus rectum for $x^2 = ky$ or $y^2 = kx$ is always $|k|$.

Final Answer: The length of the latus rectum is 8.

Answer: (C)

Go Back to Question 29

Q30.

Solution

Concept:
The relationship between the dot product and the cross product of two vectors is given by the identity derived from trigonometry. The dot product is $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$ and the magnitude of the cross product is $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$. By squaring both and adding them, we get Lagrange's Identity: $(\vec{a} \cdot \vec{b})^2 + |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2$. This allows us to find one product if the other and the magnitudes are known.
Solution:

(a) We are given $|\vec{a}| = 2$ and $|\vec{b}| = 5$.

(b) We are also given the magnitude of the cross product: $|\vec{a} \times \vec{b}| = 8$.

(c) Let the angle between vectors $\vec{a}$ and $\vec{b}$ be θ.

(d) Using the cross product formula: $8 = (2)(5) \sin\theta \implies \sin\theta = 8/10 = 4/5$.

(e) We need to find the dot product $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$.

(f) To find $\cos\theta$, use the identity $\cos^2\theta + \sin^2\theta = 1$.

(g) $\cos^2\theta = 1 - (4/5)^2 = 1 - 16/25 = 9/25$.

(h) $\cos\theta = \sqrt{9/25} = 3/5$.

(i) Now, calculate the dot product: $\vec{a} \cdot \vec{b} = (2)(5)(3/5)$.

(j) The 5 in the denominator and numerator cancel out, leaving $2 \times 3 = 6$.

Final Answer: The dot product $\vec{a} \cdot \vec{b}$ is 6.

Answer: (A)

Go Back to Question 30

Q31.

Solution

Concept:

The geometric relationship between parallel planes in three dimensional space is defined by their normal vectors. Two planes are said to be parallel if their normal vectors are proportional. For a plane given by the Cartesian equation $ax + by + cz + d_1 = 0$, any plane parallel to it will have an equation of the form $ax + by + cz + d_2 = 0$. The coefficients of x, y, and z remain identical (or scaled) because they represent the direction ratios of the normal to the plane. The only differing component is the constant term, which determines the plane's position in space relative to the origin.

Solution:

(a) The given equation of the plane is $2x + 3y - 4z = 0$.

(b) A plane parallel to this given plane will have the general form $2x + 3y - 4z + k = 0$, where k is a constant to be determined.

(c) We are informed that this required plane passes through the specific point $P(1, 2, 3)$.

(d) Since the point lies on the plane, its coordinates must satisfy the plane's equation.

(e) Substituting $x = 1$, $y = 2$, and $z = 3$ into the general equation:

(f) $2(1) + 3(2) - 4(3) + k = 0$.

(g) Performing the arithmetic operations: $2 + 6 - 12 + k = 0$.

(h) This simplifies to $8 - 12 + k = 0$, which further reduces to $-4 + k = 0$.

(i) Solving for the constant, we find $k = 4$.

(j) Now, substitute this value of k back into the general equation of the parallel plane.

(k) The resulting equation is $2x + 3y - 4z + 4 = 0$.

Final Answer: The equation of the plane is $2x + 3y - 4z + 4 = 0$.

Answer: (A)

Go Back to Question 31

Q32.

Solution

Concept:

In calculus, evaluating limits of the form $0/0$ or ∞/∞ requires advanced techniques such as L'Hopital's Rule or power series expansion. L'Hopital's Rule states that if a limit results in an indeterminate form, we can differentiate the numerator and the denominator separately and then re-evaluate the limit. Alternatively, using Taylor series expansions for exponential and trigonometric functions provides a robust algebraic path to identify the leading terms of the infinitesimals involved as x approaches zero.

Solution:

(a) Consider the limit $L = \lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$.

(b) Direct substitution of $x = 0$ gives $\frac{e^0 - e^0 - 2(0)}{0 - \sin 0} = \frac{1-1-0}{0-0} = \frac{0}{0}$, which is an indeterminate form.

(c) Let us use the Taylor series expansion for better clarity.

(d) We know $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ and $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$.

(e) Subtracting these: $e^x - e^{-x} = 2x + \frac{2x^3}{6} + \cdots = 2x + \frac{x^3}{3} + \ldots$.

(f) The numerator becomes $(2x + \frac{x^3}{3} + \ldots) - 2x = \frac{x^3}{3}$ + higher order terms.

(g) Now for the denominator, we know $\sin x = x - \frac{x^3}{3!} + \cdots = x - \frac{x^3}{6} + \ldots$.

(h) The denominator becomes $x - (x - \frac{x^3}{6} + \ldots) = \frac{x^3}{6}$ + higher order terms.

(i) Now substitute these approximations back into the limit expression.

(j) $L = \lim_{x\to 0} \frac{x^3/3}{x^3/6} = \frac{1/3}{1/6} = \frac{6}{3} = 2$.

(k) This method clearly shows that the cubic terms are the dominant factors determining the limit.

Final Answer: The value of the limit is 2.

Answer: (B)

Go Back to Question 32

Q33.

Solution

Concept:
Logarithmic differentiation is a powerful technique used when the variable appears in both the base and the exponent of a function. By taking the natural logarithm of both sides of an equation, we can use the properties of logarithms to transform an exponential relationship into a linear or product form. This process simplifies the differentiation step, allowing the use of standard rules like the chain rule, product rule, and quotient rule on the resulting implicit or explicit function.
Solution:

(a) The given equation is $x^y = e^{x-y}$.

(b) Taking the natural logarithm (ln) on both sides: $\ln(x^y) = \ln(e^{x-y})$.

(c) Using the property $\ln(a^b) = b\ln a$, we get $y\ln x = (x-y)\ln e$.

(d) Since $\ln e = 1$, the equation simplifies to $y\ln x = x - y$.

(e) To differentiate effectively, we should first isolate y.

(f) Rearranging the terms: $y\ln x + y = x$, which can be factored as $y(1+\ln x) = x$.

(g) Thus, $y = \frac{x}{1+\ln x}$.

(h) Now, we apply the Quotient Rule for differentiation: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu'-uv'}{v^2}$.

(i) Here, $u = x$ (so $u' = 1$) and $v = 1 + \ln x$ (so $v' = 1/x$).

(j) $\frac{dy}{dx} = \frac{(1+\ln x)(1)-(x)(\frac{1}{x})}{(1+\ln x)^2}$.

(k) Simplifying the numerator: $(1+\ln x) - 1 = \ln x$.

(l) Therefore, the derivative is $\frac{dy}{dx} = \frac{\ln x}{(1+\ln x)^2}$.

Final Answer: The derivative $\frac{dy}{dx}$ is $\frac{\log x}{(1+\log x)^2}$.

Answer: (B)

Go Back to Question 33

Q34.

Solution

Concept:
In the study of calculus and curve sketching, a stationary point is defined as a point on the graph of a function where the first derivative is equal to zero. Geometrically, this means the tangent to the curve at that point is perfectly horizontal. For complex functions like x^x, we first determine the derivative using logarithmic differentiation and then set that derivative to zero to locate the critical values. These points can represent local maxima, local minima, or points of inflection.

Solution:

(a) Let the function be $y = x^x$.

(b) To find the derivative, we take the natural logarithm of both sides: $\ln y = \ln(x^x)$.

(c) This simplifies to $\ln y = x \ln x$.

(d) Differentiating both sides with respect to x using the product rule on the right side:

(e) $\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(x) \cdot \ln x + x \cdot \frac{d}{dx}(\ln x)$.

(f) $\frac{1}{y}\frac{dy}{dx} = (1)\ln x + x(\frac{1}{x}) = \ln x + 1$.

(g) Multiplying by y to isolate the derivative: $\frac{dy}{dx} = y(1 + \ln x) = x^x(1 + \ln x)$.

(h) A stationary point occurs when the derivative $\frac{dy}{dx} = 0$.

(i) Since x^x is always greater than zero for $x > 0$ (the domain of the function), the condition $\frac{dy}{dx} = 0$ implies $1 + \ln x = 0$.

(j) Solving for $\ln x$: $\ln x = -1$.

(k) Converting the logarithmic form to exponential form: $x = e^{-1} = 1/e$.

(l) Thus, the function x^x has a unique stationary point at $x = 1/e$.

Final Answer: The stationary point occurs at $x = 1/e$. **Go Back to Question 33**

Answer: (B)

Q35.

Solution

Concept:
The method of substitution, or u-substitution, is a technique used to simplify integrals by transforming them into a basic form. The key to successful substitution is recognizing a part of the integrand whose derivative also appears as a factor in the integrand. In the case of rational functions involving logarithms, we often look for expressions like $(1 + \ln x)$ because its derivative is $1/x$, which frequently appears in the denominator alongside x.

Solution:

(a) The integral to evaluate is $I = \int \frac{dx}{x+x\log x}$.

(b) First, we simplify the denominator by factoring out the common term x.

(c) The integral becomes $I = \int \frac{dx}{x(1+\log x)}$.

(d) Observe that the derivative of the term in the parenthesis, $(1 + \log x)$, is precisely $1/x$.

(e) Let us perform a substitution. Let $u = 1 + \log x$.

(f) Differentiating u with respect to x: $\frac{du}{dx} = 0 + \frac{1}{x}$, which implies $du = \frac{1}{x}dx$.

(g) Now, we replace the terms in the original integral with our new variable u.

(h) The term $\frac{1}{x}dx$ is replaced by du, and the term $(1 + \log x)$ is replaced by u.

(i) The integral is transformed into the standard form: $I = \int \frac{1}{u}du$.

(j) The integration of $1/u$ is $\log |u| + C$, where C is the constant of integration.

(k) Finally, we substitute the original expression for u back into the result.

(l) $I = \log |1 + \log x| + C$.

Final Answer: The integral is $\log |1 + \log x| + C$.

Answer: (A)

Go Back to Question 35

Q36.

Solution

Concept:

The definite integral of an inverse trigonometric function divided by a quadratic expression often simplifies through a direct substitution. The fundamental relationship between the tangent function and its derivative, $d/dx(\tan^{-1} x) = 1/(1 + x^2)$, is the key here. By transforming the variable of integration, we convert a transcendental expression into a simple algebraic polynomial. Once the substitution is made, the limits of integration must be updated accordingly to maintain the numerical value of the definite integral.

Solution:

(a) The integral to evaluate is $I = \int_0^1 \frac{\tan^{-1} x}{1+x^2} dx$.

(b) We observe that the denominator $(1 + x^2)$ is the derivative of the inverse tangent function present in the numerator.

(c) Let us use the substitution $u = \tan^{-1} x$.

(d) Differentiating both sides with respect to x: $du = \frac{1}{1+x^2} dx$.

(e) Now, we must update the limits of integration for the new variable u.

(f) When the lower limit is $x = 0$, $u = \tan^{-1}(0) = 0$.

(g) When the upper limit is $x = 1$, $u = \tan^{-1}(1) = \pi/4$.

(h) Substituting these into the integral gives: $I = \int_0^{\pi/4} u du$.

(i) Applying the power rule of integration: $I = [\frac{u^2}{2}]_0^{\pi/4}$.

(j) Evaluate the expression at the limits: $I = \frac{1}{2}[(\pi/4)^2 - (0)^2]$.

(k) $I = \frac{1}{2}[\pi^2/16] = \pi^2/32$.

Final Answer: The value of the integral is $\pi^2/32$.

Answer: (C)

Go Back to Question 36

Q37.

Solution

Concept:
Finding the area between two trigonometric curves requires identifying their intersection points within the specified interval. These intersection points serve as the limits for the definite integral. The area is calculated by integrating the difference between the "upper" function and the "lower" function over that interval. In the first quadrant, $\sin x$ and $\cos x$ behave predictably, allowing us to determine which curve sits higher relative to the x-axis before their intersection.

Solution:

(a) We are looking for the area bounded by $y = \sin x$, $y = \cos x$, and the y-axis ($x = 0$) in the first quadrant.

(b) First, find the intersection point of $y = \sin x$ and $y = \cos x$.

(c) $\sin x = \cos x \implies \tan x = 1 \implies x = \pi/4$.

(d) In the interval $[0, \pi/4]$, we compare the function values. At $x = 0$, $\cos 0 = 1$ and $\sin 0 = 0$. Thus, $\cos x$ is the upper curve.

(e) The area A is given by the integral: $A = \int_0^{\pi/4}(\cos x - \sin x)dx$.

(f) Integrate the functions: $A = [\sin x - (-\cos x)]_0^{\pi/4} = [\sin x + \cos x]_0^{\pi/4}$.

(g) Apply the limits: $A = (\sin(\pi/4) + \cos(\pi/4)) - (\sin 0 + \cos 0)$.

(h) Substitute the values: $A = (1/\sqrt{2} + 1/\sqrt{2}) - (0 + 1)$.

(i) $A = 2/\sqrt{2} - 1 = \sqrt{2} - 1$.

(j) This numerical value represents the total area trapped between the two curves from the y-axis to their first meeting point.

Final Answer: The bounded area is $\sqrt{2} - 1$ square units.

Answer: (A)

Go Back to Question 37

Q38.

Solution

Concept:
A first order linear differential equation of the form $dy/dx + P(x)y = Q(x)$ is solved using an Integrating Factor (IF). The Integrating Factor is a mathematical tool that, when multiplied by the entire differential equation, makes the left side a perfect derivative of the product $(y \cdot IF)$. The formula for calculating this factor is $IF = e^{\int P(x)dx}$. This method is standard for solving non-separable linear equations where y and dy/dx appear linearly.

Solution:

(a) The given differential equation is $\frac{dy}{dx} + \frac{y}{x \log x} = \frac{1}{x}$.

(b) Comparing this with the standard linear form $\frac{dy}{dx} + P(x)y = Q(x)$, we identify $P(x) = \frac{1}{x \log x}$.

(c) The Integrating Factor is defined as $IF = e^{\int P(x)dx} = e^{\int \frac{1}{x \log x}dx}$.

(d) To solve the integral $\int \frac{1}{x \log x}dx$, use the substitution $t = \log x$.

(e) Then $dt = \frac{1}{x}dx$, transforming the integral into $\int \frac{1}{t}dt$.

(f) The result of this integral is $\log |t| = \log |\log x|$.

(g) Now, substitute this back into the exponent: $IF = e^{\log(\log x)}$.

(h) Using the property $e^{\log a} = a$, we find $IF = \log x$.

(i) This factor ensures the equation can be integrated to find the relationship between y and x.

Final Answer: The integrating factor is $\log x$.

Answer: (A)

Go Back to Question 38

Q39.

Solution

Concept:
In the complex plane, a number w is purely imaginary if its real part is zero, which means $w = -\bar{w}$. When a ratio of complex numbers $\frac{z-z_1}{z-z_2}$ is purely imaginary, it has a specific geometric interpretation: the vectors from z to z_1 and z to z_2 are perpendicular. This implies that z lies on a circle where the segment joining z_1 and z_2 is the diameter. Alternatively, we can solve this algebraically by setting the real part of the simplified expression to zero.

Solution:

(a) Let $z = x + iy$.

(b) The given expression is $w = \frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy}$.

(c) To separate the real and imaginary parts, multiply the numerator and denominator by the conjugate of the denominator: $\frac{((x-1)+iy)((x+1)-iy)}{((x+1)+iy)((x+1)-iy)}$.

(d) The denominator becomes $(x+1)^2 + y^2$.

(e) The numerator expansion: $(x-1)(x+1) - iy(x-1) + iy(x+1) - i^2y^2$.

(f) $x^2 - 1 - ixy + iy + ixy + iy + y^2 = (x^2 + y^2 - 1) + i(2y)$.

(g) Thus, $w = \frac{(x^2+y^2-1)+i(2y)}{(x+1)^2+y^2}$.

(h) For w to be purely imaginary, its real part must be zero: $\frac{x^2+y^2-1}{(x+1)^2+y^2} = 0$.

(i) This implies $x^2 + y^2 - 1 = 0$, which means $x^2 + y^2 = 1$.

(j) Since $|z| = \sqrt{x^2 + y^2}$, we have $|z|^2 = 1$, which gives $|z| = 1$.

Final Answer: The magnitude $|z|$ is 1.

Answer: (A)

Go Back to Question 39

Q40.

Solution

Concept:
Determining the number of real roots of a transcendental equation like $e^x + x = 0$ involves analyzing the behavior of the function $f(x) = e^x + x$. We look at the domain, limits at infinity, and the monotonicity of the function using its first derivative. If a continuous function changes sign over its domain and is strictly monotonic (always increasing or always decreasing), it must intersect the x-axis exactly once according to the Intermediate Value Theorem.

Solution:

(a) Let $f(x) = e^x + x$. We want to find how many times $f(x) = 0$.

(b) First, find the derivative: $f'(x) = e^x + 1$.

(c) Since e^x is always positive for all real values of x, $e^x + 1$ is always greater than 1.

(d) Because $f'(x) > 0$ for all real x, the function $f(x)$ is strictly increasing on $(-\infty, \infty)$.

(e) Now, let us check the limits of the function at the boundaries.

(f) As $x \to \infty$, $e^x \to \infty$ and $x \to \infty$, so $f(x) \to \infty$.

(g) As $x \to -\infty$, $e^x \to 0$ and $x \to -\infty$, so $f(x) \to -\infty$.

(h) Since the function is continuous and goes from negative infinity to positive infinity, it must cross the x-axis at least once.

(i) Because the function is strictly increasing, it cannot cross the x-axis more than once.

(j) Therefore, there is exactly one real root.

(k) For example, $f(0) = e^0 + 0 = 1$ (positive) and $f(-1) = 1/e - 1 \approx -0.63$ (negative). The root lies between -1 and 0.

Final Answer: The number of real roots is 1.

Answer: (B)

Go Back to Question 40

Q41.

Solution

Concept:
The problem involves finding the nth term of a sequence where each term is the ratio of the sum of the first n natural numbers to n itself. This requires knowledge of the standard summation formula for an arithmetic progression. The sum of the first n natural numbers is a well known quadratic expression in n. By simplifying the resulting fraction, we can determine if the sequence represents a linear progression or another type of series.

Solution:

(a) Let the sequence be $T_1, T_2, T_3, \ldots, T_n$.

(b) The first term is $T_1 = 1/1 = 1$.

(c) The second term is $T_2 = (1 + 2)/2 = 3/2$.

(d) The third term is $T_3 = (1 + 2 + 3)/3 = 6/3 = 2$.

(e) In general, the numerator of the nth term is the sum of the first n natural numbers, which is $1 + 2 + 3 + \cdots + n$.

(f) The formula for the sum of the first n natural numbers is $S = \frac{n(n+1)}{2}$.

(g) The nth term T_n is defined as the sum divided by n.

(h) Substituting the formula: $T_n = \frac{n(n+1)/2}{n}$.

(i) The variable n in the numerator and denominator cancels out, provided n is not zero.

(j) This leaves us with $T_n = \frac{n+1}{2}$.

(k) We can verify this for $n = 3$: $T_3 = (3 + 1)/2 = 4/2 = 2$, which matches our earlier calculation.

Final Answer: The nth term of the series is $\frac{n+1}{2}$.

Answer: (A)

Go Back to Question 41

Q42.

Solution

Concept:
The binomial expansion of $(1+x)^n$ generates a series of terms where the coefficients are given by the combination formula nCr. This problem addresses the scenario where two consecutive binomial coefficients are equal. By setting the formulas for nCr and nC(r+1) equal to each other and simplifying the factorial expressions, we can find a direct linear relationship between the power n and the index r.

Solution:

(a) In the expansion of $(1+x)^n$, the general term is $T_{k+1} = \binom{n}{k}x^k$.

(b) The coefficient of x^r is $\binom{n}{r}$.

(c) The coefficient of x^{r+1} is $\binom{n}{r+1}$.

(d) According to the problem, these two coefficients are equal: $\binom{n}{r} = \binom{n}{r+1}$.

(e) Use the factorial definition of combinations: $\frac{n!}{r!(n-r)!} = \frac{n!}{(r+1)!(n-r-1)!}$.

(f) The $n!$ on both sides cancels out.

(g) Rearrange the terms: $\frac{(r+1)!}{r!} = \frac{(n-r)!}{(n-r-1)!}$.

(h) Using the property $k! = k \cdot (k-1)!$, we can simplify the fractions.

(i) $\frac{(r+1)\cdot r!}{r!} = \frac{(n-r)\cdot(n-r-1)!}{(n-r-1)!}$.

(j) This simplifies to $r+1 = n-r$.

(k) Adding r to both sides: $2r+1 = n$.

(l) Thus, for the coefficients to be equal, the power n must be one more than twice the index r.

Final Answer: The value of n is $2r+1$.

Answer: (B)

Go Back to Question 42

Q43.

Solution

Concept:
Circular permutations differ from linear permutations because there is no fixed starting or ending point at a round table. To account for this symmetry, the total number of ways to seat n people is $(n-1)!$. When specific constraints are added, such as two people not sitting together, it is often easier to use the method of subtraction: calculate the total permutations and subtract the cases where the two people do sit together.

Solution:

(a) First, calculate the total number of ways to seat 7 people at a round table without any restrictions.

(b) Total circular permutations = $(7-1)! = 6! = 720$.

(c) Now, let us calculate the number of ways where two particular persons, say A and B, always sit together.

(d) Treat A and B as a single entity or block. Now we have 6 entities to arrange around the table (the block plus the other 5 people).

(e) The number of ways to arrange 6 entities in a circle is $(6-1)! = 5! = 120$.

(f) Within the block, A and B can swap their seats in $2! = 2$ ways.

(g) Total ways A and B sit together = $120 \times 2 = 240$.

(h) To find the number of ways they do NOT sit together, subtract the restricted case from the total.

(i) Required ways = (Total ways) - (Ways they sit together).

(j) Required ways = $720 - 240 = 480$.

(k) This logic ensures that every arrangement where they are adjacent is excluded.

Final Answer: The number of ways is 480.

Answer: (C)

Go Back to Question 43

Q44.

Solution

Concept:
The probability of events and their set theoretic relationships are governed by the Addition Theorem of Probability and De Morgan's Laws. The probability that neither of two events occurs is equivalent to the probability that "A or B" does not occur. By using the inclusion exclusion principle to find the probability of the union of two events, we can then find its complement relative to the total probability space.

Solution:

(a) We are given $P(A) = 0.25$ and $P(B) = 0.50$.

(b) The probability that both occur simultaneously is the intersection: $P(A \cap B) = 0.14$.

(c) First, find the probability that at least one of the events occurs, which is the union $P(A \cup B)$.

(d) Using the Addition Theorem: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

(e) $P(A \cup B) = 0.25 + 0.50 - 0.14 = 0.75 - 0.14 = 0.61$.

(f) We are asked for the probability that neither A nor B occurs.

(g) This is denoted as $P(A' \cap B')$, which by De Morgan's Law is $P((A \cup B)')$.

(h) The probability of a complement is $1 - P(Event)$.

(i) $P(Neither) = 1 - P(A \cup B)$.

(j) $P(Neither) = 1 - 0.61 = 0.39$.

(k) This indicates there is a 39 percent chance that none of the specified events will happen.

Final Answer: The probability is 0.39.

Answer: (A)

Go Back to Question 44

Q45.

Solution

Concept:
The relative orientation of multiple lines in a plane can be determined by examining their slopes and intercepts. If three lines pass through a single point, they are concurrent. If they have the same slope but different intercepts, they are parallel. In this problem, we compare the coefficients of the given linear equations to see if any pair of lines shares the same normal vector direction, which would indicate parallelism.

Solution:

(a) Let the three lines be $L_1 : x - 2y + 3 = 0$, $L_2 : 1x + 2y + 7 = 0$, and $L_3 : x - 2y + 9 = 0$.

(b) The slope of a line $ax + by + c = 0$ is given by $m = -a/b$.

(c) For L_1, the slope $m_1 = -1/(-2) = 1/2$.

(d) For L_2, the slope $m_2 = -1/2 = -0.5$.

(e) For L_3, the slope $m_3 = -1/(-2) = 1/2$.

(f) We observe that $m_1 = m_3 = 1/2$.

(g) Since L_1 and L_3 have the same slope, they are parallel to each other.

(h) Now check if they are the same line by looking at the constant terms. Since 3 is not equal to 9, they are distinct parallel lines.

(i) The presence of a pair of parallel lines in a set of three lines precludes the possibility of them being the sides of a triangle (which requires no lines to be parallel) or being concurrent (which requires all three to intersect at one point).

(j) Since two lines are parallel and the third has a different slope, these lines are simply parallel with a transversal.

Final Answer: The lines are parallel.

Answer: (B)

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Q46.

Solution

Concept:
The derivative of one function with respect to another function, say $f(x)$ with respect to $g(x)$, is calculated using the formula $(df/dx)/(dg/dx)$. This is essentially an application of the chain rule. By differentiating the numerator and denominator separately with respect to the independent variable x, we find the rate of change of the first transcendental function relative to the second.
Solution:

(a) Let $u = e^{\sin x}$ and $v = \cos x$. We need to find du/dv.

(b) First, calculate the derivative of u with respect to x: $du/dx = e^{\sin x} \cdot \cos x$.

(c) Next, calculate the derivative of v with respect to x: $dv/dx = -\sin x$.

(d) Now, use the ratio formula: $du/dv = (du/dx)/(dv/dx)$.

(e) $du/dv = (e^{\sin x} \cdot \cos x)/(-\sin x)$.

(f) Simplify the trigonometric ratio: $\cos x/\sin x = \cot x$.

(g) Thus, the derivative is $-e^{\sin x} \cot x$.

Final Answer: The derivative is $-e^{\sin x} \cot x$.

Answer: (A)

Go Back to Question 46

Q47.

Solution

Concept:

Infinite nested radicals of the form $y = \sqrt{f(x) + \sqrt{f(x) + \ldots}}$ can be simplified by recognizing that the nested portion is identical to the original function y. This allows us to rewrite the infinite expression as a finite algebraic equation: $y = \sqrt{f(x) + y}$. Once squared, this equation can be differentiated implicitly to find dy/dx in terms of both x and y.

Solution:

(a) The given function is $y = \sqrt{\sin x + \sqrt{\sin x + \ldots}}$.

(b) We can rewrite this as $y = \sqrt{\sin x + y}$.

(c) Squaring both sides to remove the radical: $y^2 = \sin x + y$.

(d) Differentiate both sides with respect to x: $2y(dy/dx) = \cos x + (dy/dx)$.

(e) Collect the terms involving dy/dx on one side: $2y(dy/dx) - (dy/dx) = \cos x$.

(f) Factor out dy/dx: $(2y - 1)(dy/dx) = \cos x$.

(g) Solving for the derivative: $dy/dx = \frac{\cos x}{2y-1}$.

Final Answer: The value of dy/dx is $\frac{\cos x}{2y-1}$.

Answer: (A)

Go Back to Question 47

Q48.

Solution

Concept:
Limits involving $(1 - \cos\theta)/\theta^2$ as θ approaches zero are fundamental in calculus. The standard result is $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta^2} = 1/2$. Alternatively, this can be solved using L'Hôpital's Rule, which involves differentiating the numerator and denominator until the indeterminate $0/0$ form is resolved. For an expression with $4x$, we must ensure the denominator is scaled appropriately to match the angle.

Solution:

(a) The limit to evaluate is $\lim_{x\to 0} \frac{1-\cos 4x}{x^2}$.

(b) Using the identity $1 - \cos\theta = 2\sin^2(\theta/2)$, we can write $1 - \cos 4x = 2\sin^2(2x)$.

(c) Substitute this into the limit: $\lim_{x\to 0} \frac{2\sin^2(2x)}{x^2}$.

(d) To use the standard limit $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$, we need $(2x)^2$ in the denominator.

(e) Multiply and divide the expression by 4: $\lim_{x\to 0} 2 \cdot \frac{\sin^2(2x)}{x^2} \cdot \frac{4}{4} = \lim_{x\to 0} 8 \cdot \frac{\sin^2(2x)}{(2x)^2}$.

(f) Since $\lim_{x\to 0} \frac{\sin(2x)}{2x} = 1$, the expression becomes $8 \cdot (1)^2 = 8$.

Final Answer: The value of the limit is 8.

Answer: (B)

Go Back to Question 48

Q49.

Solution

Concept:
If a tangent to a curve is parallel to a given line, then the slope of the tangent must equal the slope of that line. In calculus, the slope of the tangent to $y = f(x)$ at any point is given by the derivative dy/dx. By setting the derivative equal to the known slope and solving for x, we find the x-coordinate of the point of tangency. The y-coordinate is then found by substituting x back into the original curve equation.

Solution:

(a) The curve is $y = x^2$ and the line is $y = 4x - 5$.

(b) The slope of the line $y = mx + c$ is $m = 4$.

(c) The derivative of the curve is $dy/dx = 2x$.

(d) For the tangent to be parallel to the line, we set the slopes equal: $2x = 4$.

(e) Solving for x gives $x = 2$.

(f) To find the y-coordinate, substitute $x = 2$ into the curve equation $y = x^2$: $y = (2)^2 = 4$.

(g) The required point is $(2, 4)$.

Final Answer: The point on the curve is $(2, 4)$.

Answer: (A)

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Q50.

Solution

Concept:
If a vector $\vec{a}$ is perpendicular to both $\vec{b}$ and $\vec{c}$, it must be parallel to their cross product $\vec{b} \times \vec{c}$. This is expressed as $\vec{a} = \lambda(\vec{b} \times \vec{c})$. By using the fact that $\vec{a}$ is a unit vector ($|\vec{a}| = 1$), we can solve for the scalar λ. The magnitude of the cross product involves the magnitudes of the individual vectors and the sine of the angle between them.

Solution:

(a) Given $\vec{a}, \vec{b}, \vec{c}$ are unit vectors, so $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$.

(b) Given $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \cdot \vec{c} = 0$, meaning $\vec{a} \perp \vec{b}$ and $\vec{a} \perp \vec{c}$.

(c) Thus, $\vec{a} = \lambda(\vec{b} \times \vec{c})$.

(d) Taking magnitudes: $|\vec{a}| = |\lambda||\vec{b} \times \vec{c}| = |\lambda||\vec{b}||\vec{c}|\sin(\pi/6)$.

(e) Substitute known values: $1 = |\lambda| \cdot (1) \cdot (1) \cdot (1/2)$.

(f) This gives $|\lambda| = 2$, so $\lambda = \pm 2$.

(g) Therefore, $\vec{a} = \pm 2(\vec{b} \times \vec{c})$.

Final Answer: Vector $\vec{a}$ is equal to $\pm 2(\vec{b} \times \vec{c})$.

Answer: (A)

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Answer Key

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	B	2	B	3	B	4	B	5	C
6	B	7	B	8	D	9	A	10	C
11	A	12	B	13	C	14	A	15	C
16	A	17	B	18	A	19	A	20	B
21	C	22	B	23	A	24	B	25	C
26	A	27	A	28	B	29	C	30	A
31	A	32	B	33	B	34	B	35	A
36	C	37	A	38	A	39	A	40	B
41	A	42	B	43	C	44	A	45	B
46	A	47	A	48	B	49	A	50	A