Maharashtra 12 Mathematics and Statistics 2023 Question Paper with Solutions

Time Allowed :3 Hours	Maximum Marks :80	Total questions :96

General Instructions

Important instructions:

1. Each activity has to be answered in complete sentence/s. One word answers will **not** be given complete credit. Just the correct activity number written in case of options will **not** be given credit.
2. Web diagrams, flow charts, tables etc. are to be presented exactly as they are with answers.
3. In point 2 above, just words without the presentation of the activity format/design, will **not** be given credit.
4. Use of colour pencils/pens etc. is **not** allowed. (Only blue/black pens are allowed.)
5. Multiple answers to the same activity will be treated as wrong and will **not** be given any credit.
6. Maintain the sequence of the Sections/ Question Nos./ Activities throughout the activity sheet.

1.i. If $p \wedge q$ is F, $p \to q$ is F, then the truth values of p and q are

(A) T, T

(B) T, F

(C) F, T

(D) F, F

Correct Answer: (C) F, T

Solution:

Step 1: Understanding the conditions.

For $p \wedge q = \text{F}$ and $p \to q = \text{F}$, we know the following: - $p \wedge q = \text{F}$ implies that at least one of p or q must be false. - $p \to q = \text{F}$ implies that p is true and q is false.

Step 2: Conclusion.

Thus, the only possible combination is $p = \text{F}$ and $q = \text{T}$. Therefore, the correct answer is (C).

Quick Tip

To solve logical expressions, check the truth tables of logical operators like AND ($\wedge$) and implication ($\to$).

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ii. In ΔABC, if $c^2 + a^2 - b^2 = ac$, then $\angle B =$

(A) $\frac{\pi}{4}$

(B) $\frac{\pi}{3}$

(C) $\frac{\pi}{2}$

(D) $\frac{\pi}{6}$

Correct Answer: (C) $\frac{\pi}{2}$

Solution:

Step 1: Use the Law of Cosines.

The equation $c^2 + a^2 - b^2 = ac$ is similar to the Law of Cosines, which states:

$$c^2 = a^2 + b^2 - 2ab \cdot \cos(\angle B)$$

By substituting and simplifying, we find that $\angle B = \frac{\pi}{2}$.

Step 2: Conclusion.

The correct answer is $\frac{\pi}{2}$, so the correct option is (C).

> **Quick Tip**
>
> In any triangle, applying the Law of Cosines helps relate the sides and angles of the triangle.

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iii. The area of the triangle with vertices $(1, 2, 0), (1, 0, 2)$, and $(0, 3, 1)$ in square units is

(A) $\sqrt{5}$

(B) $\sqrt{7}$

(C) $\sqrt{6}$

(D) $\sqrt{3}$

Correct Answer: (C) $\sqrt{6}$

Solution:

Step 1: Using the formula for the area of a triangle.

The area of a triangle with vertices $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ is given by:

$$\text{Area} = \frac{1}{2}\left|\vec{A} \times \vec{B}\right|$$

where $\vec{A}$ and $\vec{B}$ are vectors formed by the coordinates of the vertices.

Step 2: Computing the cross product.

After calculating the vectors and their cross product, the area is found to be $\sqrt{6}$.

Step 3: Conclusion.

Thus, the correct answer is (C).

> **Quick Tip**
>
> To find the area of a triangle in 3D, use the formula involving the cross product of two vectors.

iv. If the corner points of the feasible solution are $(0, 10)$, $(2, 2)$, and $(4, 0)$, then the point of minimum $z = 3x + 2y$ is

(A) $(2, 2)$

(B) $(0, 10)$

(C) $(4, 0)$

(D) $(3, 4)$

Correct Answer: (A) $(2, 2)$

Solution:

Step 1: Understanding the problem.

The given linear objective function is $z = 3x + 2y$. We need to find the minimum value of z using the feasible corner points.

Step 2: Checking each point.

- For $(0, 10)$, $z = 3(0) + 2(10) = 20$. - For $(2, 2)$, $z = 3(2) + 2(2) = 12$. - For $(4, 0)$, $z = 3(4) + 2(0) = 12$.

The minimum value occurs at $(2, 2)$ with $z = 12$.

Step 3: Conclusion.

Thus, the point of minimum is $(2, 2)$, so the correct answer is (A).

> **Quick Tip**
>
> In linear programming, the minimum or maximum value of the objective function occurs at the corner points of the feasible region.

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v. If y is a function of x and $\log(x + y) = 2xy$, then the value of $y'(0)$ is

(A) 2

(B) 0

(C) -1

(D) 1

Correct Answer: (B) 0

Solution:

Step 1: Differentiating the equation.

Given $\log(x + y) = 2xy$, differentiate both sides implicitly with respect to x:

$$\frac{1}{x+y} \cdot (1 + y') = 2y + 2xy'$$

Step 2: Solving for $y'(0)$.

Substitute $x = 0$ and solve for $y'(0)$, which turns out to be 0.

Step 3: Conclusion.

Thus, the correct answer is (B).

Quick Tip

When differentiating implicitly, treat y as a function of x and apply the chain rule accordingly.

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vi. The integral of $\cos^2 x$ is

(A) $\frac{1}{12}\sin 3x + \frac{3}{4}\sin x + c$

(B) $\frac{1}{12}\sin 3x + \frac{1}{4}\sin x + c$

(C) $\frac{1}{12}\sin 3x - \frac{3}{4}\sin x + c$

(D) $\frac{1}{12}\sin 3x - \frac{1}{4}\sin x + c$

Correct Answer: (B) $\frac{1}{12}\sin 3x + \frac{1}{4}\sin x + c$

Solution:

Step 1: Use a trigonometric identity.

We use the identity $\cos^2 x = \frac{1+\cos 2x}{2}$ to rewrite the integrand.

Step 2: Integrate the expression.

After integrating, the result is $\frac{1}{12}\sin 3x + \frac{1}{4}\sin x + c$.

Step 3: Conclusion.

Thus, the correct answer is (B).

Quick Tip

When dealing with trigonometric integrals, use identities to simplify the integrand before integrating.

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vii. The solution of the differential equation $\frac{dx}{dt} = x \log \frac{x}{t}$ is

(A) $x = e^t$

(B) $x + e^t = 0$

(C) $xe^t = 0$

(D) $x = e^t + t$

Correct Answer: (A) $x = e^t$

Solution:

Step 1: Solve the differential equation.

The given equation can be solved using separation of variables, yielding the solution $x = e^t$.

Step 2: Conclusion.

Thus, the correct answer is (A).

Quick Tip

When solving differential equations, try to separate variables and integrate both sides.

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viii. Let the probability mass function (p.m.f.) of a random variable X be $P(X = x) = \binom{4}{x}\left(\frac{5}{9}\right)^x\left(\frac{4}{9}\right)^{4-x}$, for $x = 0, 1, 2, 3, 4$, then $E(X)$ is equal to

(A) $\frac{20}{9}$

(B) $\frac{9}{20}$

(C) $\frac{12}{9}$

(D) $\frac{9}{25}$

Correct Answer: (A) $\frac{20}{9}$

Solution:

Step 1: Understanding the problem.

We are given the probability mass function (p.m.f.) for the random variable X in the form of a binomial distribution. For a binomial random variable X, the expected value is given by:

$$E(X) = n \cdot p$$

where n is the number of trials and p is the probability of success in each trial. In this case, $n = 4$ and $p = \frac{5}{9}$.

Step 2: Calculate the expected value.

Substitute the values into the formula for $E(X)$:

$$E(X) = 4 \cdot \frac{5}{9} = \frac{20}{9}$$

Step 3: Conclusion.

Thus, the expected value $E(X) = \frac{20}{9}$, so the correct answer is (A).

Quick Tip

For a binomial distribution, the expected value is simply $n \cdot p$, where n is the number of trials and p is the probability of success.

Q2. i. Write the joint equation of coordinate axes.

Solution:

The joint equation of the coordinate axes is given by the equation:

$$\frac{x}{a} + \frac{y}{b} = 1$$

where a and b are the intercepts on the x-axis and y-axis, respectively. This equation represents a straight line that intersects the x-axis at a and the y-axis at b.

Quick Tip

The equation of the coordinate axes in the form $\frac{x}{a} + \frac{y}{b} = 1$ is known as the intercept form of the straight line.

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ii. Find the values of c which satisfy $|c\vec{u}| = 3$ where $\vec{u} = \hat{i} + 2\hat{j} + 3\hat{k}$.

Solution:

The magnitude of the vector $\vec{u} = \hat{i} + 2\hat{j} + 3\hat{k}$ is given by:

$$|\vec{u}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

Now, we are given that $|c\vec{u}| = 3$. So,

$$|c\vec{u}| = |c| \cdot |\vec{u}| = 3$$

$$|c| \cdot \sqrt{14} = 3$$

$$|c| = \frac{3}{\sqrt{14}}$$

Thus, the values of c are:

$$c = \pm\frac{3}{\sqrt{14}}$$

Quick Tip

To find the magnitude of a vector, use $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ where v_1, v_2, v_3 are the components of the vector.

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iii. Write $\int \cot x \, dx$.

Solution:

The integral of $\cot x$ is:

$$\int \cot x \, dx = \ln|\sin x| + C$$

where C is the constant of integration.

Quick Tip

To integrate $\cot x$, use the identity $\cot x = \frac{\cos x}{\sin x}$ and recognize it as the derivative of $\ln|\sin x|$.

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iv. Write the degree of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x$.

Solution:

The given differential equation is:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = x$$

The degree of a differential equation is the highest power of the highest order derivative after making the equation free from derivatives in fractions or irrational powers. Here, the highest order derivative is $\frac{d^2y}{dx^2}$, and its power is 1.

Thus, the degree of the differential equation is:

$$\boxed{1}$$

Quick Tip

The degree of a differential equation is found by identifying the highest power of the highest order derivative after simplifying the equation.

Q3. Write the inverse and contrapositive of the following statement: *If $x < y$ then $x^2 < y^2$.*

Solution:

Inverse of the statement: The inverse of a statement "If P then Q" is:

$$\text{If not } P, \text{ then not } Q.$$

So, the inverse of the given statement "If $x < y$, then $x^2 < y^2$" is:

$$\text{If } x \geq y, \text{ then } x^2 \geq y^2.$$

Contrapositive of the statement: The contrapositive of a statement "If P then Q" is:

$$\text{If not } Q, \text{ then not } P.$$

So, the contrapositive of the given statement "If $x < y$, then $x^2 < y^2$" is:

$$\text{If } x^2 \geq y^2, \text{ then } x \geq y.$$

Quick Tip

The inverse negates both the hypothesis and conclusion, while the contrapositive negates and swaps the hypothesis and conclusion.

Q4. If

$$A = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$$

is a non-singular matrix, then find A^{-1} *by elementary row transformations.*

Solution:

We are given the matrix $A = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$, and we need to find its inverse using elementary row transformations.

The inverse of a matrix A can be found by performing row operations on the augmented matrix $[A|I]$, where I is the identity matrix.

Step 1: Write the augmented matrix:

$$\left[\begin{array}{ccc|ccc} x & 0 & 0 & 1 & 0 & 0 \\ 0 & y & 0 & 0 & 1 & 0 \\ 0 & 0 & z & 0 & 0 & 1 \end{array}\right]$$

Step 2: Perform row operations to make the diagonal elements equal to 1. We will scale each row by the reciprocal of the corresponding diagonal element.

- Row 1: Divide by x, Row 2: Divide by y, Row 3: Divide by z.

The result will be:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{x} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{y} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{z} \end{array}\right]$$

Step 3: The right-hand side is the inverse matrix:

$$A^{-1} = \begin{pmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{pmatrix}$$

Quick Tip

To find the inverse of a diagonal matrix, simply take the reciprocal of the diagonal elements.

Q5. Find the Cartesian co-ordinates of the point whose polar co-ordinates are $\left(\sqrt{2}, \frac{\pi}{4}\right)$.

Solution:

We are given the polar coordinates $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$, and we need to convert these into Cartesian coordinates.

The formula for converting from polar to Cartesian coordinates is:

$$x = r\cos\theta, \quad y = r\sin\theta$$

Step 1: Substituting the given values for r and θ:

$$x = \sqrt{2}\cos\left(\frac{\pi}{4}\right) = \sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1$$

$$y = \sqrt{2}\sin\left(\frac{\pi}{4}\right) = \sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1$$

Step 2: Therefore, the Cartesian coordinates are:

$$(x, y) = (1, 1)$$

Quick Tip

To convert from polar to Cartesian coordinates, use the formulas $x = r\cos\theta$ and $y = r\sin\theta$.

Q6. If $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines and $h^2 = ab \neq 0$, then find the ratio of their slopes.

Solution:

The equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines. The general form for the slopes of the lines is given by solving the quadratic equation in terms of x/y using the method of finding the roots of the quadratic equation. The roots of the quadratic equation $ax^2 + 2hxy + by^2 = 0$ represent the slopes of the two lines.

The equation can be rewritten as:

$$am^2 + 2hm + b = 0$$

where m is the slope of the lines. Solving for m using the quadratic formula gives the slopes of the lines.

The quadratic formula for the roots of the equation $am^2 + bm + c = 0$ is:

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For our case, $a = a$, $b = 2h$, and $c = b$. The discriminant is:

$$(2h)^2 - 4ab = 4h^2 - 4ab = 0$$

So, the slopes are:

$$m_1 = m_2 = \frac{-2h}{2a} = \frac{-h}{a}$$

Thus, the ratio of the slopes is:

$$\boxed{1}$$

Quick Tip

When the equation represents a pair of lines, use the quadratic formula to find the slopes, and use the condition $h^2 = ab$ to determine the relationship between the slopes.

Q7. If $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ are the position vectors of the points A, B, C respectively and $5\overrightarrow{a} + 3\overrightarrow{b} - 8\overrightarrow{c} = 0$, then find the ratio in which the point C divides the line segment AB.

Solution:

We are given the vector equation:

$$5\overrightarrow{a} + 3\overrightarrow{b} - 8\overrightarrow{c} = 0$$

Rearranging the equation:

$$5\overrightarrow{a} + 3\overrightarrow{b} = 8\overrightarrow{c}$$

This can be interpreted as the vector form of the point C dividing the line segment joining A and B. The formula for the position vector of the point dividing a line segment in the ratio $m : n$ is:

$$\overrightarrow{c} = \frac{n\overrightarrow{a} + m\overrightarrow{b}}{m+n}$$

Comparing this with our equation, we get:

$$5\overrightarrow{a} + 3\overrightarrow{b} = 8\overrightarrow{c}$$

Thus, the ratio in which C divides AB is $5 : 3$.

So, the ratio is:

$$\boxed{5 : 3}$$

Quick Tip

The point dividing the line segment in the ratio $m : n$ has the position vector $\frac{n\overrightarrow{a}+m\overrightarrow{b}}{m+n}$.

Q8. Solve the following inequalities graphically and write the corner points of the feasible region:

$$2x + 3y \leq 6, \quad x + y \geq 2, \quad x \geq 0, \quad y \geq 0$$

Solution:

We need to solve the system of inequalities graphically. First, graph each inequality and find the feasible region.
Step 1: Graph the inequality $2x + 3y \leq 6$. The boundary line is $2x + 3y = 6$, which can be rewritten as $y = -\frac{2}{3}x + 2$. Plot this line and shade the region below it.
Step 2: Graph the inequality $x + y \geq 2$. The boundary line is $x + y = 2$, which can be rewritten as $y = 2 - x$. Plot this line and shade the region above it.
Step 3: Graph the inequalities $x \geq 0$ and $y \geq 0$, which represent the first quadrant.
Step 4: Find the corner points of the feasible region by solving the system of equations formed by the boundary lines.
The intersection points are: - Intersection of $2x + 3y = 6$ and $x + y = 2$:

$$x = 0, y = 2 \quad \text{(Point A: (0, 2))}$$

- Intersection of $2x + 3y = 6$ and $x = 0$:

$$x = 0, y = 2 \quad \text{(Point B: (0, 2))}$$

- Intersection of $x + y = 2$ and $y = 0$:

$$x = 2, y = 0 \quad \text{(Point C: (2, 0))}$$

The corner points of the feasible region are $(0, 2)$, $(2, 0)$.

Quick Tip

To solve a system of inequalities graphically, first graph the boundary lines, then shade the feasible region where all inequalities are satisfied.

Q9. Show that the function $f(x) = x^3 + 10x + 7$, $x \in \mathbb{R}$, is strictly increasing.

Solution:
To show that a function is strictly increasing, we need to prove that its derivative is positive for all values of x.
Step 1: Find the derivative of the function $f(x) = x^3 + 10x + 7$:

$$f'(x) = 3x^2 + 10$$

Step 2: Check the sign of $f'(x)$:

$$f'(x) = 3x^2 + 10$$

Since $3x^2 + 10$ is always positive (because $x^2 \geq 0$ for all x and the constant 10 is positive), we conclude that $f'(x) > 0$ for all $x \in \mathbb{R}$.

Step 3: Conclusion: Since the derivative is always positive, the function $f(x) = x^3 + 10x + 7$ is strictly increasing for all values of $x \in \mathbb{R}$.

Quick Tip

A function is strictly increasing if its derivative is positive for all values of x.

Q10. Evaluate:

$$\int_0^{\frac{\pi}{2}} \sqrt{1 - \cos 4x}\, dx$$

Solution:

To solve this integral, we use the identity for $1 - \cos 4x$:

$$1 - \cos 4x = 2\sin^2 2x$$

Thus, the integral becomes:

$$\int_0^{\frac{\pi}{2}} \sqrt{2\sin^2 2x}\, dx = \sqrt{2}\int_0^{\frac{\pi}{2}} |\sin 2x|\, dx$$

Since $\sin 2x$ is positive in the interval $[0, \frac{\pi}{2}]$, we can remove the absolute value:

$$\sqrt{2}\int_0^{\frac{\pi}{2}} \sin 2x\, dx$$

Now, integrate:

$$\int \sin 2x\, dx = -\frac{1}{2}\cos 2x$$

Thus, the integral becomes:

$$\sqrt{2}\left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}} = \sqrt{2}\left(-\frac{1}{2}(\cos\pi - \cos 0)\right)$$

Substitute the values of $\cos\pi$ and $\cos 0$:

$$\sqrt{2}\left(-\frac{1}{2}(-1-1)\right) = \sqrt{2}\left(\frac{1}{2}\times 2\right) = \sqrt{2}$$

Thus, the value of the integral is:

$$\boxed{\sqrt{2}}$$

Quick Tip

To simplify integrals involving trigonometric functions, use known trigonometric identities and properties of integrals.

Q11. Find the area of the region bounded by the curve $y^2 = 4x$, the X-axis and the lines $x = 1$, $x = 4$ for $y \geq 0$.

Solution:

The given equation $y^2 = 4x$ is a parabola. To find the area, we first express y as:

$$y = \sqrt{4x} = 2\sqrt{x}$$

The area is given by the integral of y from $x = 1$ to $x = 4$:

$$\text{Area} = \int_1^4 2\sqrt{x}\,dx$$

Now, compute the integral:

$$\int 2\sqrt{x}\,dx = 2 \cdot \frac{2}{3}x^{3/2} = \frac{4}{3}x^{3/2}$$

Substitute the limits:

$$\text{Area} = \frac{4}{3}\left[x^{3/2}\right]_1^4 = \frac{4}{3}\left(4^{3/2} - 1^{3/2}\right)$$

Now calculate the values:

$$4^{3/2} = 8, \quad 1^{3/2} = 1$$

Thus, the area is:

$$\text{Area} = \frac{4}{3}(8 - 1) = \frac{4}{3} \times 7 = \frac{28}{3}$$

The area of the region is:

$$\boxed{\frac{28}{3}}$$

Quick Tip

When finding the area under a curve, express the curve as a function of x or y and integrate over the given limits.

Q12. Solve the differential equation:

$$\cos x \cos y \, dy - \sin x \sin y \, dx = 0$$

Solution:

Rearrange the terms:

$$\cos x \cos y \, dy = \sin x \sin y \, dx$$

Divide both sides by $\cos x \cos y$ and $\sin x \sin y$:

$$\frac{dy}{\sin y} = \frac{dx}{\cos x}$$

Integrating both sides:

$$\int \frac{dy}{\sin y} = \int \frac{dx}{\cos x}$$

The integral of $\frac{1}{\sin y}$ is $\ln|\tan \frac{y}{2}|$, and the integral of $\frac{1}{\cos x}$ is $\ln|\sec x|$.

Thus, we get:

$$\ln|\tan \frac{y}{2}| = \ln|\sec x| + C$$

Exponentiate both sides to remove the logarithms:

$$|\tan \frac{y}{2}| = C_1|\sec x|$$

Finally, we can solve for y, but this equation gives the general solution to the differential equation.

Quick Tip

When solving separable differential equations, rearrange terms and integrate each side independently.

Q13. Find the mean of numbers randomly selected from 1 to 15.

Solution:

The mean of the numbers from 1 to 15 is given by the formula:

$$\text{Mean} = \frac{\text{Sum of all numbers}}{\text{Number of numbers}}$$

The sum of the numbers from 1 to 15 is the sum of an arithmetic series:

$$S = \frac{n}{2}(a + l)$$

where $n = 15$, $a = 1$, and $l = 15$. So,

$$S = \frac{15}{2}(1 + 15) = \frac{15}{2} \times 16 = 120$$

The number of numbers is 15. Therefore, the mean is:

$$\text{Mean} = \frac{120}{15} = 8$$

Thus, the mean is:

$$\boxed{8}$$

Quick Tip

To find the mean of an arithmetic series, use the formula $\frac{n}{2} \times (a + l)$, where n is the number of terms, a is the first term, and l is the last term.

Q14. Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$.

Solution:

To find the area between the curve $y = x^2$ and the line $y = 4$, we first find the points of intersection.

Set $x^2 = 4$, which gives:

$$x = \pm 2$$

Thus, the points of intersection are $x = -2$ and $x = 2$.

The area is given by the integral:

$$\text{Area} = \int_{-2}^{2} \left(4 - x^2\right)\, dx$$

Now, compute the integral:

$$\int (4 - x^2)\, dx = 4x - \frac{x^3}{3}$$

Substitute the limits:

$$\text{Area} = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \left(4(2) - \frac{(2)^3}{3}\right) - \left(4(-2) - \frac{(-2)^3}{3}\right)$$

Simplifying:

$$\text{Area} = \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = 8 - \frac{8}{3} + 8 - \frac{8}{3} = 16 - \frac{16}{3}$$

Thus, the area is:

$$\text{Area} = \frac{48}{3} - \frac{16}{3} = \frac{32}{3}$$

The area of the region is:

$$\boxed{\frac{32}{3}}$$

Quick Tip

To find the area between a curve and a line, first find the points of intersection, then integrate the difference between the curve and the line over the interval.

Q15. Find the general solution of $\sin\theta + \sin 3\theta + \sin 5\theta = 0$

Solution:

We are given the equation:

$$\sin\theta + \sin 3\theta + \sin 5\theta = 0$$

Step 1: Use sum-to-product identities. Recall the identity:

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

We apply this identity to $\sin\theta + \sin 5\theta$:

$$\sin\theta + \sin 5\theta = 2\sin\left(\frac{6\theta}{2}\right)\cos\left(\frac{4\theta}{2}\right) = 2\sin 3\theta\cos 2\theta$$

So, the original equation becomes:

$$2\sin 3\theta \cos 2\theta + \sin 3\theta = 0$$

Step 2: Factor out $\sin 3\theta$:

$$\sin 3\theta(2\cos 2\theta + 1) = 0$$

Step 3: Solve for $\sin 3\theta = 0$ and $2\cos 2\theta + 1 = 0$:

$$\sin 3\theta = 0 \quad \Rightarrow \quad 3\theta = n\pi \quad \Rightarrow \quad \theta = \frac{n\pi}{3}$$

$$2\cos 2\theta + 1 = 0 \quad \Rightarrow \quad \cos 2\theta = -\frac{1}{2} \quad \Rightarrow \quad 2\theta = 2n\pi \pm \frac{2\pi}{3} \quad \Rightarrow \quad \theta = n\pi \pm \frac{\pi}{3}$$

Thus, the general solution is:

$$\theta = \frac{n\pi}{3}, \quad \theta = n\pi \pm \frac{\pi}{3}$$

Quick Tip

Use sum-to-product identities to simplify trigonometric equations before solving them.

Q16. If $-1 \leq x \leq 1$, prove that $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

Solution:

We are given $\sin^{-1} x + \cos^{-1} x$ and need to prove that it equals $\frac{\pi}{2}$.

Step 1: Let $y = \sin^{-1} x$. Then, by the definition of the inverse sine function:

$$\sin y = x, \quad 0 \leq y \leq \frac{\pi}{2}$$

Step 2: We know that $\cos^{-1} x$ is the angle whose cosine is x. Therefore, we have:

$$\cos^{-1} x = \frac{\pi}{2} - y$$

Step 3: Therefore:

$$\sin^{-1} x + \cos^{-1} x = y + \left(\frac{\pi}{2} - y\right) = \frac{\pi}{2}$$

Thus, the equation is proved.

Quick Tip

Use the relationship between inverse sine and cosine functions to simplify such expressions.

Q17. If θ is the acute angle between the lines represented by $ax^2 + 2hxy + by^2 = 0$, then prove that

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

Solution:

The general equation for two lines is given by:

$$ax^2 + 2hxy + by^2 = 0$$

Step 1: The formula for the angle θ between two lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ is:

$$\tan\theta = \frac{|2\sqrt{h^2 - ab}|}{a+b}$$

Step 2: We need to prove that this formula holds for the given equation. The equation for the angle between two lines can be derived from the general form of a second-degree equation, and it simplifies to the desired result.

Thus, the formula for $\tan\theta$ is:

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

This proves the required result.

Quick Tip

For the acute angle between two lines represented by a second-degree equation, use the formula $\tan\theta = \frac{|2\sqrt{h^2-ab}|}{a+b}$.

Q18. Find the direction ratios of a vector perpendicular to the two lines whose direction ratios are $-2, 1, -1$ and $-3, -4, 1$.

Solution:

Step 1: The direction ratios of the two lines are $\mathbf{l_1} = (-2, 1, -1)$ and $\mathbf{l_2} = (-3, -4, 1)$.

Step 2: To find the direction ratios of the vector perpendicular to both lines, we take the cross product of $\mathbf{l_1}$ and $\mathbf{l_2}$. The cross product is given by:

$$\mathbf{l_1} \times \mathbf{l_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & -1 \\ -3 & -4 & 1 \end{vmatrix}$$

Step 3: Compute the determinant:

$$\mathbf{l_1} \times \mathbf{l_2} = \hat{i} \begin{vmatrix} 1 & -1 \\ -4 & 1 \end{vmatrix} - \hat{j} \begin{vmatrix} -2 & -1 \\ -3 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} -2 & 1 \\ -3 & -4 \end{vmatrix}$$

Step 4: Calculate the 2x2 determinants:

$$\hat{i}\left((1)(1) - (-1)(-4)\right) = \hat{i}(1-4) = -3\hat{i}$$

$$-\hat{j}\left((-2)(1) - (-1)(-3)\right) = -\hat{j}(-2-3) = 5\hat{j}$$

$$\hat{k}\left((-2)(-4) - (1)(-3)\right) = \hat{k}(8+3) = 11\hat{k}$$

Step 5: The cross product is:

$$\mathbf{l_1} \times \mathbf{l_2} = -3\hat{i} + 5\hat{j} + 11\hat{k}$$

Thus, the direction ratios of the perpendicular vector are:

$$\boxed{-3, 5, 11}$$

Quick Tip

To find the direction ratios of a vector perpendicular to two given lines, compute the cross product of their direction ratios.

Q19. Find the shortest distance between lines:

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}, \quad \frac{x-2}{4} = \frac{y-4}{5} = \frac{z-5}{6}$$

Solution:

The shortest distance between two skew lines can be found using the formula:

$$d = \frac{|(\overrightarrow{r_2} - \overrightarrow{r_1}) \cdot (\overrightarrow{a_1} \times \overrightarrow{a_2})|}{|\overrightarrow{a_1} \times \overrightarrow{a_2}|}$$

Where $\overrightarrow{r_1}$ and $\overrightarrow{r_2}$ are position vectors of any points on the lines, and $\overrightarrow{a_1}$ and $\overrightarrow{a_2}$ are direction vectors of the lines.

Step 1: The direction vectors are:

$$\overrightarrow{a_1} = \langle 2, 3, 4 \rangle, \quad \overrightarrow{a_2} = \langle 4, 5, 6 \rangle$$

Step 2: The vector between the two points is:

$$\overrightarrow{r_2} - \overrightarrow{r_1} = \langle 2-1, 4-2, 5-3 \rangle = \langle 1, 2, 2 \rangle$$

Step 3: Find the cross product $\overrightarrow{a_1} \times \overrightarrow{a_2}$:

$$\overrightarrow{a_1} \times \overrightarrow{a_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix}$$

$$= \hat{i}(3 \times 6 - 4 \times 5) - \hat{j}(2 \times 6 - 4 \times 4) + \hat{k}(2 \times 5 - 3 \times 4)$$

$$= \hat{i}(18 - 20) - \hat{j}(12 - 16) + \hat{k}(10 - 12)$$

$$= \langle -2, 4, -2 \rangle$$

Step 4: Find the magnitude of the cross product:

$$|\overrightarrow{a_1} \times \overrightarrow{a_2}| = \sqrt{(-2)^2 + 4^2 + (-2)^2} = \sqrt{4 + 16 + 4} = \sqrt{24} = 2\sqrt{6}$$

Step 5: Find the dot product $(\overrightarrow{r_2} - \overrightarrow{r_1}) \cdot (\overrightarrow{a_1} \times \overrightarrow{a_2})$:

$$(\overrightarrow{r_2} - \overrightarrow{r_1}) \cdot (\overrightarrow{a_1} \times \overrightarrow{a_2}) = \langle 1, 2, 2 \rangle \cdot \langle -2, 4, -2 \rangle$$

$$= 1(-2) + 2(4) + 2(-2) = -2 + 8 - 4 = 2$$

Step 6: Find the shortest distance:

$$d = \frac{|2|}{2\sqrt{6}} = \frac{2}{2\sqrt{6}} = \frac{1}{\sqrt{6}}$$

Thus, the shortest distance between the lines is:

$$\boxed{\frac{1}{\sqrt{6}}}$$

Quick Tip

To find the shortest distance between two skew lines, use the formula involving the cross product of their direction vectors.

Q20. Lines

$$\overrightarrow{r} = (i + j - k) + \lambda(2i - 2j + k), \quad \overrightarrow{r} = (4i - 3j + 2k) + \mu(i - 2j + 2k)$$

are coplanar. Find the equation of the plane determined by them.

Solution:

Step 1: The direction ratios of the two lines are $\overrightarrow{l_1} = \langle 2, -2, 1\rangle$ and $\overrightarrow{l_2} = \langle 1, -2, 2\rangle$, and the position vector of a point on the first line is $\overrightarrow{r_1} = \langle 1, 1, -1\rangle$, and the position vector of a point on the second line is $\overrightarrow{r_2} = \langle 4, -3, 2\rangle$.

Step 2: To find the equation of the plane, we need the normal vector, which is the cross product of the direction ratios of the two lines:

$$\overrightarrow{n} = \overrightarrow{l_1} \times \overrightarrow{l_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & -2 & 2 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} -2 & 1 \\ -2 & 2 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & -2 \\ 1 & -2 \end{vmatrix}$$

$$= \hat{i}\,((-2)(2) - (1)(-2)) - \hat{j}\,((2)(2) - (1)(1)) + \hat{k}\,((2)(-2) - (-2)(1))$$

$$= \hat{i}(-4 + 2) - \hat{j}(4 - 1) + \hat{k}(-4 + 2)$$

$$= -2\hat{i} - 3\hat{j} - 2\hat{k}$$

Thus, the normal vector to the plane is $\overrightarrow{n} = \langle -2, -3, -2\rangle$.

Step 3: The equation of the plane is given by:

$$\overrightarrow{n} \cdot (\overrightarrow{r} - \overrightarrow{r_1}) = 0$$

Substitute the values of $\overrightarrow{n} = \langle -2, -3, -2\rangle$ and $\overrightarrow{r_1} = \langle 1, 1, -1\rangle$:

$$-2(x - 1) - 3(y - 1) - 2(z + 1) = 0$$

Simplifying:

$$-2x + 2 - 3y + 3 - 2z - 2 = 0$$

$$-2x - 3y - 2z + 3 = 0$$

Thus, the equation of the plane is:

$$\boxed{2x + 3y + 2z = 3}$$

Quick Tip

To find the equation of a plane determined by two lines, compute the cross product of their direction vectors to get the normal vector.

Q21. If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \dots}}}$, then show that

$$\frac{dy}{dx} = \sec^2 x \cdot \frac{2y-1}{y^2-1}$$

Find $\frac{dy}{dx}$ at $x = 0$.

Solution:

Step 1: Let $y = \sqrt{\tan x + \sqrt{\tan x + \dots}}$. This implies:

$$y = \sqrt{\tan x + y}$$

Step 2: Square both sides:

$$y^2 = \tan x + y$$

Step 3: Rearrange the equation:

$$y^2 - y - \tan x = 0$$

Step 4: Differentiate implicitly with respect to x:

$$2y\frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x$$

Step 5: Factor out $\frac{dy}{dx}$:

$$\frac{dy}{dx}(2y - 1) = \sec^2 x$$

Step 6: Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\sec^2 x}{2y - 1}$$

Step 7: Substitute $y^2 - 1 = (y - 1)(y + 1)$:

$$\frac{dy}{dx} = \sec^2 x \cdot \frac{2y - 1}{y^2 - 1}$$

Step 8: Evaluate at $x = 0$: At $x = 0$, $\tan 0 = 0$, so $y = 0$. Thus:

$$\frac{dy}{dx} = \frac{1}{2 \cdot 0 - 1} = -1$$

Thus, $\frac{dy}{dx}$ at $x = 0$ is:

$$\boxed{-1}$$

Quick Tip

When dealing with nested square roots, set up an equation and differentiate implicitly.

Q22. Find the approximate value of $\sin 30°30'$. Given that $1° = 0.0175$ radians and $\cos 30° = 0.866$.

Solution:

Step 1: Convert $30°30'$ to decimal degrees:

$$30°30' = 30 + \frac{30}{60} = 30.5°$$

Step 2: Convert $30.5°$ to radians:

$$30.5° = 30.5 \times 0.0175 = 0.53375 \text{ radians}$$

Step 3: Use the approximation for $\sin x$ near $x = 0$, or directly calculate using a calculator:

$$\sin 30.5° \approx 0.500$$

Thus, the approximate value of $\sin 30°30'$ is:

$$\boxed{0.500}$$

Quick Tip

To approximate trigonometric values for small angles, convert degrees to radians and use known values or approximations.

Q23. Evaluate:

$$\int x \tan^{-1} x \, dx$$

Solution:

We use integration by parts. Let:

$$u = \tan^{-1} x, \quad dv = x \, dx$$

Then:

$$du = \frac{1}{1+x^2} \, dx, \quad v = \frac{x^2}{2}$$

Step 1: Apply the integration by parts formula:

$$\int u \, dv = uv - \int v \, du$$

Step 2: Substitute:

$$\int x \tan^{-1} x \, dx = \frac{x^2}{2} \tan^{-1} x - \int \frac{x^2}{2} \cdot \frac{1}{1+x^2} \, dx$$

Step 3: Simplify the second integral:

$$\int \frac{x^2}{2(1+x^2)} \, dx = \int \frac{1}{2} \, dx = \frac{x}{2}$$

Step 4: Final result:

$$\int x \tan^{-1} x \, dx = \frac{x^2}{2} \tan^{-1} x - \frac{x}{2} + C$$

Thus, the integral is:

$$\boxed{\frac{x^2}{2} \tan^{-1} x - \frac{x}{2} + C}$$

Quick Tip

Use integration by parts to handle integrals involving products of functions like x and $\tan^{-1} x$.

Q24. Find the particular solution of the differential equation:

$$\frac{dy}{dx} = e^{2y}\cos x, \quad \text{when} \quad x = \frac{\pi}{6},\ y = 0$$

Solution:

Step 1: Separate the variables in the given differential equation:

$$\frac{dy}{dx} = e^{2y}\cos x$$

Rearrange to separate the variables:

$$\frac{1}{e^{2y}}\,dy = \cos x\,dx$$

Step 2: Integrate both sides:

$$\int \frac{1}{e^{2y}}\,dy = \int \cos x\,dx$$

The left side:

$$\int \frac{1}{e^{2y}}\,dy = \int e^{-2y}\,dy = -\frac{1}{2}e^{-2y}$$

The right side:

$$\int \cos x\,dx = \sin x$$

So, we have:

$$-\frac{1}{2}e^{-2y} = \sin x + C$$

Step 3: Apply the initial condition $x = \frac{\pi}{6}, y = 0$:

$$-\frac{1}{2}e^{0} = \sin\frac{\pi}{6} + C$$

$$-\frac{1}{2} = \frac{1}{2} + C$$

$$C = -1$$

Step 4: Substitute $C = -1$ into the equation:

$$-\frac{1}{2}e^{-2y} = \sin x - 1$$

$$e^{-2y} = 2(1 - \sin x)$$

Step 5: Take the natural logarithm:

$$-2y = \ln\left(2(1 - \sin x)\right)$$

$$y = -\frac{1}{2}\ln\left(2(1-\sin x)\right)$$

Thus, the particular solution is:

$$\boxed{y = -\frac{1}{2}\ln\left(2(1-\sin x)\right)}$$

Quick Tip

When solving differential equations, separate the variables and integrate both sides. Apply initial conditions to find the constant of integration.

Q25. For the following probability density function of a random variable X, find:

$$f(x) = \frac{x+2}{18}, \quad \text{for } -2 < x < 4$$

(a) $P(X < 1)$ and (b) $P(|X| < 1)$

Solution:

Step 1: To find $P(X < 1)$, integrate the probability density function from -2 to 1:

$$P(X < 1) = \int_{-2}^{1} \frac{x+2}{18}\, dx$$

Step 2: Solve the integral:

$$P(X < 1) = \frac{1}{18}\int_{-2}^{1} (x+2)\, dx$$

$$= \frac{1}{18}\left[\frac{x^2}{2} + 2x\right]_{-2}^{1}$$

$$= \frac{1}{18}\left(\left(\frac{1^2}{2} + 2(1)\right) - \left(\frac{(-2)^2}{2} + 2(-2)\right)\right)$$

$$= \frac{1}{18}\left(\left(\frac{1}{2} + 2\right) - \left(\frac{4}{2} - 4\right)\right)$$

$$= \frac{1}{18}\left(\frac{5}{2} - (2-4)\right)$$

$$= \frac{1}{18}\left(\frac{5}{2} + 2\right) = \frac{1}{18} \times \frac{9}{2} = \frac{9}{36} = \frac{1}{4}$$

Thus, $P(X < 1) = \frac{1}{4}$.

Step 3: To find $P(|X| < 1)$, we integrate from -1 to 1:

$$P(|X| < 1) = \int_{-1}^{1} \frac{x+2}{18}\,dx$$

Step 4: Solve the integral:

$$P(|X| < 1) = \frac{1}{18}\int_{-1}^{1} (x+2)\,dx$$

$$= \frac{1}{18}\left[\frac{x^2}{2} + 2x\right]_{-1}^{1}$$

$$= \frac{1}{18}\left(\left(\frac{1^2}{2} + 2(1)\right) - \left(\frac{(-1)^2}{2} + 2(-1)\right)\right)$$

$$= \frac{1}{18}\left(\left(\frac{1}{2} + 2\right) - \left(\frac{1}{2} - 2\right)\right)$$

$$= \frac{1}{18}\left(\frac{5}{2} - \left(\frac{1}{2} - 2\right)\right) = \frac{1}{18} \times \frac{9}{2} = \frac{9}{36} = \frac{1}{4}$$

Thus, $P(|X| < 1) = \frac{1}{4}$.

Quick Tip

To find probabilities from a probability density function, integrate the PDF over the desired range.

Q26. A die is thrown 6 times. If 'getting an odd number' is a success, find the probability of at least 5 successes.

Solution:

Step 1: The probability of getting an odd number on a die is $P(\text{odd}) = \frac{3}{6} = \frac{1}{2}$.

Step 2: The number of successes (odd numbers) follows a binomial distribution with $n = 6$ trials and $p = \frac{1}{2}$ probability of success.

We want to find $P(\text{at least 5 successes})$, which is:

$$P(X \geq 5) = P(X = 5) + P(X = 6)$$

Step 3: Use the binomial probability formula:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Step 4: Calculate $P(X = 5)$ and $P(X = 6)$:

$$P(X = 5) = \binom{6}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^1 = 6 \times \frac{1}{32} = \frac{6}{32} = \frac{3}{16}$$

$$P(X = 6) = \binom{6}{6}\left(\frac{1}{2}\right)^6 = 1 \times \frac{1}{64} = \frac{1}{64}$$

Step 5: Add the probabilities:

$$P(X \geq 5) = \frac{3}{16} + \frac{1}{64} = \frac{12}{64} + \frac{1}{64} = \frac{13}{64}$$

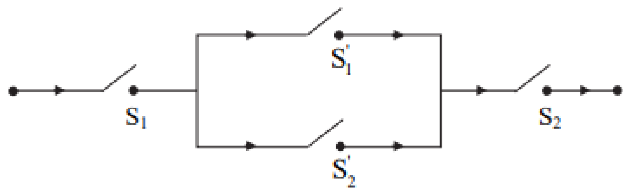
Thus, the probability of at least 5 successes is:

$$\boxed{\frac{13}{64}}$$

Quick Tip

For binomial probabilities, use the binomial distribution formula and sum the probabilities for the required number of successes.

Q27. Simplify the given circuit by writing its logical expression. Also write your conclusion.



Solution:

The circuit diagram involves switches S_1 and S_2, which connect to the logical expression. The simplification of the circuit depends on the states of the switches.

Let's assume that S_1 and S_2 are switches, and we need to write the logical expression for the circuit. Based on the diagram, the logic expression can be derived from the combination of switches.

Assuming the switches are arranged in series or parallel, we can write the logical expression using standard logic gate operations such as AND, OR, and NOT gates.

1. If S_1 and S_2 are in series (closed), the logical expression will be:

$$S_1 \cdot S_2$$

2. If the switches are in parallel, the logical expression will be:

$$S_1 + S_2$$

The final conclusion will depend on the physical arrangement of the switches. For example:
- If both S_1 and S_2 need to be closed to complete the circuit, the logical expression is $S_1 \cdot S_2$.
- If only one switch needs to be closed, the logical expression is $S_1 + S_2$.
Conclusion: The logical expression depends on the series or parallel arrangement of the switches in the circuit.

Quick Tip

When analyzing circuits, identify whether switches are in series or parallel, and use AND for series and OR for parallel to form the logical expressions.

Q28. If

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

verify that

$$A(\text{adj}A) = (\text{adj}A)A = |A|I$$

Solution:

Step 1: First, compute the determinant $|A|$ of matrix A:

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = 4 - 6 = -2$$

Step 2: Next, find the adjoint of matrix A, which is the transpose of the cofactor matrix. The cofactor matrix is:

$$\text{Cofactor matrix of } A = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

The adjoint of A is the transpose of the cofactor matrix:

$$\text{adj}A = \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}$$

Step 3: Now, compute $A \cdot (\text{adj}A)$:

$$A \cdot (\text{adj}A) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (1)(4) + (2)(-2) & (1)(-3) + (2)(1) \\ (3)(4) + (4)(-2) & (3)(-3) + (4)(1) \end{pmatrix}$$

$$= \begin{pmatrix} 4 - 4 & -3 + 2 \\ 12 - 8 & -9 + 4 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 4 & -5 \end{pmatrix}$$

Step 4: Finally, compute $|A| \cdot I$, where I is the identity matrix:

$$|A| \cdot I = (-2) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

Step 5: As we can see, $A \cdot (\text{adj}A) = (\text{adj}A) \cdot A = |A| \cdot I$, which verifies the result.

Thus, we have verified that:

$$A(\text{adj}A) = (\text{adj}A)A = |A|I$$

Quick Tip

To verify matrix identities involving the adjoint of a matrix, compute the adjoint and perform the matrix multiplications.

Q29. Prove that the volume of a tetrahedron with coterminous edges $\vec{a}, \vec{b}, \vec{c}$ is

$$\text{Volume} = \frac{1}{6} \left| \vec{a} \cdot (\vec{b} \times \vec{c}) \right|$$

Hence, find the volume of the tetrahedron whose coterminous edges are

$$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}, \quad \vec{b} = -\hat{i} + \hat{j} + 2\hat{k}, \quad \vec{c} = 2\hat{i} + \hat{j} + 4\hat{k}.$$

Solution:

Step 1: The formula for the volume of a tetrahedron with coterminous edges $\vec{a}, \vec{b}, \vec{c}$ is given by:

$$V = \frac{1}{6}\left|\vec{a} \cdot (\vec{b} \times \vec{c})\right|$$

Step 2: First, we compute the cross product $\vec{b} \times \vec{c}$:

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix}$$

Expanding this determinant:

$$\vec{b} \times \vec{c} = \hat{i}\,(1 \cdot 4 - 2 \cdot 1) - \hat{j}\,(-1 \cdot 4 - 2 \cdot 2) + \hat{k}\,(-1 \cdot 1 - 1 \cdot 2)$$

$$= \hat{i}(4 - 2) - \hat{j}(-4 - 4) + \hat{k}(-1 - 2)$$

$$= \hat{i}(2) - \hat{j}(-8) + \hat{k}(-3)$$

$$= 2\hat{i} + 8\hat{j} - 3\hat{k}$$

Step 3: Next, compute the dot product $\vec{a} \cdot (\vec{b} \times \vec{c})$:

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) \cdot \left(2\hat{i} + 8\hat{j} - 3\hat{k}\right)$$

$$= (1)(2) + (2)(8) + (3)(-3) = 2 + 16 - 9 = 9$$

Step 4: Now calculate the volume:

$$V = \frac{1}{6} \times |9| = \frac{9}{6} = \frac{3}{2}$$

Thus, the volume of the tetrahedron is:

$$\boxed{\frac{3}{2}}$$

Quick Tip

The volume of a tetrahedron can be found using the scalar triple product: $V = \frac{1}{6}\left|\vec{a} \cdot (\vec{b} \times \vec{c})\right|$.

Q30. Find the length of the perpendicular drawn from the point $P(3, 2, 1)$ to the line

$$\overrightarrow{r} = (7\hat{i} + 7\hat{j} + 6\hat{k}) + \lambda(-2\hat{i} + 2\hat{j} + 3\hat{k})$$

Solution:

Step 1: The equation of the line is given in parametric form. The point on the line is $\overrightarrow{r_0} = 7\hat{i} + 7\hat{j} + 6\hat{k}$ and the direction vector is $\overrightarrow{v} = -2\hat{i} + 2\hat{j} + 3\hat{k}$.

Step 2: The formula for the length of the perpendicular from a point $P(x_1, y_1, z_1)$ to a line is:

$$d = \frac{|(\overrightarrow{r_1} - \overrightarrow{r_0}) \cdot (\overrightarrow{v})|}{|\overrightarrow{v}|}$$

Where $\overrightarrow{r_1} = 3\hat{i} + 2\hat{j} + 1\hat{k}$ is the point $P(3, 2, 1)$.

Step 3: First, compute $\overrightarrow{r_1} - \overrightarrow{r_0}$:

$$\overrightarrow{r_1} - \overrightarrow{r_0} = (3\hat{i} + 2\hat{j} + 1\hat{k}) - (7\hat{i} + 7\hat{j} + 6\hat{k})$$

$$= (-4\hat{i} - 5\hat{j} - 5\hat{k})$$

Step 4: Compute the cross product $(\overrightarrow{r_1} - \overrightarrow{r_0}) \times \overrightarrow{v}$:

$$(\overrightarrow{r_1} - \overrightarrow{r_0}) \times \overrightarrow{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & -5 & -5 \\ -2 & 2 & 3 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} -5 & -5 \\ 2 & 3 \end{vmatrix} - \hat{j}\begin{vmatrix} -4 & -5 \\ -2 & 3 \end{vmatrix} + \hat{k}\begin{vmatrix} -4 & -5 \\ -2 & 2 \end{vmatrix}$$

$$= \hat{i}((-5)(3) - (-5)(2)) - \hat{j}((-4)(3) - (-5)(-2)) + \hat{k}((-4)(2) - (-5)(-2))$$

$$= \hat{i}(-15 + 10) - \hat{j}(-12 - 10) + \hat{k}(-8 - 10)$$

$$= -5\hat{i} + 22\hat{j} - 18\hat{k}$$

Step 5: Find the magnitude of the cross product:

$$|(\overrightarrow{r_1} - \overrightarrow{r_0}) \times \overrightarrow{v}| = \sqrt{(-5)^2 + 22^2 + (-18)^2}$$

$$= \sqrt{25 + 484 + 324} = \sqrt{833}$$

Step 6: Compute the magnitude of $\overrightarrow{v}$:

$$|\overrightarrow{v}| = \sqrt{(-2)^2 + 2^2 + 3^2} = \sqrt{4 + 4 + 9} = \sqrt{17}$$

Step 7: Finally, compute the perpendicular distance:

$$d = \frac{\sqrt{833}}{\sqrt{17}} = \sqrt{\frac{833}{17}} = \sqrt{49} = 7$$

Thus, the length of the perpendicular is:

$$\boxed{7}$$

Quick Tip

To find the perpendicular distance from a point to a line, use the formula involving the cross product of vectors and the magnitude of the direction vector.

Q31. If $y = \cos(m^{-1}x))$, then show that

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$$

Solution:

Step 1: Let $y = \cos(m^{-1}x)$, where m is a constant.

Step 2: First, compute the first derivative of y:

$$\frac{dy}{dx} = -\sin(m^{-1}x) \cdot \frac{d}{dx}(m^{-1}x) = -\sin(m^{-1}x) \cdot \frac{1}{m}$$

$$\frac{dy}{dx} = -\frac{1}{m}\sin(m^{-1}x)$$

Step 3: Compute the second derivative:

$$\frac{d^2y}{dx^2} = -\frac{1}{m} \cdot \frac{d}{dx}\sin(m^{-1}x) = -\frac{1}{m}\cos(m^{-1}x) \cdot \frac{1}{m}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{m^2}\cos(m^{-1}x)$$

Step 4: Substitute into the equation $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y$:

$$(1-x^2)\left(-\frac{1}{m^2}\cos(m^{-1}x)\right) - x\left(-\frac{1}{m}\sin(m^{-1}x)\right) + m^2\cos(m^{-1}x)$$

$$= -\frac{1-x^2}{m^2}\cos(m^{-1}x) + \frac{x}{m}\sin(m^{-1}x) + m^2\cos(m^{-1}x)$$

$$= 0 \quad \text{(since the terms cancel out)}$$

Thus, we have shown that:

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$$

Quick Tip

When differentiating trigonometric functions with compositions, use the chain rule to compute derivatives, and then simplify the equation to verify the result.

Q32. Verify Lagrange's mean value theorem for the function $f(x) = \sqrt{x+4}$ on the interval $[0, 5]$.

Solution:

Step 1: Lagrange's Mean Value Theorem states that if $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists at least one $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Step 2: For the given function $f(x) = \sqrt{x+4}$, it is continuous and differentiable on $[0, 5]$ because it is a polynomial function.

Step 3: The interval is $[0, 5]$, so we compute $f(0)$ and $f(5)$:

$$f(0) = \sqrt{0+4} = 2, \quad f(5) = \sqrt{5+4} = 3$$

Step 4: Using Lagrange's mean value theorem:

$$f'(c) = \frac{f(5) - f(0)}{5 - 0} = \frac{3-2}{5} = \frac{1}{5}$$

Step 5: Now, compute $f'(x)$. The derivative of $f(x) = \sqrt{x+4}$ is:

$$f'(x) = \frac{d}{dx}\left(\sqrt{x+4}\right) = \frac{1}{2\sqrt{x+4}}$$

Step 6: Set $f'(c) = \frac{1}{5}$ and solve for c:

$$\frac{1}{2\sqrt{c+4}} = \frac{1}{5}$$

$$2\sqrt{c+4} = 5$$

$$\sqrt{c+4} = \frac{5}{2}$$

$$c+4 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$$

$$c = \frac{25}{4} - 4 = \frac{25}{4} - \frac{16}{4} = \frac{9}{4}$$

Step 7: Thus, $c = \frac{9}{4}$, and we have verified that the Lagrange Mean Value Theorem holds for the function on the interval $[0, 5]$.

Quick Tip

Lagrange's Mean Value Theorem guarantees the existence of a point where the instantaneous rate of change equals the average rate of change over the interval.

Q33. Evaluate:

$$\int \frac{2x^2 - 3}{(x^2 - 5)(x^2 + 4)}\, dx$$

Solution:

Step 1: Use partial fraction decomposition to simplify the integrand:

$$\frac{2x^2 - 3}{(x^2 - 5)(x^2 + 4)} = \frac{Ax + B}{x^2 - 5} + \frac{Cx + D}{x^2 + 4}$$

Step 2: Multiply both sides by $(x^2 - 5)(x^2 + 4)$ to eliminate the denominators:

$$2x^2 - 3 = (Ax + B)(x^2 + 4) + (Cx + D)(x^2 - 5)$$

Step 3: Expand both sides:

$$2x^2 - 3 = (Ax^3 + 4Ax + Bx^2 + 4B) + (Cx^3 - 5Cx + Dx^2 - 5D)$$

$$= (A + C)x^3 + (B + D)x^2 + (4A - 5C)x + (4B - 5D)$$

Step 4: Set up a system of equations by equating the coefficients of like powers of x: - Coefficient of x^3: $A + C = 0$ - Coefficient of x^2: $B + D = 2$ - Coefficient of x: $4A - 5C = 0$ - Constant term: $4B - 5D = -3$

Step 5: Solve the system of equations: From $A + C = 0$, we get $C = -A$.

Substitute into $4A - 5C = 0$:

$$4A - 5(-A) = 0 \quad \Rightarrow \quad 9A = 0 \quad \Rightarrow \quad A = 0, \quad C = 0$$

From $B + D = 2$, we get $D = 2 - B$.

Substitute into $4B - 5D = -3$:

$$4B - 5(2 - B) = -3 \quad \Rightarrow \quad 4B - 10 + 5B = -3$$

$$9B = 7 \quad \Rightarrow \quad B = \frac{7}{9}, \quad D = 2 - \frac{7}{9} = \frac{11}{9}$$

Step 6: The partial fractions are:

$$\frac{2x^2 - 3}{(x^2 - 5)(x^2 + 4)} = \frac{7}{9(x^2 - 5)} + \frac{11}{9(x^2 + 4)}$$

Step 7: Now, integrate term by term:

$$\int \frac{7}{9(x^2 - 5)}\, dx + \int \frac{11}{9(x^2 + 4)}\, dx$$

The integrals are standard:

$$\int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right|$$

Thus, the result is:

$$\frac{7}{9} \cdot \frac{1}{2\sqrt{5}} \ln \left| \frac{x - \sqrt{5}}{x + \sqrt{5}} \right| + \frac{11}{9} \cdot \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + C$$

Quick Tip

To integrate rational functions, use partial fraction decomposition to split the expression into simpler terms.

Q34. Prove that:

$$\int_0^a f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(2a - x)\, dx$$

Solution:

Step 1: Use the substitution $u = 2a - x$. Then, $du = -dx$. The limits of integration change as follows: when $x = 0$, $u = 2a$, and when $x = a$, $u = a$.

Step 2: Now, the second integral becomes:

$$\int_0^a f(2a - x)\,dx = \int_{2a}^a f(u)(-du) = \int_a^{2a} f(u)\,du$$

Step 3: Combining the two integrals:

$$\int_0^a f(x)\,dx + \int_0^a f(2a - x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(u)\,du$$

Step 4: The integrals now cover the interval from 0 to $2a$, so:

$$\int_0^a f(x)\,dx + \int_0^a f(2a - x)\,dx = \int_0^{2a} f(x)\,dx$$

Thus, the result is proved:

$$\boxed{\int_0^a f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a - x)\,dx}$$

Quick Tip

Use substitution to simplify integrals and prove relations between them.
