NEST 2026

Question Paper with Solutions

Conducted by National Institute of Science Education and Research (NISER)

General Instructions

(i) The total marks for the examination is 240 marks to be attempted in 3 hours.

(ii) The question paper consists of 80 questions, divided into 4 sections- Biology, Chemistry, Mathematics, and Physics each having 20 questions.

(iii) Each correct answer carries 3 marks and an incorrect answer has a negative marking of 1.

(iv) Please note: The sequence of questions and options were differennt in the examination.

Biology

1. Mendel’s law of independent assortment implies that

(A) alleles segregate independent of one another during gamete formation.
(B) alleles segregate independent of their location on the chromosome during gamete formation.
(C) traits are encoded by independent genes.
(D) traits are always encoded by different chromosomes.

Correct Answer: (A) alleles segregate independent of one another during gamete formation.

Solution:

Step 1: Understanding the Question:

The question asks about the primary implication of Gregor Mendel’s second law, known as the

Law of Independent Assortment. This fundamental genetic principle describes how different genes and their corresponding alleles behave relative to one another during the process of gamete formation (meiosis).

Step 2: Detailed Explanation:

- The Law of Independent Assortment states that the alleles of two or more different genes sort into gametes independently of one another.
- In other words, the allele a gamete receives for one gene does not influence or depend on the allele received for another gene.
- This process occurs during Metaphase I of meiosis, where homologous chromosome pairs align randomly at the metaphase plate.
- Consequently, the segregation of one pair of alleles is completely independent of the segregation of another pair.
- Option B is incorrect because genetic linkage exists; alleles that are physically close on the same chromosome do not assort independently unless recombination occurs.
- Options C and D are incorrect because they mischaracterize the definition of traits and gene-chromosome relationships.

Step 3: Final Answer:

Therefore, the law of independent assortment implies that alleles of different genes segregate independently of one another during gamete formation.

Quick Tip: Remember that Mendel's law of independent assortment only strictly applies to genes located on different chromosomes or very far apart on the same chromosome. Physical linkage on a chromosome can restrict independent assortment.

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2. Bacteria have cell wall made of a polymer of N-acetylglucosamine and N-acetylmuramic acid connected by β-1,4 glycosidic bonds. Lysozyme, an enzyme, cleaves the glycosidic bond and disrupts the cell wall structure. In an experiment, Gram-positive and Gram-negative bacteria, which differ in terms of cell envelope and peptidoglycan thickness, are treated with lysozyme and subsequently placed in a hypotonic solution. The correct statement about the effect of the treatment is:

(A) Gram-positive bacteria are lysed, but not Gram-negative bacteria.
(B) Both Gram-positive and Gram-negative bacteria are lysed.
(C) Only Gram-negative bacteria are lysed.
(D) Neither Gram-positive nor Gram-negative bacteria are lysed.

Correct Answer: (A) Gram-positive bacteria are lysed, but not Gram-negative bacteria.

Solution:

Step 1: Understanding the Question:
The question describes the action of lysozyme on the β-1,4 glycosidic bonds in the peptidoglycan cell wall of bacteria. It asks for the outcome when Gram-positive and Gram-negative bacteria are treated with lysozyme and then placed in a hypotonic solution.

Step 2: Detailed Explanation:

- Lysozyme is an enzyme found in secretions like tears and saliva, acting as an antibacterial agent.

- It specifically targets and hydrolyzes the β-1,4 glycosidic bonds between N-acetylglucosamine (NAG) and N-acetylmuramic acid (NAM) in peptidoglycan.

- Gram-positive bacteria have a thick, outer peptidoglycan layer that is directly exposed to the external environment, making it highly susceptible to lysozyme digestion.

- Gram-negative bacteria, however, have a thin peptidoglycan layer that is protected by an outer membrane composed of lipopolysaccharides (LPS) and proteins.

- Because of this outer membrane barrier, lysozyme cannot easily access or degrade the peptidoglycan of Gram-negative bacteria unless the membrane is first disrupted.

- When Gram-positive bacteria are treated with lysozyme, their cell walls are destroyed, turning them into protoplasts that lack structural integrity.

- Upon subsequent transfer to a hypotonic solution, water rapidly enters the protoplasts due to osmosis, causing them to swell and lyse.

- In contrast, Gram-negative bacteria remain largely unaffected because their outer membrane prevents lysozyme from reaching the peptidoglycan layer.

Step 3: Final Answer:

Therefore, Gram-positive bacteria are lysed, but not Gram-negative bacteria.

Quick Tip: The presence of the outer membrane in Gram-negative bacteria serves as an effective barrier to many large macromolecules and enzymes, including lysozyme, explaining their resistance under typical experimental conditions.

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3. The fate of a cell in which pyruvate dehydrogenase is completely inhibited will be:

(A) Glycolysis continues with NAD^+ regeneration via lactate fermentation, yielding 2 ATP per glucose molecule.

(B) Pyruvate accumulates and enters the Krebs cycle via an alternative pathway, yielding 32 ATP per glucose molecule.

(C) Glycolysis stops due to NAD^+ depletion, and no ATP is produced.

(D) Pyruvate is converted to acetyl CoA, yielding 8 ATP per glucose molecule via fermentation.

Correct Answer: (A) Glycolysis continues with NAD^+ regeneration via lactate fermentation, yielding 2 ATP per glucose molecule.

Solution:

Step 1: Understanding the Question:

The question asks about the metabolic consequences of completely inhibiting the enzyme pyruvate dehydrogenase (PDH) in a cell.

Step 2: Detailed Explanation:

- Pyruvate dehydrogenase (PDH) is a key mitochondrial enzyme complex that catalyzes the link reaction.
- This reaction converts pyruvate (produced via glycolysis in the cytoplasm) into acetyl-coenzyme A (acetyl-CoA) in the mitochondrial matrix.
- Acetyl-CoA is the primary input molecule for the tricarboxylic acid (TCA) cycle (Krebs cycle).
- When PDH is completely inhibited, pyruvate cannot be converted into acetyl-CoA, effectively blocking the aerobic respiration pathway.
- As aerobic metabolism halts, the cell must rely solely on anaerobic glycolysis to produce

adenosine triphosphate (ATP).

- However, glycolysis requires a continuous supply of oxidized nicotinamide adenine dinucleotide (NAD^+) to convert glyceraldehyde 3-phosphate to 1,3-bisphosphoglycerate.

- Under aerobic conditions, NAD^+ is regenerated by the electron transport chain (ETC) in the mitochondria.

- Without PDH and aerobic respiration, the cell must regenerate NAD^+ anaerobically through lactate fermentation, where lactate dehydrogenase reduces pyruvate to lactate while oxidizing NADH back to NAD^+.

- This anaerobic pathway allows glycolysis to continue, producing a net yield of 2 ATP per glucose molecule.

Step 3: Final Answer:
Therefore, glycolysis continues with NAD^+ regeneration via lactate fermentation, yielding 2 ATP per glucose molecule.

Quick Tip: Lactate fermentation is crucial not for the energy in lactate itself, but to recycle NAD^+ so that the glycolysis pathway can continue yielding its modest 2 ATP per glucose under anaerobic or blocked conditions.

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4. Two spherical unicellular organisms, X and Y, have the same cytoplasmic composition, metabolic pathways, and live in identical nutrient-rich aerobic conditions. The diameters of X and Y are, respectively, 10 and 20 μm. When nutrients are not limiting, the ratio of total oxygen consumption per hour per cell of organisms Y to X is:

(A) 8:1

(B) 1:2

(C) 2:1

(D) 4:1

Correct Answer: (A) 8:1

Solution:

Step 1: Understanding the Question:

The question asks for the ratio of total oxygen consumption per hour between two spherical unicellular organisms, X and Y, with diameters of 10 μm and 20 μm respectively, under non-limiting nutrient and aerobic conditions.

Step 2: Key Formula or Approach:

The volume of a sphere is given by:

$$V = \frac{4}{3}\pi r^3 = \frac{1}{6}\pi d^3$$

The oxygen consumption rate of a metabolically active cell under non-limiting aerobic conditions is proportional to its total metabolic demand, which scales directly with cell volume (cytoplasmic mass) rather than surface area.

Step 3: Detailed Explanation:

- We are given two spherical cells, X and Y, with identical cytoplasmic composition and metabolic pathways.
- The diameter of cell X is $d_X = 10$ μm and the diameter of cell Y is $d_Y = 20$ μm.
- The volume of cell X is:
$$V_X = \frac{1}{6}\pi(10)^3 = \frac{1000\pi}{6}\ \mu\text{m}^3$$

- The volume of cell Y is:
$$V_Y = \frac{1}{6}\pi(20)^3 = \frac{8000\pi}{6}\ \mu\text{m}^3$$

- Since the cytoplasmic composition and metabolic pathways are identical, each unit volume of cytoplasm consumes oxygen at the same rate.

- Therefore, the total oxygen consumption per cell is directly proportional to the cell volume.

- The ratio of oxygen consumption of organism Y to organism X is:
$$\text{Ratio} = \frac{V_Y}{V_X} = \frac{8000}{1000} = 8:1$$

Step 4: Final Answer:

Therefore, the ratio of total oxygen consumption per hour per cell of organisms Y to X is 8:1.

Quick Tip: For any three-dimensional scaling problem of identical composition, volume scales as the cube of the linear dimension (d^3). Thus, doubling the diameter ($2\times$) increases the volume and overall metabolic capacity by $2^3 = 8$ times.

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5. In an experiment, bacteria were grown for 500 generations in a medium containing ^{14}N (light isotope), then transferred to a medium with ^{15}N (heavy isotope) for one generation, and finally transferred back to the ^{14}N medium for one more generation. Assuming all cells divide synchronously and replication is semi-conservative, the ratio of ^{14}N^{15}N to ^{14}N^{14}N double-stranded DNA at the end of the experiment is:

(A) 1:1

(B) 1:2

(C) 1:3

(D) 1:4

Correct Answer: (A) 1:1

Solution:

Step 1: Understanding the Question:

The question asks for the ratio of $^{14}N^{15}N$ (hybrid) to $^{14}N^{14}N$ (light) double-stranded DNA after a specific sequence of bacterial growth generations in different nitrogen isotopes.

Step 2: Key Formula or Approach:

Semiconservative replication means that during division, each of the two parental DNA strands serves as a template for the synthesis of a new complementary strand.

Step 3: Detailed Explanation:

- Initially, bacteria are grown for 500 generations in a ^{14}N medium. At this point, virtually 100% of the DNA strands are light (^{14}N), forming $^{14}N^{14}N$ double-stranded molecules. Let us denote a ^{14}N strand as L and a ^{15}N strand as H.

- These bacteria are then transferred to a ^{15}N medium for exactly one generation.

- During this replication cycle, the parent $L-L$ double helices separate. Each L strand template is paired with a newly synthesized heavy H strand. This results in 100% hybrid $L-H$ ($^{14}N^{15}N$) double-stranded DNA molecules.

- Finally, these bacteria are transferred back to a ^{14}N medium for one more generation.

- During this replication cycle, the hybrid $L-H$ parent molecules separate into individual L and H strands.

- Since the medium contains ^{14}N, all newly synthesized complementary strands will be light (L).

- The parental L strand pairs with a new L strand, forming an $L-L$ (^{14}N^{14}N) double-stranded molecule.

- The parental H strand pairs with a new L strand, forming an $L-H$ (^{14}N^{15}N) double-stranded molecule.

- Thus, for every starting hybrid molecule, we obtain one light molecule (^{14}N^{14}N) and one hybrid molecule (^{14}N^{15}N).

- The ratio of ^{14}N^{15}N to ^{14}N^{14}N double-stranded DNA molecules at the end of the experiment is therefore $1:1$.

Step 4: Final Answer:
Therefore, the ratio of ^{14}N^{15}N to ^{14}N^{14}N double-stranded DNA is 1:1.

Quick Tip: To quickly solve Meselson-Stahl type problems, keep track of the total number of individual strands. Any ^{15}N strand introduced in the second step will always remain in the population, but can only form hybrid DNA (^{14}N^{15}N) once the cells are returned to ^{14}N medium.

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6. In the roots of plants, casparian strip made of suberin forms a ring around the cells of the endodermis. It effectively blocks the entry of water and mineral ions into the stele via the apoplast. A defect in the casparian strip leads to

(A) increased solute leakage and reduced root hydraulic conductivity.

(B) decreased solute leakage and increased root hydraulic conductivity.

(C) increased solute leakage and increased root hydraulic conductivity.

(D) decreased solute leakage and reduced root hydraulic conductivity.

Correct Answer: (A) increased solute leakage and reduced root hydraulic conductivity.

Solution:

Step 1: Understanding the Question:

The question asks about the physiological consequences of a defect in the Casparian strip of plant roots.

Step 2: Detailed Explanation:

- The Casparian strip is a band of cell wall material deposited in the radial and transverse walls of the endodermis in plant roots.

- It is made of suberin, a hydrophobic, waxy substance that is impermeable to water and dissolved solutes.

- Physiologically, the Casparian strip blocks the apoplastic pathway, preventing water and mineral ions from diffusing passively into the vascular cylinder (stele).

- This forces all water and solutes to cross the selectively permeable plasma membranes of endodermal cells via the symplastic pathway.

- This mechanism allows the plant to selectively uptake nutrients and maintain a high concentration of solutes inside the stele, creating an osmotic gradient that drives root pressure.

- If the Casparian strip is defective, the barrier is compromised.

- Consequently, solutes accumulated in the stele can leak back into the cortex apoplast (increased solute leakage).

- Additionally, the root cannot maintain the osmotic pressure gradient required for the active transport and upward movement of water.

- This leads to a significant reduction in the root's hydraulic conductivity, which is its capacity to transport water from the soil to the xylem.

Step 3: Final Answer:
Therefore, a defect in the Casparian strip leads to increased solute leakage and reduced root hydraulic conductivity.

Quick Tip: Think of the Casparian strip as a "one-way check valve" for solutes in the root. If this valve leaks, solutes slip back out, and the root loses its ability to build up the osmotic pressure needed to draw water in, lowering hydraulic conductivity.

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7. The ability of phloem to distribute sugar depends on the companion cell and sieve element working as a complex. Consider the following statements with regard to possible effects of reduction in companion cell plasmodesmata.
(i) Sugar loading into sieve tubes reduces due to collapse in osmotic pressure gradient.
(ii) Water moves from companion cells into the xylem to maintain osmotic balance.
(iii) The sieve tubes collapse structurally due to loss of turgor pressure.
(iv) Reverse loading of sugar in phloem begins immediately due to pressure inversion.
Under the given condition, the correct statements are:

(A) (i) and (iii)

(B) (i) and (ii)

(C) (iii) and (iv)

(D) (ii) and (iv)

Correct Answer: (A) (i) and (iii)

Solution:

Step 1: Understanding the Question:

The question asks for the correct statements regarding the effects of reducing companion cell plasmodesmata on phloem sugar distribution.

Step 2: Detailed Explanation:

- Phloem transport is driven by a pressure-flow mechanism, where sugars are actively loaded into the sieve tubes at the source (leaves).
- This loading is facilitated by the companion cells, which are linked to the sieve elements via numerous specialized cytoplasmic channels called plasmodesmata, forming a functional sieve element-companion cell (SE-CC) complex.
- If there is a significant reduction in companion cell plasmodesmata, symplastic sugar transport into the sieve tubes is severely impaired.
- Consequently, sugar loading into the sieve tubes reduces drastically. This leads to a drop in solute concentration inside the sieve tube, preventing the establishment of the osmotic gradient needed to draw water from adjacent xylem.
- Thus, the osmotic pressure gradient collapses, confirming statement (i).
- Sieve tubes are structurally thin-walled living cells that rely heavily on internal turgor

pressure to maintain their shape and open channel structure.

- When turgor pressure is lost due to the lack of solute accumulation and water entry, the high external physical pressures in surrounding tissues cause the sieve tubes to collapse structurally, confirming statement (iii).

- Statement (ii) is incorrect because water would not move from companion cells to xylem to maintain balance under these circumstances.

- Statement (iv) is incorrect because reverse loading does not begin immediately; rather, the transport system simply breaks down due to structural collapse.

Step 3: Final Answer:
Therefore, the correct statements are (i) and (iii).

Quick Tip: Sieve elements lack nuclei and many organelles, making them dependent on companion cells. Turgor pressure is the "skeleton" of the sieve tubes; without sugar loading to drive osmosis, they lose turgor and structurally cave in.

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8. Carbonic anhydrase catalyzes the following reaction in red blood cells.

$$\mathbf{CO_2 + H_2O \rightleftharpoons H_2CO_3 \rightleftharpoons HCO_3^- + H^+}$$

This reaction acts as a buffering system in blood. Consequence of failure of the enzyme to catalyze the reverse reaction is:

(A) Blood becomes acidic and the affinity of haemoglobin for O_2 decreases.
(B) The pCO_2 increases and the affinity of haemoglobin for CO_2 increases.
(C) The pO_2 of the blood decreases and the affinity of haemoglobin for O_2 increases.

(D) Blood becomes acidic and the formation of carbamino-haemoglobin decreases.

Correct Answer: (A) Blood becomes acidic and the affinity of haemoglobin for O_2 decreases.

Solution:

Step 1: Understanding the Question:

This question asks for the physiological consequences if carbonic anhydrase fails to catalyze the reverse reaction in red blood cells.

Step 2: Detailed Explanation:

- Carbonic anhydrase is a highly efficient enzyme in red blood cells that catalyzes the reversible hydration of carbon dioxide:

$$CO_2 + H_2O \rightleftharpoons H_2CO_3 \rightleftharpoons HCO_3^- + H^+$$

- In systemic tissues, carbon dioxide produced by respiration enters the red blood cells and is converted into bicarbonate (HCO_3^-) and hydrogen ions (H^+).
- In the lungs, the enzyme must catalyze the reverse reaction to convert bicarbonate and hydrogen ions back into carbon dioxide and water so that CO_2 can be exhaled.
- If carbonic anhydrase fails to catalyze the reverse reaction, the conversion of HCO_3^- and H^+ back to CO_2 is severely hindered.
- As a result, hydrogen ions (H^+) accumulate in the blood, causing the pH of the blood to decrease significantly (the blood becomes acidic).

- According to the Bohr effect, a lower pH (acidic environment) stabilizes the T-state (deoxy-form) of hemoglobin, thereby reducing its affinity for oxygen (O_2) and facilitating oxygen release into tissues.

- This reduction in oxygen affinity at low pH is a direct consequence of the structural changes in hemoglobin triggered by proton binding.

Step 3: Final Answer:
Therefore, the consequence is that the blood becomes acidic and the affinity of hemoglobin for O_2 decreases.

Quick Tip: Remember the Bohr Effect: High H^+ (acidity) and high CO_2 shift the oxygen-hemoglobin dissociation curve to the right, which physically means hemoglobin has a decreased affinity for O_2.

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9. Events occurring during signal propagation in a chemical synapse are given below.
P: Postsynaptic membrane potential changes.
Q: Synaptic vesicle fuses with the presynaptic membrane.
R: Action potential depolarizes the presynaptic terminal.
S: Neurotransmitter binds to receptors on the postsynaptic membrane.
T: Neurotransmitter is released into the synaptic cleft.
The correct sequence of occurrence of these events is:

(A) R, Q, T, S, P
(B) Q, R, T, S, P
(C) R, T, Q, P, S
(D) Q, R, P, S, T

Correct Answer: (A) R, Q, T, S, P

Solution:

Step 1: Understanding the Question:

The question asks for the correct temporal sequence of events during chemical synaptic transmission.

Step 2: Detailed Explanation:

- Synaptic transmission is a highly coordinated process that allows neurons to communicate with each other using chemical messengers.
- The process begins when an action potential travels down the axon and reaches the presynaptic terminal, causing depolarization of the membrane (Event R).
- This depolarization opens voltage-gated calcium (Ca^{2+}) channels, allowing calcium ions to rush into the presynaptic terminal.
- The influx of Ca^{2+} triggers the fusion of neurotransmitter-containing synaptic vesicles with the presynaptic membrane (Event Q).
- Upon fusion, the neurotransmitter molecules are released into the synaptic cleft via exocytosis (Event T).
- The released neurotransmitter molecules diffuse across the narrow synaptic cleft and bind to specific receptor proteins located on the postsynaptic membrane (Event S).
- The binding of neurotransmitters to these receptors (which are often ligand-gated ion channels) causes them to open, leading to ion flow and a subsequent change in the postsynaptic membrane potential (Event P).

- Thus, the correct sequence of events is: R → Q → T → S → P.

Step 3: Final Answer:

Therefore, the correct sequence of occurrence of these events is R, Q, T, S, P.

Quick Tip: Always trace the physical pathway of the signal: electrical (action potential R) → calcium influx and vesicle fusion (Q) → release into cleft (T) → receptor binding (S) → postsynaptic electrical change (P).

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10. In a population of 1000 moths, 450 moths were brown-colored with homozygous genotype and 350 albino moths that were homozygous for the recessive allele. Assuming random mating within the population, the frequency of heterozygotes in the next generation would be:

(A) 0.495
(B) 0.200
(C) 0.800
(D) 0.505

Correct Answer: (A) 0.495

Solution:

Step 1: Understanding the Question:

The question asks for the frequency of heterozygotes in the next generation of a population of moths with given genotype numbers, assuming random mating.

Step 2: Key Formula or Approach:

Under the Hardy-Weinberg principle, allele frequencies in a population can be calculated from genotype numbers.

If p is the frequency of the dominant allele and q is the frequency of the recessive allele:

$$p = \frac{2 \times N_{AA} + N_{Aa}}{2N}$$

$$q = \frac{2 \times N_{aa} + N_{Aa}}{2N}$$

After one generation of random mating, the genotype frequencies will reach Hardy-Weinberg equilibrium, where the heterozygote frequency is given by:

$$f(Aa) = 2pq$$

Step 3: Detailed Explanation:

- We are given a population of $N = 1000$ moths.

- The number of homozygous dominant (brown-colored) moths is $N_{AA} = 450$.

- The number of homozygous recessive (albino) moths is $N_{aa} = 350$.

- The remaining moths are heterozygous, so:

$$N_{Aa} = 1000 - 450 - 350 = 200$$

- Let us calculate the allele frequencies:

$$p = f(A) = \frac{2(450) + 200}{2000} = \frac{900 + 200}{2000} = \frac{1100}{2000} = 0.55$$

$$q = f(a) = \frac{2(350) + 200}{2000} = \frac{700 + 200}{2000} = \frac{900}{2000} = 0.45$$

- Now, assuming random mating occurs, the population will establish Hardy-Weinberg equilibrium in the next generation.

- The expected frequency of heterozygous individuals (Aa) in the next generation is:

$$f(Aa) = 2pq = 2 \times 0.55 \times 0.45 = 0.495$$

Step 4: Final Answer:

Therefore, the frequency of heterozygotes in the next generation will be 0.495.

Quick Tip: Even if a population is not initially in Hardy-Weinberg equilibrium, just one generation of random mating is sufficient to bring the offspring generation into equilibrium. Always calculate allele frequencies from the parental gene pool first.

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11. In a haplodiploid ant species, sons are haploid and produced from unfertilized eggs, while daughters are diploid and produced from fertilized eggs. A single queen establishes a colony after mating with three different unrelated males. The queen uses the sperm from all three males equally to produce her female offspring. The average genetic relatedness between two randomly selected female offspring in this colony will be:

(A) 0.417
(B) 0.500
(C) 0.750
(D) 0.333

Correct Answer: (A) 0.417

Solution:

Step 1: Understanding the Question:
The question asks for the average genetic relatedness between two randomly selected female offspring in a haplodiploid ant colony where the queen mated with three unrelated males.

Step 2: Key Formula or Approach:
In haplodiploid species, females are diploid (receive 50% of their genome from the mother and 50% from the father), while males are haploid (receive 100% of their genome from the mother's unfertilized egg).
The genetic relatedness (r) between two female offspring can be calculated based on whether they share the same father (full sisters) or different fathers (half sisters).

Step 2: Detailed Explanation:

- Since the queen mates with three different unrelated males, and uses their sperm equally, any two randomly selected female offspring have a certain probability of being full sisters (sharing the same father) or half sisters (having different fathers).

- The probability that two randomly selected females share the same father is:

$$P(\text{same father}) = 3 \times \left(\frac{1}{3} \times \frac{1}{3}\right) = \frac{1}{3}$$

- The probability that they have different fathers is:

$$P(\text{different fathers}) = 1 - P(\text{same father}) = 1 - \frac{1}{3} = \frac{2}{3}$$

- Let us calculate the genetic relatedness (r) in both scenarios: 1. If they are full sisters (same father):

  - They share 50% of their genes from the mother (relatedness via mother = 0.25).
  - Since the father is haploid, he passes 100% of his genes to all of his daughters. Thus, they share 100% of their paternal genes (relatedness via father = 0.5).
  - Therefore, the relatedness between full sisters is:

$$r_{\text{full}} = 0.25 + 0.5 = 0.75$$

2. If they are half sisters (different fathers):
   - They share 50% of their genes from the mother (relatedness via mother = 0.25).
   - Since their fathers are different and unrelated, they share 0% of their paternal genes (relatedness via father = 0).
   - Therefore, the relatedness between half sisters is:

$$r_{\text{half}} = 0.25 + 0 = 0.25$$

- The average genetic relatedness (r) is the weighted average of these two scenarios:

$$r = P(\text{same father}) \times r_{\text{full}} + P(\text{different fathers}) \times r_{\text{half}}$$

$$r = \left(\frac{1}{3} \times 0.75\right) + \left(\frac{2}{3} \times 0.25\right)$$

$$r = 0.25 + \frac{0.5}{3} = 0.25 + 0.1667 \approx 0.417$$

Step 3: Final Answer:

Therefore, the average genetic relatedness between two randomly selected female offspring in this colony is 0.417.

Quick Tip: A quick way to compute this is: $r = \frac{1}{4} + \frac{1}{2}P(\text{same father})$. Since $P(\text{same father}) = 1/k$ (where k is the number of mates), $r = \frac{1}{4} + \frac{1}{2k}$. For $k = 3$, $r = 0.25 + 0.167 = 0.417$.

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12. In the axon of a giant squid neuron,

(i) The resting membrane potential is -70 mV.

(ii) At the equilibrium potential of Na^+ ($E_{Na^+} = +55$ mV) there is no net movement of Na^+ ions across the membrane.

In an experiment, when the axon is stimulated, the voltage-gated Na^+ channels open. The membrane potential peaks at +30 mV, where the resistance of the axonal membrane for Na^+ flow is $1 \times 10^6\ \Omega$. The net Na^+ current (I_{Na^+}) across the membrane and the direction of ionic movement at the peak is:

(A) $I_{Na^+} = 25$ nA; direction = into the cell

(B) $I_{Na^+} = 100$ nA; direction = out of the cell

(C) $I_{Na^+} = 25$ nA; direction = out of the cell

(D) $I_{Na^+} = 100$ nA; direction = into the cell

Correct Answer: (A) $I_{Na^+} = 25$ nA; direction = into the cell

Solution:

Step 1: Understanding the Question:

The question asks for the net sodium current (I_{Na^+}) and its direction at the peak of the action potential (+30 mV) in a giant squid axon, given the resting potential, the sodium equilibrium potential ($E_{Na^+} = +55$ mV), and the membrane resistance.

Step 2: Key Formula or Approach:

The ionic current (I) flowing through a membrane is given by Ohm's Law for ionic flow:

$$I = \frac{V_m - E_{\text{ion}}}{R}$$

where V_m is the membrane potential, E_{ion} is the equilibrium potential of the ion, and R is the membrane resistance.

Step 3: Detailed Explanation:

- We are given:
  - Membrane potential at the peak, $V_m = +30 \text{ mV} = +30 \times 10^{-3} \text{ V}$
  - Equilibrium potential of sodium, $E_{\text{Na}^+} = +55 \text{ mV} = +55 \times 10^{-3} \text{ V}$
  - Resistance of the membrane, $R = 1 \times 10^6 \ \Omega$
- The driving force acting on the sodium ions is:

$$V_m - E_{\text{Na}^+} = 30 \text{ mV} - 55 \text{ mV} = -25 \text{ mV} = -25 \times 10^{-3} \text{ V}$$

- Now, calculate the net sodium current (I_{Na^+}):

$$I_{\text{Na}^+} = \frac{V_m - E_{\text{Na}^+}}{R} = \frac{-25 \times 10^{-3} \text{ V}}{1 \times 10^6 \ \Omega} = -25 \times 10^{-9} \text{ A} = -25 \text{ nA}$$

- The magnitude of the current is 25 nA.

- The negative sign indicates that the current is inward (positive sodium ions flowing into the cell). This occurs because the membrane potential of +30 mV is less positive than the sodium equilibrium potential of +55 mV, creating an inward electrochemical gradient for Na^+.

Step 4: Final Answer:

Therefore, $I_{Na^+} = 25$ nA; direction = into the cell.

Quick Tip: Whenever the membrane potential (V_m) is more negative than the equilibrium potential (E_{ion}) of a cation (like Na^+), the driving force is negative, meaning the positive ions will flow inward (into the cell) to try to depolarize the cell toward E_{ion}.

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13. Consider a mitochondrion in which the F_0 subunit of ATP synthase is mutated, such that it can still passively conduct protons but is mechanically uncoupled from the F_1 subunit. If such a mitochondrion is provided with succinate, ADP, and Pi in aerobic conditions, the correct outcome based on chemiosmotic theory is:

(A) Continuous oxygen consumption and no ATP synthesis.
(B) No oxygen consumption and no ATP synthesis.
(C) Rapid initial oxygen consumption that plateaus at a low rate and normal ATP synthesis.
(D) Continuous oxygen consumption and normal ATP synthesis.

Correct Answer: (A) Continuous oxygen consumption and no ATP synthesis.

Solution:

Step 1: Understanding the Question:

This question asks for the physiological outcome in a mitochondrion with a mutated F_0 subunit of ATP synthase that can still conduct protons but is mechanically uncoupled from the F_1 catalytic subunit.

Step 2: Detailed Explanation:

- According to Peter Mitchell's chemiosmotic hypothesis, ATP synthesis in mitochondria is coupled to the transfer of electrons through the electron transport chain (ETC).

- As electrons are transferred, protons (H^+) are pumped from the mitochondrial matrix into the intermembrane space, creating a proton-motive force (an electrochemical proton gradient).

- Normally, these protons flow back into the matrix through the F_0 channel of the ATP synthase complex, driving the mechanical rotation of the stalk that triggers ATP synthesis in the F_1 subunit.

- In the given mutant, the F_0 subunit still allows passive proton flow, but it is mechanically uncoupled from F_1.

- Because the F_0 channel is open, protons can freely leak back into the matrix down their electrochemical gradient. This dissipates the proton-motive force.

- Since the proton gradient is constantly dissipated, there is no "back-pressure" on the electron transport chain. The ETC continues to function at its maximum rate to re-establish the gradient, consuming oxygen continuously as the terminal electron acceptor.

- However, because of the mechanical uncoupling between F_0 and F_1, the energy released by the proton flow cannot be used to synthesize ATP from ADP and P_i. Thus, no ATP is synthesized.

- This condition is functionally identical to the action of mitochondrial uncouplers like 2,4-dinitrophenol (DNP).

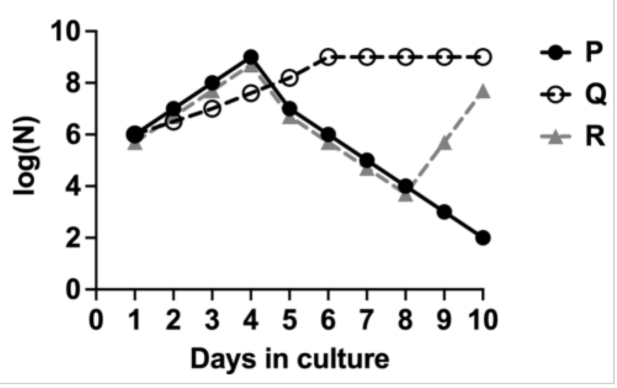
Step 3: Final Answer:

Therefore, the outcome is continuous oxygen consumption and no ATP synthesis.

Quick Tip: An uncoupled mitochondrion behaves like a short circuit in an electrical system. The flow of current (protons/electrons) is high, consuming fuel (oxygen) rapidly, but no useful work (ATP synthesis) is done.

---

14. Three different bacterial strains (P, Q, and R) when grown independently in a culture medium showed typical growth curves. Equal number of cells from all three actively growing cultures are mixed and grown together for 10 days. The number of bacteria (N) are counted each day and plotted.



Assuming nutrients are not limiting, the correct explanation for the observed trend is:

(A) Q is an antibiotic producer, P is antibiotic sensitive, and R is antibiotic sensitive but evolves resistance during the course of the experiment.

(B) Q is an antibiotic producer, P is antibiotic sensitive, and R is antibiotic resistant.

(C) R is an antibiotic producer, P is antibiotic sensitive, and Q is antibiotic resistant.

(D) P is an antibiotic producer, Q is antibiotic resistant, and R is antibiotic sensitive but evolves resistance during the course of the experiment.

Correct Answer: (A) Q is an antibiotic producer, P is antibiotic sensitive, and R is antibiotic sensitive but evolves resistance during the course of the experiment.

Solution:

Step 1: Understanding the Question:

This question asks to interpret the results of a competition experiment between three bacterial strains (P, Q, and R) grown together, based on a provided population graph.

Step 2: Detailed Explanation:

- Let us analyze the growth trajectories of the three strains over the 10-day period:
  - During the initial phase (Days 0 to 4), all three strains (P, Q, and R) show a typical increase in population size, demonstrating that they are viable in the growth medium.
  - After Day 4, the population of strain Q continues to rise and eventually stabilizes at a high carrying capacity.
  - In contrast, the populations of both P and R begin to decrease dramatically after Day 4. This simultaneous decline suggests that strain Q is producing a chemical substance, such as an antibiotic, that is toxic to the other strains.
  - Strain P continues to decline steadily throughout the remainder of the 10-day experiment, showing that it is highly sensitive to the antibiotic produced by Q.
  - Strain R also declines from Day 4 to Day 7. However, after Day 7, its population turns around and increases rapidly, showing a strong recovery.
  - Because the nutrients are not limiting, this recovery of R can be explained by the emergence of a mutation conferring resistance to the antibiotic produced by Q. This allows R to grow and replicate despite the presence of Q.

Step 3: Final Answer:

Therefore, Q is an antibiotic producer, P is antibiotic sensitive, and R is antibiotic sensitive but evolves resistance during the course of the experiment.

Quick Tip: A sharp decline followed by a subsequent rapid recovery in population size in the presence of a selective agent (like an antibiotic produced by a competitor) is a classic signature of the evolution of resistance via natural selection.

---

15. There is a difference between the predicted and observed global carbon budget. The possible reason is:

(A) Higher CO_2 levels stimulate plant growth and productivity, leading to more carbon being sequestered in plant tissues and root systems.
(B) Higher CO_2 levels lead to increased soil respiration rates, releasing more carbon into the atmosphere.
(C) Oceans are absorbing more carbon due to increased temperature, resulting in increased acidification.
(D) The thinning of the stratospheric ozone layer allows higher levels of ultraviolet radiation directly breaking down CO_2 molecules.

Correct Answer: (A) Higher CO_2 levels stimulate plant growth and productivity, leading to more carbon being sequestered in plant tissues and root systems.

Solution:

Step 1: Understanding the Question:
This question asks for the possible reason for the difference between the predicted and observed global carbon budget.

Step 2: Detailed Explanation:

- The global carbon budget describes the balance between carbon dioxide emissions and the uptake of carbon by carbon sinks (like oceans and land biosphere).

- Historically, there has been a discrepancy where the observed concentration of atmospheric CO_2 is lower than predicted by standard emission and absorption models. This discrepancy is known as the "missing carbon sink."
- A primary mechanism explaining this sink is the "CO_2 fertilization effect."
- Higher concentrations of atmospheric carbon dioxide stimulate the rate of photosynthesis in many plants (especially C3 plants), enhancing plant growth, forest productivity, and biomass accumulation.
- As a result, more carbon is sequestered in plant tissues, wood, and extensive root systems, as well as being transferred into the soil organic matter.
- This biological response acts as a negative feedback loop that temporarily mitigates the rise of atmospheric CO_2.
- Option B describes carbon release, which would increase atmospheric CO_2, not explain the missing sink.
- Option C describes ocean acidification, but warming actually decreases the solubility of CO_2 in water, reducing physical uptake.
- Option D is scientifically inaccurate because UV radiation does not efficiently photolyze CO_2 in the troposphere.

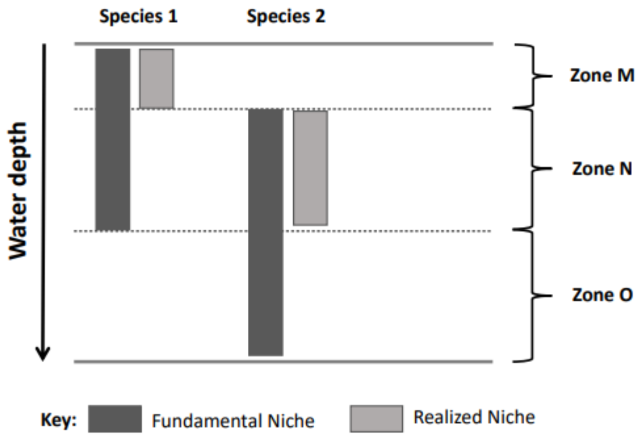
Step 3: Final Answer:

Therefore, the possible reason is that higher CO_2 levels stimulate plant growth and productivity, leading to more carbon being sequestered in plant tissues and root systems.

Quick Tip: The "missing carbon sink" is predominantly biological. Increased CO_2 acts as a fertilizer for plants, accelerating carbon capture through photosynthesis and leading to unexpected buffering of global temperature rises.

---

16. The fundamental niche of a species is the full range of environmental conditions and resources a species can potentially occupy in the absence of limitations. The realized niche is the actual, smaller subset of the fundamental niche that the species occupies due to biotic limitations. The distribution of populations of Species 1 and Species 2 over various levels towards the shore of an aquatic habitat is shown in the figure.



Consider the following statements based on the distribution.

(i) Competition for resources between Species 1 and Species 2 is mainly in the zones M and N.

(ii) Species 2 is a stronger competitor as compared to Species 1.

(iii) In absence of Species 2, Species 1 could prevail in zone O.

(iv) Species 2 probably has predators who thrive in zone O.

The correct statement(s) is(are):

(A) (ii) and (iv)

(B) (i) only

(C) (ii) only

(D) (i) and (iii)

Correct Answer: (A) (ii) and (iv)

Solution:

Step 1: Understanding the Question:
This question asks to identify the correct statements about the ecological niches of Species 1 and Species 2 based on a provided bar graph of their fundamental and realized niches.

Step 2: Detailed Explanation:

- Let us analyze the fundamental and realized niches shown in the schematic:
  - Species 1 has a fundamental niche spanning Zone M and Zone N, but its realized niche is restricted only to Zone M.
  - Species 2 has a fundamental niche spanning Zone N and Zone O, but its realized niche is restricted only to Zone N.
- Let us evaluate the statements:
  - Statement (i) is incorrect because the fundamental niches only overlap in Zone N, meaning any direct competition between them is primarily limited to Zone N.
  - Statement (ii) is correct because in Zone N (where their fundamental niches overlap), Species 2 successfully occupies its realized niche, while Species 1 is completely excluded. This demonstrates that Species 2 is a stronger competitor than Species 1.
  - Statement (iii) is incorrect because Zone O is entirely outside the fundamental niche of Species 1. Thus, even in the absence of Species 2, Species 1 cannot survive in Zone O.
  - Statement (iv) is correct because Species 2 has a fundamental niche in Zone O but is completely absent from this zone in its realized niche. Since Species 1 does not

live in Zone O, the exclusion of Species 2 must be due to another biotic factor, such as predators or other competitors that thrive in Zone O.

- Therefore, statements (ii) and (iv) are correct.

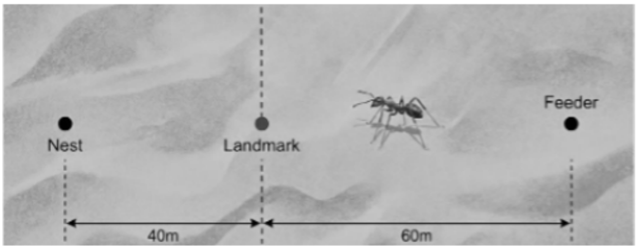
Step 3: Final Answer:

Therefore, the correct statements are (ii) and (iv).

Quick Tip: A species' realized niche is restricted compared to its fundamental niche by biotic interactions like competition (such as Species 2 excluding Species 1 from Zone N) or predation (such as predators excluding Species 2 from Zone O).

---

17. Desert ants are known to use landmarks (reference points), pedometers (memorising step counts), and optic flow (rate of passing objects across its eye to estimate navigation speed) to navigate across deserts. A desert ant is trained to walk 100 m from its nest to a feeder in an artificial arena as depicted in the schematic. The arena had a single tall landmark placed at 40 m from the nest.



At the feeder, certain manipulations were performed before allowing the ant to return to its nest with the food.

Manipulation 1: Stilts were attached to its legs, which doubled the step length.

Manipulation 2: The optic flow of the arena was doubled.

While the ant uses the landmark exclusively for long-distance (>55 m) navigation, it switches to optic flow and pedometer simultaneously and exclusively for short-distance navigation. Due to the manipulations, the distance from the feeder where the ant will search its nest is:

(A) 110 m

(B) 100 m

(C) 120 m

(D) 160 m

Correct Answer: (A) 110 m

Solution:

Step 1: Understanding the Question:

The question asks for the distance from the feeder where a trained desert ant will search for its nest, following two experimental manipulations (stilts that double step length and a doubled optic flow in the arena).

Step 2: Key Formula or Approach:

Under long-distance navigation (> 55 m), the ant relies exclusively on landmarks.

Under short-distance navigation (< 55 m), the ant integrates both optic flow and pedometer inputs simultaneously.

We need to calculate the actual distance traveled in each segment of the journey back to the nest.

Step 3: Detailed Explanation:

- The ant is at the feeder, located 100 m from the nest. The landmark is located 40 m from the nest (which is 60 m from the feeder).

- First, the ant begins its return journey from the feeder (at 100 m) to the nest (at 0 m).

- Since the initial distance to the landmark is $100 - 40 = 60$ m, which is greater than 55 m, the ant uses the landmark exclusively for long-distance navigation.

- Thus, it ignores the altered step length and doubled optic flow during this phase, and successfully reaches the landmark, covering exactly 60 m.

- Now, the ant is at the landmark (40 m from the nest). The remaining distance to the nest is 40 m, which is less than 55 m.

- For this short-distance navigation phase, the ant switches to using both the pedometer and optic flow simultaneously.

- Let us analyze the effect of each cue:
  - With stilts, the ant's step length is doubled. To cover the 40 m distance based on its internal step counter (which expects a normal step length), the ant would have to take the number of steps that would normally cover 40 m. Since each step is now twice as long, this pedometer cue on its own would cause the ant to walk $40 \times 2 = 80$ m.

  - The optic flow of the arena is doubled. This means the visual system registers double the normal motion per unit distance. To perceive a visual travel of 40 m, the ant only needs to physically move $40/2 = 20$ m.

- Since the ant integrates these two sensory inputs simultaneously (by averaging the two estimates), the distance it travels after the landmark is:

$$\text{Distance after landmark} = \frac{80\text{ m} + 20\text{ m}}{2} = 50\text{ m}$$

- Thus, the total distance traveled from the feeder is:

$$\text{Total distance} = 60\text{ m} + 50\text{ m} = 110\text{ m}$$

Step 4: Final Answer:

Therefore, the distance from the feeder where the ant will search for its nest is 110 m.

Quick Tip: Break down multi-cue navigation experiments into distinct physiological phases. Identify which cue dominates in which phase, calculate the contribution of each manipulated cue, and then combine them as specified.

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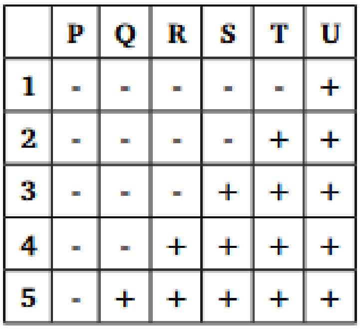
18. The table shows the presence (+) / absence (-) of features 1 - 5 for animals P - U.



	P	Q	R	S	T	U
1	-	-	-	-	-	+
2	-	-	-	-	+	+
3	-	-	-	+	+	+
4	-	-	+	+	+	+
5	-	+	+	+	+	+

Based on the table, the correct statement is:

(A) If Q and R represent lamprey and fish respectively, then feature 4 could be jaws.

(B) Feature 5 is the most recently evolved trait.

(C) If S, T and U are respectively amphibian, reptile and mammal, then feature 3 could be amniotic egg.

(D) If S, T and U respectively are salamander, turtle and leopard, then features 1 and 2 could respectively represent hair and four legs.

Correct Answer: (A) If Q and R represent lamprey and fish respectively, then feature 4 could be jaws.

Solution:

Step 1: Understanding the Question:

This question asks to identify the correct evolutionary statement based on a character matrix showing the presence (+) or absence (-) of five morphological features across six animal taxa (P to U).

Step 2: Detailed Explanation:

- Let us analyze the character matrix to reconstruct the order of trait evolution:
  - Taxon P has none of the features (all -). This represents the outgroup.
  - Taxon Q has only feature 5 (+), while all other features are absent (-).
  - Taxon R has features 4 and 5 (+).
  - Taxon S has features 3, 4, and 5 (+).
  - Taxon T has features 2, 3, 4, and 5 (+).
  - Taxon U has all five features (+).
- From this nested pattern, we can infer that the evolutionary sequence of characters is: Feature 5 → Feature 4 → Feature 3 → Feature 2 → Feature 1.
- Let us evaluate the options:
  - In Option A, Q represents a lamprey (a jawless vertebrate) and R represents a jawed fish. Lampreys have a vertebral column (Feature 5) but lack jaws. Gnathostomes (jawed vertebrates, starting with fish, R) possess jaws. Thus, Feature 4, which is present in R, S, T, U but absent in Q, perfectly represents the evolution of jaws. This is a highly consistent evolutionary scenario.
  - Option B is incorrect because Feature 5 is present in almost all taxa, indicating it is the most ancestral trait, not the most recently evolved.
  - Option C is incorrect because if S is an amphibian and T is a reptile, then Feature 3 (present in S) cannot be the amniotic egg, as amphibians are anamniotes and do

not lay amniotic eggs.

- Option D is incorrect because salamanders (S) do not have four legs and hair represented by features 1 and 2 in that order.

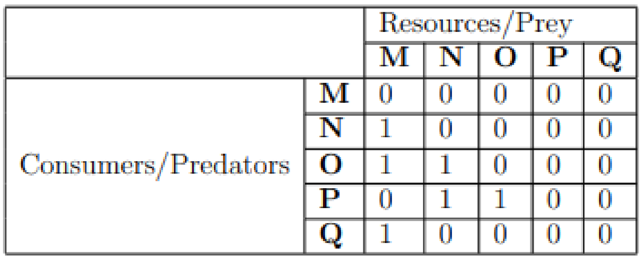
Step 3: Final Answer:

Therefore, if Q and R represent lamprey and fish respectively, then feature 4 could be jaws.

Quick Tip: In cladistics, shared derived characters (synapomorphies) are nested. A trait present in many taxa (like Feature 5) is ancestral (plesiomorphic), while a trait unique to a terminal taxon (like Feature 1) is recently derived (apomorphic).

---

19. The trophic relationships between species in a habitat can be represented in the form of a food web matrix where, rows represent consumers/ predators, and columns represent resources/prey. If the organism of a species in the row eats the organism of a particular species in the column, then a '1' is placed in the intersection. Otherwise a '0' is placed in the intersection. Such a matrix for species M - Q is given below.



		Resources/Prey				
		M	**N**	**O**	**P**	**Q**
Consumers/Predators	**M**	0	0	0	0	0
	N	1	0	0	0	0
	O	1	1	0	0	0
	P	0	1	1	0	0
	Q	1	0	0	0	0

Based on the matrix, the correct statement is:

(A) Species P is both a secondary and tertiary consumer.

(B) Scarcity of species M can directly affect primary consumers O and P.

(C) There are two producer species in this ecosystem.

(D) Species N is an omnivore.

Correct Answer: (A) Species P is both a secondary and tertiary consumer.

Solution:

Step 1: Understanding the Question:

This question asks to identify the correct statement about the trophic relationships in a food web using a provided resource-consumer interaction matrix.

Step 2: Detailed Explanation:

- Let us analyze the rows (consumers) and columns (resources/prey) of the given matrix:
  - A value of '1' at the intersection of a row and column means the species in the row eats the species in the column.
  - Species M has '0' across all columns, meaning it does not consume any other species in this matrix. Thus, Species M is a primary producer.
  - Species N consumes only Species M (indicated by '1' in the M column). Since it eats a producer, Species N is a primary consumer (herbivore).
  - Species O consumes both Species M (producer) and Species N (primary consumer). Since it eats both a producer and a consumer, Species O acts as both a primary consumer and a secondary consumer (omnivore).
  - Species P consumes Species N (primary consumer) and Species O (which can act as a secondary consumer). When P eats N, P acts as a secondary consumer. When P eats O (acting as a secondary consumer), P acts as a tertiary consumer. Thus, Species P is both a secondary and a tertiary consumer.
  - Species Q consumes only Species M, making it a primary consumer.
- Let us evaluate the options:
  - Option A matches our deduction that Species P acts as both a secondary and a

tertiary consumer.

- Option B is incorrect because Species P does not consume Species M directly.
- Option C is incorrect because only Species M acts as a producer (all 0s).
- Option D is incorrect because Species N only eats M, making it a specialist herbivore, not an omnivore.

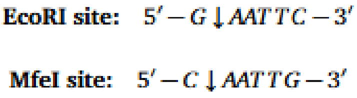
Step 3: Final Answer:

Therefore, Species P is both a secondary and tertiary consumer.

Quick Tip: To find the trophic level of any consumer, write out all the possible pathways starting from the producer (M): Pathway 1: $M \to N \to P$ (P is a secondary consumer) Pathway 2: $M \to N \to O \to P$ (P is a tertiary consumer).

---

20. A researcher wants to insert a linear DNA fragment into a plasmid DNA. The linear DNA fragment has two MfeI restriction sites at its ends. The plasmid DNA has only one EcoRI restriction site. The plasmid was digested with EcoRI and the DNA fragment was digested with MfeI. The recognition sequences of the restriction enzymes on one strand are given.



EcoRI site: $5' - G \downarrow AATTC - 3'$

MfeI site: $5' - C \downarrow AATTG - 3'$

(arrow indicates enzyme cut sites)

Based on this information, the DNA fragments generated by EcoRI and MfeI

(A) can ligate to each other, but the resultant product can neither be digested by EcoRI nor MfeI.

(B) cannot ligate to each other because the overhangs generated are from two different enzymes.

(C) can ligate to each other, and the resultant product can be digested by EcoRI but not MfeI.

(D) can ligate to each other, and the resultant product can be digested by MfeI but not EcoRI.

Correct Answer: (A) can ligate to each other, but the resultant product can neither be digested by EcoRI nor MfeI.

Solution:

Step 1: Understanding the Question:
The question asks about the feasibility of ligating DNA fragments generated by digesting a linear DNA fragment with MfeI and a plasmid DNA with EcoRI, and the properties of the resulting recombinant product.

Step 2: Key Formula or Approach:
Restriction enzymes EcoRI and MfeI recognize palindromic sequences and cut them as shown:

- EcoRI site: $5'-G\downarrow AATTC-3'$
- MfeI site: $5'-C\downarrow AATTG-3'$

We need to determine the single-stranded overhangs (sticky ends) produced by these enzymes and see if they can base-pair and ligate.

Step 3: Detailed Explanation:

- Let us analyze the cleavage of the EcoRI site:

$$5'-G \quad AATTC-3' \rightarrow 5'-G-3' \text{ and } 5'-AATTC-3'$$

  This leaves a $5'$ single-stranded overhang of $5'-AATT-3'$.

- Let us analyze the cleavage of the MfeI site:

$$5'-C \quad AATTG-3' \rightarrow 5'-C-3' \text{ and } 5'-AATTG-3'$$

  This also leaves a $5'$ single-stranded overhang of $5'-AATT-3'$.

- Because both EcoRI and MfeI generate identical single-stranded overhangs

($5' - AATT - 3'$), these cohesive ends are fully compatible and can easily hybridize and ligate to each other in the presence of DNA ligase.

- Let us look at the sequence at the junction after ligation:
  - On one strand, the EcoRI end $5' - G - 3'$ is joined to the MfeI end $5' - AATTG - 3'$, forming the sequence:
  $$5' - GAATTG - 3'$$
  - The complementary strand will have the sequence:
  $$3' - CTTAAC - 5' \quad \text{which is} \quad 5' - CAATTC - 3'$$
- The resulting hybrid junction sequence is $5' - GAATTG - 3'$.
- This sequence is neither recognized by EcoRI (which requires $5' - GAATTC - 3'$) nor by MfeI (which requires $5' - CAATTG - 3'$).
- Therefore, once ligated, the junction is destroyed and can neither be cleaved by EcoRI nor by MfeI.

Step 4: Final Answer:

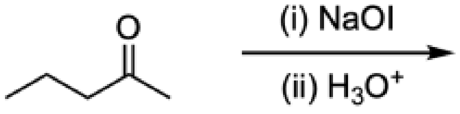
Therefore, they can ligate to each other, but the resultant product can neither be digested by EcoRI nor MfeI.

Quick Tip: Restriction enzymes that produce identical overhangs (isocaudomers) can produce compatible sticky ends that ligate together. However, the resulting hybrid site is almost always asymmetrical and cannot be cut by either of the original enzymes.

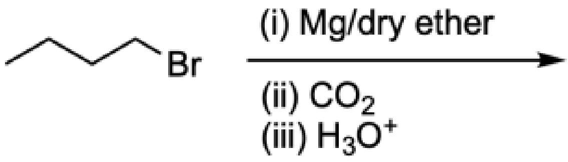
Chemistry

21. The reaction that produces butanoic acid as a product is:

A.

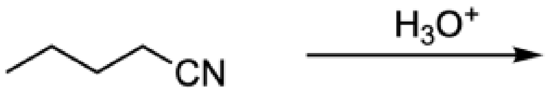


B.



C.

D.



(A) A

(B) B

(C) C

(D) D

Correct Answer: (A) A

Solution:

Step 1: Understanding the Question:

The question asks us to identify which of the given synthetic organic pathways results in the formation of butanoic acid ($CH_3CH_2CH_2COOH$) as a product.

Step 2: Key Formula or Approach:

The key to solving this problem lies in analyzing the carbon skeleton changes for each reaction type:

1. Haloform Reaction: Converts a methyl ketone (R-CO-CH_3) to a carboxylic acid with one less carbon (R-COOH) and haloform (CHX_3).

2. Grignard Carbonation: Converts an alkyl halide (R-X) to a carboxylic acid with one additional carbon (R-COOH).

3. Ester Hydrolysis: Hydrolyzes an ester (R-COOR') to form R-COOH and R'-OH.

4. Nitrile Hydrolysis: Hydrolyzes a nitrile (R-CN) to a carboxylic acid with the same number of carbons (R-COOH).

Step 3: Detailed Explanation:

- Let us evaluate each option systematically to track the carbon count of the final products.
- In Option A, the starting material is pentan-2-one, which is a methyl ketone with five carbon atoms:

$$CH_3-CH_2-CH_2-CO-CH_3$$

- Treatment of pentan-2-one with sodium hypoiodite (NaOI) triggers the haloform reaction.
- The active reagent iodinates the methyl group, which is then cleaved by the hydroxide ion to yield iodoform (CHI_3) and a carboxylate ion with four carbons:

$$CH_3-CH_2-CH_2-COO^-$$

- Subsequent protonation with hydronium ions (H_3O^+) produces butanoic acid:

$$CH_3 - CH_2 - CH_2 - COOH$$

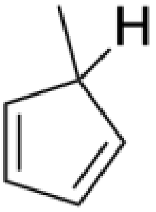
- In Option B, 1-bromobutane (4 carbons) reacts with Mg to form butylmagnesium bromide. Treatment with CO_2 followed by hydrolysis introduces a new carboxyl carbon, yielding pentanoic acid (5 carbons), not butanoic acid.
- In Option C, butyl acetate on acid hydrolysis yields acetic acid (2 carbons) and butanol (4 carbons), not butanoic acid.
- In Option D, pentanenitrile (5 carbons) on complete acid hydrolysis yields pentanoic acid (5 carbons).

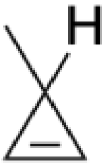
Step 4: Final Answer:
Therefore, only the haloform reaction of pentan-2-one (Option A) yields butanoic acid.

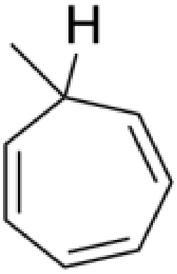
Quick Tip: The haloform reaction is a valuable synthetic tool to shorten a carbon chain by exactly one carbon, converting a methyl ketone into a carboxylic acid.

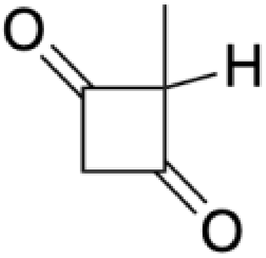
---

22. The compound that produces aromatic species upon deprotonation of the indicated hydrogen is:

A. 

B. 

C. 

D. 

(A) 5-methylcyclopentadiene

(B) 3-methylcyclopropene

(C) 7-methylcycloheptatriene

(D) cyclobutane-1,3-dione derivative

Correct Answer: (A) 5-methylcyclopentadiene

Solution:

Step 1: Understanding the Question:

The question asks us to identify which of the provided structures becomes aromatic after losing the specific proton (H^+) indicated in the drawing.

Step 2: Key Formula or Approach:

We must apply Huckel's rule of aromaticity to the conjugate bases formed after deprotonation. According to Huckel's rule, a cyclic, planar, completely conjugated system is aromatic if it contains:

$$(4n+2)\pi \text{ electrons (where } n = 0, 1, 2, 3, \dots)$$

It is antiaromatic if it contains $4n\pi$ electrons.

Step 3: Detailed Explanation:

- Let us analyze the deprotonation of the indicated sp^3 hybridized carbon in each option.
- In Option A, 5-methylcyclopentadiene contains a CH(Me) group situated between two double bonds.
- Deprotonation at this position removes the proton (H^+), converting the carbon from sp^3 to sp^2 hybridization and leaving behind a lone pair of electrons in a p-orbital.
- This creates the 5-methylcyclopentadienyl anion, which is a planar, monocyclic ring with complete conjugation.
- The total number of π electrons in this conjugated loop is:

  $$2 \text{ (from first C=C)} + 2 \text{ (from second C=C)} + 2 \text{ (from the negative charge/lone pair)} = 6\pi \text{ electrons}$$

- Since 6 is a Huckel number ($4n+2$ with $n = 1$), the resulting anion is highly stable and aromatic.

- In Option B, deprotonation of the cyclopropene derivative yields a cyclopropenyl anion with 4π electrons, which is antiaromatic and highly unstable.

- In Option C, deprotonation of the cycloheptatriene derivative yields a cycloheptatrienyl anion with 8π electrons, which is also antiaromatic.

- In Option D, the dicarbonyl derivative contains no continuous conjugated ring system that can form a planar aromatic ring of Huckel size.

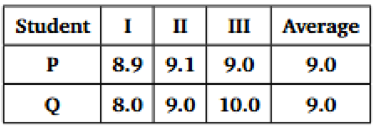
Step 4: Final Answer:

Therefore, only the 5-methylcyclopentadiene derivative (Option A) produces an aromatic anion upon deprotonation.

Quick Tip: The cyclopentadienyl anion is one of the classic examples of non-benzenoid aromatic systems. Its conjugate acid (cyclopentadiene) has an unusually low pK_a (≈ 16) for a hydrocarbon because of this aromatic stabilization of the conjugate base.

---

23. Two students (P and Q) independently made three measurements (denoted by I, II, and III) of the same property. The expected correct value is 9.0. The observed values by the students are provided in the given table.



Student	I	II	III	Average
P	8.9	9.1	9.0	9.0
Q	8.0	9.0	10.0	9.0

Based on the given data, the correct statement is:

(A) Both P and Q are equally accurate but P is more precise than Q.

(B) P is more accurate but less precise than Q.

(C) P is less accurate but more precise than Q.

(D) Both P and Q are equally precise but P is more accurate than Q.

Correct Answer: (A) Both P and Q are equally accurate but P is more precise than Q.

Solution:

Step 1: Understanding the Question:

The question asks us to compare the accuracy and precision of the experimental measurements made by two different students, P and Q, against a true value of 9.0.

Step 2: Key Formula or Approach:

1. Accuracy refers to how close the average of the measurements is to the true/expected value:

$$\text{Accuracy} \propto \frac{1}{|\text{Average} - \text{True Value}|}$$

2. Precision refers to how close the individual measurements are to one another, which is reflected by the range, standard deviation, or spread:

$$\text{Spread} = \text{Maximum Value} - \text{Minimum Value}$$

Step 3: Detailed Explanation:

- Let us first evaluate the accuracy of both students' datasets.

- The true expected value is 9.0.

- The average value calculated for Student P's measurements is:

$$\text{Average}_P = \frac{8.9 + 9.1 + 9.0}{3} = 9.0$$

- The average value calculated for Student Q's measurements is:

$$\text{Average}_Q = \frac{8.0+9.0+10.0}{3} = 9.0$$

- Since the average values for both students are exactly equal to the true value (9.0), both students are equally accurate.

- Next, let us evaluate the precision of both students' datasets by looking at the spread of their measurements.

- For Student P, the individual values are 8.9, 9.1, and 9.0. The range is:

$$\text{Range}_P = 9.1 - 8.9 = 0.2$$

- For Student Q, the individual values are 8.0, 9.0, and 10.0. The range is:

$$\text{Range}_Q = 10.0 - 8.0 = 2.0$$

- Since the spread/range of Student P's measurements (0.2) is much smaller than that of Student Q's (2.0), the individual measurements of P are much closer to one another.

- This indicates that Student P has a much higher level of precision.

Step 4: Final Answer:

Therefore, both P and Q are equally accurate, but P is more precise than Q.

Quick Tip: Remember: Accuracy is about hitting the bullseye on average. Precision is about having a tight cluster of arrows, regardless of where they land.

---

24. Species that are both isoelectronic and isostructural are:

(A) NH_4^+ and BH_4^-
(B) O_3 and NO_2^+
(C) N_2O and NO_2
(D) NH_2^- and BH_4^-

Correct Answer: (A) NH_4^+ and BH_4^-

Solution:

Step 1: Understanding the Question:
The question asks us to identify a pair of chemical species that are both isoelectronic (having the same number of electrons) and isostructural (having the same spatial molecular geometry).

Step 2: Key Formula or Approach:
1. Two species are isoelectronic if they have the same total number of electrons (or valence electrons).
2. Two species are isostructural if they have the same molecular shape and geometry as predicted by VSEPR theory.

Step 3: Detailed Explanation:

- Let us analyze Option A containing the ammonium ion (NH_4^+) and the tetrahydroborate ion (BH_4^-).

- Let us calculate the total number of electrons in each species:

  - For NH_4^+: Nitrogen has 7 electrons, and each of the 4 Hydrogens has 1 electron. Subtracting 1 electron for the positive charge:

$$\text{Total Electrons} = 7 + (4 \times 1) - 1 = 10 \text{ electrons}$$

  - For BH_4^-: Boron has 5 electrons, and each of the 4 Hydrogens has 1 electron. Adding 1 electron for the negative charge:

$$\text{Total Electrons} = 5 + (4 \times 1) + 1 = 10 \text{ electrons}$$

- Since both species have 10 electrons, they are isoelectronic.

- Now let us evaluate their molecular geometry using VSEPR theory:

  - The central nitrogen in NH_4^+ has 4 bonding pairs of electrons and 0 lone pairs, giving a steric number of 4, which corresponds to sp^3 hybridization and a tetrahedral shape.

  - The central boron in BH_4^- has 4 bonding pairs of electrons and 0 lone pairs, also giving a steric number of 4, corresponding to sp^3 hybridization and a tetrahedral shape.

- Since both have a tetrahedral geometry, they are isostructural.

- Let us briefly check the other options to verify:

  - In Option B, O_3 is bent (18 valence electrons), while NO_2^+ is linear (16 valence electrons). They are neither isoelectronic nor isostructural.

- In Option C, N_2O is linear (16 valence electrons), while NO_2 is bent (17 valence electrons).

- In Option D, NH_2^- is bent, while BH_4^- is tetrahedral.

Step 4: Final Answer:

Therefore, NH_4^+ and BH_4^- are both isoelectronic and isostructural.

Quick Tip: Species that have the same central atom group and identical ligand arrangement (like CH_4, NH_4^+, and BH_4^-) are generally isoelectronic and isostructural with each other.

---

25. Two galvanic cells, Cell I and Cell II, operate at the same temperature and involve two different redox systems. It is observed that at a particular value of the reaction quotient $Q_I = Q_{II} = Q_0$, the ratio of EMF of the two cells (E_I/E_{II}) as well as the ratio of change of Gibbs free energy for the two cells ($\Delta G_I/\Delta G_{II}$) are both 1/3. Assuming standard cell reactions written in the spontaneous direction and reversible operation, the pair of cells can satisfy these conditions is:

(A) Cell I: $Zn(s)|Zn^{2+}(aq)||Ag^+(aq)|Ag(s)$ and Cell II: $Zn(s)|Zn^{2+}(aq)||Cu^{2+}(aq)|Cu(s)$
(B) Cell I: $Ag^+(aq)|Ag(s)||Fe^{2+}(aq)|Fe^{3+}(aq)$ and Cell II: $Al(s)|Al^{3+}(aq)||Ag^+(aq)|Ag(s)$
(C) Cell I: $Zn(s)|Zn^{2+}(aq)||Cu^{2+}(aq)|Cu(s)$ and Cell II: $Al(s)|Al^{3+}(aq)||Ag^+(aq)|Ag(s)$
(D) Cell I: $Fe^{2+}(aq)|Fe^{3+}(aq)||Cu^{2+}(aq)|Cu^+(aq)$ and Cell II: $Cu(s)|Cu^{2+}(aq)||Ag^+(aq)|Ag(s)$

Correct Answer: (A) Cell I: $Zn(s)|Zn^{2+}(aq)||Ag^+(aq)|Ag(s)$ and Cell II: $Zn(s)|Zn^{2+}(aq)||Cu^{2+}(aq)|Cu(s)$

Solution:

Step 1: Understanding the Question:

The question asks us to identify which pair of galvanic cells satisfies the condition that at a particular reaction quotient, both the ratio of their cell potentials (E_I/E_{II}) and the ratio of their

Gibbs free energy changes ($\Delta G_I/\Delta G_{II}$) are exactly equal to 1/3.

Step 2: Key Formula or Approach:

The relationship between the Gibbs free energy change (ΔG) of a cell reaction and its electromotive force (EMF, E) is given by:

$$\Delta G = -nFE$$

where n is the number of moles of electrons transferred in the spontaneous cell reaction and F is Faraday's constant.

Step 3: Detailed Explanation:

- Let us express the ratio of the Gibbs free energy changes of the two cells:

$$\frac{\Delta G_I}{\Delta G_{II}} = \frac{-n_I F E_I}{-n_{II} F E_{II}} = \frac{n_I E_I}{n_{II} E_{II}}$$

- We are given that at $Q_I = Q_{II} = Q_0$, the following ratios hold:

$$\frac{E_I}{E_{II}} = \frac{1}{3} \quad \text{and} \quad \frac{\Delta G_I}{\Delta G_{II}} = \frac{1}{3}$$

- Substituting these values into our relation:

$$\frac{1}{3} = \frac{n_I}{n_{II}} \times \frac{1}{3} \implies \frac{n_I}{n_{II}} = 1 \implies n_I = n_{II}$$

- Thus, for this condition to hold, both cell reactions must involve the transfer of the exact same number of electrons ($n_I = n_{II}$) when written in their balanced, spontaneous forms.

- Let us examine the cells in Option A:
  - Cell I: $\mathrm{Zn(s)|Zn^{2+}(aq)||Ag^{+}(aq)|Ag(s)}$
    Spontaneous reaction: $\mathrm{Zn(s) + 2Ag^{+}(aq) \rightarrow Zn^{2+}(aq) + 2Ag(s)}$, which involves the transfer of $n_{\mathrm{I}} = 2$ electrons.
  - Cell II: $\mathrm{Zn(s)|Zn^{2+}(aq)||Cu^{2+}(aq)|Cu(s)}$
    Spontaneous reaction: $\mathrm{Zn(s) + Cu^{2+}(aq) \rightarrow Zn^{2+}(aq) + Cu(s)}$, which involves the transfer of $n_{\mathrm{II}} = 2$ electrons.
- Here, $n_{\mathrm{I}} = n_{\mathrm{II}} = 2$, which perfectly satisfies our condition.
- Let us check the other options:
  - In Option B: Cell I has $n_{\mathrm{I}} = 1$ ($\mathrm{Ag^{+}}$ to Ag and $\mathrm{Fe^{2+}}$ to $\mathrm{Fe^{3+}}$), and Cell II has $n_{\mathrm{II}} = 3$ (Al to $\mathrm{Al^{3+}}$). Here, $n_{\mathrm{I}} \neq n_{\mathrm{II}}$.
  - In Option C: Cell I has $n_{\mathrm{I}} = 2$, and Cell II has $n_{\mathrm{II}} = 3$.
  - In Option D: Cell I has $n_{\mathrm{I}} = 1$ ($\mathrm{Fe^{2+}}$ to $\mathrm{Fe^{3+}}$ and $\mathrm{Cu^{2+}}$ to $\mathrm{Cu^{+}}$), and Cell II has $n_{\mathrm{II}} = 2$.

Step 4: Final Answer:

Therefore, only the pair of cells in Option A satisfies the given relationship.

Quick Tip: If the ratio of energy (ΔG) and the ratio of driving force (E) are identical, then the scaling factor n (electrons transferred) must cancel out, meaning n must be equal for both processes.

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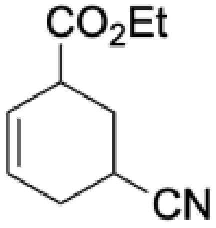
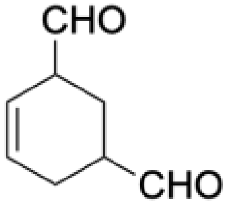
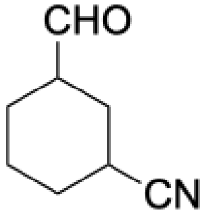
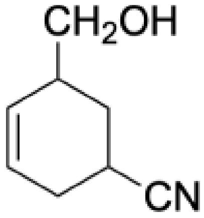
26. Consider the following compound.

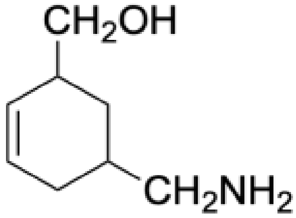


The compound undergoes reaction with two equivalents of diisobutyl aluminium hydride followed by acid hydrolysis to provide P as the major product. The structure of P is:

A. 

B. 

C. 

D. 

(A) Structure with both functional groups reduced to —CHO.

(B) Structure with ester reduced to —CHO and nitrile remaining intact as —CN.

(C) Structure with ester reduced to $-CH_2OH$ and nitrile remaining intact as —CN.

(D) Structure with ester reduced to $-CH_2OH$ and nitrile reduced to $-CH_2NH_2$.

Correct Answer: (A) Structure with both functional groups reduced to —CHO.

Solution:

Step 1: Understanding the Question:

The question asks for the major product P obtained when a cyclohexene derivative containing both an ester group ($-CO_2Et$) and a nitrile group ($-CN$) is treated with two equivalents of diisobutylaluminium hydride (DIBAL-H), followed by acid hydrolysis.

Step 2: Key Formula or Approach:

DIBAL-H is a electrophilic, sterically hindered reducing agent that is commonly used at low temperatures to selectively reduce:

1. Esters (R-COOR') to aldehydes (R-CHO).
2. Nitriles (R-CN) to imines, which upon subsequent acid hydrolysis yield aldehydes (R-CHO).

Step 3: Detailed Explanation:

- The starting material is ethyl 4-cyanocyclohex-3-ene-1-carboxylate.
- It possesses two distinct functional groups susceptible to reduction: an ester group ($-CO_2Et$) and a nitrile group ($-CN$).
- The researcher adds exactly two equivalents of DIBAL-H at a low temperature.
- One equivalent of DIBAL-H reacts with the ester group, forming a hemiacetal intermediate that is stable at low temperatures, preventing further reduction to an alcohol.
- The second equivalent of DIBAL-H coordinates to the nitrogen of the nitrile group and reduces it to an imine intermediate.
- Upon subsequent acid hydrolysis (H_3O^+), the hemiacetal intermediate collapses to release ethanol and form an aldehyde group ($-CHO$).

- At the same time, the imine intermediate undergoes complete hydrolysis to release ammonia and yield an aldehyde group (—CHO).

- Thus, both functional groups are selectively converted into aldehyde groups (—CHO), while the carbon-carbon double bond of the cyclohexene ring remains completely unaffected.

Step 4: Final Answer:

Therefore, the correct structure of P is the dialdehyde shown in Option A.

Quick Tip: At low temperatures (e.g., −78 °C), DIBAL-H is the premier reagent for stopping reductions of both esters and nitriles at the aldehyde stage.

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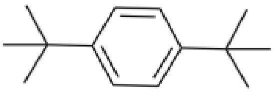
27. tert-Butylbenzene reacts with 1-chloro-2-methylpropane in the presence of anhydrous $AlCl_3$. The major product in this reaction is:

A.


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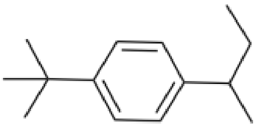
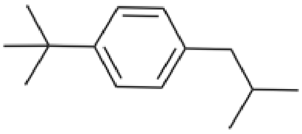
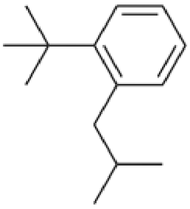


C.


D.


(A) 1,4-di-tert-butylbenzene

(B) 1-tert-butyl-4-isobutylbenzene

(C) 1-tert-butyl-4-sec-butylbenzene

(D) 1,2-di-tert-butylbenzene

Correct Answer: (A) 1,4-di-tert-butylbenzene

Solution:

Step 1: Understanding the Question:

The question asks for the major product of the Friedel-Crafts alkylation reaction of tert-butylbenzene with 1-chloro-2-methylpropane in the presence of a Lewis acid catalyst, anhydrous $AlCl_3$.

Step 2: Key Formula or Approach:

1. Friedel-Crafts alkylation involves the generation of a carbocation electrophile.
2. Primary carbocations are highly unstable and readily undergo rearrangement (hydride or alkyl shifts) to yield more stable secondary or tertiary carbocations.
3. The tert-butyl group on the benzene ring is an ortho/para-directing group due to inductive electron donation.

Step 3: Detailed Explanation:

- First, let us look at the interaction of the alkyl halide, 1-chloro-2-methylpropane (isobutyl chloride), with the Lewis acid catalyst $AlCl_3$:

$$(CH_3)_2CH\text{-}CH_2\text{-}Cl + AlCl_3 \rightarrow (CH_3)_2CH\text{-}CH_2^+ + AlCl_4^-$$

- This forms a primary isobutyl carbocation, which is relatively unstable.

- To achieve greater stability, a 1,2-hydride shift occurs immediately from the neighboring tertiary carbon to the primary carbon:

$$(CH_3)_2C(H)\text{-}CH_2^+ \xrightarrow{\text{1,2-H shift}} (CH_3)_3C^+ \text{ (tert-butyl carbocation)}$$

- This rearrangement converts the unstable primary carbocation into the highly stable, tertiary tert-butyl carbocation, which serves as the active electrophile in this reaction.
- Now, the electrophile attacks the aromatic ring of the starting material, tert-butylbenzene.
- The tert-butyl substituent already present on the ring is activating and ortho/para-directing.
- However, because the tert-butyl group is extremely bulky, steric hindrance prevents attack at the ortho-position.
- Consequently, the electrophilic aromatic substitution occurs almost exclusively at the less hindered para-position.
- This electrophilic attack and subsequent deprotonation yields 1,4-di-tert-butylbenzene as the major product.

Step 4: Final Answer:

Therefore, the major product is 1,4-di-tert-butylbenzene (Option A).

Quick Tip: In Friedel-Crafts alkylation, always check for carbocation rearrangements. Isobutyl and neopentyl halides almost always rearrange to give tert-butyl and tert-pentyl products, respectively.

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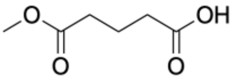
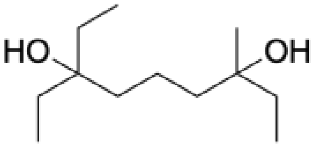
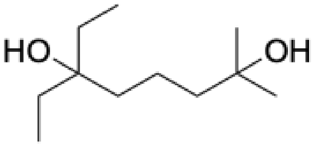
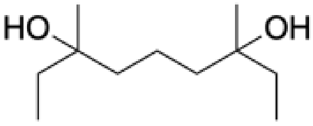
28. Consider the following compound.

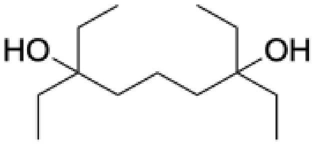


The compound reacts with $SOCl_2$ to produce P. P reacts with dimethyl cadmium to produce Q. Q upon reaction with ethyl magnesium bromide (3 equiv) followed by water workup produces R. The structure of R is:

A. 

B. 

C. 

D. 

(A) Structure with one tertiary alcohol containing two ethyl groups and another containing one methyl and one ethyl group.

(B) Structure with one secondary alcohol and one tertiary alcohol.

(C) Structure with two tertiary alcohols containing only methyl groups.

(D) Structure with two tertiary alcohols containing only ethyl groups.

Correct Answer: (A) Structure with one tertiary alcohol containing two ethyl groups and another containing one methyl and one ethyl group.

Solution:

Step 1: Understanding the Question:

The question asks us to trace a multi-step organic synthesis starting from a half-ester of glutaric acid (methyl 4-carboxybutanoate) and identify the final product R.

Step 2: Key Formula or Approach:

1. Thionyl chloride ($SOCl_2$) converts carboxylic acids to acid chlorides.

2. Organocadmium reagents (R_2Cd) are mild nucleophiles that react selectively with acid

chlorides to yield ketones, leaving esters untouched.
3. Grignard reagents (R'MgX) react with ketones (using 1 equivalent) to yield tertiary alcohols and with esters (using 2 equivalents) to yield tertiary alcohols containing two identical R' groups.

Step 3: Detailed Explanation:

- The starting material is a half-ester, methyl 4-carboxybutanoate:

  $$\text{MeOOC-CH}_2\text{-CH}_2\text{-CH}_2\text{-COOH}$$

- Step 1: Reaction with thionyl chloride ($SOCl_2$) targets the carboxylic acid group selectively, converting it to an acid chloride to produce compound P:

  $$\text{MeOOC-CH}_2\text{-CH}_2\text{-CH}_2\text{-COCl}$$

- Step 2: Reaction of P with dimethyl cadmium (Me_2Cd) selectively converts the highly reactive acid chloride group into a methyl ketone group, while the less reactive ester remains completely unreacted, forming Q:

  $$\text{MeOOC-CH}_2\text{-CH}_2\text{-CH}_2\text{-CO-CH}_3$$

- Step 3: Compound Q is treated with three equivalents of ethyl magnesium bromide (EtMgBr):

  - One equivalent of EtMgBr attacks the more electrophilic ketone group (—CO-CH_3) to form, after workup, a tertiary alcohol:

$$-\mathrm{C(OH)(Me)(Et)}$$

  - Two equivalents of EtMgBr are required to react with the ester group (—COOMe). The first equivalent replaces the methoxy group to form an ethyl ketone intermediate, and the second equivalent immediately attacks this ketone to form a tertiary alcohol containing two ethyl groups:

$$-\mathrm{C(OH)(Et)_2}$$

- Water workup protonates both alkoxide intermediates to yield the diol product R:

$$\mathrm{Et_2C(OH)\text{-}CH_2\text{-}CH_2\text{-}CH_2\text{-}C(OH)(Me)(Et)}$$

Step 4: Final Answer:

Therefore, the final product R is represented by the diol structure shown in Option A.

Quick Tip: Organocadmium and organocopper (Gilman) reagents are less nucleophilic than Grignard reagents and will not react with esters, allowing the selective conversion of acid chlorides to ketones.

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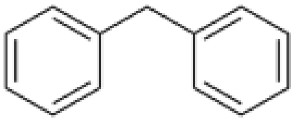
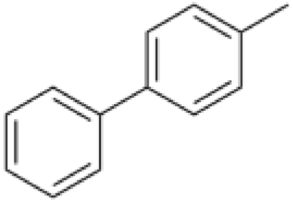
29. Acetylene is passed through red hot iron tube at 873 K to produce P. P reacts with chlorine gas in the presence of anhydrous $AlCl_3$ to produce Q. Q reacts with benzyl chloride and sodium to produce R as the major product. The structure of R is:

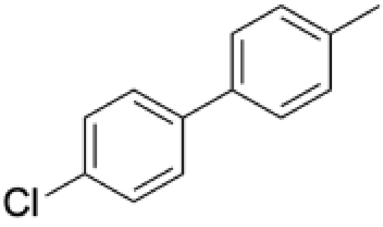
A.



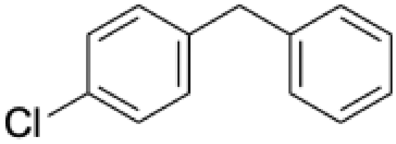
B.



C.



D.



(A) diphenylmethane

(B) biphenyl

(C) 4-chlorobiphenyl

(D) benzyl 4-chlorobenzoate

Correct Answer: (A) diphenylmethane

Solution:

Step 1: Understanding the Question:

The question asks us to identify the major organic product R after a sequence of three reactions starting with acetylene.

Step 2: Key Formula or Approach:

1. Cyclic polymerization: Passing acetylene through a red-hot iron tube at 873 K causes trimerization to form benzene.

2. Electrophilic chlorination: Benzene reacts with $Cl_2/AlCl_3$ to produce chlorobenzene.

3. Wurtz-Fittig reaction: Reaction of an aryl halide and an alkyl/arylalkyl halide with sodium metal leads to a coupling product.

Step 3: Detailed Explanation:

- Step 1: When acetylene gas ($HC \equiv CH$) is passed through a red-hot iron tube at 873 K, it undergoes cyclic trimerization to form benzene as product P:

$$3\ C_2H_2 \xrightarrow{\text{Fe, 873 K}} C_6H_6\ \text{(Benzene)}$$

- Step 2: Benzene (P) undergoes electrophilic aromatic substitution when treated with chlorine gas (Cl_2) in the presence of anhydrous aluminium chloride ($AlCl_3$). This produces chlorobenzene as product Q:

$$C_6H_6 + Cl_2 \xrightarrow{AlCl_3} C_6H_5Cl + HCl$$

- Step 3: Chlorobenzene (Q) is mixed with benzyl chloride ($C_6H_5CH_2Cl$) and treated with sodium metal (Na).

- This is a classic Wurtz-Fittig reaction.

- The sodium metal acts as a reducing agent, transferring electrons to the carbon-chlorine bonds to form highly reactive radical or carbanion intermediates from both chlorobenzene and benzyl chloride.

- These intermediates couple together to form a new carbon-carbon single bond:

$$C_6H_5\text{-}Cl + 2Na + Cl\text{-}CH_2\text{-}C_6H_5 \rightarrow C_6H_5\text{-}CH_2\text{-}C_6H_5 + 2NaCl$$

- The coupled product consists of a central methylene group (-CH_2—) bonded to two phenyl rings, which is diphenylmethane (R).

Step 4: Final Answer:

Therefore, the major product R is diphenylmethane (Option A).

Quick Tip: The Wurtz-Fittig reaction is highly effective for coupling aryl halides with alkyl or aralkyl halides using sodium in dry ether to form alkyl-substituted aromatic compounds.

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30. 1,2-dimethylbenzene on treatment with chromic oxide in acetic anhydride at 273-283 K followed by hydrolysis produces P. P on heating with concentrated NaOH followed by hydrolysis provides Q. The functional groups present in the product Q are:

(A) One carboxylic acid and one alcohol
(B) Two aldehydes
(C) One aldehyde and one carboxylic acid
(D) Two carboxylic acids

Correct Answer: (A) One carboxylic acid and one alcohol

Solution:

Step 1: Understanding the Question:

The question asks us to identify the chemical functional groups present in the final product Q of a two-step reaction starting from 1,2-dimethylbenzene (o-xylene).

Step 2: Key Formula or Approach:

1. Controlled oxidation of methylbenzenes with chromic oxide (CrO_3) in acetic anhydride

(Ac_2O) converts methyl groups into gem-diacetates, which on subsequent acid hydrolysis yield aldehydes.

2. The Cannizzaro reaction is an base-induced disproportionation of aldehydes lacking α-hydrogens, yielding an alcohol and a carboxylate salt.

Step 3: Detailed Explanation:

- The starting material is 1,2-dimethylbenzene (o-xylene).
- Treatment of o-xylene with chromic oxide (CrO_3) in acetic anhydride (Ac_2O) at 273-283 K oxidizes both methyl groups into gem-diacetate groups.
- Acid hydrolysis of these gem-diacetate intermediates yields o-phthalaldehyde as product P:

$$C_6H_4(CH_3)_2 \xrightarrow{CrO_3, Ac_2O} C_6H_4[CH(OAc)_2]_2 \xrightarrow{H_3O^+} C_6H_4(CHO)_2 \text{ (o-phthalaldehyde)}$$

- Compound P is a dialdehyde that lacks any α-hydrogen atoms relative to either carbonyl group.
- When heated with concentrated sodium hydroxide (NaOH), P undergoes an intramolecular Cannizzaro reaction.
- In this process, one of the aldehyde groups acts as a hydride donor and is oxidized to a sodium carboxylate group ($-COONa$), while the neighboring aldehyde group acts as a hydride acceptor and is reduced to a primary alcohol group ($-CH_2OH$).
- Subsequent acid hydrolysis/protonation converts the carboxylate salt to a free carboxylic acid group ($-COOH$).

- This produces 2-(hydroxymethyl)benzoic acid, which contains exactly one carboxylic acid group and one primary alcohol group.

Step 4: Final Answer:

Therefore, the functional groups present in the final product Q are one carboxylic acid and one alcohol (Option A).

Quick Tip: An intramolecular Cannizzaro reaction of a dialdehyde (like phthalaldehyde) is an elegant way to simultaneously create an alcohol and a carboxylic acid on the same molecule without needing external oxidizing or reducing agents.

---

31. The increasing order of acidic strength for the aqua complexes $[V(H_2O)_6]^{2+}$, $[Fe(H_2O)_6]^{2+}$, $[Fe(H_2O)_6]^{3+}$, and $[Co(H_2O)_6]^{3+}$ is:

(A) $[V(H_2O)_6]^{2+} < [Fe(H_2O)_6]^{2+} < [Fe(H_2O)_6]^{3+} < [Co(H_2O)_6]^{3+}$
(B) $[Fe(H_2O)_6]^{2+} < [Fe(H_2O)_6]^{3+} < [V(H_2O)_6]^{2+} < [Co(H_2O)_6]^{3+}$
(C) $[Co(H_2O)_6]^{3+} < [Fe(H_2O)_6]^{3+} < [Fe(H_2O)_6]^{2+} < [V(H_2O)_6]^{2+}$
(D) $[Fe(H_2O)_6]^{2+} < [Fe(H_2O)_6]^{3+} < [Co(H_2O)_6]^{3+} < [V(H_2O)_6]^{2+}$

Correct Answer: (A) $[V(H_2O)_6]^{2+} < [Fe(H_2O)_6]^{2+} < [Fe(H_2O)_6]^{3+} < [Co(H_2O)_6]^{3+}$

Solution:

Step 1: Understanding the Question:

The question asks us to arrange four transition metal aqua complexes in order of increasing acidic strength (ability to donate a proton from a coordinated water molecule).

Step 2: Key Formula or Approach:

The acidity of a metal aqua complex depends on the charge density of the central metal ion:

$$\text{Acidity} \propto \text{Charge Density} = \frac{\text{Ionic Charge (z)}}{\text{Ionic Radius (r)}}$$

A higher charge density polarizes the O-H bonds of the coordinated water molecules more strongly, facilitating the release of a proton (H^+).

Step 3: Detailed Explanation:

- First, let us separate the complexes by their oxidation state:
  - +2 complexes: $[V(H_2O)_6]^{2+}$ and $[Fe(H_2O)_6]^{2+}$
  - +3 complexes: $[Fe(H_2O)_6]^{3+}$ and $[Co(H_2O)_6]^{3+}$
- Because +3 metal ions have a significantly higher charge and smaller ionic radii compared to +2 metal ions, +3 complexes are much stronger acids than +2 complexes.
- Next, let us compare the +2 complexes: $[V(H_2O)_6]^{2+}$ vs $[Fe(H_2O)_6]^{2+}$.
- Going from left to right across the 3d transition series, the effective nuclear charge increases, causing a contraction in ionic radius (transition metal contraction).
- Since iron is further to the right than vanadium, the ionic radius of Fe^{2+} is smaller than that of V^{2+}.
- Therefore, Fe^{2+} has a higher charge density and is more acidic:

$$[V(H_2O)_6]^{2+} < [Fe(H_2O)_6]^{2+}$$

- Finally, let us compare the +3 complexes: $[Fe(H_2O)_6]^{3+}$ vs $[Co(H_2O)_6]^{3+}$.

- Cobalt lies to the right of iron in the periodic table, so the ionic radius of Co^{3+} is smaller than that of Fe^{3+}.

- This higher charge density on Co^{3+} makes $[Co(H_2O)_6]^{3+}$ more acidic than $[Fe(H_2O)_6]^{3+}$:

$$[Fe(H_2O)_6]^{3+} < [Co(H_2O)_6]^{3+}$$

- Combining these trends yields the complete increasing order of acidity.

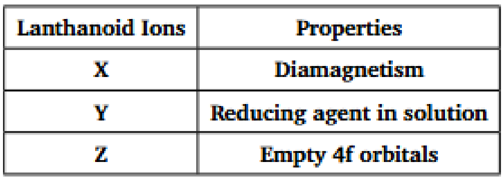
Step 4: Final Answer:

Therefore, the correct increasing order of acidic strength is Option A.

Quick Tip: For metal aqua complexes, oxidation state is the dominant factor determining acidity. If oxidation states are equal, acidity increases from left to right across a period as ionic radius decreases.

---

32. Consider the following table of three lanthanoid ions X, Y, and Z and their properties.



Lanthanoid Ions	Properties
X	Diamagnetism
Y	Reducing agent in solution
Z	Empty 4f orbitals

Atomic numbers of Ce, Eu, and Lu are 58, 63, and 71, respectively. Given these atomic numbers, the lanthanoid ions X, Y, and Z, respectively, are:

(A) Lu^{3+}, Eu^{2+}, and Ce^{4+}

(B) Lu^{3+}, Ce^{4+}, and Eu^{2+}

(C) Eu^{2+}, Ce^{4+}, and Lu^{3+}

(D) Eu^{2+}, Lu^{3+}, and Ce^{4+}

Correct Answer: (A) Lu^{3+}, Eu^{2+}, and Ce^{4+}

Solution:

Step 1: Understanding the Question:

The question asks us to identify three specific lanthanoid ions (X, Y, and Z) based on their given magnetic, chemical, and structural properties.

Step 2: Key Formula or Approach:

We must write out the electronic configurations of the neutral lanthanoids and their ionic states to evaluate:

1. Diamagnetism: Occurs when all electrons are paired (e.g., $4f^0$ or $4f^{14}$).
2. Reducing behavior: Occurs when an ion can easily lose electrons to achieve a more stable oxidation state (typically +3 for lanthanoids).
3. Empty 4f orbitals: Occurs when the configuration is $4f^0$.

Step 3: Detailed Explanation:

- Let us evaluate each lanthanoid and its ionic electronic configuration:
- Ce ($Z = 58$):
  - Neutral configuration: $[Xe]4f^1 5d^1 6s^2$
  - Ce^{4+} configuration: $[Xe]4f^0$ (all valence electrons removed)
  - This configuration has completely empty 4f orbitals, which matches the property of Z.
- Eu ($Z = 63$):

  - Neutral configuration: $[Xe]4f^76s^2$
  - Eu^{2+} configuration: $[Xe]4f^7$
  - Since the +3 oxidation state is the most stable state for all lanthanoids, Eu^{2+} readily oxidizes to Eu^{3+} by losing one electron. This makes Eu^{2+} a strong reducing agent in aqueous solution, matching property Y.

- Lu ($Z = 71$):
  - Neutral configuration: $[Xe]4f^{14}5d^16s^2$
  - Lu^{3+} configuration: $[Xe]4f^{14}$
  - The 4f subshell is completely filled with 14 electrons, meaning there are no unpaired electrons. Therefore, Lu^{3+} is diamagnetic, matching property X.

- Thus, X is Lu^{3+}, Y is Eu^{2+}, and Z is Ce^{4+}.

Step 4: Final Answer:

Therefore, the correct assignment of ions is Lu^{3+}, Eu^{2+}, and Ce^{4+} (Option A).

Quick Tip: Lanthanoids have a preferred oxidation state of +3. Any +2 lanthanoid ion (like Eu^{2+}, Yb^{2+}) will act as a reducing agent, while any +4 ion (like Ce^{4+}) will act as an oxidizing agent.

---

33. Consider the following complex ions:

$[Ni(CN)_4]^{2-}$, $[Fe(CN)_6]^{3-}$, $[Cu(H_2O)_6]^{2+}$, and $[Co(CN)_6]^{3-}$.

The complex ions that are expected to show diamagnetic behavior at room temperature are:

(A) $[Ni(CN)_4]^{2-}$ and $[Co(CN)_6]^{3-}$
(B) $[Ni(CN)_4]^{2-}$ and $[Cu(H_2O)_6]^{2+}$
(C) $[Co(CN)_6]^{3-}$ and $[Fe(CN)_6]^{3-}$
(D) $[Cu(H_2O)_6]^{2+}$ and $[Co(CN)_6]^{3-}$

Correct Answer: (A) $[Ni(CN)_4]^{2-}$ and $[Co(CN)_6]^{3-}$

Solution:

Step 1: Understanding the Question:
The question asks us to determine which of the given transition metal coordination complex ions exhibit diamagnetic behavior (no unpaired electrons) at room temperature.

Step 2: Key Formula or Approach:
We must determine the oxidation state, coordination geometry, and d-electron configuration of the central metal in each complex, and then apply Crystal Field Theory (CFT) to evaluate electron pairing.

Step 3: Detailed Explanation:

- Let us evaluate each complex systematically:

- 1. $[Ni(CN)_4]^{2-}$:

  - Nickel is in the +2 oxidation state (Ni^{2+}, a d^8 system).

  - CN^- is a very strong-field ligand, causing square planar coordination geometry.

  - In a square planar crystal field, the d_x^2-y^2 orbital is pushed very high in energy, while the other four d-orbitals are lower.

  - The 8 electrons completely fill the four lower energy d-orbitals (d_{xz}, d_{yz}, d_{z^2}, and d_{xy}), with all spins paired. Thus, it is diamagnetic.

- 2. $[Co(CN)_6]^{3-}$:

  - Cobalt is in the +3 oxidation state (Co^{3+}, a d^6 system).

  - CN^- is a strong-field ligand, forming an octahedral geometry.

  - This strong crystal field splitting causes the electrons to adopt a low-spin configuration: all 6 electrons go into the lower-energy t_{2g} orbitals, completely filling them:

$$t_{2g}^6\ e_g^0$$

  - Since all electrons are paired, this complex is diamagnetic.

- 3. $[Fe(CN)_6]^{3-}$:

  - Iron is in the +3 oxidation state (Fe^{3+}, a d^5 system).

  - With the strong-field ligand CN^-, it adopts a low-spin $t_{2g}^5\ e_g^0$ configuration, which leaves 1 unpaired electron. Thus, it is paramagnetic.

- 4. $[Cu(H_2O)_6]^{2+}$:

  - Copper is in the +2 oxidation state (Cu^{2+}, a d^9 system).

- A d^9 configuration always has exactly 1 unpaired electron, regardless of the ligand field strength. Thus, it is paramagnetic.

Step 4: Final Answer:

Therefore, only $[Ni(CN)_4]^{2-}$ and $[Co(CN)_6]^{3-}$ are diamagnetic (Option A).

Quick Tip: d^8 systems with strong-field ligands almost always form square-planar, diamagnetic complexes, while d^6 systems with strong-field ligands form low-spin, diamagnetic octahedral complexes.

---

34. Consider the following species:

(i) ClO^-

(ii) ClO_2^-

(iii) ClO_3^-

(iv) ClO_4^-

Among them, the species that undergoes(undergo) disproportionation reactions is(are):

(A) (i), (ii), and (iii) only

(B) (iv) only

(C) (i) and (ii) only

(D) (iii) and (iv) only

Correct Answer: (A) (i), (ii), and (iii) only

Solution:

Step 1: Understanding the Question:

The question asks us to identify which of the given chlorine oxyanion species are capable of undergoing disproportionation (a redox reaction where a single species is simultaneously oxidized and reduced).

Step 2: Key Formula or Approach:

For an element in a specific species to undergo disproportionation, it must exist in an intermediate oxidation state.

1. If the element is in its maximum possible oxidation state, it can only be reduced, so it cannot undergo disproportionation.
2. If the element is in its minimum possible oxidation state, it can only be oxidized, so it cannot undergo disproportionation.

Step 3: Detailed Explanation:

- Let us calculate the oxidation state of chlorine (Cl) in each of the given species:
  - (i) In ClO^-: Let the oxidation state of Cl be x.

    $$x + (-2) = -1 \implies x = +1$$

  - (ii) In ClO_2^-: Let the oxidation state of Cl be y.

    $$y + 2(-2) = -1 \implies y = +3$$

  - (iii) In ClO_3^-: Let the oxidation state of Cl be z.

    $$z + 3(-2) = -1 \implies z = +5$$

  - (iv) In ClO_4^-: Let the oxidation state of Cl be w.

    $$w + 4(-2) = -1 \implies w = +7$$

- Chlorine is a Group 17 halogen, so its possible oxidation states range from -1 to $+7$.

- In the perchlorate ion, ClO_4^-, the oxidation state of chlorine is $+7$, which is its maximum possible oxidation state. Since it cannot be oxidized any further, it cannot undergo disproportionation.

- In ClO^-, ClO_2^-, and ClO_3^-, the oxidation states ($+1$, $+3$, and $+5$) are intermediate.

- Therefore, these species can be both oxidized to higher states (like $+5$ or $+7$) and reduced to lower states (like -1), meaning they readily undergo disproportionation reactions.

Step 4: Final Answer:

Therefore, only species (i), (ii), and (iii) can undergo disproportionation (Option A).

Quick Tip: Disproportionation is thermodynamically favorable for intermediate oxidation states. If a species contains an element in its absolute maximum oxidation state, it is exclusively an oxidizing agent and cannot disproportionate.

---

35. X, Y, and Z are three volatile liquids, which when mixed make an ideal solution. At a given temperature T_0, the constituents X, Y, and Z have pure vapour pressures in the ratio 3:2:1. A mixture prepared with mole-fractions of X, Y, and Z in the ratio 1:2:3 starts to boil at temperature T_0 at 1.5 atm pressure. At temperature T_0, the partial vapour pressure (in atm) of Y is:

(A) $\frac{3}{5}$

(B) $\frac{9}{10}$

(C) $\frac{1}{6}$

(D) $\frac{9}{20}$

Correct Answer: (A) $\frac{3}{5}$

Solution:

Step 1: Understanding the Question:

The question asks us to find the partial vapour pressure of liquid Y in an ideal three-component liquid mixture that starts to boil at a given total pressure of 1.5 atm.

Step 2: Key Formula or Approach:

1. According to Raoult's Law, the total vapour pressure (P_{total}) of an ideal solution containing components X, Y, and Z is:

$$P_{\text{total}} = x_{\text{X}} P^0_{\text{X}} + x_{\text{Y}} P^0_{\text{Y}} + x_{\text{Z}} P^0_{\text{Z}}$$

2. The partial vapour pressure of any component i is:

$$P_i = x_i P^0_i$$

Step 3: Detailed Explanation:

- Let us define the pure vapour pressures of X, Y, and Z based on their given ratio of 3 : 2 : 1:

  $$P^0_{\text{X}} = 3k, \quad P^0_{\text{Y}} = 2k, \quad P^0_{\text{Z}} = 1k$$

  where k is a constant of proportionality.

- The mole fractions of the constituents in the mixture are in the ratio 1 : 2 : 3. Therefore, we can write:

$$x_X = \frac{1}{1+2+3} = \frac{1}{6}$$

$$x_Y = \frac{2}{1+2+3} = \frac{2}{6} = \frac{1}{3}$$

$$x_Z = \frac{3}{1+2+3} = \frac{3}{6} = \frac{1}{2}$$

- A liquid mixture starts to boil when its total vapour pressure equals the external pressure. Thus:

$$P_{\text{total}} = 1.5 \text{ atm}$$

- Substituting these values into Raoult's Law:

$$1.5 = \left(\frac{1}{6} \times 3k\right) + \left(\frac{1}{3} \times 2k\right) + \left(\frac{1}{2} \times 1k\right)$$

$$1.5 = \frac{1}{2}k + \frac{2}{3}k + \frac{1}{2}k$$

$$1.5 = k + \frac{2}{3}k = \frac{5}{3}k$$

- Solving for the constant k:

$$k = \frac{1.5 \times 3}{5} = \frac{4.5}{5} = 0.9 \text{ atm}$$

- Now, let us calculate the partial vapour pressure of Y:

$$P_{\text{Y}} = x_{\text{Y}} P_{\text{Y}}^0 = \frac{1}{3} \times 2k = \frac{1}{3} \times 2(0.9) = 0.6 \text{ atm}$$

- Converting 0.6 to a fraction:

$$P_{\text{Y}} = 0.6 = \frac{6}{10} = \frac{3}{5} \text{ atm}$$

Step 4: Final Answer:

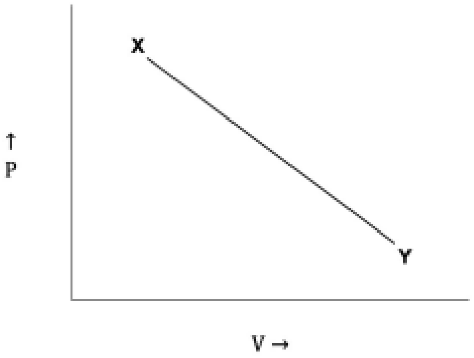
Therefore, the partial vapour pressure of Y is $\frac{3}{5}$ atm (Option A).

Quick Tip: Double check your calculations by verifying that the sum of the partial pressures equals the total pressure:

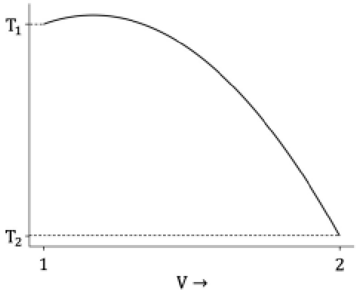
$$P_{\text{X}} = 0.45 \text{ atm}, \quad P_{\text{Y}} = 0.60 \text{ atm}, \quad P_{\text{Z}} = 0.45 \text{ atm} \implies \sum P = 1.50 \text{ atm}.$$

---

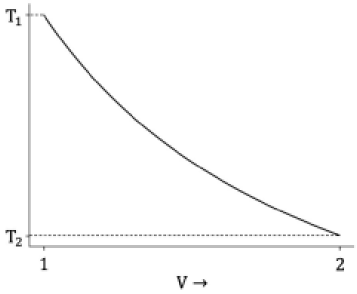
36. Consider a reversible thermodynamic process involving one mole of ideal gas represented by a line on the $P - V$ diagram connecting two states X (4 bar, 1 L, T_1) and Y (1 bar, 2 L, T_2) as shown in the figure:



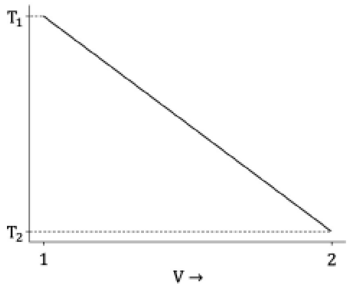
During the process the change in temperature (in K) as a function of volume (in L) is best represented as:



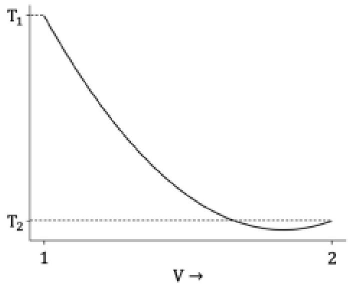
A.



B.



C.



D.

(A) A parabolic curve opening downwards with a maximum temperature.

(B) A hyperbolic curve decreasing monotonically.

(C) A straight line with a negative slope.

(D) A parabolic curve opening upwards with a minimum temperature.

Correct Answer: (A) A parabolic curve opening downwards with a maximum temperature.

Solution:

Step 1: Understanding the Question:

The question asks for the mathematical representation of the temperature (T) as a function of volume (V) for an ideal gas undergoing a linear transition from state X (1 L, 4 bar) to state Y (2 L, 1 bar).

Step 2: Key Formula or Approach:

1. The equation of a straight line in the $P-V$ plane is:

$$P = mV + c$$

2. The ideal gas equation for 1 mole of gas is:

$$PV = RT \implies T = \frac{PV}{R}$$

Step 3: Detailed Explanation:

- Let us first find the equation of the linear path connecting state X $(V_1, P_1) = (1, 4)$ and state Y $(V_2, P_2) = (2, 1)$ on the $P-V$ diagram.

- The slope (m) of this line is:

$$m = \frac{P_2 - P_1}{V_2 - V_1} = \frac{1-4}{2-1} = -3$$

- Using the point-slope form with state Y $(2,1)$:

$$P-1=-3(V-2) \implies P=-3V+7$$

- Now, let us substitute this expression for pressure into the ideal gas equation:

$$T=\frac{PV}{R}=\frac{(-3V+7)V}{R}=\frac{-3V^2+7V}{R}$$

- This equation defines the temperature as a quadratic function of volume (T $\propto -V^2$).

- Mathematically, this is a parabola that opens downwards because the coefficient of the V^2 term is negative.

- Let us find the volume at which the maximum temperature occurs:

$$\frac{dT}{dV}=\frac{-6V+7}{R}=0 \implies V=\frac{7}{6}\approx 1.17\text{ L}$$

- Since 1.17 L lies inside our experimental range (1 L to 2 L), the temperature must first increase to a peak at 1.17 L and then decrease as we continue to 2 L.

Step 4: Final Answer:

Therefore, the temperature as a function of volume is best represented by the downward-opening parabola shown in Option A.

Quick Tip: Any straight-line process on a $P - V$ diagram with a negative slope will always yield a quadratic relationship for temperature as a function of volume ($T \propto -V^2$), which manifests as a downward-opening parabola.

---

37. PQ_2 is a sparingly soluble salt with solubility product $K_{sp} = 4 \times 10^{-12}$ in aqueous medium at some given temperature. It is observed that upon addition of a highly soluble salt RQ at the same temperature, the solubility of PQ_2 drops by a factor of 100. The concentration (in millimoles per litre) of added RQ in the solution is closest to:

(A) 2
(B) 0.2
(C) 20
(D) 0.02

Correct Answer: (A) 2

Solution:

Step 1: Understanding the Question:
The question asks for the concentration of a highly soluble salt RQ that must be added to a saturated solution of the sparingly soluble salt PQ_2 to reduce its solubility by a factor of 100 via the common-ion effect.

Step 2: Key Formula or Approach:
1. The solubility product constant (K_{sp}) of PQ_2 is:

$$K_{sp} = [P^{2+}][Q^-]^2$$

2. Let S_0 be the solubility of PQ_2 in pure water:

$$K_{sp} = (S_0)(2S_0)^2 = 4S_0^3$$

Step 3: Detailed Explanation:

- Let us first calculate the solubility (S_0) of PQ_2 in pure water:

$$4S_0^3 = 4 \times 10^{-12} \implies S_0^3 = 10^{-12} \implies S_0 = 10^{-4}\ \text{M}$$

- The problem states that the addition of the highly soluble salt RQ causes the solubility of PQ_2 to drop by a factor of 100.

- Therefore, the new solubility (S) of PQ_2 is:

$$S = \frac{S_0}{100} = \frac{10^{-4}}{100} = 10^{-6}\ \text{M}$$

- Let the concentration of the added soluble salt RQ be C. Since RQ dissociates completely:

$$\text{RQ(aq)} \rightarrow \text{R}^+\text{(aq)} + \text{Q}^-\text{(aq)}$$

- This contributes a concentration C of Q^- ions to the solution.

- Now, the equilibrium concentration of ions in the solution is:

$$[\text{P}^{2+}] = S = 10^{-6}\ \text{M}$$

$$[\text{Q}^-] = 2S + C \approx C \quad \text{(since the solubility } S \text{ is extremely small, } C \gg 2S\text{)}$$

- Substituting these values into the K_{sp} expression:

$$K_{sp} = [\mathrm{P}^{2+}][\mathrm{Q}^{-}]^2$$

$$4 \times 10^{-12} = (10^{-6})(C)^2$$

$$C^2 = \frac{4 \times 10^{-12}}{10^{-6}} = 4 \times 10^{-6}$$

$$C = \sqrt{4 \times 10^{-6}} = 2 \times 10^{-3}\ \mathrm{M}$$

- Converting the concentration from moles per litre to millimoles per litre:

$$C = 2 \times 10^{-3}\ \mathrm{mol/L} = 2\ \mathrm{mmol/L}$$

Step 4: Final Answer:

Therefore, the concentration of the added RQ is closest to 2 mmol/L (Option A).

Quick Tip: In common-ion effect calculations, always assume that the concentration of the common ion comes entirely from the highly soluble salt ($C \gg 2S$). This greatly simplifies the mathematics and is highly accurate.

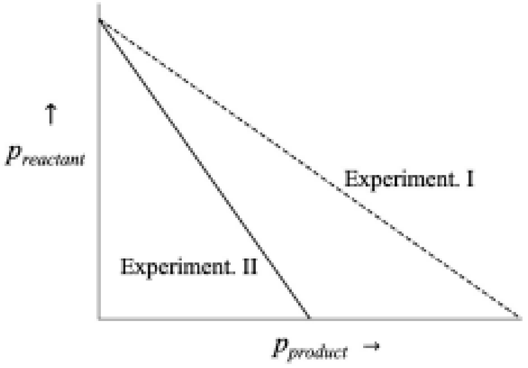
38. Consider the following chemical reactions performed at identical temperature and volume:

Experiment I: $X(g) \rightleftharpoons 2Y(g)$

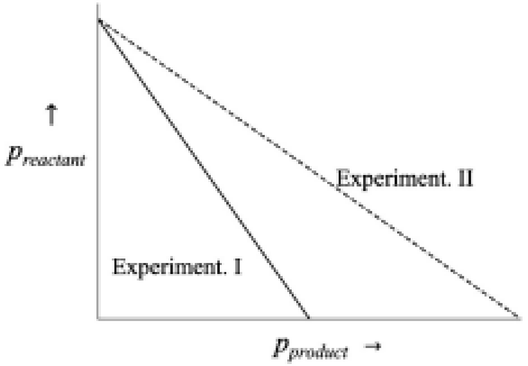
Experiment II: $X(g) \rightleftharpoons Z(g)$

The partial pressure of the reactant and product are denoted, respectively, by p_{reactant} and p_{product} during the course of the reactions. Assuming ideal gas behaviour, the correct plot of p_{reactant} versus p_{product} for Experiments I and II is:

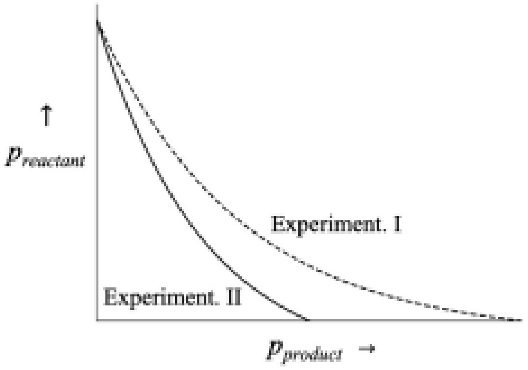
A.

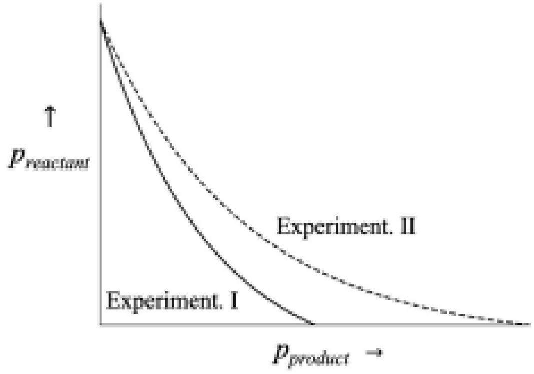


B.



C.



D. 

(A) A

(B) B

(C) C

(D) D

Correct Answer: (A) A

Solution:

Step 1: Understanding the Question:

The question asks us to identify the correct graphical representation of the partial pressure of the reactant (p_{reactant}) versus the partial pressure of the product (p_{product}) for two different gas-phase dissociation experiments.

Step 2: Key Formula or Approach:

For gas-phase reactions at constant volume and temperature, the change in partial pressure of any component is directly proportional to its stoichiometric coefficient:

$$\Delta P \propto \text{stoichiometric coefficient}$$

Step 3: Detailed Explanation:

- Let the initial partial pressure of the reactant gas X in both experiments be P_0.

- Let us analyze Experiment I: $X(g) \rightleftharpoons 2Y(g)$.

- If at any point during the reaction, a pressure x of reactant X has decomposed:

$$p_{\text{reactant}} = P_0 - x$$

$$p_{\text{product}} = 2x \implies x = \frac{p_{\text{product}}}{2}$$

- Substituting this value of x into the reactant equation:

$$p_{\text{reactant}} = P_0 - 0.5\ p_{\text{product}}$$

- This is a straight line equation with a y-intercept of P_0 and a slope of -0.5.

- Let us analyze Experiment II: $X(g) \rightleftharpoons Z(g)$.

- If at any point during the reaction, a pressure y of reactant X has decomposed:

$$p_{\text{reactant}} = P_0 - y$$

$$p_{\text{product}} = y$$

- Substituting this value of y into the reactant equation:

$$p_{\text{reactant}} = P_0 - p_{\text{product}}$$

- This is a straight line equation with a y-intercept of P_0 and a slope of -1.0.

- Comparing the two plots:
  - Both plots are straight lines that originate from the exact same initial value (P_0) on the vertical axis.
  - The slope for Experiment II (-1.0) is twice as steep as the slope for Experiment I (-0.5).

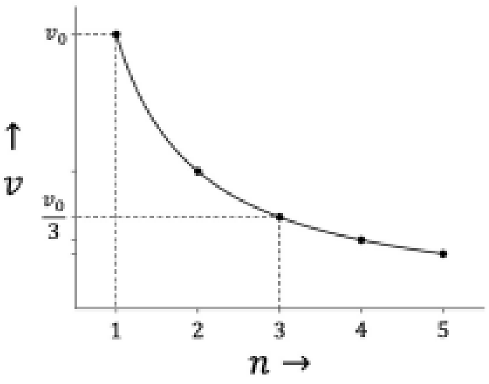
Step 4: Final Answer:

Therefore, the correct graphical representation is the one shown in Option A.

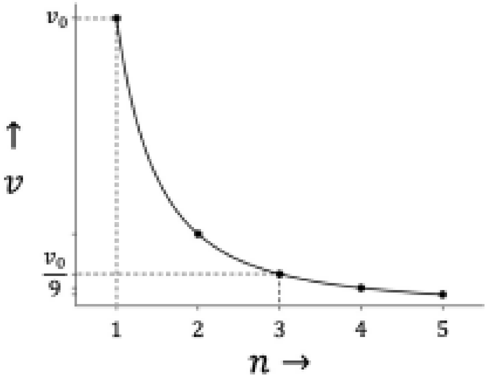
Quick Tip: Stoichiometry dictates the rate of change of pressures. Since one mole of X produces two moles of Y, the pressure of the product in Experiment I grows twice as fast as it does in Experiment II, making the slope of p_{reactant} vs p_{product} half as steep.

---

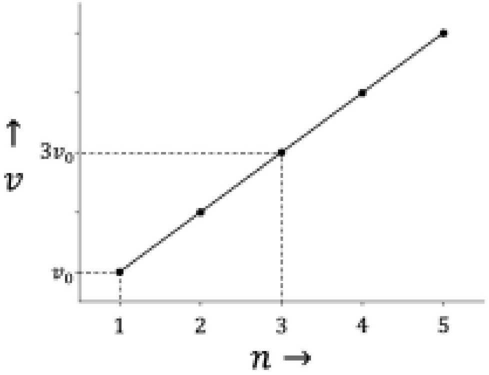
39. If velocity of an electron in the first Bohr orbit is denoted by v_0, then the velocity (v) of the electron in other orbits (as a function of principle quantum number 'n') is represented as:


v_0
↑
v
$\frac{v_0}{3}$
1
2
3
4
5
n →

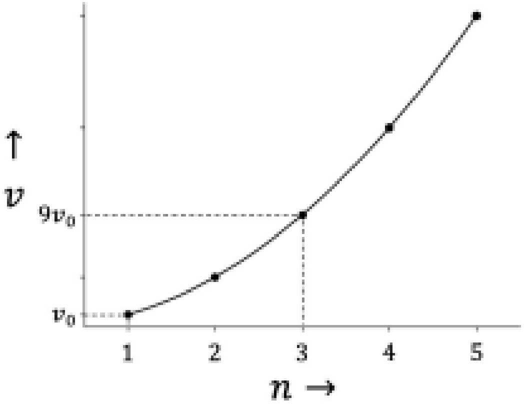
A.


v_0
↑
v
$\frac{v_0}{9}$
1
2
3
4
5
n →

B.


↑
$3v_0$
v
v_0
1
2
3
4
5
n →

C.


↑
v
$9v_0$
v_0
1
2
3
4
5
n →

D.

(A) A

(B) B

(C) C

(D) D

Correct Answer: (A) A

Solution:

Step 1: Understanding the Question:

The question asks us to identify the correct plot of the velocity (v) of an electron in a hydrogen-like atom's Bohr orbits as a function of the principal quantum number (n).

Step 2: Key Formula or Approach:

According to Bohr's model of the hydrogen atom, the velocity of an electron in the n-th orbit is given by:

$$v_n = \frac{2\pi k Z e^2}{nh} = v_0 \frac{Z}{n}$$

Step 3: Detailed Explanation:

- Let us look at the relationship between velocity (v) and the principal quantum number (n) for a given hydrogen-like atom (Z is constant):

$$v \propto \frac{1}{n}$$

- When the principal quantum number is $n = 1$ (the first Bohr orbit), the velocity is given as v_0.

- Substituting $n = 2$:

$$v_2 = \frac{v_0}{2} = 0.5\ v_0$$

- Substituting $n = 3$:

$$v_3 = \frac{v_0}{3} \approx 0.33\ v_0$$

- Substituting $n = 4$:

$$v_4 = \frac{v_0}{4} = 0.25\ v_0$$

- Mathematically, the function $y = \frac{C}{x}$ represents a rectangular hyperbola.

- As n increases, the velocity of the electron decreases asymptotically toward zero.

- Let us evaluate the options:

  - Option A shows a hyperbolic curve where the velocity at $n = 3$ is exactly equal to $\frac{v_0}{3}$, which perfectly matches our equation $v_n = \frac{v_0}{n}$.

  - Option B shows a curve where the velocity at $n = 3$ is $\frac{v_0}{9}$, which corresponds to a $1/n^2$ relationship (incorrect).

  - Options C and D show the velocity increasing as n increases, which is physically incorrect.

Step 4: Final Answer:

Therefore, the correct graph is the one shown in Option A.

Quick Tip: Remember: As an electron moves to higher energy levels (larger n), it is further from the nucleus and experiences a weaker electrostatic attraction, meaning it moves slower ($v \propto \frac{1}{n}$).

---

40. Consider two reactions.

I. A zero order reaction: $X \rightarrow P_1$.

II. A first order reaction: $Y \rightarrow P_2$.

Both reactions begin simultaneously with the same initial concentration 'C', that is $X_0 = Y_0 = C$. It is observed that the concentrations of the reactants fall to one-third of their initial values at the same time, provided the rate constants satisfy a certain ratio. If k_0 and k_1 denote, respectively, the zero and first order rate constants then the ratio k_0/k_1 necessary for this condition to be satisfied is:

(A) $\frac{2C}{3\ln 3}$

(B) $\frac{C}{\ln 3}$

(C) $\frac{C}{\ln 2}$

(D) $\frac{3C}{2\ln 2}$

Correct Answer: (A) $\frac{2C}{3\ln 3}$

Solution:

Step 1: Understanding the Question:

The question asks for the exact ratio of the rate constants (k_0/k_1) of a zero-order and a first-order reaction such that both starting reactants, with the same initial concentration C, fall to one-third of their initial concentration at the exact same time (t).

Step 2: Key Formula or Approach:

1. Integrated rate law for a zero-order reaction:

$$[\mathrm{X}]_t = [\mathrm{X}]_0 - k_0 t$$

2. Integrated rate law for a first-order reaction:

$$\ln\left(\frac{[\mathrm{Y}]_0}{[\mathrm{Y}]_t}\right) = k_1 t \implies t = \frac{1}{k_1}\ln\left(\frac{[\mathrm{Y}]_0}{[\mathrm{Y}]_t}\right)$$

Step 3: Detailed Explanation:

- Let us analyze Reaction I (zero-order): $\mathrm{X} \rightarrow P_1$ with initial concentration $[\mathrm{X}]_0 = \mathrm{C}$.
- The concentration falls to one-third of its initial value, so:

$$[\mathrm{X}]_t = \frac{\mathrm{C}}{3}$$

- Substituting this into the zero-order integrated rate law:

$$\frac{\mathrm{C}}{3} = \mathrm{C} - k_0 t$$

$$k_0 t = \mathrm{C} - \frac{\mathrm{C}}{3} = \frac{2\mathrm{C}}{3}$$

- Solving for the time (t):

$$t = \frac{2\mathrm{C}}{3k_0}$$

- Let us analyze Reaction II (first-order): $\mathrm{Y} \longrightarrow P_2$ with initial concentration $[\mathrm{Y}]_0 = \mathrm{C}$.

- The concentration falls to one-third of its initial value, so:

$$[\mathrm{Y}]_t = \frac{\mathrm{C}}{3}$$

- Substituting this into the first-order integrated rate law:

$$t = \frac{1}{k_1} \ln\left(\frac{\mathrm{C}}{\mathrm{C}/3}\right) = \frac{\ln 3}{k_1}$$

- Since both reactions fall to one-third of their initial values at the exact same time (t), we set the two expressions for t equal to each other:

$$\frac{2\mathrm{C}}{3k_0} = \frac{\ln 3}{k_1}$$

- Rearranging this equation to solve for the ratio $\frac{k_0}{k_1}$:

$$\frac{k_1}{k_0} = \frac{3\ln 3}{2\mathrm{C}} \implies \frac{k_0}{k_1} = \frac{2\mathrm{C}}{3\ln 3}$$

Step 4: Final Answer:

Therefore, the ratio of the rate constants k_0/k_1 is $\frac{2\mathrm{C}}{3\ln 3}$ (Option A).

Quick Tip: Always write out the explicit expressions for time (t) for each kinetic order first, then equate them. This avoids algebraic errors when dealing with logarithmic terms.

Mathematics

41. A possible solution of the system of equations

$$x^2 - 8xy + 16y^2 = 0$$

$$(\log_{10} x)^2 + 2(\log_{10} x)(\log_{10} y) + (\log_{10} y)^2 = 4$$

is

(A) $x = \frac{1}{5}, y = \frac{1}{20}$
(B) $x = 100, y = 25$
(C) $x = 40, y = 10$
(D) $x = \frac{4}{25}, y = \frac{1}{16}$

Correct Answer: (A) $x = \frac{1}{5}, y = \frac{1}{20}$

Solution:

Step 1: Understanding the Question:

The given problem asks us to find a possible solution to a system of two non-linear equations. The first equation is a quadratic form in two variables, while the second is a quadratic expression involving common logarithms.

We will solve this system by simplifying both equations into linear or simpler algebraic relations.

Step 2: Key Formula or Approach:

We will use the perfect square factorization formula:

$$a^2 - 2ab + b^2 = (a-b)^2$$

And the logarithmic product property:

$$\log_{10} x + \log_{10} y = \log_{10}(xy)$$

Step 3: Detailed Explanation:

- First, let us analyze the first equation:

  $$x^2 - 8xy + 16y^2 = 0$$

  We can rewrite this expression as a perfect square:

  $$(x-4y)^2 = 0 \implies x = 4y$$

  This gives us a direct linear relationship between x and y.

- Next, let us analyze the second equation:

  $$(\log_{10} x)^2 + 2(\log_{10} x)(\log_{10} y) + (\log_{10} y)^2 = 4$$

  This is also a perfect square in terms of logarithmic values:

  $$(\log_{10} x + \log_{10} y)^2 = 4$$

- Taking the square root on both sides:

$$\log_{10} x + \log_{10} y = \pm 2$$

Using the property of logarithms, we can combine the terms:

$$\log_{10}(xy) = \pm 2$$

This gives us two cases:
Case 1: $xy = 10^2 = 100$.
Case 2: $xy = 10^{-2} = \frac{1}{100}$.

- Let us substitute $x = 4y$ into Case 1:

$$(4y)y = 100 \implies 4y^2 = 100 \implies y^2 = 25 \implies y = 5$$

(Note: we only consider positive values because the domain of logarithmic functions requires $x > 0$ and $y > 0$).
Since $y = 5$, we have $x = 4(5) = 20$. So, $(20, 5)$ is a solution.

- Let us substitute $x = 4y$ into Case 2:

$$(4y)y = \frac{1}{100} \implies 4y^2 = \frac{1}{100} \implies y^2 = \frac{1}{400} \implies y = \frac{1}{20}$$

Since $y = \frac{1}{20}$, we have $x = 4\left(\frac{1}{20}\right) = \frac{1}{5}$. So, $\left(\frac{1}{5}, \frac{1}{20}\right)$ is another solution.

Step 4: Final Answer:

Comparing our solutions with the given options, we find that option (A) matches the solution $\left(\frac{1}{5}, \frac{1}{20}\right)$.

Quick Tip: Whenever you see quadratic forms, look for perfect square factorizations first.

Always verify the domain constraints of logarithmic terms, ensuring both variables remain positive.

---

42. The parabola $y = -x^2 + 16$ intersects the x-axis at points A and B. Further, the parabola intersects the line $y = 7$ at points M and N. Then the area of the quadrilateral with vertices A, B, M and N is

(A) 49

(B) 42

(C) 56

(D) 35

Correct Answer: (A) 49

Solution:

Step 1: Understanding the Question:

The question asks for the area of the quadrilateral formed by the intersection points of a parabola with the x-axis and a horizontal line.

By determining the coordinates of these four points, we can identify the shape of the quadrilateral and calculate its area.

Step 2: Key Formula or Approach:

The intersection points on the x-axis are found by setting $y = 0$.

The intersection points with the line $y = 7$ are found by setting $y = 7$.

The area of a trapezoid (since MN and AB are parallel horizontal segments) is given by:

$$\text{Area} = \frac{1}{2}(a + b)h$$

where a and b are the lengths of the parallel sides, and h is the vertical height.

Step 3: Detailed Explanation:

- First, let us find the coordinates of points A and B where the parabola $y = -x^2 + 16$ intersects the x-axis ($y = 0$):

  $$-x^2 + 16 = 0 \implies x^2 = 16 \implies x = \pm 4$$

  Thus, the points are $A(-4, 0)$ and $B(4, 0)$.
  The length of the bottom base AB is:

  $$AB = 4 - (-4) = 8 \text{ units}$$

- Next, let us find the coordinates of points M and N where the parabola intersects the line $y = 7$:

  $$-x^2 + 16 = 7 \implies x^2 = 9 \implies x = \pm 3$$

  Thus, the points are $M(-3, 7)$ and $N(3, 7)$.
  The length of the top base MN is:

  $$MN = 3 - (-3) = 6 \text{ units}$$

- Since both segments are horizontal, they are parallel to each other.
  The quadrilateral $ABNM$ is an isosceles trapezoid.
  The vertical height h is the distance between $y = 0$ and $y = 7$, which is:

  $$h = 7 - 0 = 7 \text{ units}$$

- Now, we can calculate the area of the trapezoid:

$$\text{Area} = \frac{1}{2}(AB + MN) \cdot h$$

$$\text{Area} = \frac{1}{2}(8+6) \cdot 7 = \frac{1}{2} \cdot 14 \cdot 7 = 49$$

Step 4: Final Answer:

The area of the quadrilateral with vertices A, B, M, and N is 49.

Quick Tip: Plotting or visualizing the points on a Cartesian plane quickly reveals that the quadrilateral is a symmetric trapezoid.

This allows you to bypass complex polygon area formulas and use the simple trapezoid area formula directly.

---

43. The domain of the real function $f(x) = \sin^{-1}\left(\log_2\left(\frac{x^2}{2}\right)\right)$ is

(A) $[-2, -1] \cup [1, 2]$

(B) $[-2, -1]$

(C) $[1, 2]$

(D) $[-2, 2]$

Correct Answer: (A) $[-2, -1] \cup [1, 2]$

Solution:

Step 1: Understanding the Question:

The problem requires finding the domain of the real-valued function $f(x)$.

We must find the set of all real numbers x for which both the logarithmic function and the inverse sine function are well-defined.

Step 2: Key Formula or Approach:

For a composite function $g(x) = \sin^{-1}(u)$, the domain of the inverse sine function requires:

$$-1 \le u \le 1$$

For the logarithmic function $\log_2(v)$, the argument must satisfy:

$$v > 0$$

Step 3: Detailed Explanation:

- First, let us set up the inequality for the inverse sine function:

$$-1 \le \log_2\left(\frac{x^2}{2}\right) \le 1$$

- We can eliminate the logarithm by applying the base 2 exponentiation to all parts of the inequality:

$$2^{-1} \le \frac{x^2}{2} \le 2^1 \implies \frac{1}{2} \le \frac{x^2}{2} \le 2$$

- Multiplying the entire inequality by 2 gives:

$$1 \le x^2 \le 4$$

- Now we split this inequality into two parts:

  1) $x^2 \geq 1 \implies |x| \geq 1 \implies x \in (-\infty, -1] \cup [1, \infty)$

  2) $x^2 \leq 4 \implies |x| \leq 2 \implies x \in [-2, 2]$

- Taking the intersection of these two solution sets:

$$x \in ([-2,2]) \cap ((-\infty,-1] \cup [1,\infty)) \implies x \in [-2,-1] \cup [1,2]$$

- Since $\frac{x^2}{2} \geq \frac{1}{2} > 0$ for all x in this domain, the logarithmic argument is automatically positive.

Step 4: Final Answer:

The domain of the function is $[-2,-1] \cup [1,2]$.

Quick Tip: For nested functions, work from the outside in.

Set up the standard boundaries for the outer function and solve step-by-step using inverse operations.

---

44. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $f''(x) = -f(x)$. Let $h : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $h'(x) = f(x)^2 + (f'(x))^2$ for all $x \in \mathbb{R}$. If $h(0) = 1$ and $h\left(\frac{1}{3}\right) = \frac{5}{3}$, then the value of $h(100)$ is

(A) 201

(B) 101

(C) 100

(D) 202

Correct Answer: (A) 201

Solution:

Step 1: Understanding the Question:

We are given a twice-differentiable function $f(x)$ satisfying a second-order differential relation. Another function $h(x)$ has its derivative defined in terms of $f(x)$ and $f'(x)$.

We need to analyze the expression $f(x)^2 + (f'(x))^2$ to simplify $h'(x)$ and subsequently find $h(100)$.

Step 2: Key Formula or Approach:

We use the differentiation rules:

$$\frac{d}{dx}[u^2] = 2u\frac{du}{dx}$$

And the fundamental theorem of calculus.

Step 3: Detailed Explanation:

- Let us define a auxiliary function $g(x) = f(x)^2 + (f'(x))^2$.
- Let us find the derivative of $g(x)$ with respect to x:

  $$g'(x) = \frac{d}{dx}\left[f(x)^2 + (f'(x))^2\right]$$

  Using the chain rule:

  $$g'(x) = 2f(x)f'(x) + 2f'(x)f''(x)$$

- Factor out $2f'(x)$:

  $$g'(x) = 2f'(x)\left[f(x) + f''(x)\right]$$

- We are given that $f''(x) = -f(x)$, which means $f(x) + f''(x) = 0$.
  Substituting this into our equation:

$$g'(x) = 2f'(x) \cdot 0 = 0$$

- Since the derivative of $g(x)$ is zero for all real numbers, $g(x)$ must be a constant:

$$g(x) = C \implies f(x)^2 + (f'(x))^2 = C$$

- Substituting this back into the expression for $h'(x)$:

$$h'(x) = C$$

- Integrating both sides with respect to x:

$$h(x) = Cx + D$$

  where D is the constant of integration.

- We are given $h(0) = 1$:

$$C(0) + D = 1 \implies D = 1$$

- We are also given $h\left(\frac{1}{3}\right) = \frac{5}{3}$:

$$C\left(\frac{1}{3}\right) + 1 = \frac{5}{3} \implies \frac{C}{3} = \frac{2}{3} \implies C = 2$$

- So, the function $h(x)$ is uniquely determined as:

$$h(x) = 2x + 1$$

- Now, we can find the value of $h(100)$:

$$h(100) = 2(100) + 1 = 201$$

Step 4: Final Answer:

The value of $h(100)$ is 201.

Quick Tip: The expression $f(x)^2 + (f'(x))^2$ is analogous to $\sin^2 x + \cos^2 x$, which is constant when $f''(x) = -f(x)$.

Recognizing this conservative-like quantity simplifies the derivative of $h(x)$ to a constant immediately.

---

45. Suppose $\lim_{x\to -2} \frac{bx^2+15x+15+b}{x^2+x-2} = L$, **where** b **and** L **are real numbers. Then**

(A) $b = 3$ and $L = -1$

(B) $b = -3$ and $L = -1$

(C) $b = 3$ and $L = 1$

(D) $b = -3$ and $L = 1$

Correct Answer: (A) $b = 3$ and $L = -1$

Solution:

Step 1: Understanding the Question:

The problem presents a limit of a rational function as $x \to -2$.

Since the limit is a finite real number L, we need to analyze the behavior of the numerator and denominator near $x = -2$ to find the constants b and L.

Step 2: Key Formula or Approach:

If $\lim_{x\to c} \frac{P(x)}{Q(x)} = L$ (where L is finite) and $\lim_{x\to c} Q(x) = 0$, then we must have $\lim_{x\to c} P(x) = 0$ for a valid limit to exist.

We will use factoring of polynomials to evaluate the indeterminate form.

Step 3: Detailed Explanation:

- First, let us check the limit of the denominator as $x \to -2$:

$$\lim_{x\to -2}(x^2 + x - 2) = (-2)^2 + (-2) - 2 = 4 - 2 - 2 = 0$$

- Since the denominator goes to 0 and the limit L is a real number, the numerator must also go to 0 as $x \to -2$.
  Thus, $x = -2$ must be a root of the numerator $P(x) = bx^2 + 15x + 15 + b$:

$$P(-2) = 0 \implies b(-2)^2 + 15(-2) + 15 + b = 0$$

$$4b - 30 + 15 + b = 0 \implies 5b - 15 = 0 \implies b = 3$$

- Now substitute $b = 3$ back into the limit expression:

$$\lim_{x\to-2} \frac{3x^2+15x+15+3}{x^2+x-2} = \lim_{x\to-2} \frac{3x^2+15x+18}{x^2+x-2}$$

- We can factor both the numerator and the denominator:

  Numerator:

$$3x^2+15x+18 = 3(x^2+5x+6) = 3(x+2)(x+3)$$

  Denominator:

$$x^2+x-2 = (x+2)(x-1)$$

- Now substitute these factored forms back into the limit:

$$L = \lim_{x\to-2} \frac{3(x+2)(x+3)}{(x+2)(x-1)}$$

- Since $x \to -2$, we can cancel the common non-zero term $(x+2)$:

$$L = \lim_{x\to-2} \frac{3(x+3)}{x-1}$$

- Evaluating the limit directly by substitution:

$$L = \frac{3(-2+3)}{-2-1} = \frac{3(1)}{-3} = -1$$

Step 4: Final Answer:

Therefore, $b = 3$ and $L = -1$.

Quick Tip: For any limit of the form $\frac{0}{0}$, once you establish the value of the unknown parameter, you can also use L'Hôpital's Rule to find the limit L quickly:

$$L = \lim_{x\to-2} \frac{\frac{d}{dx}(3x^2 + 15x + 18)}{\frac{d}{dx}(x^2 + x - 2)} = \lim_{x\to-2} \frac{6x + 15}{2x + 1} = \frac{-12 + 15}{-4 + 1} = -1$$

---

46. If $x \in (0,1)$ and $\sin^{-1} x - \sin^{-1} \frac{1}{4} = \frac{\pi}{3}$, then x equals

(A) $\frac{1}{8}(1 + 3\sqrt{5})$
(B) $\frac{1}{6}(1 + 2\sqrt{5})$
(C) $\frac{1}{6}(1 + 4\sqrt{3})$
(D) $\frac{1}{5}(2 + 3\sqrt{2})$

Correct Answer: (A) $\frac{1}{8}(1 + 3\sqrt{5})$

Solution:

Step 1: Understanding the Question:

The question asks us to find the value of $x \in (0,1)$ satisfying a given trigonometric equation involving inverse sine functions.
We will solve this by isolating the term $\sin^{-1} x$ and applying sine to both sides.

Step 2: Key Formula or Approach:

We will use the sine angle addition identity:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

And the basic relationship:

$$\cos(\sin^{-1}\theta) = \sqrt{1-\theta^2} \text{ for } \theta \in [0,1]$$

Step 3: Detailed Explanation:

- Let us write down the given equation:

$$\sin^{-1}x - \sin^{-1}\frac{1}{4} = \frac{\pi}{3}$$

- Rearranging the terms to isolate $\sin^{-1}x$:

$$\sin^{-1}x = \frac{\pi}{3} + \sin^{-1}\frac{1}{4}$$

- Now, take the sine of both sides of the equation:

$$x = \sin\left(\frac{\pi}{3} + \sin^{-1}\frac{1}{4}\right)$$

- Let $\theta = \sin^{-1}\frac{1}{4}$. This means $\sin\theta = \frac{1}{4}$.
  Since θ is in the first quadrant, we can find $\cos\theta$:

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{1}{4}\right)^2} = \sqrt{1-\frac{1}{16}} = \frac{\sqrt{15}}{4}$$

- Now substitute this back into the expansion formula for $\sin\left(\frac{\pi}{3}+\theta\right)$:

$$x = \sin\frac{\pi}{3}\cos\theta + \cos\frac{\pi}{3}\sin\theta$$

- Substituting the values of the trigonometric terms:

$$x = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{15}}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)$$

- Multiplying and simplifying:

$$x = \frac{\sqrt{45}}{8} + \frac{1}{8}$$

$$x = \frac{3\sqrt{5}+1}{8} = \frac{1}{8}(1+3\sqrt{5})$$

Step 4: Final Answer:

The value of x is $\frac{1}{8}(1+3\sqrt{5})$.

Quick Tip: When dealing with inverse trigonometric equations, substituting $\theta = \sin^{-1} y$ and drawing a right triangle can make finding $\cos\theta$ intuitive and less prone to errors.

---

47. Let $S = \{(x, y) \mid x, y \in \mathbb{N}, 1 \le x \le 15, 1 \le y \le 20\}$ be a set. Let $\mathscr{R}$ be the equivalence relation on S defined by $(x, y)\mathscr{R}(x', y')$ if and only if $x + y = x' + y'$. Then the number of equivalence classes of $\mathscr{R}$ on S is

(A) 34

(B) 35

(C) 15

(D) 20

Correct Answer: (A) 34

Solution:

Step 1: Understanding the Question:

The problem defines a set S consisting of ordered pairs of natural numbers within specified ranges.

An equivalence relation $\mathscr{R}$ partitions the set into distinct equivalence classes where two elements belong to the same class if their coordinates sum to the same value.

We need to find the total number of distinct possible sums.

Step 2: Key Formula or Approach:

Let $k = x + y$ be the sum of the coordinates.

An equivalence class is formed for each unique value of k that can be achieved by at least one pair $(x, y) \in S$.

The number of equivalence classes is simply the number of distinct values of k that can be formed.

Step 3: Detailed Explanation:

- The coordinates of any pair $(x, y) \in S$ must satisfy:

$$1 \le x \le 15 \quad (x \in \mathbb{N})$$

$$1 \le y \le 20 \quad (y \in \mathbb{N})$$

- Let us find the minimum possible value of $k = x + y$:

$$k_{\text{min}} = x_{\text{min}} + y_{\text{min}} = 1 + 1 = 2$$

- Let us find the maximum possible value of $k = x + y$:

$$k_{\text{max}} = x_{\text{max}} + y_{\text{max}} = 15 + 20 = 35$$

- Since x and y are natural numbers, the sum k must be an integer.
  Can every integer value from 2 to 35 be achieved?
  - For any $k \in [2,16]$, we can choose $x = 1 \in [1,15]$ and $y = k-1 \in [1,20]$.
  - For any $k \in [17,35]$, we can choose $y = 20 \in [1,20]$ and $x = k-20$. Since $17 \leq k \leq 35$, we have $1 \leq k-20 \leq 15$, which is a valid value for x.
  - Thus, every integer $k \in [2,35]$ is achievable by at least one pair in S.

- The number of distinct values of k is:

$$N = 35 - 2 + 1 = 34$$

- Each of these values corresponds to exactly one equivalence class.

Step 4: Final Answer:

The number of equivalence classes of $\mathscr{R}$ on S is 34.

Quick Tip: An equivalence class for $(x,y)\mathscr{R}(x',y')$ is determined by the output of the function $f(x,y) = x + y$.

The number of equivalence classes is equal to the size of the range of this function over the domain S.

---

48. For a complex number $z = x + iy$, where $x, y \in \mathbb{R}$, denote $\hat{z} = y + ix$. The locus of z satisfying $|z + \hat{z}| = |z - \hat{z}|$ in the complex plane is

(A) union of the real axis and the imaginary axis

(B) the real axis

(C) a circle

(D) the straight line $y = x$

Correct Answer: (A) union of the real axis and the imaginary axis

Solution:

Step 1: Understanding the Question:

The question asks for the locus of a complex number z satisfying a given modular relation.

We are given the definition of $\hat{z}$, which swaps the real and imaginary parts of z.

By substituting $z = x + iy$ and $\hat{z} = y + ix$, we can convert the equation into Cartesian coordinates.

Step 2: Key Formula or Approach:

The modulus of a complex number $w = a + ib$ (where $a, b \in \mathbb{R}$) is given by:

$$|w| = \sqrt{a^2 + b^2}$$

Step 3: Detailed Explanation:

- Let us write down the expressions for $z + \hat{z}$ and $z - \hat{z}$:

$$z+\hat{z}=(x+iy)+(y+ix)=(x+y)+i(y+x)=(x+y)(1+i)$$

$$z-\hat{z}=(x+iy)-(y+ix)=(x-y)+i(y-x)=(x-y)(1-i)$$

- Now, we find the modulus of both terms:

$$|z+\hat{z}|=|(x+y)(1+i)|=|x+y|\cdot|1+i|=|x+y|\sqrt{1^2+1^2}=|x+y|\sqrt{2}$$

$$|z-\hat{z}|=|(x-y)(1-i)|=|x-y|\cdot|1-i|=|x-y|\sqrt{1^2+(-1)^2}=|x-y|\sqrt{2}$$

- The given equation is:

$$|z+\hat{z}|=|z-\hat{z}|$$

  Substituting our simplified modulus expressions:

$$|x+y|\sqrt{2}=|x-y|\sqrt{2}\implies|x+y|=|x-y|$$

- To solve this, we square both sides:

$$(x+y)^2=(x-y)^2$$

$$x^2+2xy+y^2=x^2-2xy+y^2$$

- Canceling out x^2 and y^2 from both sides:

$$4xy = 0 \implies xy = 0$$

- The equation $xy = 0$ means:

$$x = 0 \quad \text{or} \quad y = 0$$

- $x = 0$ represents the imaginary axis.
- $y = 0$ represents the real axis.

Step 4: Final Answer:

The locus is the union of the real axis and the imaginary axis.

Quick Tip: Notice that $\hat{z}$ is related to the conjugate of z by $\hat{z} = i\bar{z}$.

Substituting this gives $|z + i\bar{z}| = |z - i\bar{z}|$.

Squaring this and using properties of complex numbers leads to the same relation $xy = 0$ very quickly.

---

49. Let m and p be real numbers such that the polynomial $f(x) = x^2 + mx + p$ has two distinct negative rational roots. Then the polynomial $g(x) = x^2 - (m^2 - 2p)x + p^2$ has distinct

(A) positive rational roots

(B) positive irrational roots

(C) negative rational roots

(D) negative irrational roots

Correct Answer: (A) positive rational roots

Solution:

Step 1: Understanding the Question:

We are given information about the roots of a quadratic polynomial $f(x)$ (distinct, negative, and rational).

We need to determine the nature of the roots (distinctness, sign, and rationality) of a related quadratic polynomial $g(x)$.

Step 2: Key Formula or Approach:

For a quadratic equation $ax^2 + bx + c = 0$:

- The discriminant is $D = b^2 - 4ac$.
- The roots are rational if and only if D is a perfect square of a rational number (assuming coefficients are rational).
- Vieta's formulas state that the sum of roots is $-b/a$ and the product of roots is c/a.

Step 3: Detailed Explanation:

- Let α and β be the roots of $f(x) = x^2 + mx + p$.
  Since α and β are rational and distinct, the coefficients m and p must be rational numbers.
  Since the roots are negative, we have $\alpha < 0$ and $\beta < 0$.

- Using Vieta's formulas:

$$m = -(\alpha + \beta) > 0$$

$$p = \alpha\beta > 0$$

- Since the roots of $f(x)$ are distinct and rational, the discriminant D_f must be a positive perfect square of a rational number, say k^2 $(k \in \mathbb{Q}, k > 0)$:

$$D_f = m^2 - 4p = k^2 > 0$$

- Now let us examine the polynomial $g(x) = x^2 - (m^2 - 2p)x + p^2$.
  Let us compute its discriminant D_g:

$$D_g = (m^2 - 2p)^2 - 4p^2$$

  Using the identity $a^2 - b^2 = (a - b)(a + b)$:

$$D_g = (m^2 - 2p - 2p)(m^2 - 2p + 2p)$$

$$D_g = (m^2 - 4p)(m^2)$$

- Substituting $m^2 - 4p = k^2$ into the expression:

$$D_g = k^2m^2 = (km)^2$$

  Since k and m are rational numbers, their product km is also rational. Thus, D_g is a perfect square of a rational number.
  Since $k > 0$ and $m > 0$, we have $D_g > 0$.
  Therefore, the roots of $g(x)$ are real, distinct, and rational.

- Let us analyze the signs of the roots of $g(x)$ using Vieta's formulas:
  - Product of roots:

$$\text{Product} = p^2 > 0$$

- Sum of roots:

$$\text{Sum} = m^2 - 2p$$

Substitute $m = -(\alpha + \beta)$ and $p = \alpha\beta$:

$$\text{Sum} = (\alpha + \beta)^2 - 2\alpha\beta = \alpha^2 + \beta^2$$

Since $\alpha \neq \beta$ and both are non-zero, $\alpha^2 + \beta^2 > 0$.

- Since both the sum and the product of the roots are positive, both roots of $g(x)$ must be positive.

Step 4: Final Answer:

The polynomial $g(x)$ has distinct positive rational roots.

Quick Tip: Using a simple example can save time. Let the roots of $f(x)$ be -1 and -2.
Then $f(x) = (x+1)(x+2) = x^2 + 3x + 2 \implies m = 3, p = 2$.
Substituting these into $g(x)$ gives $g(x) = x^2 - (9-4)x + 4 = x^2 - 5x + 4 = (x-1)(x-4)$.
The roots of $g(x)$ are 1 and 4, which are distinct, positive, and rational.

---

50. Amar, Akbar, and Anthony write a test independently. The probability that Amar passes the test and Akbar fails the test is $\frac{3}{20}$. The probability that Akbar passes the test and Anthony fails the test is $\frac{1}{4}$. The probability that Amar and Anthony both pass the test is $\frac{2}{5}$. Then the probability that at least one of Amar, Akbar and Anthony fails the test is

(A) $\frac{7}{10}$
(B) $\frac{3}{10}$
(C) $\frac{3}{5}$
(D) $\frac{4}{5}$

Correct Answer: (A) $\frac{7}{10}$

Solution:

Step 1: Understanding the Question:

The problem involves three independent events (passing a test) with given compound probabilities.
We need to determine individual probabilities to find the probability of the complement event: "at least one of them fails".

Step 2: Key Formula or Approach:

For independent events A, B, C:
- $P(A \cap B') = P(A) \cdot P(B') = P(A)(1 - P(B))$
- $P(A \cap C) = P(A) \cdot P(C)$
- $P(\text{at least one fails}) = 1 - P(\text{all pass}) = 1 - P(A)P(B)P(C)$

Step 3: Detailed Explanation:

- Let a, b, and c be the probabilities that Amar, Akbar, and Anthony pass the test, respectively.

- From the given conditions, we write the equations:
  1) $a(1-b) = \frac{3}{20}$
  2) $b(1-c) = \frac{1}{4}$
  3) $ac = \frac{2}{5} \implies c = \frac{2}{5a}$

- Substitute c into equation (2):

$$b\left(1-\frac{2}{5a}\right)=\frac{1}{4} \implies b\left(\frac{5a-2}{5a}\right)=\frac{1}{4} \implies b=\frac{5a}{4(5a-2)}$$

- Substitute this expression for b into equation (1):

$$a\left(1-\frac{5a}{4(5a-2)}\right)=\frac{3}{20}$$

$$a\left(\frac{20a-8-5a}{4(5a-2)}\right)=\frac{3}{20}$$

$$a\left(\frac{15a-8}{4(5a-2)}\right)=\frac{3}{20}$$

- Multiplying both sides by 20:

$$5a\left(\frac{15a-8}{5a-2}\right)=3$$

$$75a^2-40a=15a-6 \implies 75a^2-55a+6=0$$

- Factoring the quadratic:

$$(5a-3)(15a-2)=0 \implies a=\frac{3}{5} \text{ or } a=\frac{2}{15}$$

- Let us evaluate both cases:

Case 1: $a = \frac{3}{5}$

$$c = \frac{2}{5\left(\frac{3}{5}\right)} = \frac{2}{3} \quad \text{(valid, since } 0 \leq c \leq 1\text{)}$$

$$b = \frac{5\left(\frac{3}{5}\right)}{4\left(5\left(\frac{3}{5}\right) - 2\right)} = \frac{3}{4(3-2)} = \frac{3}{4} \quad \text{(valid)}$$

Case 2: $a = \frac{2}{15}$

$$c = \frac{2}{5\left(\frac{2}{15}\right)} = 3 \quad \text{(invalid, since probability cannot exceed 1)}$$

- Thus, the individual probabilities are $a = \frac{3}{5}$, $b = \frac{3}{4}$, and $c = \frac{2}{3}$.
- Calculate the probability that all three pass:

$$P(Amar \cap Akbar \cap Anthony) = a \cdot b \cdot c = \frac{3}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} = \frac{3}{10}$$

- The probability that at least one fails is:

$$P(\text{at least one fails}) = 1 - \frac{3}{10} = \frac{7}{10}$$

Step 4: Final Answer:

The probability that at least one fails the test is $\frac{7}{10}$.

Quick Tip: Using complementary probability is the fastest way to solve "at least one" problems. Always check the validity of solved parameters (they must lie in $[0, 1]$) to eliminate extraneous algebraic solutions.

---

51. The probability that the sum of two integers m and n, where $m, n \in \{1, 2, \ldots, 50\}$, chosen randomly and independently, being divisible by 3 is

(A) 0.3336
(B) 0.3332
(C) 0.3333
(D) 0.3338

Correct Answer: (A) 0.3336

Solution:

Step 1: Understanding the Question:
The problem asks for the probability that the sum of two integers chosen independently from the first 50 natural numbers is divisible by 3.
We will use modulo 3 arithmetic to classify the numbers and find the total number of favorable combinations.

Step 2: Key Formula or Approach:
For the sum $m + n$ to be divisible by 3, the remainders of m and n when divided by 3 must sum to a multiple of 3.
The possibilities for the remainders are:
- Both are $0 \pmod 3$
- One is $1 \pmod 3$ and the other is $2 \pmod 3$

Step 3: Detailed Explanation:

- First, let us partition the set $\{1, 2, \ldots, 50\}$ into three subsets based on their remainder modulo 3:

1) $R_0 = \{3, 6, 9, \ldots, 48\} \implies |R_0| = 16$

2) $R_1 = \{1, 4, 7, \ldots, 49\} \implies |R_1| = 17$

3) $R_2 = \{2, 5, 8, \ldots, 50\} \implies |R_2| = 17$

- Since m and n are chosen independently, the total number of possible pairs is:

$$\text{Total Pairs} = 50 \times 50 = 2500$$

- For $(m+n)$ to be divisible by 3, we have three cases:

  Case 1: Both $m, n \in R_0$.

  The number of such pairs is:

$$16 \times 16 = 256$$

  Case 2: $m \in R_1$ and $n \in R_2$.

  The number of such pairs is:

$$17 \times 17 = 289$$

  Case 3: $m \in R_2$ and $n \in R_1$.

  The number of such pairs is:

$$17 \times 17 = 289$$

- Summing the favorable pairs:

$$\text{Favorable Pairs} = 256 + 289 + 289 = 834$$

- The probability is given by:

$$P = \frac{834}{2500} = 0.3336$$

Step 4: Final Answer:

The probability is 0.3336.

Quick Tip: If the set of numbers was a perfect multiple of 3 (e.g. 51 elements), the probability would be exactly $\frac{1}{3} \approx 0.3333$.

Since we have slightly more elements in R_1 and R_2 than in R_0, the probability is slightly higher than $\frac{1}{3}$.

This immediately eliminates options B and C.

---

52. Let a variable line L meet x-axis and y-axis at points A and B, respectively. Suppose the distance of the line L from the origin is 3 units. Then the equation of the locus of the point C that divides the line segment AB internally in the ratio $2:1$ is

(A) $\frac{4}{x^2} + \frac{1}{y^2} = 1$

(B) $\frac{1}{x^2} + \frac{4}{y^2} = 1$

(C) $x^2 + 4y^2 = 9$

(D) $4x^2 + y^2 = 9$

Correct Answer: (B) $\frac{1}{x^2} + \frac{4}{y^2} = 1$

Solution:

Step 1: Understanding the Question:

A line intersects both axes forming a segment AB. Its distance from the origin is constant. We need to find the equation representing the locus of a point C dividing this segment in a 2 : 1 ratio.

Step 2: Key Formula or Approach:

- The intercept form of a line is:

$$\frac{x}{a}+\frac{y}{b}=1$$

- The perpendicular distance from the origin $(0,0)$ is:

$$d=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}$$

- The internal section formula for dividing segment joining (x_1,y_1) and (x_2,y_2) in ratio $m:n$ is:

$$\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n}\right)$$

Step 3: Detailed Explanation:

- Let the coordinates of A and B be $(a,0)$ and $(0,b)$, respectively.

- The perpendicular distance of the line $\frac{x}{a}+\frac{y}{b}=1$ from the origin is 3 units:

$$\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}=3 \implies \frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{9}$$

- Let $C(h,k)$ be the point that divides the segment AB internally in the ratio 2 : 1.
  Using the section formula with $m=2$, $n=1$:

$$h = \frac{2(0) + 1(a)}{2+1} = \frac{a}{3} \implies a = 3h$$

$$k = \frac{2(b) + 1(0)}{2+1} = \frac{2b}{3} \implies b = \frac{3k}{2}$$

- Now substitute a and b into the distance relation:

$$\frac{1}{(3h)^2} + \frac{1}{\left(\frac{3k}{2}\right)^2} = \frac{1}{9}$$

$$\frac{1}{9h^2} + \frac{4}{9k^2} = \frac{1}{9}$$

- Multiplying both sides of the equation by 9:

$$\frac{1}{h^2} + \frac{4}{k^2} = 1$$

- Replacing (h, k) with general coordinates (x, y) gives the locus of point C:

$$\frac{1}{x^2} + \frac{4}{y^2} = 1$$

Step 4: Final Answer:

The equation of the locus of point C is $\frac{1}{x^2} + \frac{4}{y^2} = 1$.

Quick Tip: Be careful with the order of points in the section formula.

Since the ratio is 2 : 1 from A to B, the coordinate h is associated with a (since $x_B = 0$), and k is associated with b (since $y_A = 0$).

---

53. In a building, an elevator starts from the ground floor (that is, 0th floor) with 10 passengers and stops at every floor until the 15th floor which is the topmost floor. No new passengers enter the elevator on any floor from the 1st through the 15th floor. If all the passengers got off the elevator then the probability that at most one passenger got off at each floor is

(A) $\frac{15!}{5!\times 15^{10}}$
(B) $\frac{10!}{15^{10}}$
(C) $\frac{15!}{10^{15}}$
(D) $\frac{15!}{5!\times 10^{15}}$

Correct Answer: (A) $\frac{15!}{5!\times 15^{10}}$

Solution:

Step 1: Understanding the Question:

We need to calculate the probability that 10 distinct passengers get off an elevator at 15 available floors such that no two passengers get off at the same floor.

Step 2: Key Formula or Approach:

- Each passenger has 15 choices for the floor to exit.
- The total number of ways for 10 passengers to exit is N^n.
- The favorable number of ways where at most one passenger exits at any floor is the permutation ${}^{N}P_n$.
- Probability is given by $\frac{\text{Favorable Outcomes}}{\text{Total Outcomes}}$.

Step 3: Detailed Explanation:

- There are 10 passengers and 15 potential stopping floors (1st through 15th).

- Each passenger chooses a floor independently.

  Thus, the total number of ways for 10 passengers to exit the elevator is:

$$\text{Total Ways} = 15 \times 15 \times \cdots \times 15 = 15^{10}$$

- For the condition "at most one passenger got off at each floor" to be satisfied, all 10 passengers must get off at 10 distinct floors.

  The number of ways to choose 10 distinct floors out of 15 and assign one passenger to each is:

$$\text{Favorable Ways} = {}^{15}P_{10} = \frac{15!}{(15-10)!} = \frac{15!}{5!}$$

- The required probability P is:

$$P = \frac{\text{Favorable Ways}}{\text{Total Ways}} = \frac{15!}{5! \times 15^{10}}$$

Step 4: Final Answer:

The probability is $\frac{15!}{5! \times 15^{10}}$.

Quick Tip: This problem is identical to distributing n distinct items into N distinct bins with a maximum capacity of 1.

Using the permutation notation ${}^{N}P_n / N^n$ makes finding the answer straightforward.

54. Let S be the set of real values of k for which the system of equations

$$kx + y + z = k$$

$$x + ky + z = k$$

$$x + y + kz = k$$

has no solution. Then $|S|$ is

(A) 1
(B) 2
(C) 0
(D) ∞

Correct Answer: (A) 1

Solution:

Step 1: Understanding the Question:

The question asks for the number of real values of k for which the given system of linear equations is inconsistent (has no solution).

Step 2: Key Formula or Approach:

For a system of linear equations to have no solution, the determinant of the coefficients (D) must be 0.

If $D = 0$, the system can either have infinitely many solutions or no solution. We must analyze both cases.

Step 3: Detailed Explanation:

- Let us write the determinant of the coefficient matrix:

$$D = \begin{vmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{vmatrix}$$

- Evaluating the determinant:

$$D = k(k^2 - 1) - 1(k - 1) + 1(1 - k)$$

$$D = k(k - 1)(k + 1) - 2(k - 1)$$

$$D = (k - 1)[k(k + 1) - 2] = (k - 1)(k^2 + k - 2)$$

$$D = (k - 1)^2(k + 2)$$

- Setting $D = 0$ gives the critical values:

$$k = 1 \quad \text{or} \quad k = -2$$

- Let us analyze Case 1: $k = 1$:

  The system of equations becomes:

$$x + y + z = 1$$

$$x + y + z = 1$$

$$x + y + z = 1$$

This represents a single plane, which has infinitely many solutions. Thus, $k = 1 \notin S$.

- Let us analyze Case 2: $k = -2$:

  The system of equations becomes:

  $$-2x + y + z = -2 \quad \text{— (Eq 1)}$$

  $$x - 2y + z = -2 \quad \text{— (Eq 2)}$$

  $$x + y - 2z = -2 \quad \text{— (Eq 3)}$$

  Adding all three equations:

  $$(-2+1+1)x + (1-2+1)y + (1+1-2)z = -2-2-2$$

  $$0x + 0y + 0z = -6 \implies 0 = -6$$

  This is a contradiction, which means the system has no solution. Thus, $k = -2 \in S$.

- Therefore, the set S contains only one element, $S = \{-2\}$, so $|S| = 1$.

Step 4: Final Answer:

The number of elements in S is 1.

Quick Tip: For symmetric systems, $k = 1$ often represents a case of identical planes (infinitely many solutions), while the other root (here, $k = -2$) represents parallel but distinct conditions (no solution).

---

55. The definite integral

$$\int_{\pi/2}^{\pi} \frac{\sin x - x\cos x}{x(x + \sin x)}\,dx$$

equals

(A) $\log\left(1 + \frac{2}{\pi}\right)$
(B) $\log\left(1 + \frac{\pi}{2}\right)$
(C) $1 + \frac{2}{\pi}$
(D) $1 + \frac{\pi}{2}$

Correct Answer: (A) $\log\left(1 + \frac{2}{\pi}\right)$

Solution:

Step 1: Understanding the Question:

The problem requires evaluating a definite integral.

We will simplify the integrand using algebraic manipulation to split it into integrable fractions.

Step 2: Key Formula or Approach:

We will use algebraic splitting:

$$\frac{\sin x - x\cos x}{x(x + \sin x)} = \frac{1}{x} - \frac{1 + \cos x}{x + \sin x}$$

And the basic integration rules:

$$\int \frac{1}{x}\,dx = \ln|x|$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|$$

Step 3: Detailed Explanation:

- Let us perform the algebraic splitting of the integrand:

$$\frac{\sin x - x\cos x}{x(x+\sin x)} = \frac{(x+\sin x) - x(1+\cos x)}{x(x+\sin x)}$$

$$= \frac{x+\sin x}{x(x+\sin x)} - \frac{x(1+\cos x)}{x(x+\sin x)}$$

$$= \frac{1}{x} - \frac{1+\cos x}{x+\sin x}$$

- Now we can write the definite integral as:

$$I = \int_{\pi/2}^{\pi} \left(\frac{1}{x} - \frac{1+\cos x}{x+\sin x} \right) dx$$

- Let us find the antiderivative:
  - For the first term: $\int \frac{1}{x}\,dx = \ln(x)$.
  - For the second term, let $u = x + \sin x$, then $du = (1+\cos x)\,dx$. This gives:

$$\int \frac{1+\cos x}{x+\sin x}\,dx = \ln(x+\sin x)$$

- Therefore, the antiderivative is:

$$F(x) = \ln(x) - \ln(x + \sin x) = \ln\left(\frac{x}{x + \sin x}\right)$$

- Evaluating at the limits $\frac{\pi}{2}$ and π:

  - At $x = \pi$:

$$F(\pi) = \ln\left(\frac{\pi}{\pi + \sin\pi}\right) = \ln\left(\frac{\pi}{\pi}\right) = \ln(1) = 0$$

  - At $x = \frac{\pi}{2}$:

$$F\left(\frac{\pi}{2}\right) = \ln\left(\frac{\pi/2}{\pi/2 + \sin(\pi/2)}\right) = \ln\left(\frac{\pi/2}{\pi/2 + 1}\right) = \ln\left(\frac{\pi}{\pi + 2}\right)$$

- Calculate the final value of the integral I:

$$I = F(\pi) - F\left(\frac{\pi}{2}\right) = 0 - \ln\left(\frac{\pi}{\pi + 2}\right)$$

$$I = \ln\left(\frac{\pi + 2}{\pi}\right) = \ln\left(1 + \frac{2}{\pi}\right)$$

Step 4: Final Answer:

The definite integral equals $\log\left(1 + \frac{2}{\pi}\right)$.

Quick Tip: Whenever the integrand contains both trigonometric terms and polynomial terms, look to group them in a way that generates the form $\frac{f'(x)}{f(x)}$.

---

56. The number of solutions of the equation $|\sin(\pi x)| = \frac{1}{50}(x^2+1)$ in $\mathbb{R}$ is

(A) 28
(B) 26
(C) 14
(D) 13

Correct Answer: (A) 28

Solution:

Step 1: Understanding the Question:

We need to find the total number of real solutions to the transcendental equation.
Since both functions are even, the graph is symmetric about the y-axis. We can count the solutions for $x > 0$ and double the result.

Step 2: Key Formula or Approach:

- Use the range of the sine function: $|\sin(\pi x)| \leq 1$.
- Determine the domain boundary where a solution is possible: $\frac{1}{50}(x^2+1) \leq 1$.

Step 3: Detailed Explanation:

- First, we establish the range of possible solutions. Since $|\sin(\pi x)| \leq 1$, we must have:

$$\frac{1}{50}(x^2+1) \leq 1 \implies x^2+1 \leq 50 \implies x^2 \leq 49 \implies x \in [-7,7]$$

- At $x = 0$, the left-hand side is $|\sin(0)| = 0$, and the right-hand side is $\frac{1}{50} = 0.02$. Since

$0 \neq 0.02$, $x = 0$ is not a solution.

- Let us analyze positive solutions in the interval $(0, 7]$.

  The function $|\sin(\pi x)|$ is periodic with period 1. It consists of 7 "humps" on the interval $(0, 7]$, one on each interval $[k, k+1]$ for $k = 0, 1, 2, 3, 4, 5, 6$.

  On each interval $[k, k+1]$:

  - At the endpoints $x = k$ and $x = k+1$, the function $|\sin(\pi x)| = 0$.
  - At the midpoint $x = k + 0.5$, the function reaches its peak value of 1.
  - The quadratic function $y = \frac{1}{50}(x^2 + 1)$ is strictly increasing for $x > 0$.
  - Let us compare the peak value and endpoint values of both functions:

  For any $k \in \{0, 1, \ldots, 6\}$, the maximum value of $y(x)$ on $[k, k+1]$ occurs at $x = k+1$, and since $x \leq 7$, we have $y(k+1) \leq y(7) = \frac{49+1}{50} = 1$.

  Specifically, at the midpoint $x = k + 0.5$, the value of the quadratic function is:

  $$y(k + 0.5) < y(7) = 1$$

  Since $|\sin(\pi x)|$ rises to 1 at the midpoint and falls to 0 at the endpoints, while the continuous function $y(x)$ is strictly less than 1 at the midpoint and greater than 0 at the endpoints, the two curves must intersect exactly twice in each of the 7 intervals $[k, k+1]$.

- Thus, the number of positive solutions is:

  $$7 \times 2 = 14$$

- Since the equation is symmetric (both sides are even functions), there are also exactly 14 negative solutions.

- The total number of solutions is:

$$14 + 14 = 28$$

Step 4: Final Answer:

The number of solutions of the equation is 28.

Quick Tip: Plotting a rough graph of the parabolic curve intersecting the absolute sine humps makes this counting problem very visual.

For each full period of the sine function that lies below the maximum bound ($y \leq 1$), there are exactly two intersection points.

---

57. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x^5 + x^3$ and let $g(x) = f^{-1}(x)$ be the inverse of f. If $g''(-2) = \frac{a}{b}$ where a and b are positive coprime integers, then the value of a is

(A) 13

(B) 26

(C) 39

(D) 256

Correct Answer: (A) 13

Solution:

Step 1: Understanding the Question:

The problem requires finding the second derivative of the inverse function $g(x) = f^{-1}(x)$ at a specific point $x = -2$.

Step 2: Key Formula or Approach:

We will use the derivative of inverse functions:

$$g'(x) = \frac{1}{f'(g(x))}$$

Differentiating this relation again gives:

$$g''(x) = -\frac{f''(g(x)) \cdot g'(x)}{[f'(g(x))]^2} = -\frac{f''(g(x))}{[f'(g(x))]^3}$$

Step 3: Detailed Explanation:

- First, we need to find $g(-2)$, which is the value of y such that $f(y) = -2$:

  $$f(y) = y^5 + y^3 = -2$$

  By inspection, we see that $y = -1$ is a solution:

  $$(-1)^5 + (-1)^3 = -1 - 1 = -2$$

  Since $f(x)$ is strictly increasing ($f'(x) = 5x^4 + 3x^2 > 0$ for $x \neq 0$), $y = -1$ is the unique real solution. Thus, $g(-2) = -1$.

- Now, we find the first and second derivatives of $f(x)$:

  $$f'(x) = 5x^4 + 3x^2$$

  $$f''(x) = 20x^3 + 6x$$

- Next, evaluate these derivatives at $y = -1$:

$$f'(-1) = 5(-1)^4 + 3(-1)^2 = 5 + 3 = 8$$

$$f''(-1) = 20(-1)^3 + 6(-1) = -20 - 6 = -26$$

- Substitute these values into the formula for $g''(-2)$:

$$g''(-2) = -\frac{f''(-1)}{[f'(-1)]^3}$$

$$g''(-2) = -\frac{-26}{8^3} = \frac{26}{512} = \frac{13}{256}$$

- We are given $g''(-2) = \frac{a}{b}$ where a and b are positive coprime integers:

$$a = 13, \quad b = 256$$

Step 4: Final Answer:

The value of a is 13.

Quick Tip: Using the identity $f(g(x)) = x$ and differentiating implicitly twice is often easier to remember than the direct formula for the derivative of an inverse function.

---

58. Let S_1 and S_2 be two circles drawn inside a unit square ABCD, touching each other externally. Further, the circle S_1 touches the sides AD and DC; and the circle S_2 touches the sides AB and

BC. If the area of S_2 is twice the area of S_1, then the radius of S_1 is

(A) $3\sqrt{2}-4$
(B) $3-2\sqrt{2}$
(C) $2-\sqrt{2}$
(D) $\frac{3}{\sqrt{2}}-2$

Correct Answer: (A) $3\sqrt{2}-4$

Solution:

Step 1: Understanding the Question:

Two circles are inscribed in opposite corners of a unit square such that they touch the adjacent sides of the square and touch each other externally.
We need to find the radius of the smaller circle S_1 using the geometric relations of the square.

Step 2: Key Formula or Approach:

Let the vertices of the unit square be positioned on a Cartesian coordinate plane:
- Let $D=(0,0)$, $A=(0,1)$, $B=(1,1)$, and $C=(1,0)$.
- The distance between the centers of two externally touching circles is $d=r_1+r_2$.

Step 3: Detailed Explanation:

- Let r_1 be the radius of S_1 and r_2 be the radius of S_2.
  - S_1 touches AD $(x=0)$ and DC $(y=0)$. Its center is:

$$C_1=(r_1,r_1)$$

  - S_2 touches AB $(y=1)$ and BC $(x=1)$. Its center is:

$$C_2=(1-r_2,1-r_2)$$

- Since the two circles touch each other externally, the distance between their centers is $r_1 + r_2$:

$$C_1C_2 = \sqrt{(1-r_2-r_1)^2+(1-r_2-r_1)^2} = \sqrt{2}(1-r_1-r_2)$$

  Set this equal to $r_1 + r_2$:

$$\sqrt{2}(1-(r_1+r_2)) = r_1 + r_2$$

$$\sqrt{2} = (r_1+r_2)(\sqrt{2}+1) \implies r_1 + r_2 = \frac{\sqrt{2}}{\sqrt{2}+1} = 2-\sqrt{2}$$

- We are given that the area of S_2 is twice the area of S_1:

$$\pi r_2^2 = 2\pi r_1^2 \implies r_2 = \sqrt{2}r_1$$

- Substitute $r_2 = \sqrt{2}r_1$ into our sum equation:

$$r_1 + \sqrt{2}r_1 = 2-\sqrt{2}$$

$$r_1(1+\sqrt{2}) = 2-\sqrt{2}$$

- Solve for r_1:

$$r_1 = \frac{2-\sqrt{2}}{\sqrt{2}+1} = \frac{\sqrt{2}(\sqrt{2}-1)}{\sqrt{2}+1}$$

Multiply numerator and denominator by $(\sqrt{2}-1)$:

$$r_1 = \sqrt{2}(\sqrt{2}-1)^2 = \sqrt{2}(2-2\sqrt{2}+1)$$

$$r_1 = \sqrt{2}(3-2\sqrt{2}) = 3\sqrt{2}-4$$

Step 4: Final Answer:

The radius of S_1 is $3\sqrt{2}-4$.

Quick Tip: Using the diagonal of the square is a great way to verify: the total diagonal of the unit square is $\sqrt{2}$.

The diagonal is also equal to $r_1\sqrt{2}+r_1+r_2+r_2\sqrt{2}$, which directly yields $(r_1+r_2)(1+\sqrt{2})=\sqrt{2}$.

---

59. Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x\left(\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then

(A) there exists a constant C such that $|f(x)| \leq C$ for all $x \in \mathbb{R}$.

(B) f is monotonically increasing in the interval (-1, 1).

(C) f is differentiable at $x = 0$.

(D) f is not continuous at $x = 0$.

Correct Answer: (A) there exists a constant C such that $|f(x)| \leq C$ for all $x \in \mathbb{R}$.

Solution:

Step 1: Understanding the Question:

The problem presents a piecewise-defined function and asks us to evaluate its continuity, differentiability, monotonicity, and boundedness.

Step 2: Key Formula or Approach:

We can write the function in terms of hyperbolic tangent:

$$f(x) = x \tanh\left(\frac{1}{x}\right) \text{ for } x \neq 0$$

We will use standard limits and inequalities for $\tanh(u)$.

Step 3: Detailed Explanation:

- Let us analyze continuity at $x = 0$:

  $$\lim_{x\to 0} f(x) = \lim_{x\to 0} x \tanh\left(\frac{1}{x}\right)$$

  Since $|\tanh(u)| < 1$ for all real numbers u, we have:

  $$|f(x)| = |x| \left|\tanh\left(\frac{1}{x}\right)\right| \leq |x|$$

  As $x \to 0$, $|x| \to 0$. By the Squeeze Theorem, $\lim_{x\to 0} f(x) = 0 = f(0)$.
  Therefore, $f(x)$ is continuous at $x = 0$. This makes option (D) false.

- Let us check differentiability at $x = 0$:
  Using the definition of the derivative:

  $$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h \tanh(1/h) - 0}{h} = \lim_{h\to 0} \tanh\left(\frac{1}{h}\right)$$

- As $h \to 0^+$, $\frac{1}{h} \to \infty$, so $\tanh(1/h) \to 1$.
- As $h \to 0^-$, $\frac{1}{h} \to -\infty$, so $\tanh(1/h) \to -1$.

Since the left-hand and right-hand limits are different, $f(x)$ is not differentiable at $x = 0$. This makes option (C) false.

- Let us check monotonicity:

  The function $f(x) = x\tanh(1/x)$ is an even function because:

$$f(-x) = (-x)\tanh(-1/x) = (-x)(-\tanh(1/x)) = x\tanh(1/x) = f(x)$$

  A non-constant even function cannot be monotonically increasing on a symmetric interval around 0. This makes option (B) false.

- Now let us check boundedness:

  For any $x \neq 0$, let $u = \frac{1}{|x|} > 0$. Then:

$$|f(x)| = |x|\tanh\left(\frac{1}{|x|}\right) = \frac{\tanh(u)}{u}$$

  Since $\tanh(u) < u$ for all $u > 0$, we have:

$$\frac{\tanh(u)}{u} < 1 \implies |f(x)| < 1 \text{ for all } x \in \mathbb{R}$$

  Thus, the function is bounded by $C = 1$. This makes option (A) correct.

Step 4: Final Answer:

There exists a constant C such that $|f(x)| \leq C$ for all $x \in \mathbb{R}$.

Quick Tip: For any function of the form $x \cdot g(x)$ where $g(x)$ is bounded, the limit as $x \to 0$ is always 0. This quickly guarantees continuity at 0.

60. The limit

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left[1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}\right]$$

(A) equals 2

(B) equals 1

(C) equals 0

(D) does not exist

Correct Answer: (A) equals 2

Solution:

Step 1: Understanding the Question:

The problem asks for the limit of a sum of square-root reciprocals scaled by $\frac{1}{\sqrt{n}}$ as n approaches infinity.

Step 2: Key Formula or Approach:

We can evaluate this limit using the Riemann Sum approximation of definite integrals:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} g\left(\frac{i}{n}\right) = \int_0^1 g(x)\,dx$$

Step 3: Detailed Explanation:

- Let us rewrite the given expression inside the limit:

$$L = \lim_{n\to\infty} \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{i}}$$

- We can rewrite this by multiplying and dividing by $\sqrt{n}$ to bring it into the standard

Riemann sum form:

$$L = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}\cdot\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$$

$$L = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{\sqrt{n}}{\sqrt{i}}$$

$$L = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sqrt{\frac{i}{n}}}$$

- This matches the standard Riemann sum structure where $g(x) = \frac{1}{\sqrt{x}}$:

$$L = \int_0^1 \frac{1}{\sqrt{x}}\, dx$$

- Let us evaluate this definite integral:

$$I = \int_0^1 x^{-1/2}\, dx = \left[2x^{1/2}\right]_0^1 = 2(1) - 2(0) = 2$$

Step 4: Final Answer:

The limit equals 2.

Quick Tip: An alternative way to solve this is using the Squeeze Theorem by bounding the sum with the integral:

$$\int_1^{n+1} \frac{1}{\sqrt{x}}\,dx < \sum_{i=1}^{n} \frac{1}{\sqrt{i}} < 1 + \int_1^{n} \frac{1}{\sqrt{x}}\,dx$$

Evaluating and dividing by $\sqrt{n}$ easily shows both sides approach 2.

Physics

61. The angular speed of the Earth's rotation is 7.3×10^{-5} rad $\cdot$ s^{-1}. Take the radius of the Earth at the equator to be 6400 km. Then the ratio (a_c/g) of the magnitude of the centripetal acceleration a_c at a point on the equator to g, is of the order

(A) 10^{-3}
(B) 10^{-5}
(C) 10^{0}
(D) 10^{-1}

Correct Answer: (A) 10^{-3}

Solution:

Step 1: Understanding the Question:

The question asks us to compute the ratio of the centripetal acceleration of a point on the Earth's equator to the standard acceleration due to gravity, and then determine its order of magnitude.

Centripetal acceleration arises from the Earth's rotation about its polar axis.

Step 2: Key Formula or Approach:

The formula for centripetal acceleration a_c of a point rotating at a distance R with angular speed ω is:

$$a_c = \omega^2 R$$

We will substitute the given values into this formula and divide by the standard acceleration due to gravity $g \approx 9.8\ \text{m/s}^2$.

Step 3: Detailed Explanation:

- The given angular speed of rotation is $\omega = 7.3 \times 10^{-5}$ rad/s.
- The equatorial radius of the Earth is $R = 6400\ \text{km} = 6.4 \times 10^6$ m.
- Let us calculate the centripetal acceleration a_c at the equator:

$$a_c = (7.3 \times 10^{-5}\ \text{rad/s})^2 \times (6.4 \times 10^6\ \text{m})$$

$$a_c = (53.29 \times 10^{-10}) \times (6.4 \times 10^6)$$

$$a_c = 341.056 \times 10^{-4} \approx 0.034\ \text{m/s}^2$$

- Now, we find the ratio of this centripetal acceleration to the acceleration due to gravity ($g \approx 9.8\ \text{m/s}^2$):

$$\frac{a_c}{g} = \frac{0.034\ \text{m/s}^2}{9.8\ \text{m/s}^2} \approx 0.00347$$

- Expressing this ratio in scientific notation:

$$\frac{a_c}{g} = 3.47 \times 10^{-3}$$

- The order of magnitude of this value is 10^{-3}.

Step 4: Final Answer:

The ratio of the centripetal acceleration to g is of the order of 10^{-3}.

Quick Tip: An order of magnitude estimate can be done quickly by rounding numbers:

$\omega \approx 7 \times 10^{-5}$, so $\omega^2 \approx 5 \times 10^{-9}$.

With $R \approx 6 \times 10^6$, we get $a_c \approx 3 \times 10^{-2} = 0.03$.

Dividing by $g \approx 10$ yields $0.003 = 3 \times 10^{-3}$, confirming the order of magnitude.

---

62. An electromagnetic wave travels from vacuum into a non-magnetic dielectric medium with permittivity $\epsilon = 4\epsilon_0$. If the ratio (E_0/B_0) of the electric field amplitude E_0 to the magnetic field amplitude B_0 in vacuum is equal to the speed of light c, then the corresponding ratio in the given medium is

(A) $c/2$
(B) c
(C) $c/\sqrt{2}$
(D) $c/4$

Correct Answer: (A) $c/2$

Solution:

Step 1: Understanding the Question:

The problem asks for the ratio of the electric field amplitude to the magnetic field amplitude of an electromagnetic wave inside a specific dielectric medium.
This ratio is fundamentally linked to the speed of wave propagation in that medium.

Step 2: Key Formula or Approach:

For any electromagnetic wave propagating in a medium, the ratio of the electric field amplitude to the magnetic field amplitude is equal to the wave speed in that medium:

$$\frac{E}{B} = v$$

The speed of light in a medium is given by:

$$v = \frac{1}{\sqrt{\mu\epsilon}}$$

Step 3: Detailed Explanation:

- In vacuum, the permeability is μ_0 and the permittivity is ϵ_0. The speed of the wave is:

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = \frac{E_0}{B_0}$$

- For the non-magnetic dielectric medium:
  - Non-magnetic means the magnetic permeability is equal to that of vacuum, i.e., $\mu = \mu_0$.
  - The permittivity is given as $\epsilon = 4\epsilon_0$.

- Let us calculate the speed of the wave in this medium:

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_0(4\epsilon_0)}}$$

$$v = \frac{1}{2\sqrt{\mu_0 \epsilon_0}}$$

- Since $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$, we can write:

$$v = \frac{c}{2}$$

- Therefore, the ratio of the amplitudes in this medium is:

$$\frac{E_{\text{medium}}}{B_{\text{medium}}} = v = \frac{c}{2}$$

Step 4: Final Answer:

The corresponding ratio in the given medium is $c/2$.

Quick Tip: The refractive index of a non-magnetic medium is $n = \sqrt{\epsilon_r} = \sqrt{\frac{\epsilon}{\epsilon_0}}$.
The speed of light in the medium is $v = \frac{c}{n}$.
Since $\epsilon_r = 4$, we get $n = 2$, which immediately yields $v = \frac{c}{2}$.

---

63. Among the following, the waves responsible for energy transfer in vacuum are

(A) infrared waves.

(B) sound waves.

(C) SONAR waves.

(D) seismic waves.

Correct Answer: (A) infrared waves.

Solution:

Step 1: Understanding the Question:

The question asks to identify which of the listed wave types can propagate and transfer energy through a vacuum.

Step 2: Key Formula or Approach:

- Waves are broadly classified into mechanical waves and electromagnetic waves.
- Mechanical waves require a material medium (solid, liquid, or gas) to propagate because they rely on elastic deformation.
- Electromagnetic waves consist of oscillating electric and magnetic fields and do not require a material medium, allowing them to propagate through a vacuum.

Step 3: Detailed Explanation:

- Infrared waves are a type of electromagnetic radiation, lying between microwaves and visible light in the electromagnetic spectrum.
  Since they are electromagnetic, they can travel through a vacuum and transfer thermal energy.

- Sound waves are mechanical longitudinal waves that propagate through the compression and rarefaction of a physical medium. They cannot travel through a vacuum.

- SONAR (Sound Navigation and Ranging) uses acoustic (sound) waves to navigate and detect objects, typically underwater. Since these are sound waves, they require a physical medium.

- Seismic waves are waves of energy that travel through the Earth's layers (resulting from earthquakes or explosions). They are mechanical waves and require a solid or liquid medium to propagate.

Step 4: Final Answer:

The waves responsible for energy transfer in a vacuum are infrared waves.

Quick Tip: Remember that all forms of light (radio, microwave, infrared, visible, ultraviolet, X-rays, gamma rays) are electromagnetic waves and can travel through the vacuum of space.

---

64. Let A be the mass number, Z be the atomic number and N be the neutron number of a nucleus. Then the statement which is always true, is

(A) $A^2 \geq NZ$
(B) $A \geq 2N$
(C) $A \geq 2Z$
(D) $AN \geq Z^2$

Correct Answer: (A) $A^2 \geq NZ$

Solution:

Step 1: Understanding the Question:

The mass number A, atomic number Z, and neutron number N describe the composition of a nucleus.
We need to find an algebraic relationship among these three variables that holds true for all possible nuclei.

Step 2: Key Formula or Approach:

For any nucleus, the mass number is the sum of protons (atomic number) and neutrons:

$$A = Z + N$$

All three quantities A, Z, and N are non-negative integers.

Step 3: Detailed Explanation:

- Let us analyze each option mathematically:

- Option (A): $A^2 \geq NZ$:

  Substitute $A = Z + N$:

$$(Z + N)^2 = Z^2 + 2NZ + N^2$$

  We want to compare this with NZ:

$$Z^2 + 2NZ + N^2 \geq NZ$$

$$Z^2 + NZ + N^2 \geq 0$$

  Since $Z \geq 0$ and $N \geq 0$, all terms Z^2, NZ, and N^2 are non-negative.

  Thus, $Z^2 + NZ + N^2 \geq 0$ is always true for any nucleus. This means $A^2 \geq NZ$ is always true.

- Option (B): $A \geq 2N$:

  This simplifies to $Z + N \geq 2N \implies Z \geq N$.

  However, for heavy stable nuclei (such as Lead-208, where $Z = 82, N = 126$), we have $N > Z$. Thus, this is not always true.

- Option (C): $A \geq 2Z$:

  This simplifies to $Z + N \geq 2Z \implies N \geq Z$.

  However, for light nuclei (such as Helium-3, where $Z = 2, N = 1$), we have $Z > N$. Thus, this is not always true.

- Option (D): $AN \geq Z^2$:

For Hydrogen-1 (^{1}H), we have $A = 1, Z = 1, N = 0$.

Evaluating the expression:

$$(1)(0) \geq 1^2 \implies 0 \geq 1$$

This is false. Thus, this option is not always true.

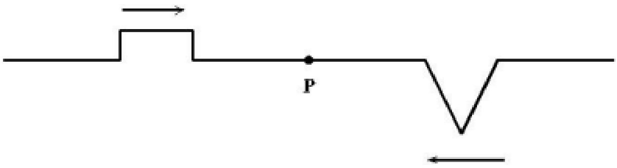
Step 4: Final Answer:

The statement which is always true is $A^2 \geq NZ$.

Quick Tip: Using simple counterexamples like Hydrogen-1 ($A = 1, Z = 1, N = 0$) and Helium-3 ($A = 3, Z = 2, N = 1$) lets you quickly eliminate incorrect options in nuclear physics relations.

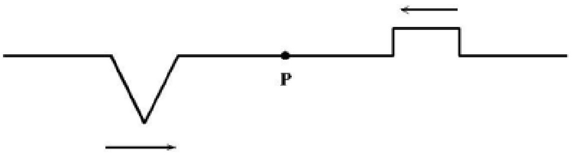
---

65. Two wave pulses, one rectangular and other triangular, approach each other as shown in the figure below.

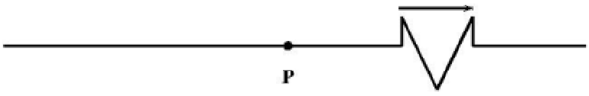


They overlap at the point P at time t. The diagram best representing the appearance of the wave pulses at a time $t' > t$ is

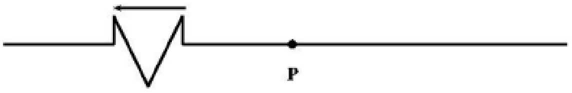
A.



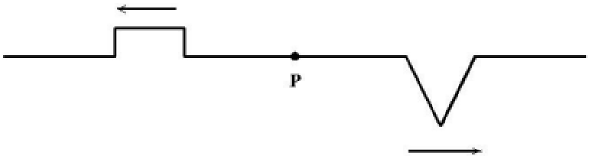
B.



C.



D.



(A) [A]

(B) [B]

(C) [C]

(D) [D]

Correct Answer: (A) [A]

Solution:

Step 1: Understanding the Question:

The question asks for the shape and position of two wave pulses (one rectangular, one triangular) traveling in opposite directions after they have crossed each other.

Step 2: Key Formula or Approach:

We apply the Principle of Superposition for waves.
When wave pulses meet, they interfere, but after they pass through each other, they emerge completely unaltered in shape, size, and direction of travel.

Step 3: Detailed Explanation:

- Let us analyze the initial state:
  - A rectangular pulse is on the left, above the reference line, traveling to the right.
  - A triangular pulse is on the right, below the reference line, traveling to the left.
- At time t, they overlap at point P.
- At time $t' > t$, the pulses have completely crossed each other.
- Since waves do not affect each other's shape or velocity permanently:
  - The rectangular pulse must continue to travel to the right, remaining above the reference line. It will now be to the right of point P.
  - The triangular pulse must continue to travel to the left, remaining below the reference

line. It will now be to the left of point P.

- Looking at the options:
  - Option A correctly shows the triangular pulse on the left pointing downwards (moving left) and the rectangular pulse on the right pointing upwards (moving right).

Step 4: Final Answer:

The diagram representing the pulses at $t' > t$ is given by option A.

Quick Tip: Remember that wave pulses are not particles; they do not bounce off each other. They pass through each other as if the other wave was not even there, keeping their original shape, speed, and orientation.

---

66. Ice on land mass melting into ocean, due to global warming, is estimated to be around 1.3×10^{15} kg per year. The density of sea water is $1025\ \text{kg} \cdot \text{m}^{-3}$. Assuming that the melting rate of ice remains constant and the effective surface area covered by the oceans is $3.6 \times 10^{14}\ \text{m}^2$, the estimated average rise in sea level (in m) in the next 75 years will be closest to

(A) 0.25

(B) 0.45

(C) 0.15

(D) 0.50

Correct Answer: (A) 0.25

Solution:

Step 1: Understanding the Question:

The problem asks for the vertical rise in the sea level over a span of 75 years, given the rate of ice melting on land, the density of seawater, and the total surface area of the oceans.

Step 2: Key Formula or Approach:

- Find the total mass of melted ice in 75 years.
- Convert this mass to volume using the density of seawater:

$$V = \frac{M}{\rho}$$

- Find the rise in height h by dividing the volume by the ocean surface area A:

$$h = \frac{V}{A}$$

Step 3: Detailed Explanation:

- The rate of ice melting is $R = 1.3 \times 10^{15}$ kg/year.

- Over a period of $t = 75$ years, the total mass of water added to the oceans is:

$$M_{\text{total}} = R \times t = (1.3 \times 10^{15} \text{ kg/year}) \times 75 \text{ years}$$

$$M_{\text{total}} = 9.75 \times 10^{16} \text{ kg}$$

- Now we convert this mass of water into equivalent volume V using the density of seawater $\rho = 1025 \text{ kg/m}^3$:

$$V = \frac{M_{\text{total}}}{\rho} = \frac{9.75 \times 10^{16} \text{ kg}}{1025 \text{ kg/m}^3}$$

$$V \approx 9.512 \times 10^{13} \text{ m}^3$$

- The rise in sea level h is the volume divided by the surface area of the oceans $A = 3.6 \times 10^{14}$ m^2:

$$h = \frac{V}{A} = \frac{9.512 \times 10^{13} \text{ m}^3}{3.6 \times 10^{14} \text{ m}^2}$$

$$h \approx 0.264 \text{ m}$$

- Looking at the options, 0.264 m is closest to 0.25 m.

Step 4: Final Answer:

The estimated average rise in sea level is closest to 0.25 m.

Quick Tip: Using quick approximation:

$1025 \approx 1000 = 10^3$.

$V \approx 9.75 \times 10^{13}$ m^3.

$h \approx \frac{9.75\times10^{13}}{3.6\times10^{14}} = \frac{9.75}{36} \approx 0.27$ m, which points directly to 0.25 m.

---

67. A spherical glass bottle having negligible wall thickness is placed in air. When the bottle is completely filled with water, its focal length is f. If the water is replaced by another transparent liquid of higher refractive index, then the focal length changes to f'. Then the correct option is

(A) $f' < f$

(B) $f' > f$

(C) $f' = f = \infty$

(D) $f' = f$, but finite

Correct Answer: (A) $f' < f$

Solution:

Step 1: Understanding the Question:

The problem presents a spherical bottle filled with liquid acting as a thick spherical lens in air. We need to determine how the focal length changes when the refractive index of the liquid inside is increased.

Step 2: Key Formula or Approach:

For a sphere of radius R and refractive index n placed in a medium of refractive index $n_0 = 1$ (air), the focal length measured from the center is given by the thick lens formula or refraction at spherical surfaces:
Alternatively, we can analyze the bending power of the sphere:
- A medium with a higher refractive index causes greater bending of light rays at the interfaces.
- More bending means the rays converge closer to the sphere, resulting in a smaller focal length.

Step 3: Detailed Explanation:

- When light enters the spherical bottle filled with a liquid of refractive index n_1 from air ($n_0 = 1$), it undergoes refraction at the front and back spherical surfaces.

- The optical power P of a lens represents its ability to converge or diverge light rays, which is inversely proportional to its focal length:

$$P = \frac{1}{f}$$

- The power of a refracting surface is directly proportional to the difference in refractive index across the interface ($\Delta n = n_{\text{liquid}} - n_{\text{air}}$).

- Since the second liquid has a higher refractive index:

$$n'_{\text{liquid}} > n_{\text{liquid}}$$

- The difference in refractive index increases:

$$(n'_{\text{liquid}} - 1) > (n_{\text{liquid}} - 1)$$

- This increases the converging power of both the front and back surfaces of the spherical lens:

$$P' > P$$

- Since $P = \frac{1}{f}$, an increase in power results in a decrease in focal length:

$$f' < f$$

Step 4: Final Answer:

The correct option is $f' < f$.

Quick Tip: A simple mnemonic: Higher refractive index $\implies$ stronger bending $\implies$ shorter focal length.

This is always true for any positive (converging) lens.

---

68. Physicist Luis Alvarez and his collaborators hypothesised that the extinction of dinosaurs was due to the impact of an asteroid with the Earth. They estimated the mass and the radius of the asteroid to be about 2×10^{15} kg and 10 km respectively. Take the mass of the Earth to

be 6×10^{24} kg. The gravitational acceleration (in SI units) of the Earth due to the asteroid just before the impact would be of the order

(A) 10^{-9}
(B) 10^{1}
(C) 10^{-1}
(D) 10^{-5}

Correct Answer: (A) 10^{-9}

Solution:

Step 1: Understanding the Question:

The question asks for the gravitational acceleration experienced by the Earth due to the gravitational pull of the asteroid just before they collide.

Step 2: Key Formula or Approach:

The gravitational acceleration a experienced by an object due to another body of mass M at a distance r is:

$$a = \frac{GM}{r^2}$$

where $G = 6.67 \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2$.
Just before impact, the distance between the center of the Earth and the center of the asteroid is approximately equal to the radius of the Earth $R_E \approx 6.4 \times 10^6$ m.

Step 3: Detailed Explanation:

- The mass of the asteroid is $M_a = 2 \times 10^{15}$ kg.
- The distance r is the distance between their centers of mass just before impact. Since the Earth's radius (6400 km) is much larger than the asteroid's radius (10 km), we can approximate $r \approx R_E \approx 6.4 \times 10^6$ m.

- The gravitational acceleration of the Earth towards the asteroid is:

$$a = \frac{GM_a}{R_E^2}$$

- Substituting the values:

$$a = \frac{(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}) \times (2 \times 10^{15}\ \mathrm{kg})}{(6.4 \times 10^6\ \mathrm{m})^2}$$

$$a = \frac{1.334 \times 10^5}{4.096 \times 10^{13}}$$

$$a \approx 3.26 \times 10^{-9}\ \mathrm{m/s^2}$$

- The order of magnitude of 3.26×10^{-9} is 10^{-9}.

Step 4: Final Answer:

The gravitational acceleration is of the order of 10^{-9}.

Quick Tip: Be careful whose acceleration is being asked!

We want the acceleration of the Earth due to the asteroid, so we must use the mass of the asteroid M_a in the formula, not the mass of the Earth.

---

69. An infinitely long straight wire with uniform line charge density λ lies at a perpendicular distance d from a point O. The total electric flux through the surface of a sphere of radius

$R > d$ centred at O, is

(A) $\frac{2\lambda}{\epsilon_0}\sqrt{R^2 - d^2}$
(B) $\frac{2\lambda}{\epsilon_0}\sqrt{Rd}$
(C) $\frac{2\lambda}{\epsilon_0}\sqrt{R^2 + d^2}$
(D) 0

Correct Answer: (A) $\frac{2\lambda}{\epsilon_0}\sqrt{R^2 - d^2}$

Solution:

Step 1: Understanding the Question:

The problem asks for the total electric flux passing through a sphere of radius R enclosing a portion of an infinitely long straight charged wire.

Step 2: Key Formula or Approach:

According to Gauss's Law, the total electric flux Φ through any closed surface is equal to the net charge enclosed by the surface divided by the permittivity of free space:

$$\Phi = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

Since the wire has a uniform linear charge density λ, the enclosed charge is:

$$Q_{\text{enclosed}} = \lambda L$$

where L is the length of the wire segment that lies inside the sphere.

Step 3: Detailed Explanation:

- The wire is straight and is located at a perpendicular distance d from the center O of the sphere.

- Since $R > d$, the wire cuts through the sphere. The intersection of the wire with the sphere forms a chord.

- Let us find the length L of this chord.
  - Draw a perpendicular from the center O to the wire. The length of this perpendicular is d.
  - The distance from the center O to the points where the wire intersects the sphere is equal to the radius R.
  - This forms a right-angled triangle where the hypotenuse is R, one leg is d, and the other leg is half the chord length $(L/2)$.

- Using the Pythagorean theorem:

$$\left(\frac{L}{2}\right)^2 + d^2 = R^2$$

$$\frac{L}{2} = \sqrt{R^2 - d^2} \implies L = 2\sqrt{R^2 - d^2}$$

- The total charge enclosed by the sphere is:

$$Q_{\text{enclosed}} = \lambda L = 2\lambda\sqrt{R^2 - d^2}$$

- Using Gauss's Law, the total electric flux through the sphere is:

$$\Phi = \frac{Q_{\text{enclosed}}}{\epsilon_0} = \frac{2\lambda}{\epsilon_0}\sqrt{R^2 - d^2}$$

Step 4: Final Answer:

The total electric flux is $\frac{2\lambda}{\epsilon_0}\sqrt{R^2 - d^2}$.

Quick Tip: For any Gauss's Law problem, focus solely on calculating the enclosed charge. Geometry of the intersection of a line with a sphere simplifies to a simple 2D circle chord problem.

---

70. A sphere made of a material with density $12\ \text{kg}\cdot\text{m}^{-3}$ and weighing 100 N in vacuum is immersed in a container of gas. Its weight in gas is 85 N. The density of the gas in $\text{kg}\cdot\text{m}^{-3}$ is closest to

(A) 1.80
(B) 0.01
(C) 80.00
(D) 0.55

Correct Answer: (A) 1.80

Solution:

Step 1: Understanding the Question:

The problem presents a solid sphere weighed first in a vacuum and then inside a gas.
The apparent loss of weight in the gas is due to the buoyant force exerted by the gas. We need to determine the density of this gas.

Step 2: Key Formula or Approach:

- Archimedes' Principle states that the buoyant force F_b is equal to the weight of the fluid displaced:

$$F_b = \rho_{\text{fluid}} V g$$

- The weight of the sphere in vacuum is:

$$W_{\text{vac}} = \rho_{\text{sphere}} V g$$

Step 3: Detailed Explanation:

- The weight in vacuum is $W_{\text{vac}} = 100$ N.

- The weight in the gas is $W_{\text{gas}} = 85$ N.

- The buoyant force F_b exerted by the gas is:

$$F_b = W_{\text{vac}} - W_{\text{gas}} = 100 - 85 = 15 \text{ N}$$

- Let us write the expressions for W_{vac} and F_b:

$$W_{\text{vac}} = \rho_s V g \implies V g = \frac{W_{\text{vac}}}{\rho_s}$$

$$F_b = \rho_g V g$$

- Substituting the expression for $V g$ into the buoyant force formula:

$$F_b = \rho_g \left(\frac{W_{\text{vac}}}{\rho_s} \right)$$

- Rearranging the equation to solve for the density of the gas ρ_g:

$$\rho_g = \frac{F_b \cdot \rho_s}{W_{\text{vac}}}$$

- Substituting the given numerical values:

$$\rho_g = \frac{15\ \text{N} \times 12\ \text{kg/m}^3}{100\ \text{N}}$$

$$\rho_g = \frac{180}{100} = 1.80\ \text{kg/m}^3$$

Step 4: Final Answer:

The density of the gas is closest to $1.80\ \text{kg} \cdot \text{m}^{-3}$.

Quick Tip: The ratio of buoyant force to weight in vacuum is equal to the ratio of the fluid density to the object density:

$$\frac{F_b}{W_{\text{vac}}} = \frac{\rho_{\text{fluid}}}{\rho_{\text{object}}}$$

This direct ratio formula bypasses the need to compute the volume V or g.

---

71. One mole of a monatomic ideal gas undergoes a transformation from an initial state with temperature 290 K and volume 30 litres to a final state with temperature 310 K and volume 16 litres. On the pressure–volume ($P-V$) diagram, this process is represented by a straight line path. The magnitude of the work done (in joules) during this process is close to

(A) 1690

(B) 1939

(C) 877

(D) 1375

Correct Answer: (A) 1690

Solution:

Step 1: Understanding the Question:

The problem describes an ideal gas undergoing a thermodynamic process along a straight line on a $P - V$ diagram.

We need to compute the work done, which corresponds to the area under this straight line.

Step 2: Key Formula or Approach:

- Use the ideal gas law to calculate the pressures at the initial and final states:

$$P = \frac{nRT}{V}$$

- The work done W along a straight-line path on a $P - V$ diagram is the area of the trapezoid:

$$W = \frac{P_i + P_f}{2}(V_f - V_i)$$

Step 3: Detailed Explanation:

- The given values are $n = 1$ mole and $R = 8.314$ J/(mol $\cdot$ K).

- Initial state:
  - $T_i = 290$ K
  - $V_i = 30$ litres $= 30 \times 10^{-3}$ m^3

$$P_i = \frac{nRT_i}{V_i} = \frac{1 \times 8.314 \times 290}{30 \times 10^{-3}} \approx 80369 \text{ Pa}$$

- Final state:

  - $T_f = 310$ K
  - $V_f = 16$ litres $= 16 \times 10^{-3}$ m^3

$$P_f = \frac{nRT_f}{V_f} = \frac{1 \times 8.314 \times 310}{16 \times 10^{-3}} \approx 161084 \text{ Pa}$$

- Since the path is a straight line, the magnitude of the work done is:

$$|W| = \frac{P_i + P_f}{2} |V_f - V_i|$$

$$|W| = \frac{80369 + 161084}{2} \times (30 - 16) \times 10^{-3} \text{ m}^3$$

$$|W| = \frac{241453}{2} \times 14 \times 10^{-3}$$

$$|W| = 241453 \times 7 \times 10^{-3} \approx 1690.17 \text{ J}$$

Step 4: Final Answer:

The magnitude of the work done during this process is close to 1690 J.

Quick Tip: Remember to convert volume from litres to m^3 (1 L $= 10^{-3}$ m^3) to keep all units in the standard SI system.

72. The electron in the hydrogen atom is in the third excited state. Its potential energy (in eV) is

(A) -1.70

(B) -1.51

(C) -0.85

(D) -3.02

Correct Answer: (A) -1.70

Solution:

Step 1: Understanding the Question:

The problem asks for the potential energy of an electron in a hydrogen atom when it is in its third excited state.

Step 2: Key Formula or Approach:

- The total energy E_n of an electron in the n-th orbit of a hydrogen atom is:

$$E_n = -\frac{13.6}{n^2}\text{ eV}$$

- For a Coulombic potential, the potential energy U_n is related to the total energy E_n by:

$$U_n = 2E_n$$

Step 3: Detailed Explanation:

- The "third excited state" corresponds to the principal quantum number $n = 4$. (Ground state is $n = 1$, first excited state is $n = 2$, second excited state is $n = 3$, third excited state is $n = 4$).
- Let us calculate the total energy E_4 in the $n = 4$ state:

$$E_4 = -\frac{13.6}{4^2} = -\frac{13.6}{16} = -0.85 \text{ eV}$$

- Using the relationship between potential energy and total energy:

$$U_4 = 2E_4 = 2 \times (-0.85 \text{ eV}) = -1.70 \text{ eV}$$

Step 4: Final Answer:

Its potential energy is −1.70 eV.

Quick Tip: Remember that:

Total Energy (E) = Kinetic Energy (K) + Potential Energy (U).

By the virial theorem for a $1/r$ potential, $K = -E$ and $U = 2E$.

---

73. A sphere is given an initial push so that it starts rolling (without slipping) up an inclined plane. During its climb

(A) the direction of the force of friction on the sphere is up the incline.

(B) the direction of the net force on the sphere is up the incline.

(C) the net torque on the sphere is zero.

(D) the work done by the force of friction on the sphere is negative.

Correct Answer: (A) the direction of the force of friction on the sphere is up the incline.

Solution:

Step 1: Understanding the Question:

The problem describes a sphere rolling without slipping up an incline after being given an initial push.
We need to determine the behavior of the forces, torque, and work done on the sphere.

Step 2: Key Formula or Approach:

For rolling without slipping:
- The contact point is instantaneously at rest, so the friction force is static.
- The relation between linear acceleration a and angular acceleration α is:

$$a = \alpha R$$

- Torque about the center of mass is:

$$\tau = I\alpha$$

Step 3: Detailed Explanation:

- As the sphere moves up the incline, gravity has a component $mg\sin\theta$ down the incline. This component slows down the linear motion.
  - Thus, the linear acceleration a is directed down the incline.

- For rolling without slipping to be maintained, as the linear speed v decreases, the angular speed ω must also decrease in proportion ($v = \omega R$).
  - Since the sphere is rotating counter-clockwise (viewed from the side where the incline goes up to the left), to decrease ω, there must be a clockwise torque.

- The force of gravity and the normal force act through the center of mass, so they cannot produce any torque about the center.
  - Therefore, the only force that can produce a torque about the center of mass to slow down the rotation is the static friction force f.

- To create a torque that opposes the rotation, the static friction force must act up the incline.
- Let us verify this: static friction up the incline creates a torque $\tau = fR$ that decelerates the rotation, which is consistent with the linear deceleration caused by gravity.
- For pure rolling, the instantaneous velocity of the point of contact is zero, so the work done by static friction is zero.

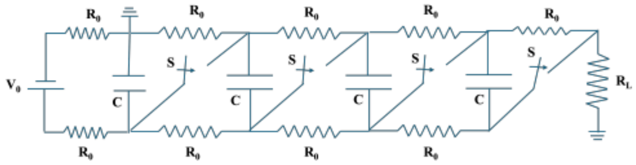
Step 4: Final Answer:

During its climb, the direction of the force of friction on the sphere is up the incline.

Quick Tip: For a rolling body decelerating on an incline, friction always acts in the direction that helps slow down both the rotation and translation in a synchronized manner. Here, that direction is up the incline.

---

74. All the capacitors in a given circuit are initially fully charged with all switches open. At a later time t, all the switches are simultaneously closed. The current flowing through the circuit at that instant is given by



(A) $4V_0/R_L$

(B) V_0/R_0

(C) $3V_0/R_0$

(D) V_0/R_L

Correct Answer: (A) $4V_0/R_L$

Solution:

Step 1: Understanding the Question:

The circuit shown is a classic Marx generator configuration.

Initially, the capacitors are charged in parallel to a voltage V_0. When all the switches are closed, the capacitors are reconfigured in series to discharge through a load resistor.

Step 2: Key Formula or Approach:

- When n capacitors, each charged to voltage V_0, are connected in series, their voltages add up.
- The total equivalent voltage of the series combination is:

$$V_{\text{total}} = nV_0$$

- The current I flowing through the load resistor R_L at the instant of connection is given by Ohm's Law:

$$I = \frac{V_{\text{total}}}{R_L}$$

Step 3: Detailed Explanation:

- The given circuit has 4 capacitors, each initially fully charged to the source voltage V_0 when the switches are open.

- When the switches S are simultaneously closed, they connect the positive plate of one capacitor to the negative plate of the next.

- This places all 4 capacitors in a series configuration across the load resistor R_L.

- The total voltage across the load resistor at the instant of closing is:

$$V_{\text{instant}} = 4V_0$$

- Thus, the initial current flowing through the load resistor R_L is:

$$I = \frac{4V_0}{R_L}$$

Step 4: Final Answer:

The current flowing through the circuit at that instant is $4V_0/R_L$.

Quick Tip: A Marx generator is a clever way to generate high-voltage pulses from a low-voltage DC supply by charging capacitors in parallel and discharging them in series.

---

75. An incandescent lamp of wattage W is filled with argon gas and has a tungsten filament. The collision frequency of evaporated tungsten atoms with the argon atoms is found to be proportional to W^{α}. The most reasonable value of α is

(A) $\frac{1}{8}$
(B) $\frac{1}{4}$
(C) $\frac{1}{2}$
(D) 0

Correct Answer: (A) $\frac{1}{8}$

Solution:

Step 1: Understanding the Question:

The problem asks us to relate the collision frequency of evaporated tungsten atoms inside an incandescent lamp to its wattage W using physical laws of radiation and kinetic theory.

Step 2: Key Formula or Approach:

- According to the Stefan-Boltzmann Law, the power radiated (wattage W) from a hot filament is proportional to the fourth power of its absolute temperature:

$$W \propto T^4 \implies T \propto W^{1/4}$$

- From kinetic theory, the average velocity of gas atoms is proportional to the square root of the temperature:

$$v \propto \sqrt{T}$$

- The collision frequency f is proportional to the average velocity v:

$$f \propto v$$

Step 3: Detailed Explanation:

- The steady-state power consumption (wattage W) of the lamp is equal to the power radiated by the filament at temperature T.
  - By Stefan's law:

$$W \propto T^4 \implies T \propto W^{1/4}$$

- The collision frequency f of the evaporated tungsten atoms with the surrounding gas depends on their thermal speed:

$$f \propto v_{\text{thermal}}$$

- According to kinetic theory of gases, the thermal speed of the atoms is:

$$v_{\text{thermal}} = \sqrt{\frac{3kT}{m}} \implies v_{\text{thermal}} \propto \sqrt{T}$$

- Therefore, substituting the relation for T in terms of W:

$$f \propto \sqrt{W^{1/4}} = W^{1/8}$$

- This gives $\alpha = \frac{1}{8}$.

Step 4: Final Answer:

The most reasonable value of α is $\frac{1}{8}$.

Quick Tip: Using dimensional and proportional relations is highly effective in scaling problems. Combine $W \propto T^4$ and $f \propto T^{1/2}$ directly to find the scaling exponent $\alpha = 1/2 \times 1/4 = 1/8$.

---

76. Two protons are separated by a distance of 1 nanometer. The magnitudes of gravitational, electromagnetic and strong nuclear forces between them are denoted by F_G, F_E and F_S respectively. Select the correct option.

(A) $F_E > F_G > F_S$

(B) $F_E > F_S > F_G$

(C) $F_S > F_G > F_E$

(D) $F_S > F_E > F_G$

Correct Answer: (A) $F_E > F_G > F_S$

Solution:

Step 1: Understanding the Question:

The problem asks us to compare the strengths of the three fundamental forces acting between two protons separated by a distance of 1 nanometer (10^{-9} m).

Step 2: Key Formula or Approach:

We must consider the range and relative strengths of the fundamental forces:

- Electromagnetic force F_E has an infinite range and is very strong between charged particles at atomic scales.
- Gravitational force F_G also has an infinite range but is extremely weak between subatomic particles.
- Strong nuclear force F_S is the strongest force but has an extremely short range (only acts up to $1-2$ fm or 10^{-15} m).

Step 3: Detailed Explanation:

- The separation distance is $r = 1 \text{ nm} = 10^{-9}$ m.
- At this distance, the strong nuclear force is virtually zero because 1 nm is about a million times larger than its range of influence (10^{-15} m):

$$F_S \approx 0$$

- The electromagnetic force is given by Coulomb's law:

$$F_E = \frac{kq^2}{r^2} \approx \frac{(9 \times 10^9) \times (1.6 \times 10^{-19})^2}{(10^{-9})^2} \approx 2.3 \times 10^{-10}\ \text{N}$$

- The gravitational force is given by Newton's law:

$$F_G = \frac{Gm^2}{r^2} \approx \frac{(6.67 \times 10^{-11}) \times (1.67 \times 10^{-27})^2}{(10^{-9})^2} \approx 1.86 \times 10^{-46}\ \text{N}$$

- Comparing these values, we find:

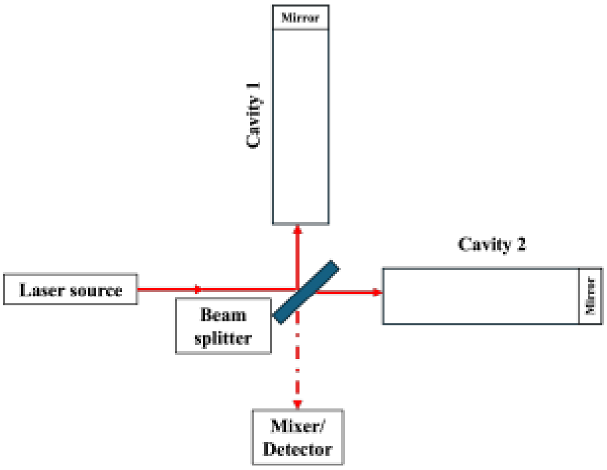
$$F_E > F_G > F_S$$

Step 4: Final Answer:

The correct option is $F_E > F_G > F_S$.

Quick Tip: Remember that the strong nuclear force falls off exponentially with distance ($F_S \propto e^{-r/r_0}$). At 1 nm, it is mathematically much smaller than even the gravitational force, which only falls off as $1/r^2$.

77. A laser beam of wavelength $1\ \mu$m is split and sent into two vacuum cavities of equal length L as shown in the figure. A detector can register an interference signal only if the phase difference between the returning beams is at least 5×10^{-11} rad. A certain physical effect changes the length of cavity 2 by an amount ΔL such that $\Delta L/L \approx 10^{-21}$. The minimum cavity length (in km) needed for measuring this physical effect is approximately



(A) 4

(B) 1

(C) 2

(D) 16

Correct Answer: (A) 4

Solution:

Step 1: Understanding the Question:

The problem describes an interferometer with two cavities of length L.

A physical change alters the length of one cavity, creating a phase difference between the two returning beams. We need to find the minimum length L required to detect this change.

Step 2: Key Formula or Approach:

- The phase difference $\Delta\phi$ caused by a path length difference Δx is:

$$\Delta\phi = \frac{2\pi}{\lambda}\Delta x$$

- Since the laser beams travel back and forth in the cavities, the path difference is:

$$\Delta x = 2\Delta L$$

Step 3: Detailed Explanation:

- The wavelength of the laser is $\lambda = 1\ \mu\text{m} = 10^{-6}$ m.

- The minimum detectable phase difference is $\Delta\phi_{\text{min}} = 5 \times 10^{-11}$ rad.

- The relative length change is:

$$\frac{\Delta L}{L} = 10^{-21} \implies \Delta L = 10^{-21} L$$

- The phase difference created by the round-trip change of length in cavity 2 is:

$$\Delta\phi = \frac{4\pi\Delta L}{\lambda}$$

- To detect this signal, we need:

$$\Delta\phi \geq \Delta\phi_{\text{min}}$$

$$\frac{4\pi(10^{-21}L)}{\lambda} \geq 5 \times 10^{-11}$$

- Substituting $\lambda = 10^{-6}$ m:

$$\frac{4\pi \cdot 10^{-21}L}{10^{-6}} \geq 5 \times 10^{-11}$$

$$4\pi \cdot 10^{-15} L \geq 5 \times 10^{-11}$$

$$L \geq \frac{5 \times 10^{-11}}{4\pi \cdot 10^{-15}}$$

$$L \geq \frac{50000}{4\pi} \text{ m} \approx 3978 \text{ m} \approx 4 \text{ km}$$

Step 4: Final Answer:

The minimum cavity length needed is approximately 4 km.

Quick Tip: Remember that in Michelson-type interferometers or cavities, the light travels a round trip, so the change in path length is $2\Delta L$ instead of just ΔL.

---

78. The critical electric field required to produce electron-positron pairs depends on the physical constants h, c, m_e and e. Use dimensional analysis and assume that the dimensionless coefficient is of order one. The magnitude of the critical electric field, in SI units, is of the order

(A) 10^{18}

(B) 10^{21}

(C) 10^{24}

(D) 10^{15}

Correct Answer: (A) 10^{18}

Solution:

Step 1: Understanding the Question:

The question asks us to estimate the order of magnitude of the critical electric field required for spontaneous electron-positron pair production using dimensional analysis with the constants h, c, m_e, and e.

Step 2: Key Formula or Approach:

The critical electric field E_c is reached when the work done on an electron by the field over a distance equal to the Compton wavelength of the electron ($\lambda_c = \frac{h}{m_e c}$) is equal to the rest mass energy of the electron ($m_e c^2$):

$$eE_c \cdot \left(\frac{h}{m_e c}\right) \approx m_e c^2 \implies E_c \approx \frac{m_e^2 c^3}{eh}$$

Step 3: Detailed Explanation:

- Let us write down the values of the physical constants in SI units:
  - Electron mass $m_e \approx 9.11 \times 10^{-31}$ kg
  - Speed of light $c \approx 3 \times 10^8$ m/s
  - Elementary charge $e \approx 1.6 \times 10^{-19}$ C
  - Planck's constant $h \approx 6.63 \times 10^{-34}$ J · s

- Now substitute these values into our derived expression for the critical electric field E_c:

$$E_c \approx \frac{(9.11 \times 10^{-31}\ \text{kg})^2 \times (3 \times 10^8\ \text{m/s})^3}{(1.6 \times 10^{-19}\ \text{C}) \times (6.63 \times 10^{-34}\ \text{J} \cdot \text{s})}$$

$$E_c \approx \frac{8.3 \times 10^{-61} \times 2.7 \times 10^{25}}{1.06 \times 10^{-52}}$$

$$E_c \approx \frac{2.24 \times 10^{-35}}{1.06 \times 10^{-52}}$$

$$E_c \approx 2.1 \times 10^{17}\ \text{V/m}$$

- This value is closest to the order of magnitude of 10^{18} (since 2.1×10^{17} is of the order of 10^{18}).

Step 4: Final Answer:

The magnitude of the critical electric field is of the order of 10^{18} V/m.

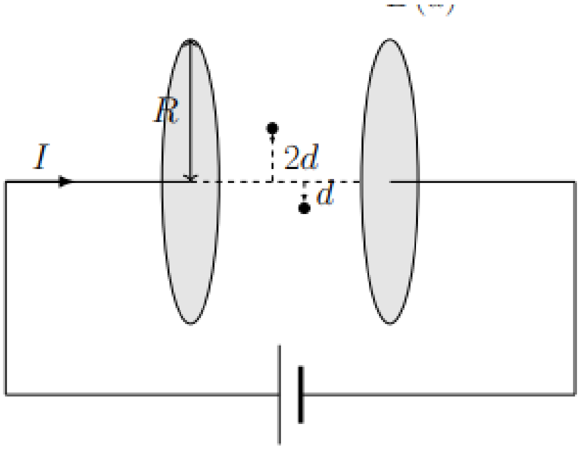
Quick Tip: Using the reduced Planck's constant $\hbar = \frac{h}{2\pi} \approx 1.05 \times 10^{-34}\ \text{J}\cdot\text{s}$ gives:

$$E_c = \frac{m_e^2 c^3}{e\hbar} \approx 1.3 \times 10^{18}\ \text{V/m}$$

which directly confirms the order of magnitude of 10^{18}.

---

79. A parallel plate capacitor with circular plates of radius R is being charged as shown in the figure. Let $B(r)$ be the induced magnetic field at a distance r from the central axis between the plates. Assuming $d \ll R$, the ratio $\frac{B(2d)}{B(d)}$, while charging, is



(A) 2

(B) $\frac{1}{2}$

(C) $\frac{1}{4}$

(D) 1

Correct Answer: (A) 2

Solution:

Step 1: Understanding the Question:

The problem asks for the ratio of the induced magnetic fields at two different distances from the central axis of a circular parallel plate capacitor during its charging phase.

Step 2: Key Formula or Approach:

According to the Ampere-Maxwell Law, the magnetic field induced inside the plates of a capacitor at a distance $r < R$ is:

$$\oint B \cdot dl = \mu_0 I_{\text{enclosed}}$$

Since there is no real conduction current between the plates, the magnetic field is created by the displacement current I_d:

$$B(r) \cdot 2\pi r = \mu_0 I_d \left(\frac{\pi r^2}{\pi R^2} \right) \implies B(r) \propto r \quad (\text{for } r < R)$$

Step 3: Detailed Explanation:

- The capacitor has circular plates of radius R.

- We are given the condition $d \ll R$. This means both d and $2d$ are much smaller than the radius R.
  - Therefore, both points are located inside the region between the plates ($r < R$).

- Since both points lie inside the capacitor plates, the magnetic field at both locations is directly proportional to the distance from the central axis:

$$B(r) = \left(\frac{\mu_0 I_d}{2\pi R^2} \right) r$$

- Let us find the ratio of the magnetic field at $r = 2d$ to that at $r = d$:

$$\frac{B(2d)}{B(d)} = \frac{2d}{d} = 2$$

Step 4: Final Answer:

The ratio $\frac{B(2d)}{B(d)}$ is 2.

Quick Tip: Inside the plates ($r < R$), the induced magnetic field grows linearly with r ($B \propto r$).
Outside the plates ($r > R$), it falls off as $1/r$ ($B \propto 1/r$).
Always check if the given distances are inside or outside the radius R!

80. Two point bodies of masses m and $3m$ are connected by a massless spring of spring constant $k = m\omega_0^2$ and kept on a frictionless horizontal surface. The spring is extended by a small distance l over its natural length at time $t = 0$ and then released so that the masses execute simple harmonic motion. The maximum speed of the particle with mass m is given by

(A) $\frac{\sqrt{3}\omega_0 l}{2}$
(B) $\frac{2\omega_0 l}{\sqrt{3}}$
(C) $\frac{\omega_0 l}{\sqrt{3}}$
(D) $\frac{3\omega_0 l}{4}$

Correct Answer: (A) $\frac{\sqrt{3}\omega_0 l}{2}$

Solution:

Step 1: Understanding the Question:

This is a two-body oscillator system. Both masses execute simple harmonic motion about their common center of mass (COM) on a frictionless surface.

Step 2: Key Formula or Approach:

- The reduced mass μ of a two-body system is:

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

- The angular frequency of oscillation is:

$$\omega = \sqrt{\frac{k}{\mu}}$$

- In the COM frame, the displacement of mass m_1 is related to the extension of the spring by:

$$x_1 = \frac{m_2}{m_1 + m_2} x$$

Step 3: Detailed Explanation:

- First, let us find the reduced mass of the system:

$$\mu = \frac{m \cdot 3m}{m + 3m} = \frac{3m^2}{4m} = \frac{3}{4}m$$

- The angular frequency of oscillation is:

$$\omega = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{m\omega_0^2}{\frac{3}{4}m}} = \frac{2}{\sqrt{3}}\omega_0$$

- Since the center of mass remains at rest, the extension x of the spring is shared between the two masses.
  The displacement x_1 of the mass m is:

$$x_1 = \frac{3m}{m + 3m}x = \frac{3}{4}x$$

- The maximum extension of the spring is l. Therefore, the amplitude of oscillation for mass m is:

$$A_1 = \frac{3}{4}l$$

- The maximum speed of the mass m is given by:

$$v_{1,\max} = \omega A_1$$

$$v_{1,\max} = \left(\frac{2}{\sqrt{3}}\omega_0\right)\left(\frac{3}{4}l\right)$$

$$v_{1,\max} = \frac{6}{4\sqrt{3}}\omega_0 l = \frac{3}{2\sqrt{3}}\omega_0 l = \frac{\sqrt{3}}{2}\omega_0 l$$

Step 4: Final Answer:

The maximum speed of the particle with mass m is $\frac{\sqrt{3}\omega_0 l}{2}$.

Quick Tip: Using conservation of momentum and conservation of energy at the point of maximum speed (equilibrium position) yields the same result:

$$\frac{1}{2}kl^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}(3m)v_2^2$$

With $mv_1 = 3mv_2 \implies v_2 = v_1/3$.
