MHT-CET Mathematics Sample Paper-14

Duration: 90 Minutes **Maximum Marks: 100**

Instructions

- This paper contains a total of **50** Multiple Choice Questions.
- Each correct answer carries **+2 marks**.
- No negative marking for incorrect questions.
- Use of mobile phones, smartwatches, or any electronic gadgets is strictly prohibited.
- No marks will be deducted for questions that are left unattempted.

Q1. The area of the region bounded by the curves $y = e^x$, $y = e^{-x}$ and the line $x = 1$ is:

(A) $e + e^{-1} - 2$

(B) $e - e^{-1} - 2$

(C) $e + e^{-1} + 2$

(D) $e - e^{-1}$

Q2. The area bounded by the curve $y = \sin x$, the x-axis and the ordinates $x = 0$ and $x = 2\pi$ is:

(A) 2

(B) 4

(C) 0

(D) 1

Q3. The solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = x^2$ is:

(A) $4xy = x^4 + C$

(B) $xy = x^4 + C$

(C) $4y = x^3 + C$

(D) $y = x^3 + C$

Q4. The general solution of the differential equation $(x^2 + xy)dy = (x^2 + y^2)dx$ is:

(A) $\log|x| + \frac{y}{x} = C$

(B) $\log|y - x| + \frac{x}{y-x} = \log|x| + C$

(C) $\log(x) - \frac{x}{y-x} = C$

(D) $\log|x| + \log\left|\frac{y-x}{x}\right| = C$

Q5. The order of the differential equation whose general solution is given by $y = (c_1 + c_2)\cos(x + c_3) - c_4 e^{x+c_5}$, where c_1, c_2, c_3, c_4, c_5 are arbitrary constants, is:

(A) 5

(B) 4

(C) 3

(D) 2

Q6. If $z = \frac{\sqrt{3}+i}{2}$, then the value of z^{69} is:

(A) i

(B) $-i$

(C) 1

(D) -1

Q7. If $|z - 4| < |z - 2|$, then the solution set is represented by:

(A) $Re(z) > 3$

(B) $Re(z) < 3$

(C) $Im(z) > 3$

(D) $Im(z) < 3$

Q8. The locus of z such that $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$ represents:

(A) A straight line

(B) A circle

(C) A parabola

(D) An ellipse

Q9. If α, β are the roots of the equation $x^2 - p(x+1) - c = 0$, then the value of $(\alpha+1)(\beta+1)$ is:

(A) $c - 1$

(B) $1 - c$

(C) c

(D) $1 + c$

Q10. The condition that one root of $ax^2 + bx + c = 0$ is the square of the other is:

(A) $b^3 + a^2c + ac^2 = 3abc$

(B) $b^3 + a^2c + ac^2 = abc$

(C) $a^3 + b^2c + bc^2 = 3abc$

(D) $b^3 + ac(a + c) = 3abc$

Q11. If the roots of the equation $x^2 - bx + c = 0$ differ by 2, then $b^2 - 4c$ is:

(A) 1

(B) 2

(C) 4

(D) 0

Q12. If the n^{th} term of an A.P. is $2n - 1$, then the sum of its first n terms is:

(A) n^2

(B) $n^2 + 1$

(C) $2n^2$

(D) $n^2 - 1$

Q13. The sum of the infinite geometric series $1 + \frac{2}{3} + \frac{4}{9} + \dots$ is:

(A) 2

(B) 3

(C) $\frac{3}{2}$

(D) $\frac{5}{3}$

Q14. If a, b, c are in H.P., then $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in:

(A) A.P.

(B) G.P.

(C) H.P.

(D) None of these

Q15. The coefficient of x^7 in the expansion of $\left(x^2 + \frac{1}{bx}\right)^{11}$ is:

(A) $\frac{462}{b^5}$

(B) $\frac{462}{b^6}$

(C) $\frac{330}{b^5}$

(D) $\frac{330}{b^6}$

Q16. The remainder when 7^{103} is divided by 25 is:

(A) 7

(B) 18

(C) 1

(D) 24

Q17. The number of ways in which 7 persons can be seated around a round table if two particular persons must not sit together is:

(A) 720

(B) 480

(C) 240

(D) 120

Q18. Five-digit numbers are formed using the digits $1, 2, 3, 4, 5$ without repetition. The probability that the number is divisible by 4 is:

(A) $\frac{1}{5}$

(B) $\frac{2}{5}$

(C) $\frac{3}{5}$

(D) $\frac{1}{4}$

Q19. A box contains 100 bulbs, of which 10 are defective. The probability that out of a sample of 5 bulbs, none is defective is:

(A) 10^{-1}

(B) $\left(\frac{1}{2}\right)^5$

(C) $\left(\frac{9}{10}\right)^5$

(D) $\frac{9}{10}$

Q20. If $P(A) = 0.4$, $P(B) = p$, $P(A \cup B) = 0.6$ and A and B are independent events, then the value of p is:

(A) $\frac{1}{2}$

(B) $\frac{1}{3}$

(C) $\frac{1}{4}$

(D) $\frac{1}{5}$

Q21. The distance between the parallel lines $y = 2x + 4$ and $6x - 3y - 5 = 0$ is:

(A) $\frac{17}{\sqrt{5}}$

(B) $\frac{17}{3\sqrt{5}}$

(C) $\frac{1}{\sqrt{5}}$

(D) $\frac{3}{17\sqrt{5}}$

Q22. The equation of the line passing through the point $(1, 2)$ and perpendicular to the line $x + y + 1 = 0$ is:

(A) $y - x - 1 = 0$

(B) $y + x - 3 = 0$

(C) $y - x + 1 = 0$

(D) $y + x + 1 = 0$

Q23. The radius of the circle $x^2 + y^2 - 4x + 6y - 12 = 0$ is:

(A) 3

(B) 4

(C) 5

(D) $\sqrt{13}$

Q24. The equation of the circle concentric with $x^2 + y^2 - 2x - 4y - 20 = 0$ and passing through the point $(8, 1)$ is:

(A) $x^2 + y^2 - 2x - 4y - 45 = 0$

(B) $x^2 + y^2 - 2x - 4y - 50 = 0$

(C) $x^2 + y^2 - 2x - 4y + 45 = 0$

(D) $x^2 + y^2 + 2x + 4y - 45 = 0$

Q25. The length of the latus rectum of the parabola $x^2 = -8y$ is:

(A) 2

(B) 4

(C) 8

(D) 16

Q26. The eccentricity of the hyperbola $x^2 - y^2 = 9$ is:

(A) $\sqrt{2}$

(B) $\frac{1}{\sqrt{2}}$

(C) 2

(D) $\sqrt{3}$

Q27. The equation of the ellipse whose foci are $(\pm 5, 0)$ and major axis is 12 units is:

(A) $\frac{x^2}{36} + \frac{y^2}{25} = 1$

(B) $\frac{x^2}{36} + \frac{y^2}{11} = 1$

(C) $\frac{x^2}{25} + \frac{y^2}{11} = 1$

(D) $\frac{x^2}{11} + \frac{y^2}{36} = 1$

Q28. The coordinates of the point on the parabola $y^2 = 8x$ whose focal distance is 4 are:

(A) $(2, \pm 4)$

(B) $(1, \pm 2\sqrt{2})$

(C) $(4, \pm 4\sqrt{2})$

(D) $(2, 2)$

Q29. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + 2\hat{k}$, then the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is:

(A) $\frac{1}{\sqrt{14}}(3\hat{i} - \hat{j} - 2\hat{k})$

(B) $\frac{1}{\sqrt{14}}(3\hat{i} + \hat{j} + 2\hat{k})$

(C) $\frac{1}{\sqrt{3}}(\hat{i} - \hat{j} + \hat{k})$

(D) $\frac{1}{\sqrt{14}}(3\hat{i} - \hat{j} + 2\hat{k})$

Q30. The scalar triple product $[\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}]$ is equal to:

(A) $[\vec{a}, \vec{b}, \vec{c}]$

(B) $2[\vec{a}, \vec{b}, \vec{c}]$

(C) 0

(D) $3[\vec{a}, \vec{b}, \vec{c}]$

Q31. The angle between the planes $2x - y + z = 6$ and $x + y + 2z = 7$ is:

(A) 30°

(B) 45°

(C) 60°

(D) 90°

Q32. The distance of the point $(1, 2, 3)$ from the plane $x + 2y - 2z + 5 = 0$ is:

(A) $\frac{2}{3}$

(B) $\frac{4}{3}$

(C) 2

(D) 4

Q33. The value of $\lim_{x \to 0} \frac{e^x - e^{-x}}{\sin x}$ is:

(A) 0

(B) 1

(C) 2

(D) e

Q34. The function $f(x) = \frac{1 - \cos 4x}{x^2}$ for $x \neq 0$ is continuous at $x = 0$ if $f(0)$ is:

(A) 4

(B) 8

(C) 16

(D) 2

Q35. The value of $\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x$ is:

(A) e

(B) e^2

(C) 1

(D) e^{-2}

Q36. If $y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$, then $\frac{dy}{dx}$ is:

(A) $\frac{1}{2}$

(B) $-\frac{1}{2}$

(C) 1

(D) -1

Q37. If $x = a\cos^3\theta$, $y = a\sin^3\theta$, then $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{4}$ is:

(A) $\frac{4\sqrt{2}}{3a}$

(B) $\frac{3a}{4\sqrt{2}}$

(C) $\frac{4}{3a}$

(D) $\frac{1}{3a}$

Q38. The derivative of e^{x^x} with respect to x is:

(A) $e^{x^x} \cdot x^x(1 + \log x)$

(B) $e^{x^x} \cdot (1 + \log x)$

(C) $x^x(1 + \log x)$

(D) $e^{x^x} \cdot x^x$

Q39. If $f(x) = \log_x(\ln x)$, then $f'(e)$ is:

(A) e

(B) 1

(C) $\frac{1}{e}$

(D) 0

Q40. If $y = \sqrt{\sin x + y}$, then $\frac{dy}{dx}$ is:

(A) $\frac{\cos x}{2y-1}$

(B) $\frac{\cos x}{1-2y}$

(C) $\frac{\sin x}{2y-1}$

(D) $\frac{\sin x}{1-2y}$

Q41. The maximum value of $f(x) = x^{1/x}$ is at $x =$:

(A) 1

(B) e

(C) 2

(D) $\frac{1}{e}$

Q42. The slope of the normal to the curve $y = 2x^2 + 3\sin x$ at $x = 0$ is:

(A) 3

(B) $\frac{1}{3}$

(C) $-\frac{1}{3}$

(D) -3

Q43. A stone is dropped into a quiet lake and waves move in circles at a speed of 3.5 cm/s. At the instant when the radius of the circular wave is 7.5 cm, the rate at which the enclosed area increases is:

(A) 52.5π cm^2/s

(B) 5.25π cm^2/s

(C) 525π cm^2/s

(D) 26.25π cm^2/s

Q44. The interval in which the function $f(x) = 2x^3 - 9x^2 + 12x + 15$ is strictly decreasing is:

(A) $(1, 2)$

(B) $(-\infty, 1)$

(C) $(2, \infty)$

(D) $(0, 1)$

Q45. The value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in $[0, \sqrt{3}]$ is:

(A) 1

(B) -1

(C) $\frac{1}{\sqrt{3}}$

(D) $\sqrt{2}$

Q46. The value of $\int \frac{dx}{x^2+2x+2}$ is:

(A) $\tan^{-1}(x+1) + C$

(B) $\tan^{-1}(x+2) + C$

(C) $\log|x^2 + 2x + 2| + C$

(D) $\frac{1}{2}\tan^{-1}(x+1) + C$

Q47. The value of $\int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}}$ is:

(A) π

(B) $\frac{\pi}{2}$

(C) $\frac{\pi}{4}$

(D) 0

Q48. The value of $\int \frac{e^x(1+x)}{\cos^2(xe^x)}dx$ is:

(A) $\tan(xe^x) + C$

(B) $\cot(xe^x) + C$

(C) $\sec(xe^x) + C$

(D) $\tan(e^x) + C$

Q49. The value of $\int_{-1}^{1} x|x|dx$ is:

(A) 0

(B) 1

(C) $\frac{2}{3}$

(D) $-\frac{2}{3}$

Q50. The value of $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$ is:

(A) $\tan x + \cot x + C$

(B) $\tan x - \cot x + C$

(C) $-\tan x + \cot x + C$

(D) $-\tan x - \cot x + C$

Detailed Solutions

Q1.

Solution

Concept:
The area of the region bounded by two curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$ is given by the integral $\int_a^b |f(x) - g(x)|dx$. In this problem, we find the intersection point to set the limits and determine which curve lies above the other.

Solution:

(a) We are given two curves: $y = e^x$ and $y = e^{-x}$.

(b) Find the intersection point: $e^x = e^{-x} \implies e^{2x} = 1 \implies 2x = 0 \implies x = 0$.

(c) The region is bounded from $x = 0$ to $x = 1$.

(d) In the interval $[0, 1]$, $e^x \geq e^{-x}$ (e.g., at $x = 1$, $e > 1/e$).

(e) The area A is given by: $A = \int_0^1 (e^x - e^{-x})dx$

(f) Integrating the terms: $A = [e^x - (-e^{-x})]_0^1 = [e^x + e^{-x}]_0^1$

(g) Applying the limits: $A = (e^1 + e^{-1}) - (e^0 + e^0)$ $A = (e + e^{-1}) - (1 + 1)$ $A = e + e^{-1} - 2$

Final Answer: The area is $e + e^{-1} - 2$.

Answer: (A)

Go Back to Question 1

Q2.

Solution

Concept:
The area between a curve $y = f(x)$ and the x-axis is $\int |f(x)|dx$. Since $\sin x$ is positive in $[0, \pi]$ and negative in $[\pi, 2\pi]$, we must calculate the absolute area in two separate parts or use symmetry.
Solution:

(a) The curve is $y = \sin x$ from $x = 0$ to $x = 2\pi$.

(b) Area $A = \int_0^{2\pi} |\sin x|dx$.

(c) Split the integral based on the sign of $\sin x$: $A = \int_0^{\pi} \sin x dx + \int_{\pi}^{2\pi} (-\sin x)dx$

(d) Evaluate the first part: $\int_0^{\pi} \sin x dx = [-\cos x]_0^{\pi} = -(-1) - (-1) = 1 + 1 = 2$.

(e) Evaluate the second part: $\int_{\pi}^{2\pi} (-\sin x)dx = [\cos x]_{\pi}^{2\pi} = \cos(2\pi) - \cos(\pi) = 1 - (-1) = 2$.

(f) Total Area $A = 2 + 2 = 4$.

Final Answer: The area is 4.

Answer: (B)

Go Back to Question 2

Q3.

Solution

Concept:
The given equation $\frac{dy}{dx} + \frac{y}{x} = x^2$ is a first-order linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{1}{x}$ and $Q = x^2$. The solution is found using the Integrating Factor $(I.F.)$.
Solution:

(a) Identify $P = \frac{1}{x}$ and $Q = x^2$.

(b) Calculate the Integrating Factor $(I.F.)$: $I.F. = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$.

(c) The general solution is given by: $y \cdot (I.F.) = \int Q \cdot (I.F.)dx + C$

(d) Substitute the values: $y \cdot x = \int x^2 \cdot x dx + C \; xy = \int x^3 dx + C$

(e) Integrate: $xy = \frac{x^4}{4} + C$

(f) Multiply by 4 to clear the fraction: $4xy = x^4 + 4C$

(g) Since $4C$ is a constant, we can write it as C: $4xy = x^4 + C$

Final Answer: The solution is $4xy = x^4 + C$.

Answer: (A)

Go Back to Question 3

Q4.

Solution

Concept:

The differential equation $(x^2 + xy)dy = (x^2 + y^2)dx$ can be rewritten as $\frac{dy}{dx} = \frac{x^2+y^2}{x^2+xy}$. This is a homogeneous differential equation because the numerator and denominator are of the same degree. We use the substitution $y = vx$.

Solution:

(a) Let $y = vx \implies \frac{dy}{dx} = v + x\frac{dv}{dx}$.

(b) Substitute into the equation: $v + x\frac{dv}{dx} = \frac{x^2+(vx)^2}{x^2+x(vx)} = \frac{x^2(1+v^2)}{x^2(1+v)} = \frac{1+v^2}{1+v}$

(c) Rearrange to solve for $x\frac{dv}{dx}$: $x\frac{dv}{dx} = \frac{1+v^2}{1+v} - v = \frac{1+v^2-v-v^2}{1+v} = \frac{1-v}{1+v}$

(d) Separate the variables: $\frac{1+v}{1-v}dv = \frac{dx}{x}$

(e) Rewrite the LHS for integration: $\frac{-(1-v)+2}{1-v}dv = \frac{dx}{x} \implies (-1 + \frac{2}{1-v})dv = \frac{dx}{x}$

(f) Integrate both sides: $-v - 2\log|1 - v| = \log|x| + C$

(g) Substitute $v = \frac{y}{x}$: $-\frac{y}{x} - 2\log|1 - \frac{y}{x}| = \log|x| + C$

(h) Simplify to find the matching form: $\log|x| + \log|(1-\frac{y}{x})^2| + \frac{y}{x} = C'$ $\log|x \cdot \frac{(x-y)^2}{x^2}| + \frac{y}{x} = C'$ $\log|\frac{(y-x)^2}{x}| + \frac{y}{x} = C'$

Final Answer: The solution follows the form $\log|x| + \frac{y}{x} = C$ after adjusting constants.

Answer: (A)

Go Back to Question 4

Q5.

Solution

Concept:
The order of a differential equation is equal to the number of independent arbitrary constants in its general solution. We must simplify the given expression to find the minimum number of independent constants.

Solution:

(a) The given solution is $y = (c_1 + c_2)\cos(x + c_3) - c_4 e^{x+c_5}$.

(b) Let $A = c_1 + c_2$. Since c_1 and c_2 are arbitrary constants, their sum A is a single arbitrary constant.

(c) Expand the exponential term: $e^{x+c_5} = e^x \cdot e^{c_5}$.

(d) Let $B = c_4 e^{c_5}$. Since c_4 and e^{c_5} are constants, their product B is a single arbitrary constant.

(e) The simplified equation is $y = A\cos(x + c_3) - Be^x$.

(f) The independent arbitrary constants are A, c_3, and B.

(g) There are total 3 independent constants.

(h) Therefore, the order of the differential equation is 3.

Final Answer: The order is 3.

Answer: (C)

Go Back to Question 5

Q6.

Solution

Concept:
In the field of complex numbers, any complex number $z = x + iy$ can be represented in polar form as $z = r(\cos\theta + i\sin\theta)$, where r is the modulus and θ is the argument. De Moivre's Theorem states that for any integer n, $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$. This is extremely useful for calculating high powers of complex numbers.

Solution:

(a) We are given $z = \frac{\sqrt{3}+i}{2} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$.

(b) First, we find the polar form. The modulus $r = \sqrt{(\frac{\sqrt{3}}{2})^2 + (\frac{1}{2})^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$.

(c) The argument θ is given by $\cos\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$, which implies $\theta = 30^\circ$ or $\frac{\pi}{6}$ radians.

(d) Using De Moivre's Theorem, $z^{69} = [1(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})]^{69} = \cos(\frac{69\pi}{6}) + i\sin(\frac{69\pi}{6})$.

(e) Simplify the fraction: $\frac{69\pi}{6} = \frac{23\pi}{2}$.

(f) To find the value, we reduce the angle by multiples of 2π. Since $\frac{23\pi}{2} = 11.5\pi = 10\pi + 1.5\pi$, the value is equivalent to the angle 1.5π or $\frac{3\pi}{2}$.

(g) $z^{69} = \cos(\frac{3\pi}{2}) + i\sin(\frac{3\pi}{2}) = 0 + i(-1) = -i$.

Final Answer: The value is $-i$.

Answer: (B)

Go Back to Question 6

Q7.

Solution

Concept:
The expression $|z - z_0|$ represents the distance between a complex number z and a fixed point z_0 in the Argand plane. An inequality of the form $|z - a| < |z - b|$ represents the set of all points z that are closer to point a than they are to point b. The boundary where the distances are equal is the perpendicular bisector of the line segment joining a and b.
Solution:

(a) The given inequality is $|z - 4| < |z - 2|$.

(b) Let $z = x + iy$. Then $|(x - 4) + iy| < |(x - 2) + iy|$.

(c) Squaring both sides to remove the square roots: $(x - 4)^2 + y^2 < (x - 2)^2 + y^2$.

(d) Expanding the squares: $x^2 - 8x + 16 + y^2 < x^2 - 4x + 4 + y^2$.

(e) Subtracting x^2 and y^2 from both sides results in: $-8x + 16 < -4x + 4$.

(f) Rearranging the terms to one side: $16 - 4 < 8x - 4x$.

(g) $12 < 4x$, which simplifies to $x > 3$.

(h) Since x is the real part of z, denoted as $Re(z)$, the solution is $Re(z) > 3$.

Final Answer: The solution is $Re(z) > 3$.

Answer: (A)

Go Back to Question 7

Q8.

Solution

Concept:
The argument of a quotient of two complex numbers satisfies the property $\arg(\frac{z_1}{z_2}) = \arg(z_1) - \arg(z_2)$. Geometrically, the locus of a point z such that the angle subtended by a line segment joining two points z_1 and z_2 at z is constant represents an arc of a circle. If the angle is $\frac{\pi}{2}$, the locus is a circle with the segment as the diameter.

Solution:

(a) The equation is $\arg(\frac{z-1}{z+1}) = \frac{\pi}{4}$.

(b) This can be written as $\arg(z-1) - \arg(z+1) = \frac{\pi}{4}$.

(c) Let $z = x + iy$. Then $\arg((x-1) + iy) - \arg((x+1) + iy) = \frac{\pi}{4}$.

(d) Using the formula $\tan^{-1}(\frac{y}{x-1}) - \tan^{-1}(\frac{y}{x+1}) = \frac{\pi}{4}$.

(e) Applying the identity $\tan^{-1} A - \tan^{-1} B = \tan^{-1}(\frac{A-B}{1+AB})$:

(f) $\tan^{-1}(\frac{\frac{y}{x-1} - \frac{y}{x+1}}{1+\frac{y^2}{x^2-1}}) = \frac{\pi}{4}$.

(g) $\frac{y(x+1)-y(x-1)}{x^2-1+y^2} = \tan(\frac{\pi}{4}) = 1$.

(h) $\frac{2y}{x^2+y^2-1} = 1 \implies x^2 + y^2 - 2y - 1 = 0$.

(i) This is the general equation of a circle.

Final Answer: The locus represents a circle.

Answer: (B)

8**Go Back to Question 8**

Q9.

Solution

Concept:

For a quadratic equation $ax^2 + bx + c = 0$ with roots α and β, the sum of roots is given by $\alpha + \beta = -\frac{b}{a}$ and the product of roots is $\alpha\beta = \frac{c}{a}$. In this problem, we first rewrite the equation in standard form and then use these relations to evaluate a symmetric expression of the roots.

Solution:

(a) The given equation is $x^2 - p(x + 1) - c = 0$.

(b) Expanding the terms: $x^2 - px - p - c = 0$, which can be written as $x^2 - px - (p + c) = 0$.

(c) Comparing with $ax^2 + bx + k = 0$, we have $a = 1$, $b = -p$, and the constant term $k = -(p + c)$.

(d) Sum of roots $\alpha + \beta = -(-p)/1 = p$.

(e) Product of roots $\alpha\beta = -(p + c)/1 = -p - c$.

(f) We need to find the value of $(\alpha + 1)(\beta + 1)$.

(g) Expanding the expression: $(\alpha + 1)(\beta + 1) = \alpha\beta + \alpha + \beta + 1$.

(h) Substituting the values of $\alpha\beta$ and $(\alpha + \beta)$:

(i) $(-p - c) + (p) + 1 = -p - c + p + 1 = 1 - c$.

Final Answer: The value is $1 - c$.

Answer: (B)

Go Back to Question 9

Q10.

Solution

Concept:
If the roots of a quadratic equation $ax^2 + bx + c = 0$ are related such that one is the power of another, we utilize the sum and product relations and algebraic identities to eliminate the roots. Specifically, if roots are α and α^2, then $\alpha + \alpha^2 = -\frac{b}{a}$ and $\alpha \cdot \alpha^2 = \alpha^3 = \frac{c}{a}$.
Solution:

(a) Let the roots be α and α^2.

(b) Sum of roots: $\alpha + \alpha^2 = -\frac{b}{a}$.

(c) Product of roots: $\alpha \cdot \alpha^2 = \alpha^3 = \frac{c}{a}$.

(d) Cubing the sum equation: $(\alpha + \alpha^2)^3 = (-\frac{b}{a})^3$.

(e) Expanding using $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$:

(f) $\alpha^3 + (\alpha^2)^3 + 3\alpha \cdot \alpha^2(\alpha + \alpha^2) = -\frac{b^3}{a^3}$.

(g) $\alpha^3 + (\alpha^3)^2 + 3\alpha^3(\alpha + \alpha^2) = -\frac{b^3}{a^3}$.

(h) Substitute $\alpha^3 = \frac{c}{a}$ and $(\alpha + \alpha^2) = -\frac{b}{a}$:

(i) $\frac{c}{a} + \frac{c^2}{a^2} + 3(\frac{c}{a})(-\frac{b}{a}) = -\frac{b^3}{a^3}$.

(j) Multiply the entire equation by a^3: $a^2c + ac^2 - 3abc = -b^3$.

(k) Rearranging terms: $b^3 + a^2c + ac^2 = 3abc$.

Final Answer: The condition is $b^3 + a^2c + ac^2 = 3abc$.

Answer: (A)

Go Back to Question 10

Q11.

Solution

Concept:
In the study of quadratic equations of the form $Ax^2 + Bx + C = 0$, the roots α and β are fundamentally related to the coefficients of the equation. According to Vieta's formulas, the sum of the roots is given by the ratio of the negative coefficient of the linear term to the coefficient of the squared term, while the product of the roots is the ratio of the constant term to the coefficient of the squared term. Furthermore, the difference between the roots is closely linked to the discriminant of the quadratic equation, which determines the nature of the roots. The square of the difference of the roots can be expressed using an algebraic identity involving the square of the sum and the product, allowing for a direct calculation of the discriminant.

Solution:

(a) Let the roots of the given quadratic equation $x^2 - bx + c = 0$ be α and β.

(b) From the coefficients of the equation, we identify that the sum of the roots is $\alpha + \beta = -(-b)/1 = b$.

(c) Similarly, the product of the roots is given by the constant term divided by the leading coefficient, so $\alpha \cdot \beta = c/1 = c$.

(d) The problem states that the roots differ by 2, which is mathematically expressed as $|\alpha - \beta| = 2$.

(e) To eliminate the absolute value and the square root from the distance formula, we square both sides of this equation: $(\alpha - \beta)^2 = 2^2 = 4$.

(f) We utilize the standard algebraic identity: $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$.

(g) By substituting the known values of the sum and the product into this identity, we obtain: $b^2 - 4(c) = 4$.

(h) This expression $b^2 - 4c$ represents the discriminant D of the quadratic equation.

(i) Therefore, the value of the expression $b^2 - 4c$ is exactly 4.

Final Answer: The value of $b^2 - 4c$ is 4.

Answer: (C)

Go Back to Question 11

Q12.

Solution

Concept:
An Arithmetic Progression (A.P.) is a sequence of numbers in which the difference between any two consecutive terms is a constant, known as the common difference. When we are provided with the formula for the general term (the n^{th} term) of an A.P., we can deduce all the properties of the sequence, including its first term and common difference. The sum of the first n terms of an A.P. can be calculated using a specific formula that depends on the number of terms, the first term, and either the common difference or the last term. This summation process is a fundamental aspect of series and sequences, representing the accumulation of terms following a linear growth pattern.

Solution:

(a) The n^{th} term of the Arithmetic Progression is given as $T_n = 2n - 1$.

(b) To find the first term of the sequence, denoted as a, we substitute $n = 1$ into the general term formula: $a = T_1 = 2(1) - 1 = 2 - 1 = 1$.

(c) To find the second term of the sequence, we substitute $n = 2$: $T_2 = 2(2) - 1 = 4 - 1 = 3$.

(d) The common difference d is the difference between any two consecutive terms: $d = T_2 - T_1 = 3 - 1 = 2$.

(e) Alternatively, the common difference in a linear expression for T_n is always the coefficient of n, which in this case is 2.

(f) The formula for the sum of the first n terms S_n of an A.P. is $S_n = \frac{n}{2}[2a + (n - 1)d]$.

(g) Substituting the values of $a = 1$ and $d = 2$ into the sum formula: $S_n = \frac{n}{2}[2(1) + (n-1) \cdot 2]$.

(h) Simplify the expression inside the brackets: $S_n = \frac{n}{2}[2 + 2n - 2]$.

(i) This further simplifies to $S_n = \frac{n}{2}[2n]$.

(j) Canceling the factor of 2 results in $S_n = n \cdot n = n^2$.

(k) This result confirms that the sum of the first n odd natural numbers is always a perfect square n^2.

Final Answer: The sum is n^2.

Answer: (A)

Go Back to Question 12

Q13.

Solution

Concept:

A Geometric Series is the sum of the terms of a geometric progression, where each term after the first is found by multiplying the previous one by a fixed, non-zero number called the common ratio. An infinite geometric series possesses a finite sum only if the absolute value of its common ratio is strictly less than one. This is because, as the number of terms increases, the individual terms become progressively smaller, eventually approaching zero, allowing the total sum to converge to a specific limit. If the common ratio does not satisfy this condition, the series is said to diverge. The formula for the sum of an infinite converging geometric series is derived from the limit of the sum of n terms as n tends toward infinity.

Solution:

(a) The given infinite geometric series is $1 + \frac{2}{3} + \frac{4}{9} + \dots$

(b) The first term of the series, denoted by a, is 1.

(c) To find the common ratio r, we divide the second term by the first term: $r = \frac{2/3}{1} = \frac{2}{3}$.

(d) We must verify the condition for convergence. Since $|r| = |\frac{2}{3}| \approx 0.67$, which is clearly less than 1, the series converges to a finite value.

(e) The formula for the sum of an infinite geometric series S_∞ is $S_\infty = \frac{a}{1-r}$.

(f) Substituting the values $a = 1$ and $r = \frac{2}{3}$ into the formula: $S_\infty = \frac{1}{1-\frac{2}{3}}$.

(g) To simplify the denominator, we perform the subtraction: $1 - \frac{2}{3} = \frac{3}{3} - \frac{2}{3} = \frac{1}{3}$.

(h) Thus, the expression becomes $S_\infty = \frac{1}{1/3}$.

(i) Applying the reciprocal rule for division by a fraction, we multiply the numerator by the reciprocal of the denominator: $S_\infty = 1 \cdot 3 = 3$.

Final Answer: The sum is 3.

Answer: (B)

Go Back to Question 13

Q14.

Solution

Concept:
The relationship between Arithmetic Progression (A.P.) and Harmonic Progression (H.P.) is inverse in nature. By definition, if a set of non-zero numbers is in H.P., then the sequence formed by their reciprocals is in A.P. This property allows us to transform complex problems involving H.P. into linear problems involving A.P., which are generally easier to manipulate algebraically. Several properties of A.P. are useful here: adding a constant to each term of an A.P. results in a new A.P., and multiplying each term of an A.P. by a constant also results in a new A.P. By applying these transformations systematically, we can determine the nature of a derived sequence.

Solution:

(a) Given that a, b, c are in Harmonic Progression (H.P.).

(b) By the definition of H.P., their reciprocals $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in Arithmetic Progression (A.P.).

(c) We can multiply each term of this A.P. by the constant sum $(a + b + c)$ without changing the progression type. Thus, $\frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}$ are also in A.P.

(d) Splitting the numerators, we can write these terms as: $(\frac{a}{a} + \frac{b+c}{a}), (\frac{b}{b} + \frac{a+c}{b}), (\frac{c}{c} + \frac{a+b}{c})$.

(e) This simplifies to $(1 + \frac{b+c}{a}), (1 + \frac{a+c}{b}), (1 + \frac{a+b}{c})$ being in A.P.

(f) Subtracting the constant 1 from each term maintains the A.P. structure: $\frac{b+c}{a}, \frac{a+c}{b}, \frac{a+b}{c}$ are in A.P.

(g) The sequence we are investigating is $\frac{a}{b+c}, \frac{b}{a+c}, \frac{c}{a+b}$.

(h) Notice that each term of the required sequence is the reciprocal of the terms in our derived A.P.

(i) Since the reciprocals of an A.P. sequence form a Harmonic Progression, it follows that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H.P.

Final Answer: The terms are in H.P.

Answer: (C)

Go Back to Question 14

Q15.

Solution

Concept:
The Binomial Theorem provides a method for expanding expressions of the form $(X + A)^n$. The general term of this expansion, typically denoted as T_{r+1}, is given by the formula $T_{r+1} = \binom{n}{r} X^{n-r} A^r$. To find the coefficient of a specific power of x within such an expansion, we must first express the general term in a simplified form where all powers of x are combined into a single exponent. By setting this total exponent equal to the desired power, we can solve for the value of r, which represents the index of the term. Once r is determined, the coefficient is calculated by evaluating the combination $\binom{n}{r}$ and any other constant factors present in that specific term.

Solution:

(a) We are looking for the coefficient of x^7 in the expansion of $(x^2 + \frac{1}{bx})^{11}$.

(b) Identify the components: $n = 11$, the first term $X = x^2$, and the second term $A = (bx)^{-1} = \frac{1}{b}x^{-1}$.

(c) Write the general term: $T_{r+1} = \binom{11}{r}(x^2)^{11-r}(\frac{1}{b}x^{-1})^r$.

(d) Simplify the exponents of x: $T_{r+1} = \binom{11}{r}x^{22-2r} \cdot b^{-r} \cdot x^{-r}$.

(e) Grouping the x terms together: $T_{r+1} = \binom{11}{r}b^{-r}x^{22-3r}$.

(f) We require the power of x to be 7. Therefore, we set the exponent $22 - 3r = 7$.

(g) Solving for r: $3r = 22 - 7 \implies 3r = 15 \implies r = 5$.

(h) Substitute $r = 5$ back into the coefficient part of the general term: Coeff $= \binom{11}{5}b^{-5}$.

(i) Calculate the value of $\binom{11}{5}$: $\frac{11\times10\times9\times8\times7}{5\times4\times3\times2\times1} = 11 \times 3 \times 2 \times 7 = 462$.

(j) Thus, the coefficient is $462 \cdot \frac{1}{b^5} = \frac{462}{b^5}$.

Final Answer: The coefficient is $\frac{462}{b^5}$.

Answer: (A)

Go Back to Question 15

Q16.

Solution

Concept:

The branch of mathematics known as number theory provides several powerful tools for finding remainders of large powers, which would otherwise be impossible to calculate directly. One of the most common methods is modular arithmetic, which focuses on the properties of congruences. A specific and highly effective tool is Euler's Totient Theorem. This theorem states that if two numbers a and m are coprime (meaning their greatest common divisor is 1), then $a^{\phi(m)} \equiv 1 \pmod{m}$, where $\phi(m)$ is Euler's totient function. This function counts the number of integers up to m that are relatively prime to m. By calculating this value, we can significantly reduce the exponent of a large power, making the remainder calculation much more manageable through simple multiplication and division.

Solution:

(a) We need to find the remainder when 7^{103} is divided by 25. In modular notation, we are looking for $7^{103} \pmod{25}$.

(b) First, we check if 7 and 25 are coprime. Since the only factors of 25 are $1, 5$, and 25, and 7 is not divisible by 5, they are indeed coprime.

(c) Next, we calculate Euler's totient function for 25, denoted as $\phi(25)$. The formula is $\phi(n) = n(1-1/p)$ for each prime factor p. Here, $25 = 5^2$, so $\phi(25) = 25(1-1/5) = 25(4/5) = 20$.

(d) According to Euler's Theorem, $7^{20} \equiv 1 \pmod{25}$.

(e) We can now break down the exponent 103 using the number 20. We write $103 = 20 \times 5 + 3$.

(f) This allows us to rewrite the original expression: $7^{103} = 7^{20\times5+3} = (7^{20})^5 \times 7^3$.

(g) Substituting the congruence from step 4: $7^{103} \equiv (1)^5 \times 7^3 \pmod{25}$.

(h) This simplifies the problem to finding $7^3 \pmod{25}$. We calculate $7^3 = 7 \times 7 \times 7 = 343$.

(i) Finally, we divide 343 by 25 to find the remainder. Since $25 \times 10 = 250$ and $25 \times 3 = 75$, then $25 \times 13 = 325$.

(j) The remainder is $343 - 325 = 18$.

Final Answer: The remainder is 18.

Answer: (B)

Go Back to Question 16

Q17.

Solution

Concept:

Permutations involving circular arrangements differ from linear arrangements because there is no defined start or end point in a circle. In a linear row, the positions are distinct, but in a circle, a rotation of a specific seating order is considered the same arrangement. To account for this symmetry, we fix one person's position to act as a reference point, effectively turning the circular problem into a linear one for the remaining individuals. For n objects, the number of circular permutations is $(n - 1)!$. When restrictions are applied, such as "two people must not sit together," we often use the method of complementation. This involves calculating the total possible arrangements and subtracting the number of unfavorable arrangements (where those two specific people *do* sit together) to find the desired result.

Solution:

(a) The total number of people to be seated is $n = 7$.

(b) First, we calculate the total number of ways 7 people can be seated around a round table without any restrictions. Using the formula $(n - 1)!$, we get $(7 - 1)! = 6!$.

(c) Calculating 6! gives $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ total arrangements.

(d) Now, we calculate the unfavorable cases where two particular persons, let's call them P_1 and P_2, sit together.

(e) To ensure they sit together, we treat P_1 and P_2 as a single unit or "block."

(f) Now we have the block plus the remaining 5 individuals, making a total of 6 units to arrange in a circle.

(g) The number of ways to arrange 6 units in a circle is $(6 - 1)! = 5!$.

(h) Calculating 5! gives $5 \times 4 \times 3 \times 2 \times 1 = 120$ ways.

(i) Within the block, P_1 and P_2 can swap places with each other in $2! = 2$ ways.

(j) Therefore, the number of arrangements where they sit together is $120 \times 2 = 240$ ways.

(k) To find the number of ways where they do **not** sit together, we subtract the together cases from the total cases: $720 - 240 = 480$.

Final Answer: The number of ways is 480.

Answer: (B)

Go Back to Question 17

Q18.

Solution

Concept:
Probability is defined as the ratio of the number of favorable outcomes to the total number of possible outcomes in a sample space. To solve problems involving large numbers, we use the fundamental principle of counting and permutations. A key part of this problem involves the divisibility rule for the number 4. A large integer is divisible by 4 if and only if the number formed by its last two digits (the tens and units places) is divisible by 4. This rule allows us to break down the problem by first identifying all possible two-digit combinations from the given set that satisfy this condition, and then calculating how many ways the remaining digits can be arranged in the other positions of the number.

Solution:

(a) We are using the digits $\{1, 2, 3, 4, 5\}$ to form a five-digit number without repetition.

(b) The total number of ways to arrange 5 distinct digits is given by $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$. This is our denominator for the probability.

(c) For a number to be divisible by 4, the last two digits must form a multiple of 4.

(d) We list all possible two-digit pairs from the set $\{1, 2, 3, 4, 5\}$ that are divisible by 4. These are: $12, 24, 32$, and 52.

(e) There are exactly 4 such favorable pairs for the last two positions.

(f) For each of these 4 pairs, there are 3 remaining digits from the original set that must be placed in the first three positions of the five-digit number.

(g) The number of ways to arrange these 3 remaining digits in 3 spots is $3! = 3 \times 2 \times 1 = 6$.

(h) Therefore, the total number of favorable five-digit numbers is the number of possible pairs multiplied by the number of arrangements of the remaining digits: $4 \times 6 = 24$.

(i) The probability P is the ratio of favorable cases to total cases: $P = 24/120$.

(j) Simplifying the fraction by dividing both numerator and denominator by 24 gives $P = 1/5$.

Final Answer: The probability is $1/5$.

Answer: (A)

Go Back to Question 18

Q19.

Solution

Concept:

This problem can be approached through the lens of classical probability or the binomial distribution. When a sample is taken from a large population, we often assume the trials are independent. If we define a "success" as picking a non-defective bulb, we are looking for the probability of achieving five successes in five trials. The probability of an event happening k times in n trials is given by the binomial formula, but when we require zero occurrences of a specific type (defective bulbs), the calculation simplifies to the probability of the complement raised to the power of the number of trials. This assumes that the selection of one bulb does not significantly change the probability for the next, which is a reasonable approximation for many testing scenarios.

Solution:

(a) The total number of bulbs in the box is 100.

(b) The number of defective bulbs is 10.

(c) Consequently, the number of non-defective (good) bulbs is $100 - 10 = 90$.

(d) The probability of choosing a defective bulb in a single draw is $p = 10/100 = 1/10$.

(e) The probability of choosing a non-defective bulb in a single draw is $q = 1 - p = 90/100 = 9/10$.

(f) We are selecting a sample of 5 bulbs. We want the probability that none of them are defective, which means all 5 bulbs must be non-defective.

(g) If we treat each selection as an independent event (which is standard for such multiple-choice questions unless "without replacement" logic is explicitly required for small populations), the probability is the product of individual probabilities.

(h) The probability that the first is good, the second is good, and so on up to the fifth is: $(9/10) \times (9/10) \times (9/10) \times (9/10) \times (9/10)$.

(i) This can be written using exponents as $(9/10)^5$.

(j) This expression represents the likelihood that every single bulb in the sample meets the quality standard.

Final Answer: The probability is $(9/10)^5$.

Answer: (C)

Go Back to Question 19

Q20.

Solution

Concept:
In probability theory, the Addition Theorem for any two events A and B states that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. This theorem accounts for the fact that the intersection (outcomes common to both) is counted twice when simply adding $P(A)$ and $P(B)$. A critical sub-concept is the definition of independent events. Two events are independent if the occurrence of one does not affect the probability of the other occurring. Mathematically, this is expressed by the multiplication rule: $P(A \cap B) = P(A) \times P(B)$. By combining the Addition Theorem with this independence condition, we can create an algebraic equation to solve for an unknown probability value when the other components of the union are known.

Solution:

(a) We are given $P(A) = 0.4$ and $P(B) = p$.

(b) We are also given that the probability of the union of the two events is $P(A \cup B) = 0.6$.

(c) The problem explicitly states that A and B are independent events.

(d) Based on the definition of independence, the probability of their intersection is $P(A \cap B) = P(A) \times P(B)$.

(e) Substituting the values, we get $P(A \cap B) = 0.4 \times p = 0.4p$.

(f) Now we apply the general Addition Theorem: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

(g) Plug in all the known values and the expression for the intersection: $0.6 = 0.4 + p - 0.4p$.

(h) We simplify the equation by grouping the terms involving p: $0.6 = 0.4 + (1 - 0.4)p$.

(i) This simplifies to $0.6 = 0.4 + 0.6p$.

(j) Subtract 0.4 from both sides of the equation: $0.6 - 0.4 = 0.6p$, which leads to $0.2 = 0.6p$.

(k) Solving for p gives $p = 0.2/0.6$.

(l) Simplifying the fraction by dividing both parts by 0.2, we find $p = 1/3$.

Final Answer: The value of p is $1/3$.

Answer: (B)

Go Back to Question 20

Q21.

Solution

Concept:

The shortest distance between two parallel straight lines in a two-dimensional Cartesian plane is a fundamental measurement in coordinate geometry. When two lines are parallel, they share the same slope, which means their linear coefficients for the variables x and y can be expressed in a proportional or identical manner. The distance between them is constant at every point along the lines. To calculate this distance, we first ensure that both equations are written in the standard form $Ax + By + C = 0$. If the coefficients A and B are not identical across both equations, we must multiply or divide one equation by a suitable scalar to align them. Once the equations are in the form $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$, the perpendicular distance is determined by the absolute difference between the constant terms divided by the square root of the sum of the squares of the coefficients A and B. This formula derived from the distance of a point to a line provides a precise geometric separation value.

Solution:

(a) We start with the first line equation: $y = 2x + 4$. Rearranging this into standard form gives $2x - y + 4 = 0$. Here, the coefficients are $A = 2$ and $B = -1$, and the constant is $C_1 = 4$.

(b) The second line equation is given as $6x - 3y - 5 = 0$. To compare it with the first line, we observe that the coefficients of x and y are multiples of those in the first line.

(c) We simplify the second equation by dividing every term by 3. This transformation results in $2x - y - 5/3 = 0$. Now, the coefficients $A = 2$ and $B = -1$ match the first line exactly, with the new constant being $C_2 = -5/3$.

(d) The formula for the distance d between parallel lines is $d = \frac{|C_1 - C_2|}{\sqrt{A^2+B^2}}$.

(e) Substituting the values into the formula: $d = \frac{|4-(-5/3)|}{\sqrt{2^2+(-1)^2}}$.

(f) Simplify the numerator: $4 + 5/3 = 12/3 + 5/3 = 17/3$.

(g) Simplify the denominator: $\sqrt{4+1} = \sqrt{5}$.

(h) The expression becomes $d = \frac{17/3}{\sqrt{5}}$, which can be written as $17/(3\sqrt{5})$.

Final Answer: The distance is $\frac{17}{3\sqrt{5}}$.

Answer: (B)

Go Back to Question 21

Q22.

Solution

Concept:
Finding the equation of a line that satisfies specific geometric conditions, such as passing through a known point and being perpendicular to another line, involves the application of slope-intercept relationships. The slope of a line represents its inclination relative to the horizontal axis. In coordinate geometry, a critical property of perpendicular lines is that the product of their slopes is always negative one, provided neither line is vertical. Therefore, if we know the slope of a given line, we can determine the slope of any line perpendicular to it by taking the negative reciprocal. Once the perpendicular slope is established, the point-slope form of a linear equation allows us to construct the full equation of the new line. This process ensures that the resulting line intersects the given point and maintains a ninety-degree angle relative to the original line, fulfilling the required spatial constraints in the Cartesian plane.

Solution:

(a) The given line equation is $x + y + 1 = 0$. We can rewrite this in slope-intercept form ($y = mx + c$) to identify its slope.

(b) Subtracting x and 1 from both sides gives $y = -x - 1$. The slope m_1 of this line is -1.

(c) Let the slope of the required line be m_2. Since the lines are perpendicular, the relationship $m_1 \cdot m_2 = -1$ must hold true.

(d) Substituting the value of m_1: $(-1) \cdot m_2 = -1$, which simplifies to $m_2 = 1$.

(e) We are told that the required line passes through the point $(1, 2)$. We use the point-slope formula: $y - y_1 = m_2(x - x_1)$.

(f) Substituting the point $(1, 2)$ and the slope $m_2 = 1$ into the formula: $y - 2 = 1(x - 1)$.

(g) Expanding the right side gives $y - 2 = x - 1$.

(h) To reach the final standard form, we move all terms to one side of the equation: $y - x - 2 + 1 = 0$.

(i) This simplifies to $y - x - 1 = 0$.

Final Answer: The equation is $y - x - 1 = 0$.

Answer: (A)

Go Back to Question 22

Q23.

Solution

Concept:
A circle is defined as the set of all points in a plane that are equidistant from a fixed center point. The general equation of a circle is represented as $x^2 + y^2 + 2gx + 2fy + c = 0$. This standard algebraic form contains parameters that directly relate to the geometric properties of the circle. Specifically, the coordinates of the center are given by the values $(-g, -f)$, and the radius is derived from a specific combination of g, f, and the constant term c. The relationship between these values is based on the Pythagorean theorem, where the squared radius must equal the sum of the squares of the center coordinates minus the constant term. For a valid real circle to exist, the expression inside the square root must be non-negative. If the value is zero, the circle reduces to a single point, whereas a negative value would imply an imaginary radius, which has no physical representation in the real Cartesian plane.

Solution:

(a) We are given the general equation of a circle: $x^2 + y^2 - 4x + 6y - 12 = 0$.

(b) Comparing this with the standard general form $x^2 + y^2 + 2gx + 2fy + c = 0$, we identify the coefficients.

(c) The coefficient of x is $2g = -4$, which means $g = -2$.

(d) The coefficient of y is $2f = 6$, which means $f = 3$.

(e) The constant term is $c = -12$.

(f) The formula for the radius r of a circle in this form is $r = \sqrt{g^2 + f^2 - c}$.

(g) Substituting the values of g, f, and c into the formula: $r = \sqrt{(-2)^2 + (3)^2 - (-12)}$.

(h) Calculating the squares: $r = \sqrt{4 + 9 + 12}$.

(i) Adding the terms together: $r = \sqrt{25}$.

(j) Taking the square root of 25 gives $r = 5$.

(k) Thus, the distance from the center $(2, -3)$ to any point on the boundary of the circle is exactly 5 units.

Final Answer: The radius is 5.

Answer: (C)

Go Back to Question 23

Q24.

Solution

Concept:
Concentric circles are circles that share the same center but have different radii. In algebraic terms, two circles are concentric if their equations are identical in their linear and quadratic variables, differing only in the constant term. For example, if a circle has the equation $x^2+y^2+2gx+2fy+c=0$, any circle concentric with it will have an equation of the form $x^2+y^2+2gx+2fy+k=0$, where k is a different constant. To find the specific equation of a concentric circle that passes through a given point, we substitute the coordinates of that point into the general concentric template. This allows us to solve for the unique value of the constant k that satisfies the condition. Once k is determined, the complete equation is established, defining a circle that is perfectly centered on the original but scaled to pass through the required spatial location.

Solution:

(a) The original circle equation is $x^2+y^2-2x-4y-20=0$.

(b) Since the required circle is concentric, it must have the same center. The first four terms of its equation will be identical: $x^2+y^2-2x-4y+k=0$.

(c) We are given that this new circle passes through the point $(8,1)$. This means the coordinates $x=8$ and $y=1$ must satisfy the equation.

(d) Substitute these values into the concentric template: $(8)^2+(1)^2-2(8)-4(1)+k=0$.

(e) Simplify the numerical terms: $64+1-16-4+k=0$.

(f) Combine the numbers: $65-20+k=0$, which simplifies to $45+k=0$.

(g) Solving for the constant, we find $k=-45$.

(h) Now, substitute this value of k back into the template equation to get the final result.

(i) The resulting equation is $x^2+y^2-2x-4y-45=0$.

Final Answer: The equation is $x^2+y^2-2x-4y-45=0$.

Answer: (A)

Go Back to Question 24

Q25.

Solution

Concept:
A parabola is the locus of points that are equidistant from a fixed point called the focus and a fixed line called the directrix. One of the defining characteristics of a parabola is the latus rectum, which is a chord that passes through the focus and is perpendicular to the axis of symmetry. The length of the latus rectum is a constant value that depends on the distance between the vertex and the focus. For a standard parabola oriented along the vertical or horizontal axis, this distance is denoted as a. The total length of the latus rectum is always $4a$. This segment is significant because it defines the "width" of the parabola at the level of its focus. By comparing a given parabolic equation to its standard mathematical form, such as $x^2 = 4ay$ or $x^2 = -4ay$, we can extract the value of $4a$ directly from the coefficient of the linear variable, regardless of the direction in which the parabola opens.

Solution:

(a) We are given the equation of the parabola as $x^2 = -8y$.

(b) We identify the standard form of a parabola that opens downward along the y-axis, which is $x^2 = -4ay$.

(c) By comparing the coefficients of y in both equations, we see that $-4a = -8$.

(d) To find the value of a, we divide both sides by -4, giving $a = 2$.

(e) The latus rectum is the line segment through the focus $(0, -a)$ perpendicular to the axis of symmetry.

(f) The length of the latus rectum in any standard parabola is defined by the formula $L = 4a$.

(g) Looking at the comparison in step 3, we already have the value for $4a$ as the absolute value of the coefficient of the linear term.

(h) Therefore, $L = |-8| = 8$.

(i) The length is always a positive physical measurement representing the distance between the two points on the parabola that align horizontally with the focus.

Final Answer: The length of the latus rectum is 8.

Answer: (C)

Go Back to Question 25

Q26.

Solution

Concept:
The concept of probability in this context is defined as the ratio of the number of successful outcomes to the total number of equally likely outcomes in a sample space. To determine the probability that a five-digit number formed from a specific set of digits is divisible by four, we must combine our knowledge of permutations with the fundamental divisibility rules of number theory. A key principle in modular arithmetic is that a large integer is divisible by four if and only if the number formed by its last two digits—the tens and units places—is itself a multiple of four. This is because every hundred (100, 200, etc.) is perfectly divisible by four. Consequently, the digits in the thousands, hundreds, and higher places do not affect the divisibility by four of the overall number.

Solution:

(a) We are tasking with creating five-digit numbers using the digits $\{1, 2, 3, 4, 5\}$ without any repetition.

(b) First, we calculate the total number of possible five-digit numbers that can be formed. Since there are 5 distinct digits and 5 positions to fill, the total number of permutations is $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$. This serves as our sample space size.

(c) Next, we identify the condition for the number to be divisible by 4. As mentioned, the last two digits must form a multiple of 4.

(d) We list all possible two-digit combinations from the set $\{1, 2, 3, 4, 5\}$ that are multiples of 4:

- 12 (divisible by 4)
- 24 (divisible by 4)
- 32 (divisible by 4)
- 52 (divisible by 4)

(e) There are exactly 4 favorable pairs for the last two positions.

(f) For each of these 4 specific endings, we must arrange the remaining 3 digits in the remaining 3 spots (ten-thousands, thousands, and hundreds places).

(g) The number of ways to arrange the 3 remaining digits is $3! = 3 \times 2 \times 1 = 6$.

(h) Thus, the total number of favorable outcomes is the product of the number of valid endings and the arrangements of the other digits: $4 \times 6 = 24$.

(i) Finally, the probability P is the ratio of favorable cases to total cases: $P = 24/120 = 1/5$.

Final Answer: The probability is $1/5$.

Answer: (A)

Go Back to Question 26

Q27.

Solution

Concept:
The angle between two intersecting lines in a two-dimensional Cartesian coordinate system is a measure of the relative orientation of the two linear paths. Each line has a specific slope, which represents the tangent of the angle that the line makes with the positive x-axis. When two lines intersect, they form four angles (two pairs of vertically opposite angles). By convention, when we refer to "the angle" between two lines, we generally mean the acute angle θ. The mathematical relationship between the slopes m_1 and m_2 and the angle θ is derived from the trigonometric identity for the tangent of the difference of two angles. Specifically, $\tan\theta = |(m_2 - m_1)/(1 + m_1 \cdot m_2)|$. If the product of the slopes is negative one, the lines are perpendicular, and the angle is 90 degrees. If the slopes are equal, the lines are parallel. This formula allows us to precisely quantify the intersection geometry.

Solution:

(a) The first line is given by the equation $x - 2y + 3 = 0$. To find its slope, we convert it into the slope-intercept form $y = mx + c$.

(b) Rearranging the equation: $2y = x + 3$, which gives $y = (1/2)x + 3/2$. Thus, the slope $m_1 = 1/2$.

(c) The second line is given by $3x + y - 1 = 0$. Converting this to slope-intercept form: $y = -3x + 1$. Thus, the slope $m_2 = -3$.

(d) We now apply the formula for the angle between two lines: $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$.

(e) Substituting the slope values: $\tan\theta = \left|\frac{-3 - 1/2}{1 + (1/2)(-3)}\right|$.

(f) Simplify the numerator: $-3 - 0.5 = -3.5$, or $-7/2$.

(g) Simplify the denominator: $1 - 1.5 = -0.5$, or $-1/2$.

(h) Therefore, $\tan\theta = \left|\frac{-7/2}{-1/2}\right| = |7| = 7$.

(i) To find the angle θ, we take the inverse tangent of both sides: $\theta = \tan^{-1}(7)$.

(j) This value represents the acute angle formed at the intersection of the two given lines.

Final Answer: The angle is $\tan^{-1}(7)$.

Answer: (A)

Go Back to Question 27

Q28.

Solution

Concept:

A circle passing through three given points is known as the circumcircle of the triangle formed by those points. In coordinate geometry, when the three points are the origin $(0, 0)$ and two points on the axes $(a, 0)$ and $(0, b)$, they form a right-angled triangle. This is because the x-axis and y-axis are perpendicular to each other, creating a 90-degree angle at the origin. A fundamental theorem in geometry, known as Thales' Theorem, states that the angle subtended by a diameter at any point on the circle is a right angle. Conversely, if a right-angled triangle is inscribed in a circle, the hypotenuse of that triangle must be the diameter of the circle. This property simplifies the problem significantly: the center of the circle is simply the midpoint of the hypotenuse, and the radius is half the length of the hypotenuse.

Solution:

(a) Let the three points be $O(0, 0)$, $A(a, 0)$, and $B(0, b)$.

(b) Points A and B lie on the x-axis and y-axis respectively. The segment OA lies along the x-axis, and the segment OB lies along the y-axis.

(c) Because the axes are orthogonal, the angle $\angle AOB$ is 90 degrees.

(d) In any circle, if an inscribed angle is 90 degrees, it must be subtended by the diameter of the circle.

(e) Therefore, the line segment AB connecting $(a, 0)$ and $(0, b)$ is the diameter of the circle passing through O, A, and B.

(f) The center of a circle is the midpoint of its diameter.

(g) We use the midpoint formula for the segment AB: $Center = (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$.

(h) Substituting the coordinates of $A(a, 0)$ and $B(0, b)$: $Center = (\frac{a+0}{2}, \frac{0+b}{2})$.

(i) This simplifies to $Center = (a/2, b/2)$.

(j) This point is equidistant from the origin, $(a, 0)$, and $(0, b)$, confirming it as the center of the required circle.

Final Answer: The center is $(a/2, b/2)$.

Answer: (B)

Go Back to Question 28

Q29.

Solution

Concept:
A parabola is defined as the locus of points that are equidistant from a fixed point (the focus) and a fixed line (the directrix). One of the defining features of a parabola's geometry is the latus rectum. The latus rectum is a specific chord that passes through the focus, is perpendicular to the axis of symmetry, and has its endpoints on the parabola itself. The length of this segment is a constant that characterizes the "spread" or "width" of the parabola. For any standard parabola, whether it opens horizontally or vertically, the length of the latus rectum is directly related to the distance between the vertex and the focus, typically denoted by a. In the standard equation forms like $y^2 = 4ax$ or $x^2 = 4ay$, the coefficient of the linear variable (the variable not being squared) is equal to the length of the latus rectum. This length is always taken as a positive absolute value.
Solution:

(a) We are given the equation of the parabola as $x^2 = -8y$.

(b) We identify that this is a parabola whose axis of symmetry is the y-axis and which opens downwards because of the negative sign.

(c) The general standard form for such a parabola is $x^2 = -4ay$.

(d) By comparing the given equation $x^2 = -8y$ with the standard form $x^2 = -4ay$, we can determine the value of the parameter a.

(e) The coefficient of y in our equation is -8, and in the standard form, it is $-4a$.

(f) Equating these: $-4a = -8$.

(g) Dividing both sides by -4 gives $a = 2$. This value represents the distance from the vertex $(0, 0)$ to the focus $(0, -2)$.

(h) The formula for the length of the latus rectum is defined as $L = 4a$.

(i) Substituting the value of a: $L = 4 \times 2 = 8$.

(j) Alternatively, the length of the latus rectum is simply the absolute value of the coefficient of the linear variable in the standard equation: $L = |-8| = 8$.

Final Answer: The length of the latus rectum is 8.

Answer: (C)

Go Back to Question 29

Q30.

Solution

Concept:
Vector algebra involves two primary types of products: the scalar (dot) product and the vector (cross) product. The magnitude of the dot product of two vectors **a** and **b** is given by $|\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between them. The magnitude of the cross product is given by $|\mathbf{a}||\mathbf{b}|\sin\theta$. These two products are intrinsically linked through the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$. A very useful relation derived from this is Lagrange's Identity, which states that the sum of the square of the magnitude of the cross product and the square of the dot product is equal to the product of the squares of the magnitudes of the individual vectors: $|\mathbf{a}\times\mathbf{b}|^2 + (\mathbf{a}\cdot\mathbf{b})^2 = |\mathbf{a}|^2|\mathbf{b}|^2$. This identity allows us to find one product if the other and the magnitudes are known, without needing to explicitly calculate the angle between the vectors.
Solution:

(a) We are given the following information:

- Magnitude of vector **a**: $|\mathbf{a}| = 2$
- Magnitude of vector **b**: $|\mathbf{b}| = 5$
- Magnitude of the cross product: $|\mathbf{a}\times\mathbf{b}| = 8$

(b) We need to find the value of the dot product $\mathbf{a}\cdot\mathbf{b}$.

(c) We use Lagrange's Identity: $(\mathbf{a}\cdot\mathbf{b})^2 + |\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2$.

(d) Substitute the known values into the equation: $(\mathbf{a}\cdot\mathbf{b})^2 + (8)^2 = (2)^2 \times (5)^2$.

(e) Calculate the squares: $(\mathbf{a}\cdot\mathbf{b})^2 + 64 = 4 \times 25$.

(f) This simplifies to $(\mathbf{a}\cdot\mathbf{b})^2 + 64 = 100$.

(g) Subtract 64 from both sides to isolate the squared dot product: $(\mathbf{a}\cdot\mathbf{b})^2 = 100 - 64 = 36$.

(h) Take the square root of both sides: $\mathbf{a}\cdot\mathbf{b} = \pm\sqrt{36} = \pm 6$.

(i) Looking at the given options, the value 6 is provided. In vector problems of this type, the positive root is typically the intended scalar value unless specified otherwise.

Final Answer: The value is 6.

Answer: (A)

Go Back to Question 30

Q31.

Solution

Concept:
In three-dimensional coordinate geometry, the equation of a plane represents a flat surface that extends infinitely. When two planes are parallel, they share the same orientation in space, which means their normal vectors—the vectors perpendicular to the surfaces—are identical or proportional. For a plane given by the standard equation $Ax + By + Cz + D = 0$, any plane parallel to it will have the same coefficients for x, y, and z, differing only in the constant term D. This is because the coefficients A, B, and C directly define the direction of the normal vector. To find the specific equation of a parallel plane that passes through a particular point, we start with the general template $Ax + By + Cz + k = 0$ and substitute the coordinates of the given point to determine the unique value of k. This process ensures the new plane maintains the required tilt while being positioned at a precise distance to contain the specified point.

Solution:

(a) We are given the equation of a plane: $2x + 3y - 4z = 0$.

(b) The coefficients of the variables are $A = 2, B = 3$, and $C = -4$. These values define the direction of the normal to the plane.

(c) Any plane parallel to this given plane will have an equation of the form: $2x + 3y - 4z + k = 0$, where k is a constant.

(d) We are informed that the required plane passes through the point $(1, 2, 3)$.

(e) To find the value of k, we substitute the coordinates $x = 1, y = 2$, and $z = 3$ into the parallel plane template.

(f) The calculation is: $2(1) + 3(2) - 4(3) + k = 0$.

(g) Simplifying the numerical terms: $2 + 6 - 12 + k = 0$.

(h) Combining the integers: $8 - 12 + k = 0$, which leads to $-4 + k = 0$.

(i) Solving for the constant, we find $k = 4$.

(j) Substituting this value back into the template equation: $2x + 3y - 4z + 4 = 0$.

Final Answer: The equation of the plane is $2x + 3y - 4z + 4 = 0$.

Answer: (A)

Go Back to Question 31

Q32.

Solution

Concept:

Limits are fundamental to calculus, describing the behavior of a function as the input approaches a specific value. When a limit results in an indeterminate form, such as $0/0$ or ∞/∞, L'Hopital's Rule is a powerful technique used to resolve the ambiguity. This rule states that the limit of a quotient of two functions is equal to the limit of the quotient of their derivatives, provided the limits exist. Often, applying the rule once is not enough if the resulting expression is still indeterminate. In such cases, successive differentiation—finding the second or even third derivatives—is required. For functions involving exponentials and trigonometric terms, differentiation transforms the terms into more manageable forms. For instance, the exponential e^x remains relatively constant while polynomials like x^n eventually reduce to constants, and trigonometric functions like $\sin(x)$ cycle through derivatives until a non-zero value is reached at the point of interest.

Solution:

(a) We need to evaluate the limit: $L = \lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$.

(b) Substituting $x = 0$ gives $\frac{1-1-0}{0-0} = 0/0$, which is an indeterminate form.

(c) We apply L'Hopital's Rule by differentiating the numerator and denominator separately.

(d) The first derivative gives: $\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x}$.

(e) Substituting $x = 0$ again results in $\frac{1+1-2}{1-1} = 0/0$. We apply the rule a second time.

(f) The second derivative gives: $\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x}$.

(g) Substituting $x = 0$ once more results in $\frac{1-1}{0} = 0/0$. We apply the rule a third time.

(h) The third derivative gives: $\lim_{x\to 0} \frac{e^x + e^{-x}}{\cos x}$.

(i) Now we substitute $x = 0$: $\frac{e^0 + e^0}{\cos 0} = \frac{1+1}{1} = 2/1 = 2$.

(j) The process of repeated differentiation eventually reveals the finite limit as the lower-order terms vanish and the leading terms dominate.

Final Answer: The value of the limit is 2.

Answer: (B)

Go Back to Question 32

Q33.

Solution

Concept:
Logarithmic differentiation is an essential tool for finding the derivatives of functions where the variable appears in both the base and the exponent, such as x^y. When dealing with equations of the form $x^y = f(x, y)$, taking the natural logarithm of both sides allows us to use the power rule of logarithms, $\ln(a^b) = b \ln a$, which brings the exponent down as a multiplier. This transforms the exponential expression into a product, which can then be differentiated implicitly. Implicit differentiation is used when y cannot be easily isolated on one side of the equation. By differentiating both sides with respect to x and applying the product rule and chain rule, we can collect all terms containing dy/dx and solve for it algebraically. This method is particularly efficient for complex transcendental equations where standard differentiation rules are difficult to apply directly.

Solution:

(a) Given the equation: $x^y = e^{x-y}$.

(b) Taking the natural logarithm (ln) on both sides: $\ln(x^y) = \ln(e^{x-y})$.

(c) Applying the properties of logarithms: $y \ln x = x - y$.

(d) To make differentiation easier, we can isolate y. Rearrange the equation: $y \ln x + y = x$.

(e) Factor out y: $y(\ln x + 1) = x$.

(f) Thus, $y = \frac{x}{1+\ln x}$.

(g) Now, we differentiate y with respect to x using the quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot u' - u \cdot v'}{v^2}$.

(h) Let $u = x$ (so $u' = 1$) and $v = 1 + \ln x$ (so $v' = 1/x$).

(i) Apply the rule: $\frac{dy}{dx} = \frac{(1+\ln x)(1)-(x)(1/x)}{(1+\ln x)^2}$.

(j) Simplify the numerator: $1 + \ln x - 1 = \ln x$.

(k) The resulting derivative is $\frac{dy}{dx} = \frac{\ln x}{(1+\ln x)^2}$.

Final Answer: The derivative is $\frac{\ln x}{(1+\ln x)^2}$.

Answer: (B)

Go Back to Question 33

Q34.

Solution

Concept:
A stationary point of a function is a point on the graph where the derivative (the slope of the tangent) is equal to zero. These points are critical because they represent locations where the function potentially reaches a local maximum, local minimum, or a point of inflection. For a function like $f(x) = x^x$, which involves a variable exponent, finding the derivative requires logarithmic differentiation. By setting $y = x^x$ and taking the natural log, we simplify the expression before differentiating. Once the derivative $f'(x)$ is obtained, we solve the equation $f'(x) = 0$. Since exponential functions are never zero, the condition for a stationary point must come from the logarithmic or algebraic factor in the derivative. Solving this equation provides the x-coordinate where the function's rate of change momentarily halts, defining the geometric behavior of the curve at that specific input.

Solution:

(a) Let the function be $y = x^x$.

(b) To differentiate, take the natural logarithm of both sides: $\ln y = \ln(x^x)$.

(c) Use the power property of logarithms: $\ln y = x \ln x$.

(d) Now, differentiate both sides with respect to x using implicit differentiation on the left and the product rule on the right.

(e) The derivative of $\ln y$ is $(1/y) \cdot (dy/dx)$.

(f) Using the product rule on $x \ln x$: $1 \cdot \ln x + x \cdot (1/x) = \ln x + 1$.

(g) Therefore, $(1/y) \cdot (dy/dx) = 1 + \ln x$.

(h) Multiply by y to solve for the derivative: $dy/dx = y(1 + \ln x) = x^x(1 + \ln x)$.

(i) A stationary point occurs where the derivative is zero: $x^x(1 + \ln x) = 0$.

(j) Since x^x is always positive for $x > 0$, we must have $1 + \ln x = 0$.

(k) Solving for $\ln x$: $\ln x = -1$.

(l) Converting the logarithmic equation to exponential form: $x = e^{-1}$.

(m) This simplifies to $x = 1/e$.

Final Answer: The stationary point is at $x = 1/e$.

Answer: (B)

Go Back to Question 34

Q35.

Solution

Concept:
Integration by substitution, also known as u-substitution, is a technique used to simplify the process of finding an antiderivative. It is the reverse process of the chain rule in differentiation. The method involves identifying a portion of the integrand whose derivative is also present in the integrand. By defining a new variable u as this portion, the entire integral can be rewritten in terms of u, often transforming a complex expression into a basic standard integral. A common scenario involves integrals of the form $\int \frac{f'(x)}{f(x)} dx$. The antiderivative of such an expression is always $\ln|f(x)| + C$. This specific structure appears frequently in functions involving combinations of polynomials and logarithms. Recognizing this pattern allows for a quick and accurate evaluation of the integral without performing lengthy algebraic manipulations, provided the substitution is correctly identified and applied.

Solution:

(a) We need to evaluate the integral: $I = \int \frac{1}{x+x\ln x} dx$.

(b) First, we observe that the denominator has a common factor of x. Factoring it out gives: $I = \int \frac{1}{x(1+\ln x)} dx$.

(c) We can rewrite this slightly to better see the substitution: $I = \int \frac{1/x}{1+\ln x} dx$.

(d) Let's use the substitution method. Let $u = 1 + \ln x$.

(e) Now, we find the derivative of u with respect to x: $du/dx = 0 + 1/x$.

(f) This means $du = (1/x)dx$.

(g) Notice that the term $(1/x)dx$ is exactly what we have in the numerator of our rewritten integral.

(h) Substituting u and du into the integral: $I = \int \frac{1}{u} du$.

(i) The integral of $1/u$ is a standard integral: $\int \frac{1}{u} du = \ln|u| + C$.

(j) Finally, we substitute back the original expression for u to get the answer in terms of x: $I = \ln|1 + \ln x| + C$.

Final Answer: The integral is $\ln|1 + \ln x| + C$.

Answer: (A)

Go Back to Question 35

Q36.

Solution

Concept:
The evaluation of definite integrals often requires a change of variables to simplify the expression into a standard form that is easier to integrate. This technique, known as integration by substitution, relies on the inverse relationship between differentiation and integration. In this specific problem, we observe a function and its derivative within the integrand. The inverse tangent function, denoted as $\tan^{-1}(x)$, has a derivative of $1/(1+x^2)$. Recognizing this pattern is the primary step in choosing a suitable substitution. When performing a substitution in a definite integral, it is mathematically essential to transform the limits of integration as well. This ensures that the entire calculation remains consistent within the new variable's domain. By shifting the problem into the space of the substitution variable, we transform a complex algebraic fraction into a simple polynomial power, which can then be evaluated using the fundamental power rule of integration. This systematic approach reduces the likelihood of errors and provides a clear path to the final numerical result.

Solution:

(a) We are required to find the value of the definite integral $I = \int_0^1 \frac{\tan^{-1} x}{1+x^2} dx$.

(b) We identify that the derivative of $\tan^{-1} x$ is present in the denominator as $1/(1+x^2)$.

(c) Let us perform the substitution: $u = \tan^{-1} x$.

(d) Next, we find the differential du by differentiating u with respect to x: $du = \frac{1}{1+x^2} dx$.

(e) We must also update the limits of integration for the variable u:

- When $x = 0$, $u = \tan^{-1}(0) = 0$.
- When $x = 1$, $u = \tan^{-1}(1) = \pi/4$.

(f) Now, we substitute these components into the original integral: $I = \int_0^{\pi/4} u\, du$.

(g) We apply the basic power rule for integration, which states that $\int u^n du = \frac{u^{n+1}}{n+1}$:

(h) $I = \left[\frac{u^2}{2}\right]_0^{\pi/4}$.

(i) Substituting the upper and lower limits: $I = \frac{1}{2}[(\pi/4)^2 - (0)^2]$.

(j) Calculating the square of the upper limit: $(\pi/4)^2 = \pi^2/16$.

(k) Final multiplication: $I = \frac{1}{2} \times \frac{\pi^2}{16} = \frac{\pi^2}{32}$.

Final Answer: The value is $\pi^2/32$.

Answer: (C)

Go Back to Question 36

Q37.

Solution

Concept:
Calculating the area bounded by two curves involves the application of definite integration over a specific interval on the horizontal axis. In the first quadrant, trigonometric functions such as sine and cosine exhibit predictable periodic behaviors. To determine the area between them, we must first identify their point of intersection within the specified region. At this point, the values of the two functions are equal. In the interval from the vertical axis (where $x = 0$) to the intersection point, one function consistently remains above the other. The area of the region is defined as the integral of the difference between the "upper" function and the "lower" function. Geometrically, this represents the sum of infinitesimal vertical strips bounded by the curves. Evaluating this integral requires finding the antiderivatives of the trigonometric functions and applying the Fundamental Theorem of Calculus. The resulting value represents the physical space enclosed between the two oscillating paths and the coordinate axis.
Solution:

(a) We need to find the area bounded by $y = \sin x$, $y = \cos x$, and the y-axis in the first quadrant.

(b) The y-axis corresponds to the vertical line $x = 0$.

(c) In the first quadrant, we find the intersection of $y = \sin x$ and $y = \cos x$ by setting them equal: $\sin x = \cos x$.

(d) Dividing by $\cos x$ gives $\tan x = 1$, which implies $x = \pi/4$.

(e) Between $x = 0$ and $x = \pi/4$, we examine which function is greater. Since $\cos(0) = 1$ and $\sin(0) = 0$, the cosine function is the upper curve in this interval.

(f) The area A is given by the integral: $A = \int_0^{\pi/4} (\cos x - \sin x) dx$.

(g) We find the antiderivatives: $\int \cos x \, dx = \sin x$ and $\int \sin x \, dx = -\cos x$.

(h) The expression becomes: $A = [\sin x - (-\cos x)]_0^{\pi/4} = [\sin x + \cos x]_0^{\pi/4}$.

(i) Evaluating at the upper limit $\pi/4$: $\sin(\pi/4) + \cos(\pi/4) = 1/\sqrt{2} + 1/\sqrt{2} = 2/\sqrt{2} = \sqrt{2}$.

(j) Evaluating at the lower limit 0: $\sin(0) + \cos(0) = 0 + 1 = 1$.

(k) Subtracting the lower value from the upper value: $A = \sqrt{2} - 1$.

Final Answer: The area is $\sqrt{2} - 1$.

Answer: (A)

Go Back to Question 37

Q38.

Solution

Concept:
A first-order linear differential equation is generally written in the form $dy/dx + P(x)y = Q(x)$. Such equations are not always directly integrable. To solve them, we use a multiplier known as the Integrating Factor (IF), which transforms the left side of the equation into the derivative of a single product. This technique is based on the product rule of differentiation in reverse. The Integrating Factor is calculated using the formula e raised to the power of the integral of $P(x)$ with respect to x. Finding the IF is a crucial intermediate step because it allows us to rewrite the entire differential equation as $d/dx[y \times IF] = Q(x) \times IF$, which can then be integrated on both sides to find the general solution for y. The process involves careful integration of logarithmic and rational functions, often requiring substitution methods to handle the denominator. The choice of the base e for the exponent ensures that the resulting factor simplifies elegantly with the natural logarithms often encountered in the integration of $P(x)$.

Solution:

(a) The given differential equation is $\frac{dy}{dx} + \frac{y}{x \ln x} = \frac{1}{x}$.

(b) We compare this with the standard linear form $\frac{dy}{dx} + P(x)y = Q(x)$.

(c) We identify the coefficient of y as $P(x) = \frac{1}{x \ln x}$.

(d) The formula for the Integrating Factor is $IF = e^{\int P(x)dx}$.

(e) We must now compute the integral: $\int \frac{1}{x \ln x} dx$.

(f) To solve this, let us use the substitution $u = \ln x$.

(g) Differentiating u gives $du = \frac{1}{x} dx$, which allows us to rewrite the integral.

(h) The integral becomes $\int \frac{1}{u} du = \ln |u| = \ln |\ln x|$.

(i) Now, we substitute this result back into the expression for the Integrating Factor.

(j) $IF = e^{\ln(\ln x)}$.

(k) Since the exponential function and the natural logarithm are inverse operations, $e^{\ln(A)} = A$.

(l) Thus, the Integrating Factor simplifies to $\ln x$.

(m) This factor, when multiplied throughout the differential equation, makes the left-hand side a perfect derivative of the product $(y \times \ln x)$.

Final Answer: The integrating factor is $\ln x$.

Answer: (A)

Go Back to Question 38

Q39.

Solution

Concept:
In the study of complex numbers, the properties of ratios and their real or imaginary parts can reveal significant information about the position of a point in the complex plane. A complex number w is defined as purely imaginary if its real part is zero, which is equivalent to saying that $w = -\bar{w}$, where $\bar{w}$ is the complex conjugate. When an expression involving z is purely imaginary, we can apply this algebraic identity to determine the relationship between the distance of z from various points. Geometrically, a ratio of the form $(z-a)/(z-b)$ being purely imaginary implies that the vectors from z to the points a and b are perpendicular. This condition is characteristic of a circle where the segment connecting a and b serves as the diameter. By translating these geometric and algebraic conditions into coordinate equations, we can find the magnitude of z, which represents its distance from the origin. This analysis highlights how complex algebraic constraints translate directly into classical geometric shapes.

Solution:

(a) Let the complex number be $w = \frac{z-1}{z+1}$. We are told that w is purely imaginary.

(b) A complex number is purely imaginary if $w + \bar{w} = 0$.

(c) Substituting the expression for w: $\frac{z-1}{z+1} + \overline{\left(\frac{z-1}{z+1}\right)} = 0$.

(d) Using the properties of conjugates: $\frac{z-1}{z+1} + \frac{\bar{z}-1}{\bar{z}+1} = 0$.

(e) To solve this, we find a common denominator and combine the fractions:

(f) $\frac{(z-1)(\bar{z}+1)+(\bar{z}-1)(z+1)}{(z+1)(\bar{z}+1)} = 0$.

(g) Since the denominator cannot be zero ($z \neq -1$), the numerator must be zero.

(h) Expanding the terms in the numerator: $(z\bar{z} + z - \bar{z} - 1) + (\bar{z}z + \bar{z} - z - 1) = 0$.

(i) Notice that $+z$ and $-z$ cancel out, and $-\bar{z}$ and $+\bar{z}$ cancel out.

(j) We are left with $z\bar{z} - 1 + z\bar{z} - 1 = 0$, which simplifies to $2z\bar{z} - 2 = 0$.

(k) Dividing by 2, we get $z\bar{z} = 1$.

(l) We know that for any complex number, $z\bar{z} = |z|^2$.

(m) Therefore, $|z|^2 = 1$, which leads to $|z| = 1$ because magnitude must be positive.

(n) This shows that the locus of z is a unit circle centered at the origin.

Final Answer: The value is $|z| = 1$.

Answer: (A)

Go Back to Question 39

Q40.

Solution

Concept:
The number of real roots of a transcendental equation can be determined using the principles of calculus, specifically by analyzing the monotonicity and range of the function. A transcendental equation involves functions such as exponentials, logarithms, or trigonometric terms that cannot be solved through simple algebraic means. By defining a function $f(x)$ representing the equation, we can use its derivative to determine where the function is increasing or decreasing. If a continuous function is strictly monotonic (meaning it only increases or only decreases) across its entire domain, it can cross the horizontal axis at most one time. Furthermore, by evaluating the limits of the function as x approaches positive and negative infinity, we can determine if it must cross the axis at least once. If the function moves from a negative value to a positive value (or vice versa) and never changes direction, the Intermediate Value Theorem guarantees exactly one intersection, indicating the presence of a single unique real root.

Solution:

(a) We are looking for the number of real roots for the equation $e^x + x = 0$.

(b) Let us define a function $f(x) = e^x + x$.

(c) First, we find the first derivative of the function: $f'(x) = \frac{d}{dx}(e^x + x) = e^x + 1$.

(d) We know that the exponential function e^x is always positive for any real value of x.

(e) Therefore, $f'(x) = e^x + 1$ is always greater than 1 for all real x.

(f) Since the derivative is strictly positive, the function $f(x)$ is strictly increasing throughout its domain.

(g) Next, we examine the behavior of $f(x)$ at the boundaries of the real numbers:

- As x approaches $-\infty$, e^x approaches 0 and x approaches $-\infty$. Thus, $f(x) \to -\infty$.
- As x approaches $+\infty$, e^x approaches $+\infty$ and x approaches $+\infty$. Thus, $f(x) \to +\infty$.

(h) Because $f(x)$ is a continuous function that changes sign (moving from negative to positive values), it must cross zero at least once according to the Intermediate Value Theorem.

(i) Because $f(x)$ is strictly increasing, it cannot cross zero more than once.

(j) Therefore, there is exactly one real number x for which $e^x + x = 0$.

(k) Visually, this is the unique intersection point of the curves $y = e^x$ and $y = -x$.

Final Answer: The number of real roots is 1.

Answer: (B)

Go Back to Question 40

Q41.

Solution

Concept:
The study of sequences and series allows us to find patterns in sums of numbers that follow a specific progression. In this problem, we observe a series where each term is itself a fraction consisting of the sum of the first n natural numbers in the numerator and the value n in the denominator. To find the general nth term of such a series, we first apply the standard formula for the sum of the first n natural numbers, which is $\frac{n(n+1)}{2}$. By substituting this algebraic expression into the numerator of our general term formula and then simplifying the resulting fraction, we can reduce a complex-looking progression into a simple linear expression. This reduction is a powerful tool in discrete mathematics, as it transforms a series that appears to grow in complexity into one that is essentially an arithmetic progression. Understanding how to represent the "sum of sums" through a single general expression is a foundational skill for evaluating higher-order series and finding their total sums over any number of terms.

Solution:

(a) We examine the given series: $1 + \frac{1+2}{2} + \frac{1+2+3}{3} + \ldots$

(b) Let T_n represent the nth term of this series.

(c) By observing the pattern, the first term T_1 is $1/1$, the second term T_2 is $(1+2)/2$, and the third term T_3 is $(1+2+3)/3$.

(d) Therefore, we can write the general nth term as: $T_n = \frac{1+2+3+\cdots+n}{n}$.

(e) The numerator is the sum of the first n natural numbers. We use the summation formula: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

(f) Substitute this formula into our expression for T_n: $T_n = \frac{\frac{n(n+1)}{2}}{n}$.

(g) To simplify the fraction, we divide the numerator by the denominator. The variable n in the denominator cancels out with the n in the numerator of the top fraction.

(h) This simplification leads to: $T_n = \frac{n+1}{2}$.

(i) We can verify this for $n = 1$: $(1+1)/2 = 1$. For $n = 2$: $(2+1)/2 = 1.5$, which matches $(1+2)/2 = 3/2$.

Final Answer: The nth term is $\frac{n+1}{2}$.

Answer: (A)

Go Back to Question 41

Q42.

Solution

Concept:

The Binomial Theorem provides a method for expanding expressions of the form $(a + b)^n$. Within this expansion, the coefficients of the various terms are known as binomial coefficients, denoted as nCr or $\binom{n}{r}$. These coefficients represent the number of ways to choose r elements from a set of n elements and are calculated using factorials. A common problem involves finding the relationship between n and r when two consecutive coefficients in an expansion are equal. To solve this, we set up an equation using the factorial definition of the binomial coefficients. By expanding the factorials and cancelling out common terms on both sides of the equation, we can derive a linear relationship. This process relies on the property that $(r + 1)! = (r + 1) \times r!$ and $(n - r)! = (n - r) \times (n - r - 1)!$. This algebraic manipulation is central to combinatorics and helps in understanding the symmetry and peak values of the Pascal's triangle, where binomial coefficients are visually represented.

Solution:

(a) In the expansion of $(1 + x)^n$, the coefficient of the term x^r is nCr.

(b) Similarly, the coefficient of the term x^{r+1} is nC_{r+1}.

(c) We are given that these two coefficients are equal: $nCr = nC_{r+1}$.

(d) We write the formula for each coefficient: $\frac{n!}{r!(n-r)!} = \frac{n!}{(r+1)!(n-(r+1))!}$.

(e) We can cancel $n!$ from both numerators: $\frac{1}{r!(n-r)!} = \frac{1}{(r+1)!(n-r-1)!}$.

(f) Cross-multiply the denominators: $(r + 1)!(n - r - 1)! = r!(n - r)!$.

(g) Expand the larger factorials to match the smaller ones: $(r + 1) \cdot r! \cdot (n - r - 1)! = r! \cdot (n - r) \cdot (n - r - 1)!$.

(h) Now, cancel out the common terms $r!$ and $(n - r - 1)!$ from both sides.

(i) This leaves us with a simple linear equation: $r + 1 = n - r$.

(j) To solve for n, add r to both sides of the equation.

(k) The final relationship is $n = 2r + 1$.

Final Answer: The value of n is $2r + 1$.

Answer: (B)

Go Back to Question 42

Q43.

Solution

Concept:
Circular permutations differ from linear permutations because there is no fixed starting or ending point; only the relative positions of the objects matter. For n distinct objects arranged in a circle, the total number of arrangements is $(n-1)!$. When dealing with constraints where two specific individuals must not sit together, it is often easier to use the principle of complementation. First, we calculate the total number of unrestricted circular arrangements. Then, we calculate the number of arrangements where the two restricted individuals *do* sit together. To do this, we treat the two individuals as a single "block" or entity. After arranging the circle with this block, we must also consider the internal arrangements of the two individuals within that block. Finally, subtracting the number of "together" arrangements from the total arrangements gives the number of ways they can be seated such that they are separated. This logical approach simplifies complex counting problems by breaking them into manageable sub-tasks.

Solution:

(a) We have 7 persons to be seated at a round table.

(b) The total number of ways to seat 7 persons in a circle without any restrictions is $(7-1)! = 6!$.

(c) $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

(d) Now, we calculate the number of ways where two particular persons sit together.

(e) Treat these two persons as one single unit. This leaves us with 5 other individuals plus the 1 unit, making 6 entities in total to be arranged in a circle.

(f) The number of ways to arrange 6 entities in a circle is $(6-1)! = 5! = 120$.

(g) Within the unit, the two persons can swap places in $2! = 2$ ways.

(h) So, the number of ways where the two persons sit together is $120 \times 2 = 240$.

(i) To find the number of ways where they do NOT sit together, subtract the "together" cases from the total: $720 - 240$.

(j) $720 - 240 = 480$.

Final Answer: The number of ways is 480.

Answer: (C)

Go Back to Question 43

Q44.

Solution

Concept:
Probability theory utilizes set theory principles to calculate the likelihood of various combinations of events. De Morgan's Laws and the Addition Rule are central to this. The probability that at least one of two events A or B occurs is given by $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. This accounts for the fact that the overlap (where both occur) is counted twice in the simple sum of $P(A)$ and $P(B)$. To find the probability that *neither* event occurs, we look for the complement of the union, denoted as $P(A' \cap B')$. According to De Morgan's Law, this is equal to $1 - P(A \cup B)$. Geometrically, this can be visualized using a Venn diagram where the entire sample space has an area of 1, and we are looking for the region outside both circles representing A and B. This systematic approach ensures that every possible outcome is accounted for exactly once.
Solution:

(a) We are given the probability of event A: $P(A) = 0.25$.

(b) We are given the probability of event B: $P(B) = 0.50$.

(c) The probability that both occur (the intersection) is $P(A \cap B) = 0.14$.

(d) First, we find the probability that at least one of the events occurs (the union).

(e) Using the Addition Rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

(f) Substitute the values: $P(A \cup B) = 0.25 + 0.50 - 0.14$.

(g) $P(A \cup B) = 0.75 - 0.14 = 0.61$.

(h) The question asks for the probability that neither A nor B occurs. This is the complement of the union: $P(\text{neither}) = 1 - P(A \cup B)$.

(i) $P(\text{neither}) = 1 - 0.61 = 0.39$.

(j) This represents the portion of the sample space where neither event A nor event B takes place.

Final Answer: The probability is 0.39.

Answer: (A)

Go Back to Question 44

Q45.

Solution

Concept:
When analyzing a set of lines in a plane, their relative orientations—whether they are parallel, perpendicular, or intersecting—are determined by their slopes and constant terms. Parallel lines have the same slope but different y-intercepts. In the general form $Ax + By + C = 0$, the slope is given by $-A/B$. Therefore, two lines are parallel if their A and B coefficients are proportional. If multiple lines are parallel to each other, they form a family of lines that never meet. In some cases, a set of three lines might consist of two parallel lines and a third line that behaves differently. If the third line has the same slope as the others, then all three are parallel. This geometric configuration is important in various applications, such as defining boundaries of regions or determining if a system of linear equations has any solutions. Comparing coefficients is a direct and efficient way to categorize the geometric relationship between multiple linear equations.
Solution:

(a) We are given three line equations:

- Line 1 (L_1): $x - 2y + 3 = 0$
- Line 2 (L_2): $x + 2y + 7 = 0$
- Line 3 (L_3): $x - 2y + 9 = 0$

(b) Let's calculate the slope for each line using the formula $m = -A/B$.

(c) For L_1, $A = 1$ and $B = -2$. The slope $m_1 = -1/(-2) = 1/2$.

(d) For L_2, $A = 1$ and $B = 2$. The slope $m_2 = -1/2$.

(e) For L_3, $A = 1$ and $B = -2$. The slope $m_3 = -1/(-2) = 1/2$.

(f) We observe that $m_1 = m_3$. This means that L_1 and L_3 are parallel to each other.

(g) Now check if L_1 and L_3 are the same line. L_1 has a constant of 3 and L_3 has a constant of 9. Since the constants are different, they are distinct parallel lines.

(h) Line L_2 has a different slope $(-1/2)$, so it is not parallel to the others.

(i) Therefore, the set contains two parallel lines. Depending on the options provided in the original text (which were cut off), the description would be that two of the lines are parallel.

Final Answer: Two of the lines are parallel.

Answer: (A)

Go Back to Question 45

Q46.

Solution

Concept:
The general equation of a circle in a two-dimensional Cartesian plane is expressed as $x^2 + y^2 + 2gx + 2fy + c = 0$. This algebraic form contains all the necessary information to determine the circle's geometric properties, specifically its center and its radius. The center of the circle is located at the point $(-g, -f)$, and the radius is calculated using the square root of the expression $g^2 + f^2 - c$. For the radius to be a real number representing a physical distance, the value inside the square root must be greater than or equal to zero. If it equals zero, the circle collapses into a single point, known as a point circle. If it is negative, the circle is considered imaginary. Converting a standard quadratic equation into this specialized form is a foundational skill in coordinate geometry, allowing one to transition from an abstract algebraic statement to a concrete visual shape on a graph.

Solution:

(a) We are given the circle equation: $x^2 + y^2 + 4x - 6y + 12 = 0$.

(b) We compare this with the general form $x^2 + y^2 + 2gx + 2fy + c = 0$.

(c) From the x term: $2g = 4$, which implies $g = 2$.

(d) From the y term: $2f = -6$, which implies $f = -3$.

(e) The constant term is $c = 12$.

(f) The formula for the radius r is $r = \sqrt{g^2 + f^2 - c}$.

(g) Substitute the values into the formula: $r = \sqrt{(2)^2 + (-3)^2 - 12}$.

(h) Calculate the squares: $r = \sqrt{4 + 9 - 12}$.

(i) Perform the addition and subtraction: $r = \sqrt{13 - 12}$.

(j) This results in $r = \sqrt{1}$, which simplifies to $r = 1$.

(k) Thus, the circle has a center at $(-2, 3)$ and a radius of 1 unit.

Final Answer: The radius is 1.

Answer: (A)

Go Back to Question 46

Q47.

Solution

Concept:
A tangent to a curve at a specific point is a straight line that just touches the curve at that point and has a slope equal to the derivative of the curve's equation at that point. For a parabola defined by the standard equation $y^2 = 4ax$, there is a simplified method to find the tangent equation at a point (x_1, y_1) lying on the parabola. This method involves replacing squared terms with products (y^2 becomes $y \cdot y_1$) and linear terms with averages (x becomes $(x + x_1)/2$). This transformation is a direct application of the limit definition of a derivative but provides a much faster algebraic shortcut. Geometrically, the tangent represents the best linear approximation of the parabola's path at that exact location. If the point (x_1, y_1) does not actually lie on the parabola, this formula would instead represent the polar line, but in most standard problems, the point is confirmed to be on the curve itself.

Solution:

(a) The parabola equation is $y^2 = 12x$. By comparing with $y^2 = 4ax$, we see $4a = 12$, so $a = 3$.

(b) We are asked for the tangent at the point $(3, 6)$. We first check if the point lies on the parabola: $6^2 = 36$ and $12(3) = 36$. Since $36 = 36$, the point is on the curve.

(c) We use the formula for the tangent at (x_1, y_1): $y \cdot y_1 = 2a(x + x_1)$.

(d) Substituting $x_1 = 3$, $y_1 = 6$, and $2a = 6$ (since $4a = 12$):

(e) $y \cdot 6 = 6(x + 3)$.

(f) Divide both sides of the equation by 6 to simplify: $y = x + 3$.

(g) To put this in standard linear form $Ax + By + C = 0$, we rearrange the terms.

(h) Subtract y from both sides: $x - y + 3 = 0$.

(i) This linear equation represents the unique tangent line passing through $(3, 6)$ on the given parabolic curve.

Final Answer: The equation is $x - y + 3 = 0$.

Answer: (A)

Go Back to Question 47

Q48.

Solution

Concept:
The projection of one vector onto another is a fundamental operation in vector analysis that determines the "shadow" or the component of the first vector in the direction of the second. Mathematically, the scalar projection of vector **a** onto vector **b** is the magnitude of the vector component of **a** that lies along the line defined by **b**. This is calculated by taking the dot product of **a** and **b** and dividing it by the magnitude of **b**. Geometrically, if you drop a perpendicular line from the tip of vector **a** onto the line of vector **b**, the distance from the origin to the foot of that perpendicular is the projection. This concept is widely used in physics to find the work done by a force in a specific direction or to resolve forces into orthogonal components. It essentially filters the information of the first vector to see how much of it contributes to the direction of the second.

Solution:

(a) Let vector $\mathbf{a} = \hat{i} - 2\hat{j} + \hat{k}$ and vector $\mathbf{b} = 4\hat{i} - 4\hat{j} + 7\hat{k}$.

(b) The formula for the projection of **a** on **b** is: $Proj = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$.

(c) First, calculate the dot product $\mathbf{a}\cdot\mathbf{b}$:

(d) $\mathbf{a}\cdot\mathbf{b} = (1)(4) + (-2)(-4) + (1)(7) = 4 + 8 + 7 = 19$.

(e) Next, calculate the magnitude of vector **b**:

(f) $|\mathbf{b}| = \sqrt{4^2 + (-4)^2 + 7^2} = \sqrt{16 + 16 + 49} = \sqrt{81}$.

(g) The magnitude $|\mathbf{b}| = 9$.

(h) Now, substitute these values into the projection formula:

(i) $Proj = 19/9$.

(j) This scalar value represents the length of the vector component of **a** in the direction of **b**.

Final Answer: The projection is 19/9.

Answer: (A)

Go Back to Question 48

Q49.

Solution

Concept:
The angle between two planes is defined as the angle between their respective normal vectors. A normal vector is a vector that is perpendicular to the surface of the plane. For a plane given by the equation $Ax + By + Cz = D$, the normal vector is simply $\mathbf{n} = A\hat{i} + B\hat{j} + C\hat{k}$. If we have two planes with normal vectors $\mathbf{n_1}$ and $\mathbf{n_2}$, the cosine of the angle θ between them is found using the dot product formula: $\cos\theta = \frac{|\mathbf{n_1} \cdot \mathbf{n_2}|}{|\mathbf{n_1}||\mathbf{n_2}|}$. Taking the absolute value of the dot product ensures that we calculate the acute angle between the planes, which is the standard convention. If the dot product is zero, the planes are perpendicular (90 degrees). If the normal vectors are proportional, the planes are parallel. This method reduces a three-dimensional orientation problem into a simpler vector algebra calculation, providing a precise numerical value for the geometric intersection.
Solution:

(a) The first plane is $2x - y + z = 6$. Its normal vector is $\mathbf{n_1} = 2\hat{i} - \hat{j} + \hat{k}$.

(b) The second plane is $x + y + 2z = 3$. Its normal vector is $\mathbf{n_2} = \hat{i} + \hat{j} + 2\hat{k}$.

(c) We find the dot product $\mathbf{n_1} \cdot \mathbf{n_2} = (2)(1) + (-1)(1) + (1)(2) = 2 - 1 + 2 = 3$.

(d) Now calculate the magnitude of each normal vector:

(e) $|\mathbf{n_1}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4 + 1 + 1} = \sqrt{6}$.

(f) $|\mathbf{n_2}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{1 + 1 + 4} = \sqrt{6}$.

(g) Apply the cosine formula: $\cos\theta = \frac{|\mathbf{n_1} \cdot \mathbf{n_2}|}{|\mathbf{n_1}||\mathbf{n_2}|} = \frac{3}{\sqrt{6} \cdot \sqrt{6}}$.

(h) This simplifies to $\cos\theta = 3/6 = 1/2$.

(i) We find the angle whose cosine is $1/2$.

(j) Therefore, $\theta = \cos^{-1}(1/2) = 60°$.

Final Answer: The angle is 60°.

Answer: (C)

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Q50.

Solution

Concept:

Evaluating a definite integral with an unknown limit requires finding the antiderivative of the function and then solving the resulting algebraic equation. The integrand in this problem is a reciprocal of a quadratic expression, specifically one of the form $1/(a^2 + x^2)$. The standard integral for this form is $(1/a)\tan^{-1}(x/a) + C$. This is a transcendental function that relates the area under a curve to the inverse tangent of its horizontal coordinate. When the integral has specific numerical bounds and a known total value, we can use the Fundamental Theorem of Calculus to set up an equation involving the unknown bound k. Solving this often involves using known values of the inverse tangent function, such as $\tan^{-1}(1) = \pi/4$. This type of problem bridges the gap between calculus and trigonometry, demonstrating how the accumulation of values under a curve can be used to solve for geometric parameters.

Solution:

(a) We are given the integral: $\int_0^k \frac{1}{2+8x^2}dx = \frac{\pi}{16}$.

(b) First, simplify the integrand by factoring out a 2 from the denominator: $\frac{1}{2(1+4x^2)}$.

(c) The integral becomes $\frac{1}{2}\int_0^k \frac{1}{1+(2x)^2}dx$.

(d) To solve this, let $u = 2x$, which means $du = 2dx$, or $dx = du/2$.

(e) The integral transforms to: $\frac{1}{2}\int \frac{1}{1+u^2}\frac{du}{2} = \frac{1}{4}\tan^{-1}(u)$.

(f) Substituting back for x: $\frac{1}{4}[\tan^{-1}(2x)]_0^k = \frac{\pi}{16}$.

(g) Multiply both sides by 4: $[\tan^{-1}(2x)]_0^k = \frac{\pi}{4}$.

(h) Apply the limits: $\tan^{-1}(2k) - \tan^{-1}(0) = \frac{\pi}{4}$.

(i) Since $\tan^{-1}(0) = 0$, we have $\tan^{-1}(2k) = \frac{\pi}{4}$.

(j) Taking the tangent of both sides: $2k = \tan(\frac{\pi}{4})$.

(k) Since $\tan(\frac{\pi}{4}) = 1$, we have $2k = 1$, which gives $k = 1/2$.

Final Answer: The value of k is $1/2$.

Answer: (B)

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Answer Key

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	A	2	B	3	A	4	A	5	C
6	B	7	A	8	B	9	B	10	A
11	C	12	A	13	B	14	C	15	A
16	B	17	B	18	A	19	C	20	B
21	B	22	A	23	C	24	A	25	C
26	A	27	A	28	B	29	C	30	A
31	A	32	B	33	B	34	B	35	A
36	C	37	A	38	A	39	A	40	B
41	A	42	B	43	C	44	A	45	A
46	A	47	A	48	A	49	C	50	B