NIOS Class 12 Mathematics Sample Paper – 1

Duration: 180 Minutes **Maximum Marks: 100**

Instructions

- This paper contains **45** Questions. The paper is divided into two sections: **Section A – 50** marks, **Section B – 50** marks.
- **Section A** consists of
  -**Q.No. 1 to 20** – Multiple Choice type questions (MCQs) carrying **+1 mark** each. Select and write the most appropriate option out of the four options given in each of these questions.
  -**Q.No. 21 to 29** – **Objective type questions**.
  -**Q.No. 21 to 24** carry **02 marks** each (with 2 sub-parts of 1 mark each).
  -**Q.No. 25 to 28** carry **04 marks** each (with 4 sub-parts of 1 mark each).
  -**Q.No. 29** carries **06 marks** (with 6 sub-parts of 1 mark each). Attempt these questions as per the instructions given for each of the questions 21–29.
- **Section B** consists of
  -**Q.No. 30 to 38**– Very Short questions carrying **02 marks** each.
  -**Q.No. 39 to 43** – Short Answer type questions carrying **04 marks** each.
  -**Q.No. 44 to 45** – Long Answer type questions carrying **06 marks** each.
  (An internal choice has been provided in some of the questions in Section B. You have to attempt only one of the given choices in such questions.)
- There is **No Negative marking**.
- Use of mobile phones, smartwatches, calculators, or any electronic gadgets is strictly prohibited.

Section: A

Q1. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, then the value of $A^2 - 6A$ is equal to: **(1)**

(A) $-11I$

(B) $11I$

(C) $-5I$

(D) $5I$

Q2. The distance between the parallel lines $3x + 4y - 9 = 0$ and $6x + 8y + 12 = 0$ is: **(1)**

(A) 3 units

(B) $\frac{21}{10}$ units

(C) 6 units

(D) 2 units

Q3. The domain of the function $f(x) = \frac{1}{\sqrt{x^2-5x+6}}$ is: **(1)**

(A) $(-\infty, 2) \cup (3, \infty)$

(B) $(2, 3)$

(C) $(-\infty, 2] \cup [3, \infty)$

(D) $(-\infty, \infty)$

Q4. Let R be a relation defined on the set of natural numbers $\mathbb{N}$ as aRb if $a + 3b = 12$. The range of this relation is: **(1)**

(A) $\{1, 2, 3\}$

(B) $\{3, 6, 9\}$

(C) $\{2, 4, 6\}$

(D) $\{1, 2, 3, 4\}$

Q5. $\lim_{x\to 0} \frac{e^{\sin x}-1}{x}$ is equal to: **(1)**

(A) 0

(B) 1

(C) e

(D) Does not exist

Q6. If $y = \log(\sec x + \tan x)$, then $\frac{dy}{dx}$ is: **(1)**

(A) $\sec x$

(B) $\sec x + \tan x$

(C) $\frac{1}{\sec x}$

(D) $\tan x$

Q7. The value of $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx$ is: **(1)**

(A) π

(B) $\pi/2$

(C) $\pi/4$

(D) 0

Q8. The degree of the differential equation $\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2} = k\frac{d^2y}{dx^2}$ is: **(1)**

(A) 1

(B) 2

(C) 3

(D) 4

Q9. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $|\vec{a} + \vec{b}| = 1$, then the angle between $\vec{a}$ and $\vec{b}$ is: **(1)**

(A) $\pi/3$

(B) $\pi/2$

(C) $2\pi/3$

(D) $\pi/4$

Q10. The direction cosines of a line making equal angles with the coordinate axes are: **(1)**

(A) $\pm(1, 1, 1)$

(B) $\pm\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

(C) $\pm\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

(D) $\pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Q11. The value of the determinant $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ is: **(1)**

(A) $a + b + c$

(B) abc

(C) 0

(D) 1

Q12. If $\sin^{-1} x + \sin^{-1} y = \frac{2\pi}{3}$, then the value of $\cos^{-1} x + \cos^{-1} y$ is: **(1)**

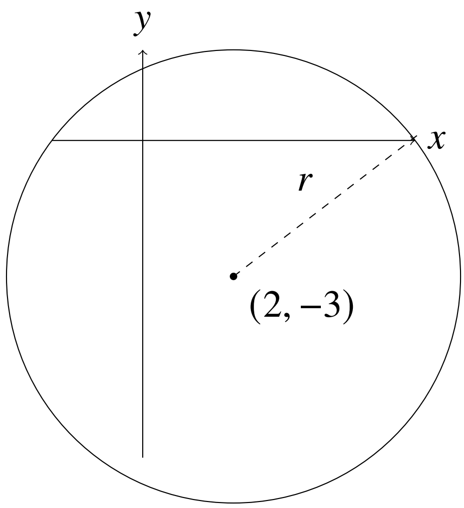
(A) $\frac{\pi}{3}$

(B) $\frac{2\pi}{3}$

(C) π

(D) $\frac{\pi}{6}$

Q13. The radius of the circle $x^2 + y^2 - 4x + 6y - 12 = 0$ is:



(1)

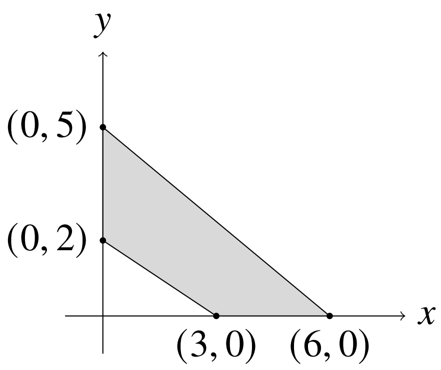
(A) 5 units

(B) $\sqrt{13}$ units

(C) 12 units

(D) 25 units

Q14. The corner points of the feasible region for an LPP are $(0, 2)$, $(3, 0)$, $(6, 0)$, and $(0, 5)$. If the objective function is $Z = 4x + 3y$, the minimum value of Z occurs at:



(1)

(A) $(0, 2)$

(B) $(3, 0)$

(C) $(0, 5)$

(D) Both (A) and (B)

Q15. The contrapositive of the statement "If a number is divisible by 9, then it is divisible by 3" is: **(1)**

(A) If a number is not divisible by 9, then it is not divisible by 3.

(B) If a number is divisible by 3, then it is divisible by 9.

(C) If a number is not divisible by 3, then it is not divisible by 9.

(D) A number is divisible by 3 if and only if it is divisible by 9.

Q16. The projection of vector $\vec{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ on vector $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$ is: **(1)**

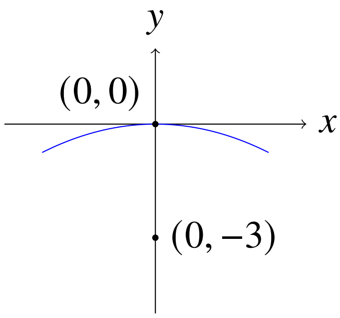
(A) $\frac{10}{\sqrt{6}}$

(B) $\frac{10}{\sqrt{17}}$

(C) $\frac{5}{\sqrt{6}}$

(D) $\sqrt{6}$

Q17. The equation of the parabola with vertex at $(0, 0)$ and focus at $(0, -3)$ is:



(1)

(A) $y^2 = -12x$

(B) $x^2 = -12y$

(C) $x^2 = 12y$

(D) $y^2 = 12x$

Q18. If $f(x) = x^3 - 3x^2 + 4x$, then $f(x)$ is: **(1)**

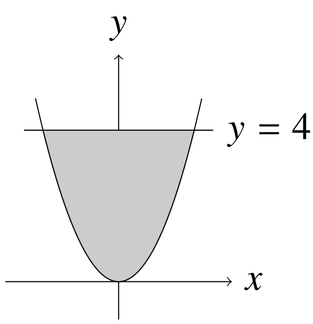
(A) Strictly increasing on $\mathbb{R}$

(B) Strictly decreasing on $\mathbb{R}$

(C) Increasing in $(-\infty, 1)$ and decreasing in $(1, \infty)$

(D) Decreasing in $(-\infty, 1)$ and increasing in $(1, \infty)$

Q19. The area bounded by the curve $y = x^2$ and the line $y = 4$ is:



(1)

(A) $\frac{32}{3}$ sq. units

(B) $\frac{16}{3}$ sq. units

(C) $\frac{8}{3}$ sq. units

(D) 16 sq. units

Q20. If A is a square matrix of order 3 such that $|A| = 5$, then the value of $|\text{adj } A|$ is: **(1)**

(A) 5

(B) 25

(C) 125

(D) $\frac{1}{5}$

Q21. Match the functions in **Column I** with their respective derivatives in **Column II**:

Column I	**Column II**
(i) $\tan^{-1} x$	(a) $\frac{-1}{\sqrt{1-x^2}}$
(ii) $\cos^{-1} x$	(b) $\frac{1}{1+x^2}$
(iii) $\sec^{-1} x$	(c) $\frac{1}{x\sqrt{x^2-1}}$

(2)

Q22. Complete the following:

(i) If a line makes angles α, β, γ with the positive directions of x, y, z axes respectively, then $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma =$ __________.

(ii) The unit vector perpendicular to both $\hat{i} + \hat{j}$ and $\hat{j} + \hat{k}$ having positive $\hat{j}$ component is __________. **(2)**

Q23. State whether the following statements are True or False:

(i) Every identity relation on a set is also an equivalence relation.

(ii) If A and B are symmetric matrices of the same order, then $AB - BA$ is a symmetric matrix. **(2)**

Q24. Provide the statements for the following:

(i) Write the negation of the statement: "All prime numbers are odd numbers."

(ii) Write the converse of the statement: "If a triangle is equilateral, then it is isosceles." **(2)**

Q25. Let $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ satisfy the equation $A^2 - 5A + kI = O$. Answer the following sub-parts:

(i) Find the matrix A^2.

(ii) Determine the scalar value of k.

(iii) Express A^{-1} in terms of A and I.

(iv) Find the exact matrix value of A^{-1}. **(4)**

Q26. An object moves along a straight line such that its displacement s (in meters) at time t (in seconds) is given by the function $s(t) = 2t^3 - 9t^2 + 12t + 4$. Answer the following questions:

(i) Find the velocity $v(t)$ of the object at any time t.

(ii) At what time intervals is the object moving forward (velocity is positive)?

(iii) Find the acceleration $a(t)$ of the object when it momentarily comes to rest.

(iv) What is the minimum velocity attained by the object? **(4)**

Q27. Consider the integral $I = \int_0^{\pi} x \sin x \, dx$. Evaluate using the following steps:

(i) Apply the property $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$ to rewrite the integral I.

(ii) By adding the original integral and the rewritten integral, eliminate the variable x to get an expression for $2I$.

(iii) Evaluate the resulting integral $\int_0^{\pi} \sin x \, dx$.

(iv) Find the final value of I. **(4)**

Q28. Three points in a space layout have position vectors $\vec{A} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{B} = 2\hat{i} + 3\hat{j} + \hat{k}$, and $\vec{C} = 3\hat{i} + \hat{j} + 2\hat{k}$. Find the following vector attributes:

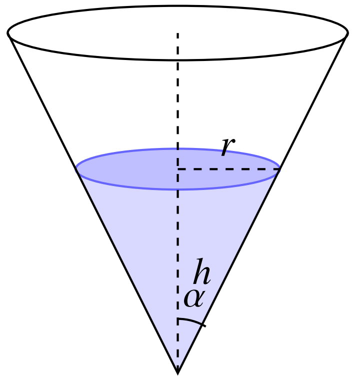
(i) Write the vector $\vec{AB}$.

(ii) Write the vector $\vec{AC}$.

(iii) Calculate the cross product $\vec{AB} \times \vec{AC}$.

(iv) Find the area of the triangle formed by these three points. **(4)**

Q29. A water tank has the shape of an inverted right circular cone with its axis vertical and vertex lowermost. Its semi-vertical angle is $\alpha = \tan^{-1}(0.5)$. Water is poured into it at a constant rate of 5 cubic meters per minute. Let r be the radius, h be the depth of water at time t, and V be the volume of water in the tank.



Based on this setup, answer the following six questions:

(i) Express the radius r explicitly in terms of depth h.

(ii) Write the volume V of water in the tank as a function of depth h alone.

(iii) Differentiate V with respect to time t to establish a relationship connecting $\frac{dV}{dt}$ and $\frac{dh}{dt}$.

(iv) Find the rate at which the depth of water h is increasing when the depth is 4 meters.

(v) Find the corresponding rate of change of the water surface area at this same depth.

(vi) If the total height of the cone is 10 meters, how much total water can the tank hold? **(6)**

Section: B

Q30. Find the equation of the ellipse whose focus is at $(0, \pm 4)$ and vertices are at $(0, \pm 5)$. **(2)**

Q31. Evaluate the determinant:

$$\begin{vmatrix} x+1 & x+2 & x+4 \\ x+3 & x+5 & x+8 \\ x+7 & x+10 & x+14 \end{vmatrix}$$

(2)

Q32. Find the principal value of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) + \cos^{-1}\left(\cos\frac{7\pi}{6}\right)$. **(2)**

Q33. Evaluate the limit:

$$\lim_{x\to 0}\frac{1-\cos 4x}{x^2}$$

(2)

Q34. Find $\frac{dy}{dx}$ if $x^y = e^{x-y}$. **(2)**

Q35. Find the intervals in which the function $f(x) = 2x^3 - 3x^2 - 36x + 7$ is strictly decreasing. **(2)**

Q36. Find the general solution of the differential equation:

$$\frac{dy}{dx} + 2y = e^{-x}$$

(2)

Q37. Find the shortest distance between the lines whose vector equations are:

$$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$$

$$\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$$

(2)

Q38. Verify if Lagrange's Mean Value Theorem is applicable for the function $f(x) = x(x-2)$ in the interval $[1, 3]$, and find the corresponding value of c. **(2)**

Q39. Show that the semi-latus rectum of the parabola $y^2 = 4ax$ is a harmonic mean between the segments of any focal chord. **(4)**

Q40. Solve the following system of equations using matrix inversion method:

$$2x - 3y + 5z = 11$$

$$3x + 2y - 4z = -5$$

$$x + y - 2z = -3$$

(4)

Q41. Prove that:

$$\tan^{-1}\left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right) = \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x, \quad -\frac{1}{\sqrt{2}} \leq x \leq 1$$

(4)

Q42. Evaluate the definite integral:

$$\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx$$

(4)

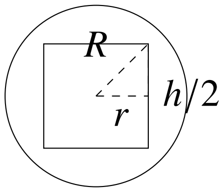
Q43. Find the equation of the plane passing through the intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$, and which is perpendicular to the plane $x - y + z = 0$. **(4)**

Q44. A furniture manufacturer makes two products: chairs and tables. Processing of these products is done on two machines A and B. A chair requires 2 hours on machine A and 6 hours on machine B. A table requires 4 hours on machine A and 2 hours on machine B. There are 16 hours of time available on machine A and 30 hours on machine B per day. Profit earned on a chair is ₹ 300 and on a table is ₹ 500. Formulate this problem as a Linear Programming Problem and solve it graphically to maximize daily profit.

OR

A company produces two types of goods, G_1 and G_2. Production requires resources of raw materials and labor. To make one unit of G_1, 3 kg of raw material and 2 hours of labor are needed. To make one unit of G_2, 2 kg of raw material and 4 hours of labor are needed. The total availability of raw material is 18 kg and labor is 20 hours. If the profit on a unit of G_1 is ₹ 400 and on G_2 is ₹ 600, find the optimal product mix using graphical optimization to maximize total profit. **(6)**

Q45. Show that the height of a cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also, find the maximum volume of this cylinder.



OR

An open rectangular box with a square base is to be made from a given quantity of sheet metal of area C^2 sq. units. Show that the maximum volume of the box is $\frac{C^3}{6\sqrt{3}}$ cubic units. **(6)**

Detailed Solutions

Q1.

Solution

Concept: For a square matrix A, a matrix polynomial is evaluated by substituting A into the expression using matrix multiplication ($A^2 = A \cdot A$) and scalar multiplication ($6A$), then simplifying to find a scalar multiple of the identity matrix I.

Solution: Step 1: Write down the matrix A:

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$$

Step 2: Compute A^2 by multiplying matrix A with itself:

$$A^2 = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4-3 & -2-4 \\ 6+12 & -3+16 \end{bmatrix} = \begin{bmatrix} 1 & -6 \\ 18 & 13 \end{bmatrix}$$

Step 3: Compute $6A$ by multiplying each element of A by 6:

$$6A = \begin{bmatrix} 12 & -6 \\ 18 & 24 \end{bmatrix}$$

Step 4: Substitute A^2 and $6A$ into the expression $A^2 - 6A$:

$$A^2 - 6A = \begin{bmatrix} 1 & -6 \\ 18 & 13 \end{bmatrix} - \begin{bmatrix} 12 & -6 \\ 18 & 24 \end{bmatrix} = \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix}$$

Step 5: Factor out -11 to express the result in terms of the identity matrix I:

$$A^2 - 6A = -11 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -11I$$

Therefore, the value of $A^2 - 6A$ is equal to $-11I$.

Final Answer: $\boxed{-11I}$

Answer: (A)

Go Back to Question 1

Q2.

Solution

Concept: The shortest perpendicular distance d between two parallel straight lines given by the general linear equations $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ is evaluated using the formula:

$$d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$$

Before applying the formula, the coefficients of x and y must be made identical in both equations.

Solution: Step 1: Write down the equations of the two parallel lines:

$$\text{Line 1: } 3x + 4y - 9 = 0$$

$$\text{Line 2: } 6x + 8y + 12 = 0$$

Step 2: Divide the equation of Line 2 by 2 so that its x and y coefficients match those of Line 1:

$$\frac{6x + 8y + 12}{2} = \frac{0}{2}$$

$$3x + 4y + 6 = 0$$

Step 3: Identify the common coefficients A and B, and the constant terms C_1 and C_2 from both modified equations:

$$A = 3, \quad B = 4$$

$$C_1 = -9, \quad C_2 = 6$$

Step 4: Substitute these values into the standard distance formula for parallel lines:

$$d = \frac{|-9 - 6|}{\sqrt{3^2 + 4^2}}$$

$$d = \frac{|-15|}{\sqrt{9 + 16}}$$

$$d = \frac{15}{\sqrt{25}}$$

$$d = \frac{15}{5} = 3 \text{ units}$$

Thus, the distance between the two parallel lines is 3 units.

Final Answer: 3 units

Answer: (A)

Go Back to Question 2

Q3.

Solution

Concept: The domain of a real-valued function $f(x) = \frac{1}{\sqrt{g(x)}}$ consists of all real numbers x for which the expression inside the square root is strictly positive, because division by zero is undefined and the square root of a negative number is non-real. Hence, we must solve the inequality $g(x) > 0$.

Solution: Step 1: Identify the expression inside the square root under the denominator:

$$g(x) = x^2 - 5x + 6$$

Step 2: Set up the strict inequality condition for the domain to be valid:

$$x^2 - 5x + 6 > 0$$

Step 3: Factorize the quadratic expression by splitting the middle term:

$$x^2 - 2x - 3x + 6 > 0$$

$$x(x - 2) - 3(x - 2) > 0$$

$$(x - 2)(x - 3) > 0$$

Step 4: Find the critical points where the expression equals zero, which are $x = 2$ and $x = 3$. Use the sign scheme (wavy curve method) to determine intervals where the product is positive: * For $x \in (-\infty, 2)$, both factors $(x - 2)$ and $(x - 3)$ are negative, so their product is positive. * For $x \in (2, 3)$, $(x - 2)$ is positive and $(x - 3)$ is negative, so their product is negative. * For $x \in (3, \infty)$, both factors are positive, so their product is positive.
Step 5: Select the positive intervals to establish the final domain:

$$x \in (-\infty, 2) \cup (3, \infty)$$

Therefore, the domain of the function is $(-\infty, 2) \cup (3, \infty)$.

Final Answer: $(-\infty, 2) \cup (3, \infty)$

Answer: (A)

Go Back to Question 3

Q4.

Solution

Concept: A relation R is defined on the set of natural numbers $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$. The range of the relation is the set of all second elements $b \in \mathbb{N}$ such that there exists a first element $a \in \mathbb{N}$ satisfying the given conditional equation $a + 3b = 12$.

Solution: Step 1: Write down the definition of the relation equation:

$$a + 3b = 12, \quad \text{where } a, b \in \mathbb{N}$$

Step 2: Express the variable a explicitly in terms of b to analyze constraints easily:

$$a = 12 - 3b$$

Step 3: Since a must be a natural number, it must satisfy the condition $a > 0$. Substitute the expression for a into this inequality:

$$12 - 3b > 0$$

$$12 > 3b$$

$$b < 4$$

Step 4: List the positive integer values of $b \in \mathbb{N}$ that satisfy $b < 4$:

$$b = 1, 2, 3$$

Step 5: Verify if corresponding values of a belong to $\mathbb{N}$ for each chosen value of b: * If $b = 1 \implies a = 12 - 3(1) = 9 \in \mathbb{N}$ * If $b = 2 \implies a = 12 - 3(2) = 6 \in \mathbb{N}$ * If $b = 3 \implies a = 12 - 3(3) = 3 \in \mathbb{N}$

Step 6: Collect all valid values of b to write down the range of the relation R:

$$\text{Range}(R) = \{1, 2, 3\}$$

Hence, the range of the given relation is the set $\{1, 2, 3\}$.

Final Answer: $\boxed{\{1, 2, 3\}}$

Answer: (A)

Go Back to Question 4

Q5.

Solution

Concept: To find the limit of an indeterminate form $\frac{0}{0}$, we can use algebraic manipulation to transform the expression into standard limits. The fundamental limit rules used here are $\lim_{t\to 0} \frac{e^t - 1}{t} = 1$ and $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Solution: Step 1: Write down the given limit problem and verify the indeterminate form:

$$L = \lim_{x\to 0} \frac{e^{\sin x} - 1}{x}$$

As $x \to 0$, $\sin x \to 0$, so the numerator approaches $e^0 - 1 = 0$ and the denominator approaches 0. This is a $\frac{0}{0}$ form.

Step 2: Multiply and divide the expression by $\sin x$ to introduce the standard exponential limit form:

$$L = \lim_{x\to 0} \left(\frac{e^{\sin x} - 1}{\sin x} \cdot \frac{\sin x}{x} \right)$$

Step 3: Apply the product rule for limits to evaluate each part separately, provided both individual limits exist:

$$L = \left(\lim_{x\to 0} \frac{e^{\sin x} - 1}{\sin x} \right) \cdot \left(\lim_{x\to 0} \frac{\sin x}{x} \right)$$

Step 4: Substitute $t = \sin x$. As $x \to 0$, the new variable $t \to 0$. The first limit becomes:

$$\lim_{t\to 0} \frac{e^t - 1}{t} = 1$$

Step 5: Recall the standard trigonometric limit value for the second part:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Step 6: Multiply the results of the two individual limits together to get the final answer:

$$L = 1 \cdot 1 = 1$$

Thus, the value of the given limit is 1.

Final Answer: $\boxed{1}$

Answer: (B)

Go Back to Question 5

Q6.

Solution

Concept: The derivative of a composite function is found using the chain rule. For $y = \log(u)$, the derivative is $\frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx}$. The basic standard derivatives needed are $\frac{d}{dx}(\sec x) = \sec x \tan x$ and $\frac{d}{dx}(\tan x) = \sec^2 x$.

Solution: Step 1: Write down the given logarithmic function:

$$y = \log(\sec x + \tan x)$$

Step 2: Apply the chain rule of differentiation with respect to x:

$$\frac{dy}{dx} = \frac{1}{\sec x + \tan x} \cdot \frac{d}{dx}(\sec x + \tan x)$$

Step 3: Differentiate each individual trigonometric term inside the brackets:

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

Step 4: Substitute these derivatives back into the chain rule expression:

$$\frac{dy}{dx} = \frac{1}{\sec x + \tan x} \cdot (\sec x \tan x + \sec^2 x)$$

Step 5: Factor out the common term $\sec x$ from the numerator:

$$\frac{dy}{dx} = \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$$

Step 6: Cancel out the common factor $(\sec x + \tan x)$ from both the numerator and the denominator:

$$\frac{dy}{dx} = \sec x$$

The derivative of the given function is therefore $\sec x$.

Final Answer: $\boxed{\sec x}$

Answer: (A)

Go Back to Question 6

Q7.

Solution

Concept: To evaluate the definite integral, we use the definite integral reflection property:

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

Adding the original equation and the transformed equation together helps simplify the integrand.

Solution: Step 1: Let the given definite integral be denoted as equation (1):

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx \quad \text{— (1)}$$

Step 2: Apply the integration property by replacing the variable x with $(\frac{\pi}{2} - x)$:

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin(\frac{\pi}{2} - x)}}{\sqrt{\sin(\frac{\pi}{2} - x)} + \sqrt{\cos(\frac{\pi}{2} - x)}}\,dx$$

Step 3: Use the standard complementary trigonometric identities $\sin(\frac{\pi}{2} - x) = \cos x$ and $\cos(\frac{\pi}{2} - x) = \sin x$ to simplify the expression:

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx \quad \text{— (2)}$$

Step 4: Add equation (1) and equation (2) together:

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$$

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$$

Step 5: Cancel the common terms in the numerator and denominator to simplify the integrand to 1:

$$2I = \int_0^{\pi/2} 1\,dx$$

Step 6: Integrate and substitute the upper and lower limits:

$$2I = [x]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

The final value of the definite integral is $\frac{\pi}{4}$.

Final Answer: $\pi/4$

Answer: (C)

Go Back to Question 7

Q8.

Solution

Concept: The degree of a differential equation is the highest power (exponent) of the highest-order derivative appearing in it, after the equation has been cleared of any fractional indices or radicals regarding the derivatives.

Solution: Step 1: Write down the given differential equation containing fractional exponents:

$$\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2} = k\frac{d^2y}{dx^2}$$

Step 2: Identify the highest order derivative present in the equation. The term $\frac{d^2y}{dx^2}$ is a second-order derivative, while $\frac{dy}{dx}$ is first-order. Therefore, the order of the differential equation is 2.
Step 3: Eliminate the fractional power $\frac{3}{2}$ on the left-hand side by squaring both sides of the entire differential equation:

$$\left[\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}\right]^2 = \left(k\frac{d^2y}{dx^2}\right)^2$$

$$\left(1+\left(\frac{dy}{dx}\right)^2\right)^3 = k^2\left(\frac{d^2y}{dx^2}\right)^2$$

Step 4: Now that all derivatives have integer exponents, observe the power raised on the highest order derivative $\frac{d^2y}{dx^2}$.
Step 5: The highest power raised on $\frac{d^2y}{dx^2}$ is 2. Consequently, the degree of the differential equation is 2.

Final Answer: $\boxed{2}$

Answer: (B)

Go Back to Question 8

Q9.

Solution

Concept: For any two vectors $\vec{a}$ and $\vec{b}$, the magnitude of their sum satisfies the identity $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta$, where θ represents the angle between the two vectors. Unit vectors always have a magnitude equal to 1.

Solution: Step 1: State the values given in the problem statement:

$$|\vec{a}| = 1, \quad |\vec{b}| = 1, \quad |\vec{a}+\vec{b}| = 1$$

Step 2: Square both sides of the given magnitude sum equation:

$$|\vec{a}+\vec{b}|^2 = 1^2 = 1$$

Step 3: Expand the left-hand side using the vector dot product expansion formula:

$$|\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a}\cdot\vec{b}) = 1$$

$$|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta = 1$$

Step 4: Substitute the known values of the magnitudes $|\vec{a}| = 1$ and $|\vec{b}| = 1$ into the expanded equation:

$$(1)^2 + (1)^2 + 2(1)(1)\cos\theta = 1$$

$$1 + 1 + 2\cos\theta = 1$$

$$2 + 2\cos\theta = 1$$

Step 5: Rearrange terms to isolate the trigonometric function $\cos\theta$:

$$2\cos\theta = 1 - 2$$

$$2\cos\theta = -1$$

$$\cos\theta = -\frac{1}{2}$$

Step 6: Determine the principal value of the angle θ within the standard interval $[0, \pi]$:

$$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

Hence, the angle between the two unit vectors is $\frac{2\pi}{3}$.

Final Answer: $2\pi/3$

Answer: (C)

Go Back to Question 9

Q10.

Solution

Concept: The direction cosines (l, m, n) of a straight line are defined as $l = \cos\alpha$, $m = \cos\beta$, and $n = \cos\gamma$, where α, β, γ are the angles made with the x, y, z axes respectively. They satisfy the geometric constraint equation:

$$l^2 + m^2 + n^2 = 1$$

Solution: Step 1: Use the given condition that the line makes equal angles with all three coordinate axes:

$$\alpha = \beta = \gamma \implies \cos\alpha = \cos\beta = \cos\gamma \implies l = m = n$$

Step 2: Substitute $l = m = n$ into the fundamental identity $l^2 + m^2 + n^2 = 1$:

$$3l^2 = 1 \implies l^2 = \frac{1}{3} \implies l = \pm\frac{1}{\sqrt{3}}$$

Step 3: Since all direction cosines are equal, assign the same value to m and n:

$$l = m = n = \pm\frac{1}{\sqrt{3}}$$

Therefore, the direction cosines of the line are $\pm\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

Final Answer: $\boxed{\pm\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}$

Answer: (C)

Go Back to Question 10

Q11.

Solution

Concept: The value of a determinant remains unchanged if a scalar multiple of one column (or row) is added to another column (or row). If any two rows or columns of a determinant become identical or proportional, the value of the determinant is automatically zero.

Solution: Step 1: Write down the given determinant matrix expression:

$$\Delta = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

Step 2: Apply the column operation $C_3 \rightarrow C_3 + C_2$ to combine the elements of the second and third columns:

$$\Delta = \begin{vmatrix} 1 & a & (b+c)+a \\ 1 & b & (c+a)+b \\ 1 & c & (a+b)+c \end{vmatrix}$$

Step 3: Rearrange the terms inside the third column elements to make them look uniform:

$$\Delta = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix}$$

Step 4: Factor out the common polynomial term $(a + b + c)$ from the third column:

$$\Delta = (a+b+c) \begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}$$

Step 5: Observe that column 1 (C_1) and column 3 (C_3) are completely identical:

$$C_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Step 6: Since two columns are identical, the value of that determinant is 0:

$$\Delta = (a+b+c) \cdot 0 = 0$$

The value of the determinant is 0.

Final Answer: 0

Answer: (C)

Go Back to Question 11

Q12.

Solution

Concept: For any real number k lying within the domain $[-1, 1]$, the complementary inverse trigonometric identity states that $\sin^{-1} k + \cos^{-1} k = \frac{\pi}{2}$. We can add the equations for both variables to find the required sum.

Solution: Step 1: Write down the basic inverse trigonometric identity for both variables x and y:

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \quad \text{— (1)}$$

$$\sin^{-1} y + \cos^{-1} y = \frac{\pi}{2} \quad \text{— (2)}$$

Step 2: Add equations (1) and (2) together:

$$\left(\sin^{-1} x + \cos^{-1} x\right) + \left(\sin^{-1} y + \cos^{-1} y\right) = \frac{\pi}{2} + \frac{\pi}{2}$$

Step 3: Group the sine terms together and the cosine terms together on the left-hand side:

$$\left(\sin^{-1} x + \sin^{-1} y\right) + \left(\cos^{-1} x + \cos^{-1} y\right) = \pi$$

Step 4: Substitute the given value $\sin^{-1} x + \sin^{-1} y = \frac{2\pi}{3}$ into this equation:

$$\frac{2\pi}{3} + \left(\cos^{-1} x + \cos^{-1} y\right) = \pi$$

Step 5: Isolate the required cosine sum term by subtracting $\frac{2\pi}{3}$ from π:

$$\cos^{-1} x + \cos^{-1} y = \pi - \frac{2\pi}{3}$$

$$\cos^{-1} x + \cos^{-1} y = \frac{3\pi - 2\pi}{3} = \frac{\pi}{3}$$

The value of $\cos^{-1} x + \cos^{-1} y$ is equal to $\frac{\pi}{3}$.

Final Answer: $\boxed{\pi/3}$

Answer: (A)

Go Back to Question 12

Q13.

Solution

Concept: The general second-degree equation of a circle is written as $x^2 + y^2 + 2gx + 2fy + c = 0$. The radius r of this circle can be calculated using the standard formula:

$$r = \sqrt{g^2 + f^2 - c}$$

Solution: Step 1: Write down the given equation of the circle:

$$x^2 + y^2 - 4x + 6y - 12 = 0$$

Step 2: Compare this with the general circle equation $x^2 + y^2 + 2gx + 2fy + c = 0$ to find the values of g, f, and c:

$$2g = -4 \implies g = -2$$

$$2f = 6 \implies f = 3$$

$$c = -12$$

Step 3: Substitute these coefficients into the standard radius formula:

$$r = \sqrt{(-2)^2 + (3)^2 - (-12)}$$

Step 4: Simplify the values inside the square root:

$$r = \sqrt{4 + 9 + 12}$$

$$r = \sqrt{25}$$

$$r = 5 \text{ units}$$

Thus, the radius of the given circle is 5 units.

Final Answer: 5 units

Answer: (A)

Go Back to Question 13

Q14.

Solution

Concept: According to the fundamental theorem of linear programming, the optimal (minimum or maximum) value of a linear objective function $Z = ax + by$ always occurs at one of the corner points (vertices) of the bounded feasible region. We find it by evaluating Z at each vertex.

Solution: Step 1: Write down the linear objective function to be minimized:

$$Z = 4x + 3y$$

Step 2: List all the four given corner points of the feasible region boundary:

$$\text{Points: } (0, 2),\ (3, 0),\ (6, 0),\ (0, 5)$$

Step 3: Evaluate the value of Z at the first corner point $(0, 2)$:

$$Z_{(0,2)} = 4(0) + 3(2) = 0 + 6 = 6$$

Step 4: Evaluate the value of Z at the second corner point $(3, 0)$:

$$Z_{(3,0)} = 4(3) + 3(0) = 12 + 0 = 12$$

Step 5: Evaluate the value of Z at the third corner point $(6, 0)$:

$$Z_{(6,0)} = 4(6) + 3(0) = 24 + 0 = 24$$

Step 6: Evaluate the value of Z at the fourth corner point $(0, 5)$:

$$Z_{(0,5)} = 4(0) + 3(5) = 0 + 15 = 15$$

Step 7: Compare all the calculated values of Z:

$$\{6, 12, 24, 15\}$$

The smallest value among these is 6, which belongs to the corner point $(0, 2)$.
Therefore, the minimum value of Z occurs at the point $(0, 2)$.

Final Answer: $\boxed{(0, 2)}$

Answer: (A)

Go Back to Question 14

Q15.

Solution

Concept: For a conditional logical statement of the conditional form *"If p, then q"* (symbolized as $p \implies q$), its contrapositive statement is logically equivalent and is defined as *"If not q, then not p"* (symbolized as $\sim q \implies \sim p$).

Solution: Step 1: Break down the given conditional statement into its component propositions p and q: * Statement p: A number is divisible by 9. * Statement q: A number is divisible by 3.
Step 2: Formulate the negations ($\sim$) of both individual component statements: * Negation $\sim q$: A number is not divisible by 3. * Negation $\sim p$: A number is not divisible by 9.
Step 3: Combine these negated statements into a new conditional statement template using the conditional structure *"If $\sim q$, then $\sim p$"*:

"If a number is not divisible by 3, then it is not divisible by 9."

Step 4: Match this derived statement with the given options. It corresponds exactly to option (C).

Final Answer: If a number is not divisible by 3, then it is not divisible by 9.

Answer: (C)

Go Back to Question 15

Q16.

Solution

Concept: The scalar projection of a vector $\vec{a}$ onto another vector $\vec{b}$ is given by the formula:

$$\text{Projection} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

where $\vec{a} \cdot \vec{b}$ represents the standard dot product and $|\vec{b}|$ represents the magnitude of vector $\vec{b}$.

Solution: Step 1: Write down the components of both given vectors:

$$\vec{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$$

$$\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$$

Step 2: Calculate the dot product $\vec{a} \cdot \vec{b}$ by multiplying corresponding components:

$$\vec{a} \cdot \vec{b} = (2)(1) + (3)(2) + (2)(1)$$

$$\vec{a} \cdot \vec{b} = 2 + 6 + 2 = 10$$

Step 3: Calculate the magnitude of vector $\vec{b}$:

$$|\vec{b}| = \sqrt{(1)^2 + (2)^2 + (1)^2}$$

$$|\vec{b}| = \sqrt{1 + 4 + 1} = \sqrt{6}$$

Step 4: Substitute the dot product value and magnitude into the scalar projection formula:

$$\text{Projection} = \frac{10}{\sqrt{6}}$$

Thus, the projection of vector $\vec{a}$ on vector $\vec{b}$ is equal to $\frac{10}{\sqrt{6}}$.

Final Answer: $\boxed{\frac{10}{\sqrt{6}}}$

Answer: (A)

Go Back to Question 16

Q17.

Solution

Concept: A standard vertical parabola symmetric about the y-axis with its vertex at the origin $(0, 0)$ and opening downwards has a focus located at $(0, -a)$, where $a > 0$. The standard geometric equation for such a parabola is given by $x^2 = -4ay$.

Solution: Step 1: Identify the given coordinates of the vertex and focus:

$$\text{Vertex} = (0, 0)$$

$$\text{Focus} = (0, -3)$$

Step 2: Analyze the position of the focus. Since the focus lies on the negative y-axis, the axis of symmetry is the y-axis, and the parabola opens downwards.
Step 3: Match this with the standard focus form $(0, -a)$ to find the value of parameter a:

$$-a = -3 \implies a = 3$$

Step 4: Write down the standard equation format for a downward-opening vertical parabola:

$$x^2 = -4ay$$

Step 5: Substitute the value $a = 3$ into this standard equation formula:

$$x^2 = -4(3)y$$

$$x^2 = -12y$$

Therefore, the equation of the parabola is $x^2 = -12y$.

Final Answer: $\boxed{x^2 = -12y}$

Answer: (B)

Go Back to Question 17

Q18.

Solution

Concept: A differentiable function $f(x)$ is strictly increasing on an interval if its first derivative satisfies $f'(x) > 0$ for all values of x within that interval. We can check this by completing the square for the derivative function.

Solution: Step 1: Write down the given cubic polynomial function:

$$f(x) = x^3 - 3x^2 + 4x$$

Step 2: Differentiate the function with respect to x using the power rule to find $f'(x)$:

$$f'(x) = \frac{d}{dx}(x^3) - \frac{d}{dx}(3x^2) + \frac{d}{dx}(4x)$$

$$f'(x) = 3x^2 - 6x + 4$$

Step 3: Analyze the sign behavior of the quadratic derivative expression $3x^2 - 6x + 4$ by completing the square:

$$f'(x) = 3(x^2 - 2x) + 4$$

$$f'(x) = 3(x^2 - 2x + 1 - 1) + 4$$

$$f'(x) = 3(x - 1)^2 - 3 + 4$$

$$f'(x) = 3(x - 1)^2 + 1$$

Step 4: Since the term $(x - 1)^2 \geq 0$ for all real numbers x, multiplying it by 3 and adding 1 ensures that the expression is always strictly greater than zero:

$$3(x - 1)^2 + 1 \geq 1 > 0 \quad \text{for all } x \in \mathbb{R}$$

Step 5: Because $f'(x) > 0$ for every real number, the function is strictly increasing on the entire set of real numbers $\mathbb{R}$.

Final Answer: Strictly increasing on $\mathbb{R}$

Answer: (A)

Go Back to Question 18

Q19.

Solution

Concept: The area bounded by a symmetric curve $y = x^2$ and a horizontal line $y = c$ can be evaluated using definite integration. Integrating with respect to y along the vertical axis from $y = 0$ to $y = c$ gives the area as $\int_0^c 2x\,dy$.

Solution: Step 1: Write down the equations defining the boundaries of the region:

$$\text{Curve: } y = x^2 \implies x = \pm\sqrt{y}$$

$$\text{Line: } y = 4$$

Step 2: Identify the limits of integration along the y-axis, which go from the origin vertex $y = 0$ up to the line $y = 4$.
Step 3: Set up the definite integral expression. Due to symmetry about the y-axis, the total area is twice the area in the first quadrant:

$$\text{Area} = 2\int_0^4 x\,dy = 2\int_0^4 \sqrt{y}\,dy$$

Step 4: Express the square root as a fractional exponent and apply the power rule for integration:

$$\text{Area} = 2\int_0^4 y^{1/2}\,dy = 2\left[\frac{y^{3/2}}{3/2}\right]_0^4$$

$$\text{Area} = 2\cdot\frac{2}{3}\left[y^{3/2}\right]_0^4 = \frac{4}{3}\left(4^{3/2} - 0^{3/2}\right)$$

Step 5: Calculate the value of the fractional power $4^{3/2}$:

$$4^{3/2} = (\sqrt{4})^3 = 2^3 = 8$$

Step 6: Multiply the values together to obtain the final area calculation:

$$\text{Area} = \frac{4}{3}\cdot 8 = \frac{32}{3}\text{ sq. units}$$

The area bounded by the curve and the line is $\frac{32}{3}$ square units.

Final Answer: $\boxed{\frac{32}{3}\text{ sq. units}}$

Answer: (A)

Go Back to Question 19

Q20.

Solution

Concept: For any square matrix A of order n, the determinant of its adjugate matrix satisfies the fundamental matrix determinant identity property:

$$|\text{adj } A| = |A|^{n-1}$$

Solution: Step 1: Identify the given values from the problem description:

$$\text{Order of matrix } A \implies n = 3$$

$$\text{Determinant value } |A| = 5$$

Step 2: Recall the standard determinant identity property relating the adjugate matrix determinant to the original determinant:

$$|\text{adj } A| = |A|^{n-1}$$

Step 3: Substitute the known values of $n = 3$ and $|A| = 5$ into this formula:

$$|\text{adj } A| = 5^{3-1}$$

Step 4: Simplify the exponent value in the equation:

$$|\text{adj } A| = 5^2$$

Step 5: Compute the final numerical value:

$$5^2 = 25$$

Therefore, the value of $|\text{adj } A|$ is equal to 25.

Final Answer: $\boxed{25}$

Answer: (B)

Go Back to Question 20

Q21.

Solution

Concept: This question requires matching standard inverse trigonometric functions with their basic derivative formulas derived from calculus principles.

Solution: Step 1: Analyze item (i) in Column I, which is the standard inverse tangent function:

$$f(x) = \tan^{-1} x$$

From standard calculus formulas, the derivative of $\tan^{-1} x$ with respect to x is:

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$$

This matches exactly with entry (b) in Column II.

Step 2: Analyze item (ii) in Column I, which is the standard inverse cosine function:

$$f(x) = \cos^{-1} x$$

From standard calculus formulas, the derivative of $\cos^{-1} x$ with respect to x is:

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1 - x^2}}$$

This matches exactly with entry (a) in Column II.

Step 3: Combine both individual matching pairs to form the complete answer set:

$$\text{(i)} \longrightarrow \text{(b)}, \quad \text{(ii)} \longrightarrow \text{(a)}$$

Final Answer: (i) → (b), (ii) → (a)

Go Back to Question 21

Q22.

Solution

Concept: (i) Direction cosines satisfy $l^2 + m^2 + n^2 = 1$, where $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$. We use $\sin^2\theta = 1 - \cos^2\theta$ to change terms to sine.
(ii) A vector perpendicular to two given vectors $\vec{u}$ and $\vec{v}$ is parallel to their cross product $\vec{u} \times \vec{v}$.

Solution: Step 1: Solve sub-part (i). Write down the direction cosine identity:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Substitute $\cos^2\theta = 1 - \sin^2\theta$ for each term:

$$(1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) = 1$$

$$3 - (\sin^2\alpha + \sin^2\beta + \sin^2\gamma) = 1$$

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 3 - 1 = 2$$

Step 2: Solve sub-part (ii). Let $\vec{u} = \hat{i} + \hat{j}$ and $\vec{v} = \hat{j} + \hat{k}$. Find their cross product using a determinant:

$$\vec{w} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$$\vec{w} = \hat{i}(1-0) - \hat{j}(1-0) + \hat{k}(1-0) = \hat{i} - \hat{j} + \hat{k}$$

Step 3: Check the $\hat{j}$ component of $\vec{w}$, which is -1. The question requires a positive $\hat{j}$ component, so multiply the vector by -1:

$$\vec{w}' = -\vec{w} = -\hat{i} + \hat{j} - \hat{k}$$

Step 4: Find the unit vector by dividing $\vec{w}'$ by its magnitude:

$$|\vec{w}'| = \sqrt{(-1)^2 + (1)^2 + (-1)^2} = \sqrt{3}$$

$$\hat{n} = \frac{-\hat{i} + \hat{j} - \hat{k}}{\sqrt{3}}$$

Final Answer: (i) 2, (ii) $\dfrac{-\hat{i} + \hat{j} - \hat{k}}{\sqrt{3}}$

Go Back to Question 22

Q23.

Solution

Concept: (i) An identity relation $I_A = \{(a, a) : a \in A\}$ is always reflexive, symmetric, and transitive, making it an equivalence relation.
(ii) A matrix M is symmetric if $M^T = M$. We apply the transpose properties $(A - B)^T = A^T - B^T$ and $(AB)^T = B^T A^T$ to check the symmetry of $AB - BA$.

Solution: Step 1: Analyze statement (i). The identity relation I_A on a set A contains only pairs of elements matched with themselves. * It is reflexive because $(a, a) \in I_A$ for all $a \in A$. * It is symmetric because if $(a, b) \in I_A$, then $a = b$, so $(b, a) = (a, a) \in I_A$. * It is transitive because if $(a, b) \in I_A$ and $(b, c) \in I_A$, then $a = b$ and $b = c$, which means $a = c$, so $(a, c) \in I_A$. Since it satisfies all three properties, it is an equivalence relation. Statement (i) is True.
Step 2: Analyze statement (ii). Given that A and B are symmetric matrices, we have:

$$A^T = A, \quad B^T = B$$

Let $M = AB - BA$. Find the transpose of matrix M:

$$M^T = (AB - BA)^T = (AB)^T - (BA)^T$$

Using the product rule for transposes:

$$M^T = B^T A^T - A^T B^T$$

Substitute $A^T = A$ and $B^T = B$ into the expression:

$$M^T = BA - AB = -(AB - BA) = -M$$

Since $M^T = -M$, the matrix $AB - BA$ is skew-symmetric, not symmetric. Statement (ii) is False.

Final Answer: (i) True, (ii) False

Go Back to Question 23

Q24.

Solution

Concept: (i) The negation of a quantified statement of the form *"All X are Y"* is logically equivalent to *"Some X are not Y"* or *"There exists an X that is not Y"*.
(ii) The converse of a conditional statement *"If p, then q"* is formed by swapping the hypothesis and conclusion to get *"If q, then p"*.

Solution: Step 1: Find the negation for statement (i): *"All prime numbers are odd numbers."* The logical negation changes the universal quantifier "All" into an existential quantifier stating that at least one counterexample exists. Thus, the negation is:

"Some prime numbers are not odd numbers."

Alternatively: "There exists a prime number which is not an odd number."
Step 2: Find the converse for statement (ii): *"If a triangle is equilateral, then it is isosceles."* Identify the component parts: * Hypothesis p: A triangle is equilateral. * Conclusion q: A triangle is isosceles. Swap the positions of p and q to form the converse statement structure *"If q, then p"*:

"If a triangle is isosceles, then it is equilateral."

Final Answer: (i) Some prime numbers are not odd numbers.
(ii) If a triangle is isosceles, then it is equilateral.

Go Back to Question 24

Q25.

Solution

Concept: Matrix multiplication is performed row by column. A matrix equation can be solved for a scalar k by substituting terms. To express A^{-1} in terms of A and I, we multiply the matrix equation by A^{-1}.

Solution: Step 1: Solve (i). Compute A^2:

$$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

Step 2: Solve (ii). Substitute A^2 and A into $A^2 - 5A + kI = O$:

$$\begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \implies \begin{bmatrix} -7+k & 0 \\ 0 & -7+k \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating corresponding elements gives $-7 + k = 0 \implies k = 7$.
Step 3: Solve (iii). Multiply $A^2 - 5A + 7I = O$ by A^{-1}:

$$A - 5I + 7A^{-1} = O \implies 7A^{-1} = 5I - A \implies A^{-1} = \frac{1}{7}(5I - A)$$

Step 4: Solve (iv). Compute the matrix value of A^{-1}:

$$A^{-1} = \frac{1}{7}\left(\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\right) = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ \frac{1}{7} & \frac{3}{7} \end{bmatrix}$$

Final Answer: (i) $\begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$ (ii) $k = 7$ (iii) $A^{-1} = \frac{1}{7}(5I - A)$ (iv) $\begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ \frac{1}{7} & \frac{3}{7} \end{bmatrix}$

Go Back to Question 25

Q26.

Solution

Concept: Velocity is the derivative of displacement, $v(t) = s'(t)$. Acceleration is the derivative of velocity, $a(t) = v'(t)$. An object is at rest when $v(t) = 0$, and minimum velocity occurs when $a(t) = 0$.

Solution: Step 1: Solve (i). Find velocity $v(t)$ by differentiating $s(t) = 2t^3 - 9t^2 + 12t + 4$:

$$v(t) = \frac{ds}{dt} = 6t^2 - 18t + 12$$

Step 2: Solve (ii). The object moves forward when $v(t) > 0$:

$$6(t^2 - 3t + 2) > 0 \implies 6(t-1)(t-2) > 0 \implies t \in [0, 1) \cup (2, \infty)$$

Step 3: Solve (iii). Find when the object is at rest by setting $v(t) = 0$:

$$6(t-1)(t-2) = 0 \implies t = 1 \text{ s or } t = 2 \text{ s}$$

Differentiate $v(t)$ to find acceleration $a(t) = 12t - 18$. Evaluate at $t = 1$ and $t = 2$: * $a(1) = 12(1) - 18 = -6$ m/s^2 * $a(2) = 12(2) - 18 = 6$ m/s^2
Step 4: Solve (iv). Minimum velocity occurs when $a(t) = 12t - 18 = 0 \implies t = 1.5$ s:

$$v(1.5) = 6(1.5)^2 - 18(1.5) + 12 = 13.5 - 27 + 12 = -1.5 \text{ m/s}$$

Final Answer:

(i) $v(t) = 6t^2 - 18t + 12$ (ii) $t \in [0, 1) \cup (2, \infty)$
(iii) -6 m/s^2 and 6 m/s^2 (iv) -1.5 m/s

Go Back to Question 26

Q27.

Solution

Concept: The reflection property of definite integrals states that $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$. We can use this property to eliminate the algebraic variable x from the integrand. The remaining trigonometric function is integrated using standard rules.

Solution: Step 1: Solve sub-part (i). Write down the original integral expression:

$$I = \int_0^{\pi} x \sin x \, dx \quad \text{— (1)}$$

Apply the reflection property by replacing x with $(\pi - x)$:

$$I = \int_{0pi} (\pi - x) \sin(\pi - x) \, dx$$

Since $\sin(\pi - x) = \sin x$, the integral becomes:

$$I = \int_0^{\pi} (\pi - x) \sin x \, dx \quad \text{— (2)}$$

Step 2: Solve sub-part (ii). Add equation (1) and equation (2) together:

$$2I = \int_0^{\pi} x \sin x \, dx + \int_0^{\pi} (\pi - x) \sin x \, dx$$

$$2I = \int_0^{\pi} (x + \pi - x) \sin x \, dx$$

$$2I = \pi \int_0^{\pi} \sin x \, dx$$

Step 3: Solve sub-part (iii). Evaluate the definite integral of $\sin x$ from 0 to π:

$$\int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi} = -(\cos\pi - \cos 0) = -(-1 - 1) = 2$$

Step 4: Solve sub-part (iv). Substitute this value back into the expression for $2I$ to find I:

$$2I = \pi \cdot 2 = 2\pi$$

$$I = \pi$$

Final Answer: (i) $\int_0^{\pi} (\pi - x) \sin x \, dx$, (ii) $2I = \pi \int_0^{\pi} \sin x \, dx$, (iii) 2, (iv) $I = \pi$

Go Back to Question 27

Q28.

Solution

Concept: The vector connecting point P to point Q is found using their position vectors: $\vec{PQ} = \vec{Q} - \vec{P}$. The cross product of two vectors is computed using a determinant, and the geometric area of a triangle with adjacent sides $\vec{AB}$ and $\vec{AC}$ is given by $\frac{1}{2}|\vec{AB} \times \vec{AC}|$.

Solution: Step 1: Solve sub-part (i). Find vector $\vec{AB}$ by subtracting position vector $\vec{A}$ from $\vec{B}$:

$$\vec{AB} = \vec{B} - \vec{A} = (2\hat{i} + 3\hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$\vec{AB} = (2-1)\hat{i} + (3-2)\hat{j} + (1-3)\hat{k} = \hat{i} + \hat{j} - 2\hat{k}$$

Step 2: Solve sub-part (ii). Find vector $\vec{AC}$ by subtracting position vector $\vec{A}$ from $\vec{C}$:

$$\vec{AC} = \vec{C} - \vec{A} = (3\hat{i} + \hat{j} + 2\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$\vec{AC} = (3-1)\hat{i} + (1-2)\hat{j} + (2-3)\hat{k} = 2\hat{i} - \hat{j} - \hat{k}$$

Step 3: Solve sub-part (iii). Calculate the cross product $\vec{AB} \times \vec{AC}$ using a matrix determinant:

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -2 \\ 2 & -1 & -1 \end{vmatrix}$$

$$\vec{AB} \times \vec{AC} = \hat{i}(1(-1) - (-2)(-1)) - \hat{j}(1(-1) - (-2)(2)) + \hat{k}(1(-1) - 1(2))$$

$$\vec{AB} \times \vec{AC} = \hat{i}(-1-2) - \hat{j}(-1+4) + \hat{k}(-1-2) = -3\hat{i} - 3\hat{j} - 3\hat{k}$$

Step 4: Solve sub-part (iv). Find the magnitude of the cross product vector:

$$|\vec{AB} \times \vec{AC}| = \sqrt{(-3)^2 + (-3)^2 + (-3)^2} = \sqrt{9+9+9} = \sqrt{27} = 3\sqrt{3}$$

The area of the triangle is half of this magnitude:

$$\text{Area} = \frac{1}{2}|\vec{AB} \times \vec{AC}| = \frac{3\sqrt{3}}{2} \text{ sq. units}$$

Final Answer:

(i) $\hat{i} + \hat{j} - 2\hat{k}$
(ii) $2\hat{i} - \hat{j} - \hat{k}$
(iii) $-3\hat{i} - 3\hat{j} - 3\hat{k}$
(iv) $\frac{3\sqrt{3}}{2}$ sq. units

Go Back to Question 28

Q29.

Solution

Concept: The volume of a cone is $V = \frac{1}{3}\pi r^2 h$. Using the semi-vertical angle relationship $\tan\alpha = \frac{r}{h}$, the volume can be expressed as a function of depth h alone. Differentiating with respect to time relates the rates of change.

Solution: Step 1: Solve (i). Given semi-vertical angle $\alpha = \tan^{-1}(0.5) \implies \tan\alpha = \frac{1}{2}$. Since $\tan\alpha = \frac{r}{h}$, we have:

$$\frac{r}{h} = \frac{1}{2} \implies r = \frac{h}{2}$$

Step 2: Solve (ii). Substitute $r = \frac{h}{2}$ into the cone volume formula:

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$$

Step 3: Solve (iii). Differentiate V with respect to time t using the chain rule:

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2 \frac{dh}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt}$$

Step 4: Solve (v). The water surface area is $A = \pi r^2 = \pi\left(\frac{h}{2}\right)^2 = \frac{\pi h^2}{4}$. Differentiating with respect to t:

$$\frac{dA}{dt} = \frac{\pi h}{2}\frac{dh}{dt}$$

Given constant inflow $\frac{dV}{dt} = 5$, from Step 3 we get $\frac{dh}{dt} = \frac{20}{\pi h^2}$. Substituting this gives:

$$\frac{dA}{dt} = \frac{\pi h}{2}\left(\frac{20}{\pi h^2}\right) = \frac{10}{h} \text{ m}^2/\text{min}$$

Step 5: Solve (vi). The total capacity when depth reaches full height $h = 10$ m is:

$$V_{\text{total}} = \frac{\pi(10)^3}{12} = \frac{250\pi}{3} \text{ m}^3$$

Final Answer:

$$\text{(i) } r = \frac{h}{2} \quad \text{(ii) } V = \frac{\pi h^3}{12} \quad \text{(iii) } \frac{dV}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt}$$
$$\text{(v) } \frac{10}{h} \text{ m}^2/\text{min} \quad \text{(vi) } \frac{250\pi}{3} \text{ m}^3$$

Go Back to Question 29

Q30.

Solution

Concept: Since the vertices and foci lie on the y-axis, this is a vertical ellipse. The standard equation for a vertical ellipse centered at the origin is:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \quad \text{where } a > b$$

The coordinates of the vertices are $(0, \pm a)$ and the foci are $(0, \pm c)$, where the parameters satisfy the relationship $c^2 = a^2 - b^2$.

Solution: Step 1: Identify the values of a and c from the given coordinates:

$$\text{Vertices: } (0, \pm 5) \implies a = 5$$

$$\text{Foci: } (0, \pm 4) \implies c = 4$$

Step 2: Use the standard relationship connecting the semi-axes and the focal distance to find b^2:

$$c^2 = a^2 - b^2$$

Substitute $a = 5$ and $c = 4$ into the equation:

$$4^2 = 5^2 - b^2$$

$$16 = 25 - b^2$$

Step 3: Isolate and solve for b^2:

$$b^2 = 25 - 16$$

$$b^2 = 9$$

Step 4: Calculate the value of a^2:

$$a^2 = 5^2 = 25$$

Step 5: Substitute the values of $b^2 = 9$ and $a^2 = 25$ into the standard equation format for a vertical ellipse:

$$\frac{x^2}{9} + \frac{y^2}{25} = 1$$

Thus, the equation of the ellipse is $\frac{x^2}{9} + \frac{y^2}{25} = 1$.

Final Answer: $\boxed{\frac{x^2}{9} + \frac{y^2}{25} = 1}$

Go Back to Question 30

Q31.

Solution

Concept: The evaluation of a matrix determinant can be simplified by applying elementary row operations to reduce entries to constants and create zeros before expanding.

Solution: Step 1: Write down the given determinant:

$$\Delta = \begin{vmatrix} x+1 & x+2 & x+4 \\ x+3 & x+5 & x+8 \\ x+7 & x+10 & x+14 \end{vmatrix}$$

Step 2: Apply row operations $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ to eliminate x from the lower rows:

$$\Delta = \begin{vmatrix} x+1 & x+2 & x+4 \\ 2 & 3 & 4 \\ 6 & 8 & 10 \end{vmatrix}$$

Step 3: Apply $R_3 \to R_3 - 3R_2$ to further simplify the third row:

$$\Delta = \begin{vmatrix} x+1 & x+2 & x+4 \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix}$$

Step 4: Expand the determinant along the third row (R_3):

$$\Delta = 0 + 1 \cdot \begin{vmatrix} x+1 & x+4 \\ 2 & 4 \end{vmatrix} - 2 \cdot \begin{vmatrix} x+1 & x+2 \\ 2 & 3 \end{vmatrix}$$

$$\Delta = [4(x+1) - 2(x+4)] - 2[3(x+1) - 2(x+2)]$$

Step 5: Simplify the algebraic expressions:

$$\Delta = (2x - 4) - 2(x - 1) = 2x - 4 - 2x + 2 = -2$$

Therefore, the value of the determinant is -2.

Final Answer: $\boxed{-2}$

Go Back to Question 31

Q32.

Solution

Concept: The principal value branch of $\tan^{-1} x$ is $(-\frac{\pi}{2}, \frac{\pi}{2})$ and for $\cos^{-1} x$ is $[0, \pi]$. If the given angle lies outside these branches, we must use trigonometric identities to find the equivalent angle within the principal value interval.

Solution: Step 1: Simplify the first term $\tan^{-1}\left(\tan \frac{3\pi}{4}\right)$. The angle $\frac{3\pi}{4}$ does not lie in the principal interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Rewrite the angle using a standard identity:

$$\tan \frac{3\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right) = -\tan \frac{\pi}{4} = \tan\left(-\frac{\pi}{4}\right)$$

Since $-\frac{\pi}{4} \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we get:

$$\tan^{-1}\left(\tan \frac{3\pi}{4}\right) = -\frac{\pi}{4}$$

Step 2: Simplify the second term $\cos^{-1}\left(\cos \frac{7\pi}{6}\right)$. The angle $\frac{7\pi}{6}$ does not lie in the principal interval $[0, \pi]$. Rewrite the angle using a standard identity:

$$\cos \frac{7\pi}{6} = \cos\left(2\pi - \frac{5\pi}{6}\right) = \cos \frac{5\pi}{6}$$

Since $\frac{5\pi}{6} \in [0, \pi]$, we get:

$$\cos^{-1}\left(\cos \frac{7\pi}{6}\right) = \frac{5\pi}{6}$$

Step 3: Add the two calculated principal values together:

$$\text{Sum} = -\frac{\pi}{4} + \frac{5\pi}{6}$$

Step 4: Find a common denominator to add the fractions, which is 12:

$$\text{Sum} = \frac{-3\pi + 10\pi}{12} = \frac{7\pi}{12}$$

The total principal value of the expression is $\frac{7\pi}{12}$.

Final Answer: $\boxed{7\pi/12}$

Go Back to Question 32

Q33.

Solution

Concept: Using the trigonometric double-angle identity, the numerator can be rewritten as $1 - \cos 4x = 2\sin^2 2x$. The limit can then be evaluated by transforming it into the standard limit form $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.

Solution: Step 1: Write down the given limit expression:

$$L = \lim_{x\to 0} \frac{1-\cos 4x}{x^2}$$

Step 2: Apply the standard trigonometric identity $1 - \cos 2\theta = 2\sin^2\theta$, setting $\theta = 2x$:

$$1 - \cos 4x = 2\sin^2 2x$$

Step 3: Substitute this identity back into the limit expression:

$$L = \lim_{x\to 0} \frac{2\sin^2 2x}{x^2}$$

Step 4: Group the terms to create the standard $\frac{\sin\theta}{\theta}$ form by multiplying and dividing the denominator by 4:

$$L = 2\lim_{x\to 0} \left(\frac{\sin 2x}{x}\right)^2$$

$$L = 2\lim_{x\to 0} \left(\frac{2\sin 2x}{2x}\right)^2$$

$$L = 2\cdot 4\lim_{x\to 0} \left(\frac{\sin 2x}{2x}\right)^2$$

Step 5: Let $\theta = 2x$. As $x \to 0$, we have $\theta \to 0$. Use the standard limit value:

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

Step 6: Complete the numerical calculation:

$$L = 8\cdot (1)^2 = 8$$

The evaluated value of the limit is 8.

Final Answer: $\boxed{8}$

Go Back to Question 33

Q34.

Solution

Concept: For equations with functional exponents, simplify differentiation by taking the natural logarithm on both sides and then applying the quotient rule.

Solution: Step 1: Write down the given equation and take the natural logarithm (log) on both sides:

$$x^y = e^{x-y} \implies \log(x^y) = \log(e^{x-y})$$

Step 2: Apply logarithm properties ($\log a^b = b \log a$ and $\log e = 1$) to simplify:

$$y \log x = x - y$$

Step 3: Rearrange the terms to solve for y explicitly in terms of x:

$$y \log x + y = x \implies y(1 + \log x) = x \implies y = \frac{x}{1 + \log x}$$

Step 4: Differentiate y with respect to x using the quotient rule:

$$\frac{dy}{dx} = \frac{(1 + \log x) \cdot \frac{d}{dx}(x) - x \cdot \frac{d}{dx}(1 + \log x)}{(1 + \log x)^2}$$

$$\frac{dy}{dx} = \frac{(1 + \log x)(1) - x\left(\frac{1}{x}\right)}{(1 + \log x)^2}$$

Step 5: Simplify the numerator:

$$\frac{dy}{dx} = \frac{1 + \log x - 1}{(1 + \log x)^2} = \frac{\log x}{(1 + \log x)^2}$$

Therefore, the derivative $\frac{dy}{dx}$ is $\frac{\log x}{(1+\log x)^2}$.

Final Answer: $\boxed{\frac{\log x}{(1 + \log x)^2}}$

Go Back to Question 34

Q35.

Solution

Concept: A function $f(x)$ is strictly decreasing on an interval if its first derivative satisfies $f'(x) < 0$ for all values of x within that interval. We find these intervals by finding the critical points and testing the intervals.

Solution: Step 1: Write down the given polynomial function:

$$f(x) = 2x^3 - 3x^2 - 36x + 7$$

Step 2: Differentiate the function with respect to x to find $f'(x)$:

$$f'(x) = 6x^2 - 6x - 36$$

Step 3: Set up the strict inequality condition for a decreasing function:

$$6x^2 - 6x - 36 < 0$$

Step 4: Divide the entire inequality by the positive scalar 6 and factorize the remaining quadratic expression:

$$x^2 - x - 6 < 0$$

$$x^2 - 3x + 2x - 6 < 0$$

$$x(x - 3) + 2(x - 3) < 0$$

$$(x + 2)(x - 3) < 0$$

Step 5: Identify the critical points where the expression equals zero, which are $x = -2$ and $x = 3$. Analyze the sign behavior across intervals: * For $x \in (-\infty, -2)$, both factors are negative, so their product is positive ($f'(x) > 0$). * For $x \in (-2, 3)$, $(x + 2)$ is positive and $(x - 3)$ is negative, so their product is negative ($f'(x) < 0$). * For $x \in (3, \infty)$, both factors are positive, so their product is positive ($f'(x) > 0$).
Step 6: Choose the interval where the derivative is strictly negative:

$$x \in (-2, 3)$$

Therefore, the function is strictly decreasing in the interval $(-2, 3)$.

Final Answer: $\boxed{(-2, 3)}$
Go Back to Question 35

Q36.

Solution

Concept: This is a first-order linear differential equation of the standard form $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x or constants. The general solution is found using an integrating factor I.F. $= e^{\int P\,dx}$, given by the formula $y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dx + C$.

Solution: Step 1: Identify P and Q by comparing the given equation with the standard linear form:

$$\frac{dy}{dx} + 2y = e^{-x} \implies P = 2, \quad Q = e^{-x}$$

Step 2: Calculate the Integrating Factor (I.F.):

$$\text{I.F.} = e^{\int P\,dx} = e^{\int 2\,dx} = e^{2x}$$

Step 3: Write down the general solution formula for a linear differential equation:

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dx + C$$

Step 4: Substitute the values of I.F. and Q into this formula:

$$y \cdot e^{2x} = \int e^{-x} \cdot e^{2x}\,dx + C$$

Step 5: Simplify the integrand using exponent laws ($e^a \cdot e^b = e^{a+b}$):

$$y \cdot e^{2x} = \int e^{x}\,dx + C$$

Step 6: Integrate the right-hand side expression:

$$y \cdot e^{2x} = e^{x} + C$$

Step 7: Divide the entire equation by e^{2x} to express y explicitly:

$$y = \frac{e^x}{e^{2x}} + \frac{C}{e^{2x}}$$

$$y = e^{-x} + Ce^{-2x}$$

The general solution of the differential equation is $y = e^{-x} + Ce^{-2x}$.

Final Answer: $\boxed{y = e^{-x} + Ce^{-2x}}$

Go Back to Question 36

Q37.

Solution

Concept: The shortest distance d between two skew lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is calculated using the formula:

$$d = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$$

Solution: Step 1: Identify the position vectors and direction vectors from the given line equations:

$$\vec{a}_1 = \hat{\imath} + 2\hat{\jmath} + \hat{k}, \quad \vec{b}_1 = \hat{\imath} - \hat{\jmath} + \hat{k}$$

$$\vec{a}_2 = 2\hat{\imath} - \hat{\jmath} - \hat{k}, \quad \vec{b}_2 = 2\hat{\imath} + \hat{\jmath} + 2\hat{k}$$

Step 2: Calculate the displacement vector $(\vec{a}_2 - \vec{a}_1)$:

$$\vec{a}_2 - \vec{a}_1 = (2-1)\hat{\imath} + (-1-2)\hat{\jmath} + (-1-1)\hat{k} = \hat{\imath} - 3\hat{\jmath} - 2\hat{k}$$

Step 3: Calculate the cross product vector $(\vec{b}_1 \times \vec{b}_2)$ using a matrix determinant:

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$\vec{b}_1 \times \vec{b}_2 = \hat{\imath}(-2-1) - \hat{\jmath}(2-2) + \hat{k}(1-(-2)) = -3\hat{\imath} + 3\hat{k}$$

Step 4: Compute the magnitude of the cross product vector $|\vec{b}_1 \times \vec{b}_2|$:

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{(-3)^2 + (0)^2 + (3)^2} = \sqrt{9+0+9} = \sqrt{18} = 3\sqrt{2}$$

Step 5: Calculate the dot product of $(\vec{a}_2 - \vec{a}_1)$ and $(\vec{b}_1 \times \vec{b}_2)$:

$$(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = (1)(-3) + (-3)(0) + (-2)(3) = -3 + 0 - 6 = -9$$

Step 6: Substitute these values into the shortest distance formula:

$$d = \frac{|-9|}{3\sqrt{2}} = \frac{9}{3\sqrt{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2} \text{ units}$$

The shortest distance between the lines is $\frac{3}{\sqrt{2}}$ units.

Final Answer: $\boxed{\frac{3}{\sqrt{2}} \text{ units}}$

Go Back to Question 37

Q38.

Solution

Concept: Lagrange's Mean Value Theorem (LMVT) is applicable for a function $f(x)$ on a closed interval $[a, b]$ if the function is continuous on $[a, b]$ and differentiable on the open interval (a, b). If these conditions are met, there exists at least one value $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Solution: Step 1: Write down the given function and expand it into standard polynomial form:

$$f(x) = x(x-2) = x^2 - 2x, \quad \text{on } [1, 3]$$

Step 2: Check the conditions for LMVT. Since $f(x)$ is a polynomial function, it is continuous everywhere on $\mathbb{R}$, which means it is continuous on the closed interval $[1, 3]$. It is also differentiable everywhere, so it is differentiable on the open interval $(1, 3)$. Therefore, LMVT is applicable.
Step 3: Evaluate the function values at the boundary endpoints $a = 1$ and $b = 3$:

$$f(1) = 1(1-2) = -1$$

$$f(3) = 3(3-2) = 3$$

Step 4: Find the first derivative function $f'(x)$:

$$f'(x) = 2x - 2 \implies f'(c) = 2c - 2$$

Step 5: Apply the LMVT formula equation to solve for c:

$$2c - 2 = \frac{f(3) - f(1)}{3 - 1}$$

$$2c - 2 = \frac{3 - (-1)}{2} = \frac{4}{2} = 2$$

Step 6: Isolate and solve for variable c:

$$2c = 2 + 2 = 4 \implies c = 2$$

Verify that $c = 2$ lies within the open interval $(1, 3)$. Since it does, this value is verified.

Final Answer: LMVT is applicable and $c = 2$

Go Back to Question 38

Q39.

Solution

Concept: For a standard parabola $y^2 = 4ax$, the focus is located at $S(a, 0)$ and the length of the semi-latus rectum is $2a$. Any focal chord can be defined by its endpoints using parametric coordinates $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$, where the parameters satisfy the property $t_1t_2 = -1$.

Solution: Step 1: State the coordinates of the focus S and the endpoints of a focal chord P and Q:

$$S = (a, 0), \quad P = (at^2, 2at), \quad Q = \left(\frac{a}{t^2}, -\frac{2a}{t}\right) \quad \text{since } t_2 = -\frac{1}{t}$$

Step 2: Calculate the lengths of the focal segments SP and SQ using the distance formula or standard geometric properties ($SP = x_P + a$):

$$SP = at^2 + a = a(t^2 + 1)$$

$$SQ = \frac{a}{t^2} + a = a\left(\frac{1 + t^2}{t^2}\right)$$

Step 3: Find the harmonic mean (H.M.) between the lengths of these two segments SP and SQ:

$$\text{H.M.} = \frac{2 \cdot SP \cdot SQ}{SP + SQ}$$

Step 4: Substitute the expressions for SP and SQ into the numerator of the harmonic mean formula:

$$\text{Numerator} = 2 \cdot [a(t^2 + 1)] \cdot \left[a\left(\frac{1 + t^2}{t^2}\right)\right] = \frac{2a^2(t^2 + 1)^2}{t^2}$$

Step 5: Substitute the expressions into the denominator of the harmonic mean formula:

$$\text{Denominator} = a(t^2 + 1) + a\left(\frac{1 + t^2}{t^2}\right) = a(t^2 + 1)\left(1 + \frac{1}{t^2}\right) = \frac{a(t^2 + 1)^2}{t^2}$$

Step 6: Divide the numerator expression by the denominator expression to find the harmonic mean value:

$$\text{H.M.} = \frac{\frac{2a^2(t^2+1)^2}{t^2}}{\frac{a(t^2+1)^2}{t^2}} = \frac{2a^2}{a} = 2a$$

Since $2a$ is exactly the length of the semi-latus rectum, the geometric property is proved.

Final Answer: Proved that the harmonic mean is $2a$

Go Back to Question 39

Q40.

Solution

Concept: A system of linear equations $AX = B$ has a unique solution given by $X = A^{-1}B$ if the determinant $|A| \neq 0$, where $A^{-1} = \frac{1}{|A|}\text{adj } A$.

Solution: Step 1: Express the linear system in the matrix form $AX = B$:

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

Step 2: Calculate the determinant of the coefficient matrix A:

$$|A| = 2(-4+4) + 3(-6+4) + 5(3-2) = 0 - 6 + 5 = -1$$

Since $|A| = -1 \neq 0$, the inverse matrix A^{-1} exists.
Step 3: Find the cofactors A_{ij} and construct the adjugate matrix adj A:

$$\begin{matrix} A_{11} = 0, & A_{12} = 2, & A_{13} = 1 \\ A_{21} = -1, & A_{22} = -9, & A_{23} = -5 \\ A_{31} = 2, & A_{32} = 23, & A_{33} = 13 \end{matrix} \implies \text{adj } A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

Step 4: Compute the inverse matrix $A^{-1} = \frac{1}{|A|}\text{adj } A$:

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

Step 5: Multiply A^{-1} by matrix B to find the solution vector X:

$$X = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 - 5 + 6 \\ -22 - 45 + 69 \\ -11 - 25 + 39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Therefore, the solutions are $x = 1$, $y = 2$, and $z = 3$.

Final Answer: $\boxed{x = 1, y = 2, z = 3}$

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Q41.

Solution

Concept: To prove this inverse trigonometric identity, we use algebraic substitution. Setting $x = \cos 2\theta$ allows us to simplify the square root terms using the standard half-angle identities $1 + \cos 2\theta = 2\cos^2\theta$ and $1 - \cos 2\theta = 2\sin^2\theta$.

Solution: Step 1: Let $x = \cos 2\theta$, which means $2\theta = \cos^{-1} x \implies \theta = \frac{1}{2}\cos^{-1} x$.
Step 2: Substitute $x = \cos 2\theta$ into the expressions under the square roots on the left-hand side (LHS):

$$\sqrt{1 + x} = \sqrt{1 + \cos 2\theta} = \sqrt{2\cos^2\theta} = \sqrt{2}\cos\theta$$

$$\sqrt{1 - x} = \sqrt{1 - \cos 2\theta} = \sqrt{2\sin^2\theta} = \sqrt{2}\sin\theta$$

Here, we use the fact that for the given domain, $\cos\theta$ and $\sin\theta$ are positive.
Step 3: Substitute these simplified square root expressions back into the original LHS formula:

$$\text{LHS} = \tan^{-1}\left(\frac{\sqrt{2}\cos\theta - \sqrt{2}\sin\theta}{\sqrt{2}\cos\theta + \sqrt{2}\sin\theta}\right)$$

Step 4: Factor out and cancel the common constant factor $\sqrt{2}$ from both the numerator and denominator:

$$\text{LHS} = \tan^{-1}\left(\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\right)$$

Step 5: Divide both the numerator and the denominator by $\cos\theta$ to convert the expression into terms of the tangent function:

$$\text{LHS} = \tan^{-1}\left(\frac{1 - \tan\theta}{1 + \tan\theta}\right)$$

Step 6: Use the standard tangent subtraction identity $\frac{\tan(\pi/4) - \tan\theta}{1 + \tan(\pi/4)\tan\theta} = \tan\left(\frac{\pi}{4} - \theta\right)$:

$$\text{LHS} = \tan^{-1}\left(\tan\left(\frac{\pi}{4} - \theta\right)\right) = \frac{\pi}{4} - \theta$$

Step 7: Substitute $\theta = \frac{1}{2}\cos^{-1} x$ back into the expression:

$$\text{LHS} = \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x = \text{RHS}$$

Hence, the identity is proved.

Final Answer: Proved

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Q42.

Solution

Concept: To evaluate this definite integral, use the reflection property $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$. Adding the original and modified equations eliminates x, simplifying the integrand.

Solution: Step 1: Let the given integral be:

$$I = \int_0^{\pi} \frac{x\tan x}{\sec x + \tan x}\,dx \quad \text{— (1)}$$

Step 2: Apply the reflection property by replacing x with $(\pi - x)$:

$$I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}\,dx = \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \tan x}\,dx \quad \text{— (2)}$$

Step 3: Add equations (1) and (2), then convert to sine and cosine:

$$2I = \pi \int_0^{\pi} \frac{\tan x}{\sec x + \tan x}\,dx = \pi \int_0^{\pi} \frac{\sin x}{1 + \sin x}\,dx$$

Step 4: Rationalize the integrand by multiplying the numerator and denominator by $(1 - \sin x)$:

$$2I = \pi \int_0^{\pi} \frac{\sin x - \sin^2 x}{\cos^2 x}\,dx = \pi \int_0^{\pi} (\sec x \tan x - \sec^2 x + 1)\,dx$$

Step 5: Integrate and substitute the limits:

$$2I = \pi\,[\sec x - \tan x + x]_0^{\pi} = \pi[(-1 - 0 + \pi) - (1 - 0 + 0)] = \pi(\pi - 2)$$

$$I = \frac{\pi(\pi - 2)}{2}$$

Final Answer: $\boxed{\dfrac{\pi(\pi - 2)}{2}}$

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Q43.

Solution

Concept: The plane passing through the intersection of planes $P_1 = 0$ and $P_2 = 0$ is given by $P_1 + \lambda P_2 = 0$. The parameter λ is determined using the perpendicularity condition ($\vec{n}_1 \cdot \vec{n}_2 = 0$).

Solution: Step 1: Write down the family of planes equation:

$$(x + y + z - 1) + \lambda(2x + 3y + 4z - 5) = 0$$

$$(1 + 2\lambda)x + (1 + 3\lambda)y + (1 + 4\lambda)z - (1 + 5\lambda) = 0$$

Step 2: Identify the normal vector $\vec{n} = (1 + 2\lambda)\hat{i} + (1 + 3\lambda)\hat{j} + (1 + 4\lambda)\hat{k}$.
Step 3: Apply the perpendicularity condition with the given plane $x - y + z = 0$ ($\vec{n}_{\text{perp}} = \hat{i} - \hat{j} + \hat{k}$):

$$(1 + 2\lambda)(1) + (1 + 3\lambda)(-1) + (1 + 4\lambda)(1) = 0$$

$$1 + 2\lambda - 1 - 3\lambda + 1 + 4\lambda = 0 \implies 3\lambda + 1 = 0 \implies \lambda = -\frac{1}{3}$$

Step 4: Substitute $\lambda = -\frac{1}{3}$ back into the grouped equation:

$$\left(1 - \frac{2}{3}\right)x + \left(1 - \frac{3}{3}\right)y + \left(1 - \frac{4}{3}\right)z - \left(1 - \frac{5}{3}\right) = 0$$

$$\frac{1}{3}x + 0y - \frac{1}{3}z + \frac{2}{3} = 0$$

Step 5: Multiply by 3 to simplify:

$$x - z + 2 = 0$$

Final Answer: $\boxed{x - z + 2 = 0}$

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Q44.

Solution

Concept: A Linear Programming Problem can be solved graphically by defining variables, writing down the objective function and linear constraints, shading the feasible region on a coordinate grid, and evaluating the objective function at all corner points.

Solution: Step 1: Solve the first problem option. Let x be the number of chairs and y be the number of tables produced daily. Objective function to maximize profit:

$$\text{Maximize } Z = 300x + 500y$$

Subject to linear machine constraints:

$$\text{Machine A: } 2x + 4y \leq 16 \implies x + 2y \leq 8$$

$$\text{Machine B: } 6x + 2y \leq 30 \implies 3x + y \leq 15$$

$$\text{Non-negativity: } x \geq 0, \quad y \geq 0$$

Step 2: Find the boundary intercepts to graph the constraint lines: * For $x + 2y = 8$: Intercepts are $(8, 0)$ and $(0, 4)$. * For $3x + y = 15$: Intercepts are $(5, 0)$ and $(0, 15)$.
Step 3: Solve the two boundary equations simultaneously to find their point of intersection: Multiply the second equation by 2: $6x + 2y = 30$. Subtract the first equation from it:

$$(6x + 2y) - (x + 2y) = 30 - 8 \implies 5x = 22 \implies x = 4.4$$

$$y = 15 - 3(4.4) = 15 - 13.2 = 1.8$$

The intersection point is $(4.4, 1.8)$.
Step 4: Identify all the corner points of the shaded feasible region:

$$\text{Corner Points: } (0, 0), \ (5, 0), \ (4.4, 1.8), \ (0, 4)$$

Step 5: Evaluate the profit objective function Z at each corner point: * At $(0, 0)$: $Z = 0$ * At $(5, 0)$: $Z = 300(5) + 500(0) = 1500$ * At $(0, 4)$: $Z = 300(0) + 500(4) = 2000$ * At $(4.4, 1.8)$: $Z = 300(4.4) + 500(1.8) = 1320 + 900 = 2220$
The maximum daily profit is ₹ 2220, which occurs when producing 4.4 chairs and 1.8 tables. (Note: For absolute integer constraints, nearby integer points can be tested, but standard graphical optimization gives this vertex).

Final Answer: Maximum profit is ₹ 2220 at $(4.4, 1.8)$

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Q45.

Solution

Concept: To find the maximum volume of an inscribed cylinder, we use geometric relationships to express the volume as a single-variable function. We then find the critical points by setting the first derivative to zero and verify the maximum using the second derivative test.

Solution: Step 1: Solve the first problem option. Let r be the radius and h be the height of the cylinder inscribed in a sphere of fixed radius R. From the right triangle geometry inside the sphere, the Pythagorean theorem gives:

$$R^2 = r^2 + \left(\frac{h}{2}\right)^2 \implies r^2 = R^2 - \frac{h^2}{4}$$

Step 2: Write down the formula for the volume V of a cylinder:

$$V = \pi r^2 h$$

Step 3: Substitute the expression for r^2 into the volume formula to express V as a function of height h alone:

$$V(h) = \pi \left(R^2 - \frac{h^2}{4}\right) h = \pi R^2 h - \frac{\pi h^3}{4}$$

Step 4: Differentiate V with respect to h and set it equal to zero to find the critical values:

$$\frac{dV}{dh} = \pi R^2 - \frac{3\pi h^2}{4} = 0$$

$$\frac{3\pi h^2}{4} = \pi R^2 \implies h^2 = \frac{4R^2}{3} \implies h = \frac{2R}{\sqrt{3}}$$

Step 5: Apply the second derivative test to verify if this critical point is a maximum:

$$\frac{d^2V}{dh^2} = -\frac{6\pi h}{4} = -\frac{3\pi h}{2}$$

Since $h > 0$, the value of the second derivative is negative ($\frac{d^2V}{dh^2} < 0$), confirming a local maximum. Step 6: Calculate the maximum volume by substituting $h = \frac{2R}{\sqrt{3}}$ back into the volume function:

$$V_{\max} = \pi \left(R^2 - \frac{\frac{4R^2}{3}}{4}\right) \cdot \frac{2R}{\sqrt{3}} = \pi \left(R^2 - \frac{R^2}{3}\right) \cdot \frac{2R}{\sqrt{3}}$$

$$V_{\max} = \pi \left(\frac{2R^2}{3}\right) \cdot \frac{2R}{\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}}$$

Final Answer: Height is $\frac{2R}{\sqrt{3}}$ and maximum volume is $\frac{4\pi R^3}{3\sqrt{3}}$

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Answer Key

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	A	2	A	3	A	4	A	5	B
6	A	7	C	8	B	9	C	10	C
11	C	12	A	13	A	14	A	15	C
16	A	17	B	18	A	19	A	20	B