NIOS Class 12 Mathematics Sample Paper – 7

Duration: 180 Minutes **Maximum Marks: 100**

Instructions

- This paper contains **45** Questions. The paper is divided into two sections: **Section A – 50** marks, **Section B – 50** marks.
- **Section A** consists of
  -**Q.No. 1 to 20** – Multiple Choice type questions (MCQs) carrying **+1 mark** each. Select and write the most appropriate option out of the four options given in each of these questions.
  -**Q.No. 21 to 29** – **Objective type questions**.
  -**Q.No. 21 to 24** carry **02 marks** each (with 2 sub-parts of 1 mark each).
  -**Q.No. 25 to 28** carry **04 marks** each (with 4 sub-parts of 1 mark each).
  -**Q.No. 29** carries **06 marks** (with 6 sub-parts of 1 mark each). Attempt these questions as per the instructions given for each of the questions 21–29.
- **Section B** consists of
  -**Q.No. 30 to 38**– Very Short questions carrying **02 marks** each.
  -**Q.No. 39 to 43** – Short Answer type questions carrying **04 marks** each.
  -**Q.No. 44 to 45** – Long Answer type questions carrying **06 marks** each.
  (An internal choice has been provided in some of the questions in Section B. You have to attempt only one of the given choices in such questions.)
- There is **No Negative marking**.
- Use of mobile phones, smartwatches, calculators, or any electronic gadgets is strictly prohibited.

Section: A

Q1. If $\lim_{x \to 0} \dfrac{\sin(ax) + \sin(bx)}{x} = 7$, where a and b are positive integers with $a > b$, then the value of $a - b$ is: **(1)**

(A) 2

(B) 3

(C) 1

(D) 5

Q2. The domain of the function $f(x) = \sqrt{\log_{0.5}(x-2)}$ is: **(1)**

(A) $(2, 3]$

(B) $[2, 3]$

(C) $(2, \infty)$

(D) $[3, \infty)$

Q3. If A is a 3×3 matrix and $|A| = 2$, then the value of $|\text{adj}(\text{adj}(A))|$ is: **(1)**

(A) 64

(B) 16

(C) 8

(D) 4

Q4. The value of $\int_{-\pi/2}^{\pi/2} \frac{\cos^5 x}{1 + e^{\sin x}}\, dx$ is: **(1)**

(A) $\frac{8}{15}$

(B) $\frac{16}{15}$

(C) $\frac{4}{15}$

(D) $\frac{2}{15}$

Q5. A line makes angles $60°$, $60°$, and γ with the positive directions of x, y, and z axes respectively. The value of $\cos^2 \gamma$ is: **(1)**

(A) $\frac{1}{2}$

(B) $\frac{1}{4}$

(C) $\frac{3}{4}$

(D) 0

Q6. The equation of the line passing through the intersection of $2x + 3y - 5 = 0$ and $x - 2y + 1 = 0$ and parallel to the y-axis is: **(1)**

(A) $x = 1$

(B) $y = 1$

(C) $x = 2$

(D) $y = -1$

Q7. The distance of the point $(3, -4, 5)$ from the yz-plane is: **(1)**

(A) 3

(B) 5

(C) $\sqrt{41}$

(D) 4

Q8. The slope of the tangent to the curve $y = e^{2x} - 3x$ at $x = 0$ is: **(1)**

(A) 1

(B) -1

(C) -2

(D) 2

Q9. The equation of the circle passing through the points $(1, 0)$, $(-1, 0)$, and $(0, 1)$ is: **(1)**

(A) $x^2 + y^2 = 1$

(B) $x^2 + y^2 - y = 1$

(C) $x^2 + y^2 = 2$

(D) $x^2 + y^2 + y = 1$

Q10. The length of the latus rectum of the parabola $2y^2 + 12x - 8y + 2 = 0$ is: **(1)**

(A) 6

(B) 3

(C) 12

(D) 8

Q11. The order and degree of the differential equation $\left(\frac{d^3y}{dx^3}\right)^2 - 3\left(\frac{dy}{dx}\right)^5 + 7y = 0$ are respectively: **(1)**

(A) 3 and 2

(B) 2 and 3

(C) 3 and 5

(D) 5 and 3

Q12. If $\vec{a} = 2\hat{i} - \hat{j} + \lambda\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ are perpendicular, then the value of λ is: **(1)**

(A) 0

(B) 1

(C) 2

(D) -2

Q13. Let R be a relation on $\mathbb{R}$ defined by aRb if $|a - b| \leq 1$. The relation R is: **(1)**

(A) Reflexive and symmetric but not transitive

(B) Reflexive and transitive but not symmetric

(C) Symmetric and transitive but not reflexive

(D) An equivalence relation

Q14. If $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$, then A^{-1} equals: **(1)**

(A) $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

(B) $\begin{pmatrix} -2 & 3 \\ 1 & -2 \end{pmatrix}$

(C) $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$

(D) $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

Q15. The statement $p \to (q \to r)$ is logically equivalent to: **(1)**

(A) $(p \wedge q) \to r$

(B) $p \to (r \to q)$

(C) $(p \vee q) \to r$

(D) $p \wedge (q \to r)$

Q16. The feasible region for an LPP with constraints $x + y \leq 4$, $x \geq 0$, $y \geq 0$ is always: **(1)**

(A) Unbounded

(B) A triangle

(C) A quadrilateral

(D) Bounded and convex

Q17. The contrapositive of "If a triangle is equilateral, then it is isosceles" is: **(1)**

(A) If a triangle is not isosceles, then it is not equilateral

(B) If a triangle is isosceles, then it is equilateral

(C) If a triangle is not equilateral, then it is not isosceles

(D) A triangle is equilateral if and only if it is isosceles

Q18. The principal value of $\sin^{-1}\left(\sin \frac{5\pi}{3}\right)$ is: **(1)**

(A) $\dfrac{5\pi}{3}$

(B) $-\dfrac{\pi}{3}$

(C) $\frac{\pi}{3}$

(D) $\frac{2\pi}{3}$

Q19. The eccentricity of the ellipse $9x^2 + 5y^2 - 30y = 0$ is: **(1)**

(A) $\frac{2}{3}$

(B) $\frac{4}{5}$

(C) $\frac{3}{5}$

(D) $\frac{\sqrt{5}}{3}$

Q20. If $y = (\sin x)^{\cos x}$, then $\frac{dy}{dx}$ at $x = \frac{\pi}{2}$ is: **(1)**

(A) 0

(B) −1

(C) 1

(D) Does not exist

Q21. Match the entries in Column-I with their correct values in Column-II: **(2)**

Column-I	**Column-II**
(i) If $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, then $\det(A^2)$ is	(A) **1**
(ii) The projection of $\vec{a} = 3\hat{i} + 4\hat{k}$ on $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$ is	(B) $\mathbf{\frac{14}{3}}$

(A) (i) → (A), (ii) → (B)

(B) (i) → (B), (ii) → (A)

Q22. Answer the following fill-in-the-blank items: **(2)**

(i) The degree of the differential equation $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{d^2y}{dx^2}$ is ________.

(ii) The distance between the parallel planes $x - 2y + 2z + 3 = 0$ and $2x - 4y + 4z - 6 = 0$ is ________ units.

Q23. Determine the truth value of the statements below: **(2)**

(i) A relation that is symmetric and transitive must also be reflexive.

(ii) The function $f(x) = x|x|$ is differentiable for all $x \in \mathbb{R}$.

Q24. Write down the logical negation for the following sentences: **(2)**

(i) Every rational number has a terminating decimal representation.

(ii) There exists a real number x such that $x^2 + 1 = 0$.

Q25. Fill in the blanks with appropriate answers for the following limits: **(4)**

(i) $\lim_{x\to 0} \frac{\tan 5x}{\sin 2x} =$ ________.

(ii) $\lim_{x\to 0} \frac{e^{3x}-1}{x} =$ ________.

(iii) $\lim_{x\to \infty} \left(1+\frac{2}{x}\right)^x =$ ________.

(iv) $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} =$ ________.

Q26. Fill in the blanks with the properties of Conic Sections: **(4)**

(i) The centre of the circle $x^2 + y^2 - 6x + 4y - 3 = 0$ is ________.

(ii) The eccentricity of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ is ________.

(iii) The focus of the parabola $x^2 = -20y$ is ________.

(iv) The length of the minor axis of the ellipse $\frac{x^2}{49} + \frac{y^2}{36} = 1$ is ________.

Q27. Write TRUE or FALSE for the following properties: **(4)**

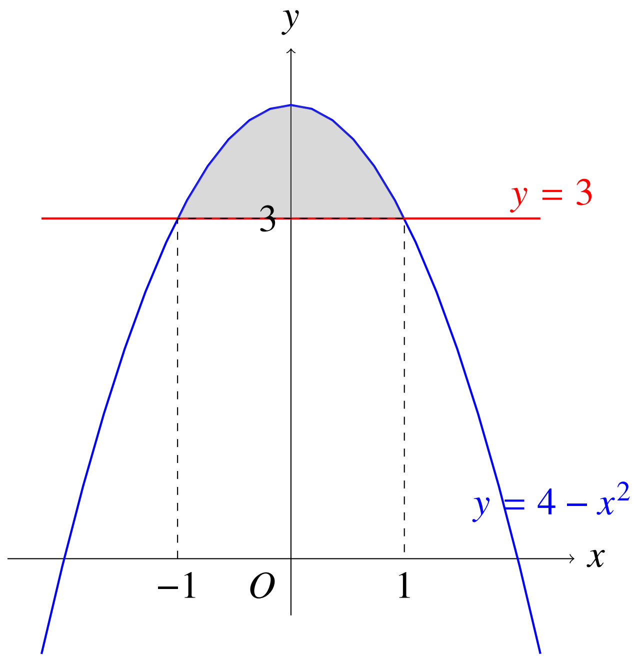
(i) If A and B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

(ii) The cross product of two unit vectors is always a unit vector.

(iii) A diagonal matrix is always a scalar matrix.

(iv) If $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \times \vec{b} = \vec{0}$, then at least one of $\vec{a}$ or $\vec{b}$ must be a zero vector.

Q28. Study the geometric representation below showing the parabola $y = 4 - x^2$ and the line $y = 3$ in the first two quadrants, then answer the following 4 questions: **(4)**



(i) Find the x-coordinates of the points of intersection of the parabola $y = 4 - x^2$ and the line $y = 3$.

(ii) Write the expression for the integrand representing the area of the shaded region.

(iii) Set up the definite integral required to compute the shaded area.

(iv) Calculate the final numeric value of the shaded area in square units.

Q29. Read the passage given below and answer the six sub-questions (i to vi) that follow:

A toy company manufactures two types of toys – Type X and Type Y. Each Type X toy requires 2 hours on Machine A and 1 hour on Machine B, and yields a profit of Rs. 30. Each Type Y toy requires 1 hour on Machine A and 3 hours on Machine B, and yields a profit of Rs. 20. Machine A is available for at most 12 hours per day, and Machine B is available for at most 15 hours per day. Let x be the number of Type X toys and y be the number of Type Y toys produced per day. **(6)**

(i) Which of the following is a constraint for Machine A?

(A) $x + 3y \leq 15$

(B) $2x + y \leq 12$

(C) $2x + 3y \leq 12$

(D) $x + 2y \leq 15$

(ii) The objective profit function Z for this LPP is:

(A) $Z = 20x + 25y$

(B) $Z = 30x + 20y$

(C) $Z = 12x + 15y$

(D) $Z = 2x + 3y$

(iii) The corner point of the feasible region lying on the positive x-axis (excluding origin) is:

(A) $(0, 5)$

(B) $(6, 0)$

(C) $(0, 4)$

(D) $(12, 0)$

(iv) The corner point obtained by solving $2x + y = 12$ and $x + 3y = 15$ simultaneously is:

(A) $\left(\frac{21}{5}, \frac{18}{5}\right)$

(B) $(4, 4)$

(C) $(3, 6)$

(D) $\left(\frac{18}{5}, \frac{21}{5}\right)$

(v) The value of Z at the corner point $(6, 0)$ is:

(A) 120

(B) 150

(C) 180

(D) 210

(vi) The maximum profit the company can earn per day is:

(A) Rs. 150

(B) Rs. 180

(C) Rs. 198

(D) Rs. 210

Section: B

Q30. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x - 3$ is a bijective function. **(2)**

Q31. Find the value of p if the line $\frac{x-2}{3} = \frac{y+1}{p} = \frac{z-4}{-2}$ is perpendicular to the line $\frac{x+3}{4} = \frac{y-2}{1} = \frac{z+5}{3}$. **(2)**

Q32. Differentiate $e^{\cos(x^3)}$ with respect to x. **(2)**

Q33. Form the differential equation of the family of parabolas $y^2 = 4ax$, where a is an arbitrary constant. **(2)**

Q34. Find the projection of the vector $\vec{a} = -\hat{i} + 3\hat{j} + 5\hat{k}$ on the vector $\vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}$.

OR

Find a unit vector perpendicular to both the vectors $\vec{a} = 2\hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = \hat{i}-\hat{j}+2\hat{k}$. **(2)**

Q35. Evaluate the integral: $\int \frac{\sin(\log x)}{x}\, dx$. **(2)**

Q36. Find the equation of the normal to the curve $y = x^2 + 2e^x$ at the point $(0, 2)$. **(2)**

Q37. Find the area of a parallelogram whose adjacent sides are represented by the vectors $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - 2\hat{k}$. **(2)**

Q38. Find the Cartesian equation of the plane passing through the points $(1, -1, 2)$, $(2, 0, -1)$, and $(0, 2, 1)$. **(2)**

Q39. Solve the following system of linear equations using Cramer's Rule:

$$3x - 2y = 5$$

$$2x + y = 8$$

(4)

Q40. Prove that $\cos^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \sin^{-1}\left(\frac{56}{65}\right)$. **(4)**

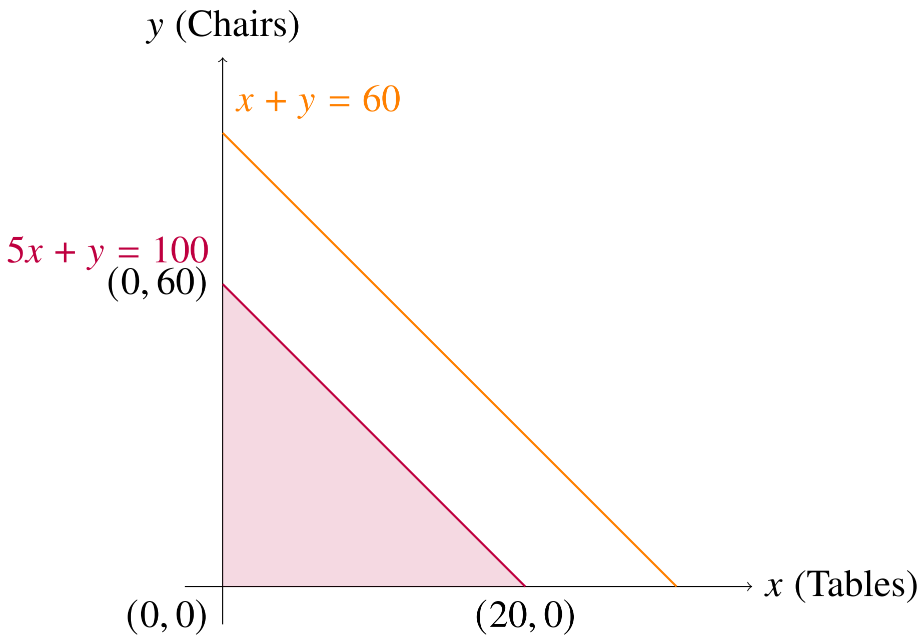
Q41. Find the vector and Cartesian equations of the line passing through the point $(1, 2, -1)$ and perpendicular to the two lines $\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z}{1}$ and $\frac{x+1}{1} = \frac{y-3}{2} = \frac{z-2}{3}$. **(4)**

Q42. Evaluate: $\int \frac{x^2+1}{(x-1)^2(x+3)}\,dx$.

OR

Evaluate the definite integral: $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x}\,dx$. **(4)**

Q43. A furniture dealer deals in only two items: tables and chairs. He has Rs. 10,000 to invest and a storage capacity of 60 pieces. A table costs Rs. 500 and a chair costs Rs. 100. He estimates that from the sale of one table, he can make a profit of Rs. 50 and from the sale of one chair, a profit of Rs. 15. Formulate this as a Linear Programming Problem and solve it graphically to find the maximum profit. **(4)**



Q44. Find the shortest distance between the two skew lines whose vector equations are:

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + \hat{j} - 2\hat{k})$$

$$\vec{r} = (2\hat{i} + 4\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 4\hat{j} - 5\hat{k})$$

OR

Find the equation of the plane passing through the intersection of the planes $x + y + z = 1$ and $2x + 3y - z + 4 = 0$ and perpendicular to the plane $x - y + z = 0$.

(6)

Q45. Find the intervals in which the function $f(x) = \dfrac{x}{\log x}$ is:

(a) Strictly increasing

(b) Strictly decreasing

Also, find the point of local extremum of the function. **(6)**

Detailed Solutions

Q1.

Solution

Concept:
The fundamental trigonometric limit states that $\lim_{u\to 0}\frac{\sin u}{u} = 1$. For a limit involving $\frac{\sin(kx)}{x}$, we multiply and divide by k to rewrite it as $k \cdot \frac{\sin(kx)}{kx}$, which converges to $k \cdot 1 = k$ as $x \to 0$. This technique converts limits of sine ratios into simple algebraic sums of the coefficients.

Solution:

(a) The given limit is: $\displaystyle\lim_{x\to 0}\frac{\sin(ax)+\sin(bx)}{x} = 7.$

(b) Split the numerator into two separate limits using the additive property:

$$\lim_{x\to 0}\frac{\sin(ax)}{x} + \lim_{x\to 0}\frac{\sin(bx)}{x} = 7$$

(c) For each term, multiply and divide by the argument of the sine function:

$$\lim_{x\to 0}\left(a \cdot \frac{\sin(ax)}{ax}\right) + \lim_{x\to 0}\left(b \cdot \frac{\sin(bx)}{bx}\right) = 7$$

(d) As $x \to 0$, both $ax \to 0$ and $bx \to 0$. Using the standard limit $\lim_{u\to 0}\frac{\sin u}{u} = 1$:

$$a \cdot 1 + b \cdot 1 = 7 \implies a + b = 7$$

(e) Given a and b are positive integers with $a > b$, the possible pairs are $(6, 1)$, $(5, 2)$, and $(4, 3)$. In each case $a - b$ evaluates to 5, 3, and 1 respectively. The question asks: among the given options, what is $a - b$? With $a + b = 7$ and both positive integers, the smallest possible difference satisfying $a > b$ is $a = 4$, $b = 3$, giving $a - b = 1$.

Final Answer: 1

Answer: (C)

Go Back to Question 1

Q2.

Solution

Concept:
For $f(x) = \sqrt{\log_a(g(x))}$ with $0 < a < 1$, two conditions must hold: (i) $g(x) > 0$ for the logarithm to be defined, and (ii) $\log_a(g(x)) \geq 0$ for the square root to be real. Since $0 < a < 1$, the logarithmic function is strictly decreasing, so $\log_a(g(x)) \geq 0 \iff g(x) \leq a^0 = 1$.
Solution:

(a) Given $f(x) = \sqrt{\log_{0.5}(x-2)}$ with base $0.5 \in (0, 1)$.

(b) Condition 1 – argument of logarithm must be positive: $x - 2 > 0 \implies x > 2$.

(c) Condition 2 – radicand must be non-negative. Since base < 1, the inequality reverses:

$$\log_{0.5}(x-2) \geq 0 \implies x - 2 \leq (0.5)^0 = 1 \implies x - 2 \leq 1 \implies x \leq 3$$

(d) Intersecting $x > 2$ and $x \leq 3$ gives the domain: $x \in (2, 3]$.

(e) Verification: At $x = 2.5$, $\log_{0.5}(0.5) = 1$, $\sqrt{1} = 1$ (valid). At $x = 3$, $\log_{0.5}(1) = 0$, $\sqrt{0} = 0$ (valid). At $x = 3.1$, $\log_{0.5}(1.1) < 0$, radicand negative (invalid).

Final Answer: $(2, 3]$

Answer: (A)

Go Back to Question 2

Q3.

Solution

Concept:
For an $n \times n$ matrix A, the adjoint satisfies $|\text{adj}(A)| = |A|^{n-1}$. Applying this iteratively, $|\text{adj}(\text{adj}(A))| = |\text{adj}(A)|^{n-1} = (|A|^{n-1})^{n-1} = |A|^{(n-1)^2}$. For a 3×3 matrix, $(3-1)^2 = 4$.
Solution:

(a) Given: A is 3×3 with $|A| = 2$. So $n = 3$.

(b) First, compute $|\text{adj}(A)|$:

$$|\text{adj}(A)| = |A|^{n-1} = 2^{3-1} = 2^2 = 4$$

(c) Now compute $|\text{adj}(\text{adj}(A))|$ by applying the same formula to $\text{adj}(A)$, which is also 3×3:

$$|\text{adj}(\text{adj}(A))| = |\text{adj}(A)|^{3-1} = 4^2 = 16$$

(d) Direct formula verification: $|\text{adj}(\text{adj}(A))| = |A|^{(n-1)^2} = 2^4 = 16$.

(e) Trap check: Common error is using $|\text{adj}(\text{adj}(A))| = |A|^n$ or confusing the exponents. The correct exponent is always $(n-1)^2$.

Final Answer: 16

Answer: (B)

Go Back to Question 3

Q4.

Solution

Concept:

For a definite integral over a symmetric interval, $\int_{-a}^{a} f(x)\,dx = \int_0^a [f(x) + f(-x)]\,dx$. This is powerful when $f(-x)$ relates to $f(x)$ via a reciprocal in the denominator, as with $e^{\sin x}$ because $e^{\sin(-x)} = e^{-\sin x} = 1/e^{\sin x}$.

Solution:

(a) Let $I = \displaystyle\int_{-\pi/2}^{\pi/2} \frac{\cos^5 x}{1 + e^{\sin x}}\,dx$.

(b) Apply the symmetric interval property:

$$I = \int_0^{\pi/2} \left[\frac{\cos^5 x}{1 + e^{\sin x}} + \frac{\cos^5(-x)}{1 + e^{\sin(-x)}} \right] dx$$

(c) Since $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$:

$$I = \int_0^{\pi/2} \left[\frac{\cos^5 x}{1 + e^{\sin x}} + \frac{\cos^5 x}{1 + e^{-\sin x}} \right] dx$$

(d) Simplify the second term: $\frac{\cos^5 x}{1+e^{-\sin x}} = \frac{\cos^5 x \cdot e^{\sin x}}{e^{\sin x}+1}$.

(e) Add the two fractions (they now share the denominator $1 + e^{\sin x}$):

$$\frac{\cos^5 x}{1 + e^{\sin x}} + \frac{\cos^5 x \cdot e^{\sin x}}{1 + e^{\sin x}} = \frac{\cos^5 x(1 + e^{\sin x})}{1 + e^{\sin x}} = \cos^5 x$$

(f) Thus $I = \int_0^{\pi/2} \cos^5 x\,dx$. Using the Wallis reduction formula:

$$\int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} \text{ (if } n \text{ is even), or } \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot 1 \text{ (if } n \text{ is odd)}$$

For $n = 5$ (odd): $\int_0^{\pi/2} \cos^5 x\,dx = \frac{4}{5} \cdot \frac{2}{3} \cdot 1 = \frac{8}{15}$.

Final Answer: $\dfrac{8}{15}$

Answer: (A)

Go Back to Question 4

Q5.

Solution

Concept:

The direction cosines $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$ of any line in 3D space satisfy the fundamental identity $l^2 + m^2 + n^2 = 1$, i.e., $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. This always holds regardless of the line's orientation.

Solution:

(a) Given: $\alpha = 60^\circ$, $\beta = 60^\circ$, and the angle with the z-axis is γ.

(b) Apply the direction cosine identity:

$$\cos^2 60^\circ + \cos^2 60^\circ + \cos^2\gamma = 1$$

(c) Substitute $\cos 60^\circ = \frac{1}{2}$:

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \cos^2\gamma = 1$$

$$\frac{1}{4} + \frac{1}{4} + \cos^2\gamma = 1$$

(d) Solve for $\cos^2\gamma$:

$$\frac{1}{2} + \cos^2\gamma = 1 \implies \cos^2\gamma = \frac{1}{2}$$

(e) This means $\cos\gamma = \pm\frac{1}{\sqrt{2}}$, corresponding to $\gamma = 45^\circ$ or $\gamma = 135^\circ$, both geometrically valid.

Final Answer: $\frac{1}{2}$

Answer: (A)

Go Back to Question 5

Q6.

Solution

Concept:
The intersection point of two lines is found by solving their equations simultaneously. A line parallel to the y-axis has an undefined slope and its equation is of the form x = constant, meaning all points on the line share the same x-coordinate.

Solution:

(a) Solve the system for the intersection point:

$$2x + 3y - 5 = 0 \quad (1)$$
$$x - 2y + 1 = 0 \quad (2)$$

(b) From (2): $x = 2y - 1$. Substitute into (1):

$$2(2y - 1) + 3y - 5 = 0 \implies 4y - 2 + 3y - 5 = 0 \implies 7y - 7 = 0 \implies y = 1$$

(c) Then $x = 2(1) - 1 = 1$. The intersection point is $(1, 1)$.

(d) A line through $(1, 1)$ parallel to the y-axis has equation $x = 1$ (fixed x-coordinate, y varies freely).

(e) Verification: Every point on $x = 1$ has $x = 1$, so it is vertical (parallel to y-axis) and passes through $(1, 1)$.

Final Answer: $x = 1$

Answer: (A)

Go Back to Question 6

Q7.

Solution

Concept:
The distance of a point $P(x, y, z)$ from the yz-plane is simply $|x|$, because the yz-plane has equation $x = 0$, and the perpendicular distance from any point to this plane is measured parallel to the x-axis. Similarly, distance from the xz-plane is $|y|$, and from the xy-plane is $|z|$.
Solution:

(a) Given point: $P(3, -4, 5)$. Coordinates: $x = 3$, $y = -4$, $z = 5$.

(b) The yz-plane is defined by $x = 0$. The perpendicular distance from P to this plane is:

$$\text{Distance} = |x_P - 0| = |3 - 0| = 3$$

(c) Geometrically, the foot of the perpendicular from P to the yz-plane is $(0, -4, 5)$. The distance is $\sqrt{(3-0)^2 + (-4+4)^2 + (5-5)^2} = \sqrt{9} = 3$.

(d) Trap check: Do not confuse the yz-plane with the x-axis. The distance from a point to a plane differs from the distance to an axis.

Final Answer: 3

Answer: (A)

Go Back to Question 7

Q8.

Solution

Concept:
The slope of the tangent line to a curve $y = f(x)$ at $x = x_0$ equals the derivative $f'(x_0)$. For exponential functions, $\frac{d}{dx}(e^{kx}) = ke^{kx}$. The derivative of a linear term cx is simply c.
Solution:

(a) Curve: $y = e^{2x} - 3x$. Differentiate with respect to x:

$$\frac{dy}{dx} = \frac{d}{dx}(e^{2x}) - \frac{d}{dx}(3x) = 2e^{2x} - 3$$

(b) Evaluate at $x = 0$:

$$\left.\frac{dy}{dx}\right|_{x=0} = 2e^{2(0)} - 3 = 2e^0 - 3 = 2(1) - 3 = -1$$

(c) Thus the tangent slope at $x = 0$ is -1.

(d) Verification: At $x = 0$, $y = e^0 - 0 = 1$. The point is $(0, 1)$. The tangent line equation: $y - 1 = -1(x - 0) \implies y = -x + 1$.

Final Answer: -1

Answer: (B)

Go Back to Question 8

Q9.

Solution

Concept:
The general equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$, with centre $(-g, -f)$. When a circle passes through given points, each point must satisfy the equation. This gives a system of linear equations in g, f, and c.

Solution:

(a) Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.

(b) Substitute $(1, 0)$: $1 + 0 + 2g(1) + 2f(0) + c = 0 \implies 2g + c = -1$ (1)

(c) Substitute $(-1, 0)$: $1 + 0 + 2g(-1) + 2f(0) + c = 0 \implies -2g + c = -1$ (2)

(d) Subtract (2) from (1): $(2g + c) - (-2g + c) = -1 - (-1) \implies 4g = 0 \implies g = 0$.

(e) From (1): $2(0) + c = -1 \implies c = -1$.

(f) Substitute $(0, 1)$: $0+1+2g(0)+2f(1)+c = 0 \implies 1+2f-1 = 0 \implies 2f = 0 \implies f = 0$.

(g) The circle equation is $x^2 + y^2 + 0 \cdot x + 0 \cdot y - 1 = 0$, i.e., $x^2 + y^2 = 1$, the unit circle centred at origin with radius 1.

Final Answer: $x^2 + y^2 = 1$

Answer: (A)

Go Back to Question 9

Q10.

Solution

Concept:
The latus rectum of a parabola is the chord through the focus perpendicular to the axis. For a parabola in standard form $(y-k)^2 = 4a(x-h)$ (opens right/left) or $(x-h)^2 = 4a(y-k)$ (opens up/down), the latus rectum length is $|4a|$. Complete the square to convert a general second-degree equation into standard form.

Solution:

(a) Given: $2y^2 + 12x - 8y + 2 = 0$. Divide by 2:

$$y^2 + 6x - 4y + 1 = 0$$

(b) Complete the square for y: $y^2 - 4y = (y-2)^2 - 4$.

(c) Substitute and simplify:

$$(y-2)^2 - 4 + 6x + 1 = 0 \implies (y-2)^2 + 6x - 3 = 0$$

$$(y-2)^2 = -6x + 3 = -6(x - \tfrac{1}{2})$$

(d) This is of the form $(Y)^2 = -4a(X)$ with $Y = y-2$, $X = x - \frac{1}{2}$, and $-4a = -6 \implies 4a = 6$.

(e) The parabola opens to the left. Length of latus rectum $= |4a| = 6$.

(f) Trap check: The latus rectum length is $4a$, not a. Many students mistakenly take $a = \frac{3}{2}$ as the answer.

Final Answer: 6

Answer: (A)

Go Back to Question 10

Q11.

Solution

Concept:
The order of a differential equation is the highest derivative present. The degree is the power of the highest-order derivative, provided the equation is expressed as a polynomial in all derivatives, free from radicals and fractional powers.
Solution:

(a) Given: $\left(\frac{d^3y}{dx^3}\right)^2 - 3\left(\frac{dy}{dx}\right)^5 + 7y = 0.$

(b) Identify the highest-order derivative: $\frac{d^3y}{dx^3}$ is third order. No higher derivative exists, so order = 3.

(c) The equation is already a polynomial in derivatives – each derivative appears with a positive integer exponent (2, 5, and 0 for y). No radicals present.

(d) The power of the highest-order derivative $\frac{d^3y}{dx^3}$ is 2. Therefore, degree = 2.

(e) Answer: Order = 3, Degree = 2.

Final Answer: Order 3 and Degree 2

Answer: (A)

Go Back to Question 11

Q12.

Solution

Concept:
Two non-zero vectors are perpendicular if and only if their dot product equals zero: $\vec{a} \cdot \vec{b} = 0$. For component form vectors, the dot product is the sum of products of corresponding components.
Solution:

(a) $\vec{a} = 2\hat{i} - \hat{j} + \lambda\hat{k}, \vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.

(b) Set dot product to zero:

$$\vec{a} \cdot \vec{b} = (2)(1) + (-1)(2) + (\lambda)(3) = 0$$

(c) Simplify: $2 - 2 + 3\lambda = 0 \implies 3\lambda = 0 \implies \lambda = 0$.

(d) Verification: When $\lambda = 0$, $\vec{a} = 2\hat{i} - \hat{j}$, and $\vec{a} \cdot \vec{b} = 2(1) + (-1)(2) + 0(3) = 2 - 2 = 0$. Confirmed perpendicular.

Final Answer: $\lambda = 0$

Answer: (A)

Go Back to Question 12

Q13.

Solution

Concept:
A relation R on a set is reflexive if $|a - a| \leq k$ for all a (always true since $0 \leq k$). It is symmetric because $|a - b| = |b - a|$. It is not necessarily transitive because $|a - b| \leq 1$ and $|b - c| \leq 1$ does not guarantee $|a - c| \leq 1$ (by the triangle inequality, $|a - c| \leq |a - b| + |b - c| \leq 2$).

Solution:

(a) Test reflexivity: For any $a \in \mathbb{R}$, $|a - a| = 0 \leq 1$, so $(a, a) \in R$. Reflexive: YES.

(b) Test symmetry: If $(a, b) \in R$, then $|a - b| \leq 1$. Since $|b - a| = |a - b| \leq 1$, $(b, a) \in R$. Symmetric: YES.

(c) Test transitivity: Counterexample – let $a = 0, b = 1, c = 2$. $|0 - 1| = 1 \leq 1 \implies (0, 1) \in R$. $|1 - 2| = 1 \leq 1 \implies (1, 2) \in R$. But $|0 - 2| = 2 > 1$, so $(0, 2) \notin R$. Transitive: NO.

(d) Conclusion: R is reflexive and symmetric but not transitive.

Final Answer: Reflexive and symmetric but not transitive

Answer: (A)

Go Back to Question 13

Q14.

Solution

Concept:

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the inverse is $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, provided $ad - bc \neq 0$. This is obtained by swapping the diagonal entries, negating the off-diagonal entries, and dividing by the determinant.

Solution:

(a) $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$. Here $a = 2, b = 3, c = 1, d = 2$.

(b) Compute determinant: $\det(A) = ad - bc = (2)(2) - (3)(1) = 4 - 3 = 1$.

(c) Since $\det(A) = 1 \neq 0$, inverse exists. The adjugate matrix is $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$.

(d) $A^{-1} = \frac{1}{1}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$.

(e) Verification: $AA^{-1} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Final Answer: $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

Answer: (A)

Go Back to Question 14

Q15.

Solution

Concept:
The conditional $p \to q$ is logically equivalent to $\neg p \vee q$. Using this transformation combined with De Morgan's laws and associativity of disjunction, complex logical expressions can be reduced to simpler equivalent forms.
Solution:

(a) Start with $p \to (q \to r)$.

(b) Apply $A \to B \equiv \neg A \vee B$ to the inner conditional:

$$p \to (\neg q \vee r)$$

(c) Apply the same equivalence to the outer conditional:

$$\neg p \vee (\neg q \vee r)$$

(d) Disjunction is associative, so drop parentheses: $\neg p \vee \neg q \vee r$.

(e) By De Morgan's law: $\neg p \vee \neg q \equiv \neg(p \wedge q)$.

(f) So: $\neg(p \wedge q) \vee r$.

(g) Apply the conditional equivalence in reverse: $\neg A \vee B \equiv A \to B$, with $A = p \wedge q$ and $B = r$:

$$(p \wedge q) \to r$$

(h) Thus $p \to (q \to r) \equiv (p \wedge q) \to r$, which is the Exportation Law.

Final Answer: $(p \wedge q) \to r$

Answer: (A)

Go Back to Question 15

Q16.

Solution

Concept:
In Linear Programming, the feasible region is the intersection of all constraint half-planes. With constraints $x + y \leq 4$, $x \geq 0$, $y \geq 0$, the region is bounded by three lines: $x = 0$, $y = 0$, and $x + y = 4$. This forms a triangle – a bounded, convex polygon. Convexity means the line segment joining any two points in the region lies entirely within it.

Solution:

(a) The constraints define: first quadrant ($x \geq 0$, $y \geq 0$) and below the line $x + y = 4$.

(b) Vertices of the feasible region: $(0, 0)$, $(4, 0)$, $(0, 4)$ – a right triangle.

(c) The region is bounded (all points lie within a finite distance from the origin) and convex (intersection of half-planes is always convex).

(d) A triangle is a special case of a bounded convex polygon. Among the options, "Bounded and convex" best describes the region.

Final Answer: Bounded and convex

Answer: (D)

Go Back to Question 16

Q17.

Solution

Concept:
For a conditional $P \to Q$ ("If P, then Q"), the contrapositive is $\neg Q \to \neg P$ ("If not Q, then not P"). The contrapositive is logically equivalent to the original statement. The converse ($Q \to P$) and inverse ($\neg P \to \neg Q$) are not equivalent.
Solution:

(a) Original: "If a triangle is equilateral, then it is isosceles."

- P: A triangle is equilateral.
- Q: It (the triangle) is isosceles.

(b) Form negations:

- $\neg Q$: A triangle is not isosceles.
- $\neg P$: A triangle is not equilateral.

(c) Contrapositive: "If $\neg Q$, then $\neg P$" $\implies$ "If a triangle is not isosceles, then it is not equilateral."

(d) This is logically sound: every equilateral triangle has (at least) two equal sides, making it isosceles. So if a triangle lacks even two equal sides, it cannot be equilateral.

Final Answer: If a triangle is not isosceles, then it is not equilateral.

Answer: (A)

Go Back to Question 17

Q18.

Solution

Concept:
The function $\sin^{-1}(\sin\theta)$ returns the principal value in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. To find it, adjust θ using periodicity and symmetry to land in the principal range while preserving the sine value.
Solution:

(a) Evaluate $\sin^{-1}(\sin\frac{5\pi}{3})$.

(b) $\frac{5\pi}{3} = 2\pi - \frac{\pi}{3}$, which lies in the fourth quadrant.

(c) In QIV, $\sin(2\pi - \alpha) = -\sin\alpha$: $\sin\frac{5\pi}{3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$.

(d) Find $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ with $\sin\phi = -\frac{\sqrt{3}}{2}$.

(e) $\sin(-\frac{\pi}{3}) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$, and $-\frac{\pi}{3} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

(f) Therefore $\sin^{-1}(\sin\frac{5\pi}{3}) = -\frac{\pi}{3}$.

(g) Trap check: Do not output $\frac{5\pi}{3}$ directly. The principal branch of $\sin^{-1}$ restricts output to $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Final Answer: $-\dfrac{\pi}{3}$

Answer: (B)

Go Back to Question 18

Q19.

Solution

Concept:

The eccentricity e of an ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ is $e = \sqrt{1 - \frac{b^2}{a^2}}$ when $a > b$ (horizontal major axis). When $b > a$ (vertical major axis), $e = \sqrt{1 - \frac{a^2}{b^2}}$. Complete the square to put the equation in standard form.

Solution:

(a) Given: $9x^2 + 5y^2 - 30y = 0$.

(b) Complete the square for y:

$$9x^2 + 5(y^2 - 6y) = 0 \implies 9x^2 + 5[(y-3)^2 - 9] = 0$$

$$9x^2 + 5(y-3)^2 - 45 = 0 \implies 9x^2 + 5(y-3)^2 = 45$$

(c) Divide by 45: $\frac{x^2}{5} + \frac{(y-3)^2}{9} = 1$.

(d) Centre is $(0, 3)$. Here $b^2 = 5$ (under x^2), $a^2 = 9$ (under $(y-3)^2$). Since $a^2 > b^2$, the major axis is vertical.

(e) For a vertically oriented ellipse: $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{5}{9}} = \sqrt{\frac{4}{9}} = \frac{2}{3}$.

(f) Trap check: The larger denominator is a^2 regardless of position. Do not assume a^2 always goes under x^2.

Final Answer: $\frac{2}{3}$

Answer: (A)

Go Back to Question 19

Q20.

Solution

Concept:

For $y = [f(x)]^{g(x)}$, take log of both sides: $\log y = g(x) \cdot \log[f(x)]$. Then differentiate implicitly using the chain rule and product rule. This technique, logarithmic differentiation, is essential when both base and exponent are functions of x.

Solution:

(a) $y = (\sin x)^{\cos x}$. Take log on both sides:

$$\log y = \cos x \cdot \log(\sin x)$$

(b) Differentiate implicitly with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = (-\sin x) \cdot \log(\sin x) + \cos x \cdot \frac{1}{\sin x} \cdot \cos x$$

$$\frac{1}{y}\frac{dy}{dx} = -\sin x \cdot \log(\sin x) + \frac{\cos^2 x}{\sin x}$$

(c) Multiply by y:

$$\frac{dy}{dx} = (\sin x)^{\cos x}\left[-\sin x \cdot \log(\sin x) + \frac{\cos^2 x}{\sin x}\right]$$

(d) Evaluate at $x = \frac{\pi}{2}$: $\sin\frac{\pi}{2} = 1$, $\cos\frac{\pi}{2} = 0$, $\log(1) = 0$.

$$\left.\frac{dy}{dx}\right|_{x=\pi/2} = (1)^0\left[-\sin\frac{\pi}{2} \cdot 0 + \frac{0^2}{1}\right] = 1 \cdot [0 + 0] = 0$$

(e) At $x = \frac{\pi}{2}$, $y = 1^0 = 1$, and the function is locally flat, so the derivative is zero.

Final Answer: 0

Answer: (A)

Go Back to Question 20

Q21.

Solution

Concept:

Each Column-I item requires independent evaluation. For item (i), the determinant of a rotation matrix and its powers involves $\cos^2\theta + \sin^2\theta = 1$. For item (ii), the scalar projection formula is $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

Solution:

(a) Item (i): $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. $\det(A) = \cos^2\theta + \sin^2\theta = 1$.

Since A^2 represents rotation by 2θ, $\det(A^2) = [\det(A)]^2 = 1^2 = 1$. So (i) → (A).

(b) Item (ii): $\vec{a} = 3\hat{i} + 4\hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$.

$\vec{a}\cdot\vec{b} = 3(2) + 0(1) + 4(2) = 6 + 0 + 8 = 14.$

$|\vec{b}| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3.$

Projection $= \frac{14}{3}$. So (ii) → (B).

(c) Thus: (i) → (A), (ii) → (B), which is option (A).

Final Answer: (i) → (A), (ii) → (B)

Go Back to Question 21

Q22.

Solution

Concept:

For the degree of a DE, first eliminate radicals; the degree is the power of the highest-order derivative. For distance between parallel planes, normalize coefficients and use $d = \frac{|D_1 - D_2|}{\sqrt{A^2+B^2+C^2}}$.

Solution:

(a) For (i): $\sqrt{1 + (\frac{dy}{dx})^2} = \frac{d^2y}{dx^2}$. Square both sides:

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{d^2y}{dx^2}\right)^2$$

Now it's a polynomial in derivatives. Highest order = 2 (from $\frac{d^2y}{dx^2}$). Its power = 2. So degree = 2.

(b) For (ii): Planes: $x - 2y + 2z + 3 = 0$ and $2x - 4y + 4z - 6 = 0$.

Divide the second by 2: $x - 2y + 2z - 3 = 0$.

Now coefficients match: $A = 1, B = -2, C = 2, D_1 = 3, D_2 = -3$.

$$d = \frac{|3-(-3)|}{\sqrt{1^2 + (-2)^2 + 2^2}} = \frac{6}{\sqrt{1+4+4}} = \frac{6}{3} = 2$$

Final Answer: (i) 2, (ii) 2

Go Back to Question 22

Q23.

Solution

Concept:

Symmetry and transitivity together do NOT imply reflexivity unless every element appears in at least one ordered pair. For differentiability of $f(x) = x|x|$, check left-hand and right-hand derivatives at $x = 0$ and confirm the function is smooth elsewhere.

Solution:

(a) Statement (i): Counterexample – On $A = \{1, 2\}$, define $R = \{(1, 1)\}$. R is symmetric (trivially) and transitive (trivially), but not reflexive since $(2, 2) \notin R$. Hence, statement (i) is FALSE.

(b) Statement (ii): $f(x) = x|x|$ can be written piecewise: $f(x) = x^2$ for $x \geq 0$, $f(x) = -x^2$ for $x < 0$.

At $x = 0$: LHD $= \lim_{h\to 0^-} \frac{-(0+h)^2-0}{h} = \lim_{h\to 0^-} \frac{-h^2}{h} = \lim_{h\to 0^-}(-h) = 0$.

RHD $= \lim_{h\to 0^+} \frac{(0+h)^2-0}{h} = \lim_{h\to 0^+} \frac{h^2}{h} = \lim_{h\to 0^+} h = 0$.

LHD = RHD, so f is differentiable at 0. For $x \neq 0$, f is a polynomial, hence differentiable. Statement (ii) is TRUE.

(c) Answer: (i) FALSE, (ii) TRUE.

Final Answer: (i) FALSE, (ii) TRUE

Go Back to Question 23

Q24.

Solution

Concept:
The negation of "Every P is Q" is "There exists some P that is not Q." The negation of "There exists x such that $P(x)$" is "For all x, not $P(x)$." These follow from the logical equivalences $\neg(\forall x, P(x)) \equiv \exists x, \neg P(x)$ and $\neg(\exists x, P(x)) \equiv \forall x, \neg P(x)$.

Solution:

(a) Statement (i): "Every rational number has a terminating decimal representation."

This is a universal claim. Its negation: "There exists a rational number that does NOT have a terminating decimal representation" (e.g., $\frac{1}{3} = 0.333\ldots$).

(b) Statement (ii): "There exists a real number x such that $x^2 + 1 = 0$."

This is an existential claim. Its negation: "For every real number x, $x^2 + 1 \neq 0$" (which is true since $x^2 + 1 \geq 1$ for all real x).

(c) Answer: (i) There exists a rational number without a terminating decimal representation. (ii) For every real number x, $x^2 + 1 \neq 0$.

Final Answer: (i) Some rational numbers have non-terminating decimals. (ii) For all $x \in \mathbb{R}$, $x^2 + 1 \neq 0$.

Go Back to Question 24

Q25.

Solution

Concept:

Standard limit formulas are essential: $\lim_{u\to0}\frac{\sin u}{u}=1$, $\lim_{u\to0}\frac{\tan u}{u}=1$, $\lim_{x\to0}\frac{e^{kx}-1}{x}=k$, $\lim_{x\to\infty}(1+\frac{a}{x})^x=e^a$, and rationalising techniques for $\frac{0}{0}$ forms.

Solution:

(a) (i) $\lim_{x\to0}\frac{\tan 5x}{\sin 2x}=\lim_{x\to0}\frac{\tan 5x}{5x}\cdot\frac{2x}{\sin 2x}\cdot\frac{5x}{2x}=1\cdot1\cdot\frac{5}{2}=\frac{5}{2}$.

(b) (ii) $\lim_{x\to0}\frac{e^{3x}-1}{x}$. Let $u=3x$; as $x\to0$, $u\to0$: $\lim_{u\to0}\frac{e^u-1}{u/3}=3\lim_{u\to0}\frac{e^u-1}{u}=3\cdot1=3$.

(c) (iii) $\lim_{x\to\infty}(1+\frac{2}{x})^x=e^2$, using $\lim_{x\to\infty}(1+\frac{a}{x})^x=e^a$.

(d) (iv) $\lim_{x\to4}\frac{\sqrt{x}-2}{x-4}$ is $0/0$ form. Rationalize: $\lim_{x\to4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)}=\lim_{x\to4}\frac{x-4}{(x-4)(\sqrt{x}+2)}=\lim_{x\to4}\frac{1}{\sqrt{x}+2}=\frac{1}{2+2}=\frac{1}{4}$.

Final Answer: (i) $\frac{5}{2}$, (ii) 3, (iii) e^2, (iv) $\frac{1}{4}$

Go Back to Question 25

Q26.

Solution

Concept:

Standard forms of conic sections reveal their geometric properties. For a circle $x^2 + y^2 + 2gx + 2fy + c = 0$, centre $= (-g, -f)$. For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $e = \sqrt{1 + \frac{b^2}{a^2}}$. For parabola $x^2 = -4ay$, focus $= (0, -a)$. For ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a > b$, minor axis length $= 2b$.

Solution:

(a) (i) $x^2 + y^2 - 6x + 4y - 3 = 0$. $2g = -6 \implies g = -3$, $2f = 4 \implies f = 2$. Centre $= (-g, -f) = (3, -2)$.

(b) (ii) $\frac{x^2}{9} - \frac{y^2}{16} = 1$. $a^2 = 9$, $b^2 = 16$. $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{16}{9}} = \sqrt{\frac{25}{9}} = \frac{5}{3}$.

(c) (iii) $x^2 = -20y$. Compare with $x^2 = -4ay$: $-4a = -20 \implies a = 5$. Focus $= (0, -5)$.

(d) (iv) $\frac{x^2}{49} + \frac{y^2}{36} = 1$. $a^2 = 49 \implies a = 7$, $b^2 = 36 \implies b = 6$. Since $a > b$, major axis is along x-axis, minor axis length $= 2b = 12$.

Final Answer: (i) $(3, -2)$, (ii) $\frac{5}{3}$, (iii) $(0, -5)$, (iv) 12

Go Back to Question 26

Q27.

Solution

Concept:

Matrix inversion reversal: $(AB)^{-1} = B^{-1}A^{-1}$ always holds. Cross product magnitude: $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$, so unit vectors produce unit cross product only when $\theta = 90°$. A diagonal matrix may have different entries; a scalar matrix is a special diagonal matrix with equal diagonal entries. The conditions $\vec{a} \cdot \vec{b} = 0$ (orthogonal) and $\vec{a} \times \vec{b} = \vec{0}$ (parallel) together force at least one vector to be zero.

Solution:

(a) (i) $(AB)^{-1} = B^{-1}A^{-1}$ is a fundamental theorem for invertible matrices. TRUE.

(b) (ii) Counterexample: $\hat{i} \times \hat{i} = \vec{0}$, not a unit vector. Even for distinct unit vectors, e.g., $\hat{i} \times \frac{\hat{i}+\hat{j}}{\sqrt{2}} = \frac{\hat{k}}{\sqrt{2}}$, magnitude is $\frac{1}{\sqrt{2}} \neq 1$. FALSE.

(c) (iii) $\text{diag}(1, 2, 3)$ is diagonal but not scalar (entries are unequal). FALSE.

(d) (iv) $\vec{a} \cdot \vec{b} = 0 \implies$ perpendicular or at least one is zero. $\vec{a} \times \vec{b} = \vec{0} \implies$ parallel or at least one is zero. For both to hold, at least one vector must be zero. TRUE.

Final Answer: (i) TRUE, (ii) FALSE, (iii) FALSE, (iv) TRUE

Go Back to Question 27

Q28.

Solution

Concept:

The area bounded between two curves $y = f(x)$ (upper) and $y = g(x)$ (lower) from $x = a$ to $x = b$ is $\int_a^b [f(x) - g(x)]\, dx$. Intersection points are found by equating the two functions. The integrand is the vertical distance between the upper and lower curves.

Solution:

(a) (i) Set $4 - x^2 = 3 \implies x^2 = 1 \implies x = \pm 1$. Intersection x-coordinates: -1 and 1.

(b) (ii) On $[-1, 1]$, the parabola $y = 4 - x^2$ is above the line $y = 3$. The integrand is the difference: Integrand $= (4 - x^2) - 3 = 1 - x^2$.

(c) (iii) The area integral: Area $= \displaystyle\int_{-1}^{1} (1 - x^2)\, dx$.

(d) (iv) Using symmetry (even function):

$$\text{Area} = 2\int_0^1 (1 - x^2)\, dx = 2\left[x - \frac{x^3}{3}\right]_0^1$$
$$= 2\left(1 - \frac{1}{3}\right) = 2 \cdot \frac{2}{3} = \frac{4}{3} \text{ square units.}$$

Final Answer: (i) $x = \pm 1$, (ii) $1 - x^2$, (iii) $\int_{-1}^{1} (1 - x^2)\, dx$, (iv) $\frac{4}{3}$

Go Back to Question 28

Q29.

Solution

Concept: In Linear Programming Problems (LPP), constraints arise from production resources such as available machine hours. The objective function Z aggregates individual item profits to determine the absolute maximum revenue attainable across the corner points of the bounded feasible region.

Solution:

(a) Identify resource constraints: Machine A requires 2 hours per unit of Type X and 1 hour per unit of Type Y, with a ceiling of 12 hours, yielding $2x + y \leq 12$.

(b) Express the total profit function: Combining profit margins of Rs. 25 for Type X and Rs. 20 for Type Y results in the objective function $Z = 25x + 20y$.

(c) Determine the positive x-axis boundary corner point by setting $y = 0$ inside the main binding resource inequality, which evaluates to $(6, 0)$.

(d) Find the internal intersection point by solving the simultaneous system $2x + y = 12$ and $x + 3y = 15$. Substitution reveals the coordinates as $\left(\frac{21}{5}, \frac{18}{5}\right)$.

(e) Compute the revenue at vertex $(6, 0)$, providing a profit value of $Z = 25(6) + 20(0) = 150$.

(f) Evaluate Z at the intersection point: $Z = 25\left(\frac{21}{5}\right) + 20\left(\frac{18}{5}\right) = 105 + 72 = 177$, identifying it as the maximum profit over the region.

Final Answer: The maximum profit achieved is Rs. 177.

Go Back to Question 29

Q30.

Solution

Concept:
A function $f : A \to B$ is bijective if it is both injective (one-to-one: $f(x_1) = f(x_2) \implies x_1 = x_2$) and surjective (onto: for each $y \in B$, there exists $x \in A$ with $f(x) = y$). Linear functions with non-zero slope are always bijective on $\mathbb{R}$.
Solution:

(a) Given $f(x) = 5x - 3$, with domain and codomain both $\mathbb{R}$.

(b) Test injectivity: Assume $f(x_1) = f(x_2)$.

$$5x_1 - 3 = 5x_2 - 3 \implies 5x_1 = 5x_2 \implies x_1 = x_2$$

Thus f is injective (one-to-one).

(c) Test surjectivity: Let $y \in \mathbb{R}$ be any real number. We need $x \in \mathbb{R}$ such that $f(x) = y$.

$$5x - 3 = y \implies 5x = y + 3 \implies x = \frac{y+3}{5}$$

Since $y \in \mathbb{R}$, $x = \frac{y+3}{5} \in \mathbb{R}$. Moreover, $f(\frac{y+3}{5}) = 5(\frac{y+3}{5}) - 3 = y + 3 - 3 = y$. Thus f is surjective.

(d) Since f is both injective and surjective, f is bijective.

Final Answer: The function is bijective (both one-to-one and onto).
Go Back to Question 30

Q31.

Solution

Concept:
Two lines in 3D with direction ratios $\langle a_1, b_1, c_1\rangle$ and $\langle a_2, b_2, c_2\rangle$ are perpendicular if and only if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ (dot product of direction vectors equals zero).

Solution:

(a) Line 1: $\frac{x-2}{3} = \frac{y+1}{p} = \frac{z-4}{-2}$. Direction ratios: $\langle 3, p, -2\rangle$.

(b) Line 2: $\frac{x+3}{4} = \frac{y-2}{1} = \frac{z+5}{3}$. Direction ratios: $\langle 4, 1, 3\rangle$.

(c) Perpendicular condition: dot product = 0.

$$(3)(4) + (p)(1) + (-2)(3) = 0$$

$$12 + p - 6 = 0 \implies p + 6 = 0 \implies p = -6$$

(d) Verification: $\langle 3, -6, -2\rangle \cdot \langle 4, 1, 3\rangle = 12 - 6 - 6 = 0$. Confirmed.

Final Answer: $p = -6$

Go Back to Question 31

Q32.

Solution

Concept:
The chain rule for a composite function: $\frac{d}{dx}[f(g(h(x)))] = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$. Here, $f(u) = e^u$, $g(v) = \cos v$, $h(x) = x^3$. Derivatives: $\frac{d}{du}(e^u) = e^u$, $\frac{d}{dv}(\cos v) = -\sin v$, $\frac{d}{dx}(x^3) = 3x^2$.

Solution:

(a) Let $y = e^{\cos(x^3)}$. Apply the chain rule from outermost to innermost function.

(b) Differentiate the outer exponential:

$$\frac{dy}{dx} = e^{\cos(x^3)} \cdot \frac{d}{dx}[\cos(x^3)]$$

(c) Differentiate the middle cosine function:

$$\frac{d}{dx}[\cos(x^3)] = -\sin(x^3) \cdot \frac{d}{dx}(x^3)$$

(d) Differentiate the inner power function:

$$\frac{d}{dx}(x^3) = 3x^2$$

(e) Combine all parts:

$$\frac{dy}{dx} = e^{\cos(x^3)} \cdot (-\sin(x^3)) \cdot (3x^2) = -3x^2 e^{\cos(x^3)} \sin(x^3)$$

(f) Trap check: Do not forget the negative sign from the derivative of cosine.

Final Answer: $-3x^2 e^{\cos(x^3)} \sin(x^3)$

Go Back to Question 32

Q33.

Solution

Concept:

To form the differential equation of a family of curves, differentiate the equation with respect to x as many times as there are arbitrary constants, then eliminate the constants. For one arbitrary constant, one differentiation suffices.

Solution:

(a) Given family: $y^2 = 4ax$, where a is the arbitrary constant.

(b) Differentiate both sides with respect to x:

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(4ax) \implies 2y\frac{dy}{dx} = 4a$$

(c) Express a from the differentiated equation: $a = \frac{y}{2}\frac{dy}{dx}$.

(d) Substitute this a back into the original equation $y^2 = 4ax$:

$$y^2 = 4 \cdot \left(\frac{y}{2}\frac{dy}{dx}\right) \cdot x = 2xy\frac{dy}{dx}$$

(e) Divide both sides by y (assuming $y \neq 0$):

$$y = 2x\frac{dy}{dx}$$

(f) Rearrange into standard form: $2x\frac{dy}{dx} - y = 0$.

(g) Alternatively, $\frac{dy}{dx} = \frac{y}{2x}$, which is the differential equation of the family of parabolas with vertex at the origin and axis along the x-axis.

Final Answer: $y = 2x\frac{dy}{dx}$ or $2x\frac{dy}{dx} - y = 0$

Go Back to Question 33

Q34.

Solution

Concept:

The scalar projection of $\vec{a}$ on $\vec{b}$ is $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$. For a unit vector perpendicular to two given vectors, compute $\vec{a}\times\vec{b}$ and normalize it: $\hat{n} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.

Solution:

(a) First choice – Projection: $\vec{a} = -\hat{i} + 3\hat{j} + 5\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}$.

$\vec{a}\cdot\vec{b} = (-1)(2) + (3)(-1) + (5)(2) = -2 - 3 + 10 = 5.$

$|\vec{b}| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3.$

Projection $= \frac{5}{3}$.

(b) OR choice – Unit vector perpendicular to both $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + 2\hat{k}$.

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \hat{i}(2 - (-1)) - \hat{j}(4 - 1) + \hat{k}(-2 - 1) = 3\hat{i} - 3\hat{j} - 3\hat{k}.$$

$|\vec{a}\times\vec{b}| = \sqrt{3^2 + (-3)^2 + (-3)^2} = \sqrt{9 + 9 + 9} = \sqrt{27} = 3\sqrt{3}.$

Unit vector $= \frac{3\hat{i} - 3\hat{j} - 3\hat{k}}{3\sqrt{3}} = \frac{\hat{i} - \hat{j} - \hat{k}}{\sqrt{3}}$.

Final Answer: Projection $= \dfrac{5}{3}$ (or $\dfrac{\hat{i} - \hat{j} - \hat{k}}{\sqrt{3}}$ for OR)

Go Back to Question 34

Q35.

Solution

Concept:
Integration by substitution simplifies an integral when the integrand contains a function and its derivative. Here, $\frac{d}{dx}(\log x) = \frac{1}{x}$, which appears as a factor alongside $\sin(\log x)$. Substitute $u = \log x$, so $du = \frac{1}{x}dx$.

Solution:

(a) $I = \int \frac{\sin(\log x)}{x}\, dx.$

(b) Substitute $u = \log x$. Then $du = \frac{1}{x}dx \implies \frac{dx}{x} = du$.

(c) The integral becomes:

$$I = \int \sin u \, du = -\cos u + C$$

(d) Substitute back $u = \log x$:

$$I = -\cos(\log x) + C$$

(e) Verification: Differentiate $-\cos(\log x) + C$ using the chain rule: $\frac{d}{dx}[-\cos(\log x)] = \sin(\log x) \cdot \frac{1}{x} = \frac{\sin(\log x)}{x}$. Confirmed.

Final Answer: $-\cos(\log x) + C$

Go Back to Question 35

Q36.

Solution

Concept:
The slope of the tangent at (x_0, y_0) is $m_t = f'(x_0)$. The normal is perpendicular to the tangent, so its slope is $m_n = -\frac{1}{m_t}$ (provided $m_t \neq 0$). The equation of the normal line is $y - y_0 = m_n(x - x_0)$.
Solution:

(a) Curve: $y = x^2 + 2e^x$. Point: $(0, 2)$. Verify: $0^2 + 2e^0 = 0 + 2 = 2$. Correct.

(b) Differentiate: $\frac{dy}{dx} = 2x + 2e^x$.

(c) Slope of tangent at $x = 0$: $m_t = 2(0) + 2e^0 = 0 + 2 = 2$.

(d) Slope of normal: $m_n = -\frac{1}{m_t} = -\frac{1}{2}$.

(e) Equation of normal through $(0, 2)$:

$$y - 2 = -\frac{1}{2}(x - 0) \implies y - 2 = -\frac{x}{2}$$

$$2y - 4 = -x \implies x + 2y - 4 = 0$$

(f) Verification: The normal passes through $(0, 2)$ and has slope $-\frac{1}{2}$. At $x = 2$, $y = \frac{4-x}{2} = 1$, and the slope is indeed $-\frac{1}{2}$.

Final Answer: $x + 2y = 4$ or $x + 2y - 4 = 0$
Go Back to Question 36

Q37.

Solution

Concept:

The area of a parallelogram with adjacent sides represented by vectors $\vec{a}$ and $\vec{b}$ equals the magnitude of their cross product: Area = $|\vec{a} \times \vec{b}|$. The cross product is computed via the determinant of a 3×3 matrix with $\hat{i}, \hat{j}, \hat{k}$ in the first row.

Solution:

(a) $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}, \vec{b} = \hat{i} + \hat{j} - 2\hat{k}$.

(b) Compute $\vec{a} \times \vec{b}$:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 1 \\ 1 & 1 & -2 \end{vmatrix}$$

(c) Expand along the first row:

$$\begin{aligned} \vec{a} \times \vec{b} &= \hat{i}[(-3)(-2) - (1)(1)] - \hat{j}[(2)(-2) - (1)(1)] + \hat{k}[(2)(1) - (-3)(1)] \\ &= \hat{i}(6 - 1) - \hat{j}(-4 - 1) + \hat{k}(2 + 3) \\ &= 5\hat{i} + 5\hat{j} + 5\hat{k} \end{aligned}$$

(d) Magnitude: $|\vec{a} \times \vec{b}| = \sqrt{5^2 + 5^2 + 5^2} = \sqrt{75} = 5\sqrt{3}$.

(e) Area of parallelogram = $5\sqrt{3}$ square units.

Final Answer: $5\sqrt{3}$ square units

Go Back to Question 37

Q38.

Solution

Concept:

The equation of a plane passing through three non-collinear points A, B, C can be found using the determinant form: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$, taking A as the reference point (x_1, y_1, z_1).

Solution:

(a) Points: $A(1, -1, 2)$, $B(2, 0, -1)$, $C(0, 2, 1)$.

(b) Vectors in the plane: $\vec{AB} = B - A = \langle 1, 1, -3 \rangle$, $\vec{AC} = C - A = \langle -1, 3, -1 \rangle$.

(c) Normal vector $\vec{n} = \vec{AB} \times \vec{AC}$:

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix}$$

(d) Expand:

$$\begin{aligned} \vec{n} &= \hat{i}[1(-1) - (-3)(3)] - \hat{j}[1(-1) - (-3)(-1)] + \hat{k}[1(3) - 1(-1)] \\ &= \hat{i}(-1 + 9) - \hat{j}(-1 - 3) + \hat{k}(3 + 1) \\ &= 8\hat{i} + 4\hat{j} + 4\hat{k} \end{aligned}$$

(e) Simplify normal: $\vec{n} = 4(2\hat{i} + \hat{j} + \hat{k})$. Use $\vec{n} = \langle 2, 1, 1 \rangle$.

(f) Plane equation using point $A(1, -1, 2)$: $2(x - 1) + 1(y + 1) + 1(z - 2) = 0$.

$$2x - 2 + y + 1 + z - 2 = 0 \implies 2x + y + z - 3 = 0$$

(g) Verify: At $B(2, 0, -1)$: $4 + 0 - 1 - 3 = 0$. At $C(0, 2, 1)$: $0 + 2 + 1 - 3 = 0$. Confirmed.

Final Answer: $2x + y + z = 3$

Go Back to Question 38

Q39.

Solution

Concept:

Cramer's Rule solves a system of n linear equations in n variables using determinants. For $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$, we compute $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $\Delta_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$, $\Delta_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$. Then $x = \frac{\Delta_x}{\Delta}$, $y = \frac{\Delta_y}{\Delta}$, provided $\Delta \neq 0$.

Solution:

(a) System: $3x - 2y = 5$, $2x + y = 8$.

(b) Compute the main determinant:

$$\Delta = \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = (3)(1) - (-2)(2) = 3 + 4 = 7$$

(c) Compute Δ_x (replace x-column with constants):

$$\Delta_x = \begin{vmatrix} 5 & -2 \\ 8 & 1 \end{vmatrix} = (5)(1) - (-2)(8) = 5 + 16 = 21$$

(d) Compute Δ_y (replace y-column with constants):

$$\Delta_y = \begin{vmatrix} 3 & 5 \\ 2 & 8 \end{vmatrix} = (3)(8) - (5)(2) = 24 - 10 = 14$$

(e) Apply Cramer's Rule:

$$x = \frac{\Delta_x}{\Delta} = \frac{21}{7} = 3$$

$$y = \frac{\Delta_y}{\Delta} = \frac{14}{7} = 2$$

(f) Verification: $3(3) - 2(2) = 9 - 4 = 5$, $2(3) + 2 = 6 + 2 = 8$. Confirmed.

Final Answer: $x = 3$, $y = 2$

Go Back to Question 39

Q40.

Solution

Concept:
To prove inverse trigonometric identities, convert each $\cos^{-1}$ term into $\sin^{-1}$ using $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2}$ (for $x \in [0, 1]$). Then apply the addition formula: $\sin^{-1} u + \sin^{-1} v = \sin^{-1}\left(u\sqrt{1-v^2} + v\sqrt{1-u^2}\right)$, valid when $u, v > 0$ and $u^2 + v^2 \leq 1$.

Solution:

(a) Let LHS $= \cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right)$.

(b) Convert each term to $\sin^{-1}$:

- $\cos^{-1}\left(\frac{4}{5}\right) = \sin^{-1}\left(\sqrt{1-\frac{16}{25}}\right) = \sin^{-1}\left(\sqrt{\frac{9}{25}}\right) = \sin^{-1}\left(\frac{3}{5}\right)$.
- $\cos^{-1}\left(\frac{12}{13}\right) = \sin^{-1}\left(\sqrt{1-\frac{144}{169}}\right) = \sin^{-1}\left(\sqrt{\frac{25}{169}}\right) = \sin^{-1}\left(\frac{5}{13}\right)$.

(c) So LHS $= \sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right)$.

(d) Apply the addition formula with $u = \frac{3}{5}$, $v = \frac{5}{13}$:

- $\sqrt{1-u^2} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$.
- $\sqrt{1-v^2} = \sqrt{1-\frac{25}{169}} = \frac{12}{13}$.

(e) Compute $u\sqrt{1-v^2} + v\sqrt{1-u^2}$:

$$\frac{3}{5}\cdot\frac{12}{13} + \frac{5}{13}\cdot\frac{4}{5} = \frac{36}{65} + \frac{20}{65} = \frac{56}{65}$$

(f) Therefore, LHS $= \sin^{-1}\left(\frac{56}{65}\right)$ = RHS. The identity is proved.

(g) Verification: $u^2 + v^2 = \frac{9}{25} + \frac{25}{169} = \frac{2146}{4225} < 1$, so the addition formula is valid.

Final Answer: $\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \sin^{-1}\left(\frac{56}{65}\right)$

Go Back to Question 40

Q41.

Solution

Concept:

A line perpendicular to two given lines has its direction vector parallel to the cross product of the direction vectors of the given lines. The Cartesian equation uses the point-direction form: $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$, where $\langle a, b, c\rangle$ is the direction vector.

Solution:

(a) Direction vectors of given lines: Line 1: $\vec{d_1} = \langle 3, -2, 1\rangle$. Line 2: $\vec{d_2} = \langle 1, 2, 3\rangle$.

(b) The required line is perpendicular to both, so its direction $\vec{d} = \vec{d_1} \times \vec{d_2}$:

$$\vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

(c) Expand:

$$\begin{aligned}\vec{d} &= \hat{i}[(-2)(3) - (1)(2)] - \hat{j}[(3)(3) - (1)(1)] + \hat{k}[(3)(2) - (-2)(1)] \\ &= \hat{i}(-6 - 2) - \hat{j}(9 - 1) + \hat{k}(6 + 2) \\ &= -8\hat{i} - 8\hat{j} + 8\hat{k} = 8(-\hat{i} - \hat{j} + \hat{k})\end{aligned}$$

(d) Simplify direction ratios: $\vec{d} = \langle -1, -1, 1\rangle$ or equivalently $\langle 1, 1, -1\rangle$.

(e) Point: $(1, 2, -1)$. Vector equation: $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(-\hat{i} - \hat{j} + \hat{k})$.

(f) Cartesian equation: $\frac{x-1}{-1} = \frac{y-2}{-1} = \frac{z+1}{1}$, or $\frac{x-1}{1} = \frac{y-2}{1} = \frac{z+1}{-1}$.

(g) Trap check: The direction of the required line must be perpendicular to both given lines. Verify: $\vec{d} \cdot \vec{d_1} = (-1)(3) + (-1)(-2) + (1)(1) = -3 + 2 + 1 = 0$. $\vec{d} \cdot \vec{d_2} = (-1)(1) + (-1)(2) + (1)(3) = -1 - 2 + 3 = 0$. Confirmed.

Final Answer: Vector: $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(-\hat{i} - \hat{j} + \hat{k})$; Cartesian: $\frac{x-1}{-1} = \frac{y-2}{-1} = \frac{z+1}{1}$

Go Back to Question 41

Q42.

Solution

Concept:
Partial fraction decomposition breaks a rational function into simpler fractions that can be integrated individually. For the definite integral option, trigonometric substitutions or standard forms may be used. For $\int \frac{\sin x+\cos x}{9+16\sin 2x}dx$, note that $\sin 2x = 2\sin x\cos x$, and set up a substitution.
Solution:

(a) First choice – Partial fractions for $\frac{x^2+1}{(x-1)^2(x+3)}$.

Decompose: $\frac{x^2+1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$.

Multiply by the denominator: $x^2 + 1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2$.

Set $x = 1$: $1 + 1 = B(4) \implies 2 = 4B \implies B = \frac{1}{2}$.

Set $x = -3$: $9 + 1 = C(16) \implies 10 = 16C \implies C = \frac{5}{8}$.

Set $x = 0$: $1 = A(-1)(3) + B(3) + C(1) = -3A + 3(\frac{1}{2}) + \frac{5}{8}$. $1 = -3A + \frac{3}{2} + \frac{5}{8} = -3A + \frac{12}{8} + \frac{5}{8} = -3A + \frac{17}{8}$. $-3A = 1 - \frac{17}{8} = -\frac{9}{8} \implies A = \frac{3}{8}$.

Now integrate:

$$\int \frac{x^2+1}{(x-1)^2(x+3)}dx = \int \left[\frac{3/8}{x-1} + \frac{1/2}{(x-1)^2} + \frac{5/8}{x+3}\right] dx$$
$$= \frac{3}{8}\log|x-1| - \frac{1}{2(x-1)} + \frac{5}{8}\log|x+3| + C$$

(b) OR choice – For $I = \int_0^{\pi/4} \frac{\sin x+\cos x}{9+16\sin 2x}dx$:

Note $\sin 2x = 2\sin x\cos x$. Let $u = \sin x - \cos x$, so $du = (\cos x + \sin x)dx$. When $x = 0$, $u = -1$. When $x = \pi/4$, $u = 0$.

Also $u^2 = (\sin x - \cos x)^2 = \sin^2 x + \cos^2 x - 2\sin x\cos x = 1 - \sin 2x$.

So $\sin 2x = 1 - u^2$.

The denominator: $9 + 16(1-u^2) = 9 + 16 - 16u^2 = 25 - 16u^2$.

The integral becomes:

$$I = \int_{-1}^{0} \frac{du}{25-16u^2} = \frac{1}{40}\int_{-1}^{0}\left[\frac{1}{5-4u} + \frac{1}{5+4u}\right] du$$

$$= \frac{1}{40}\left[-\frac{1}{4}\log|5-4u| + \frac{1}{4}\log|5+4u|\right]_{-1}^{0}$$

$$= \frac{1}{160}\left[\log\left|\frac{5+4u}{5-4u}\right|\right]_{-1}^{0} = \frac{1}{160}\left[\log(1) - \log\left(\frac{1}{9}\right)\right] = \frac{1}{160}\cdot\log 9 = \frac{\log 3}{80}$$

Final Answer: $\frac{3}{8}\log|x-1| - \frac{1}{2(x-1)} + \frac{5}{8}\log|x+3| + C$

Go Back to Question 42

Q43.

Solution

Concept: According to the fundamental theorem of linear programming, the optimal value of the linear objective function must occur at one of the corner points (vertices) of this region.

Solution:

(a) Define x as the number of tables and y as the number of chairs to be produced.

(b) Write down the linear inequality constraints based on budget and storage capacities:

$$\text{Budget constraint:} \quad 500x + 100y \leq 10000 \implies 5x + y \leq 100$$

$$\text{Storage constraint:} \quad x + y \leq 60$$

$$\text{Non-negativity parameters:} \quad x \geq 0, \quad y \geq 0$$

(c) State the linear objective function representing the total profit to be maximized:

$$Z = 50x + 15y$$

(d) Determine the boundaries of the feasible region by finding the intersection points of the limiting lines. Solving the simultaneous equations $5x + y = 100$ and $x + y = 60$ by subtraction yields $4x = 40 \implies x = 10$, which gives $y = 50$. This results in the intersection vertex $C(10, 50)$.

(e) Identify the four valid corner points enclosing the shaded feasible region in the first quadrant:

$$O(0,0), \quad A(20,0), \quad B(0,60), \quad \text{and} \quad C(10,50)$$

Note that the intercept (0, 100) is outside the region because it violates the storage capacity.

(f) Evaluate the profit objective function Z at each of these boundary vertices:

$$\text{At } O(0,0): \quad Z = 50(0) + 15(0) = 0$$

$$\text{At } A(20,0): \quad Z = 50(20) + 15(0) = 1000$$

$$\text{At } B(0,60): \quad Z = 50(0) + 15(60) = 900$$

$$\text{At } C(10,50): \quad Z = 50(10) + 15(50) = 500 + 750 = 1250$$

(g) Compare the values to find the absolute maximum profit, which is Rs. 1250 at coordinate $(10, 50)$.

Final Answer: Maximum profit = Rs. 1250, producing 10 tables and 50 chairs.

Go Back to Question 43

Q44.

Solution

Concept:
The shortest distance between two skew lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is given by $d = \frac{|(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)|}{|\vec{b}_1\times\vec{b}_2|}$. For the plane through the intersection of two planes and perpendicular to a third, use the family of planes through the intersection line: $\pi_1 + k\pi_2 = 0$, then impose the perpendicularity condition.
Solution:

(a) First choice – Skew lines:

$\vec{a}_1 = \langle 1, 2, 3\rangle$, $\vec{b}_1 = \langle 2, 1, -2\rangle$. $\vec{a}_2 = \langle 2, 4, 5\rangle$, $\vec{b}_2 = \langle 3, 4, -5\rangle$.

$\vec{a}_2 - \vec{a}_1 = \langle 1, 2, 2\rangle$.

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 3 & 4 & -5 \end{vmatrix} = \hat{i}(-5+8) - \hat{j}(-10+6) + \hat{k}(8-3) = 3\hat{i} + 4\hat{j} + 5\hat{k}.$$

$|\vec{b}_1 \times \vec{b}_2| = \sqrt{9+16+25} = \sqrt{50} = 5\sqrt{2}$.

$(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2) = 1(3) + 2(4) + 2(5) = 3 + 8 + 10 = 21$.

Shortest distance $d = \frac{|21|}{5\sqrt{2}} = \frac{21}{5\sqrt{2}} = \frac{21\sqrt{2}}{10}$.

(b) OR choice – Plane through intersection:

Family of planes: $(x + y + z - 1) + k(2x + 3y - z + 4) = 0$.

Simplify: $(1 + 2k)x + (1 + 3k)y + (1 - k)z + (-1 + 4k) = 0$.

Normal vector: $\vec{n} = \langle 1 + 2k, 1 + 3k, 1 - k\rangle$.

For perpendicularity to plane $x - y + z = 0$ (normal $\langle 1, -1, 1\rangle$): $\vec{n} \cdot \langle 1, -1, 1\rangle = (1 + 2k)(1) + (1 + 3k)(-1) + (1 - k)(1) = 0$.

$1 + 2k - 1 - 3k + 1 - k = 0 \implies 1 - 2k = 0 \implies k = \frac{1}{2}$.

Substitute $k = \frac{1}{2}$: $(1+1)x + (1+\frac{3}{2})y + (1-\frac{1}{2})z + (-1+2) = 0 \implies 2x + \frac{5}{2}y + \frac{1}{2}z + 1 = 0$.

Multiply by 2: $4x + 5y + z + 2 = 0$.

(c) Verification: The plane passes through the intersection line of the given planes and is perpendicular to $x - y + z = 0$.

Final Answer: Shortest distance $= \dfrac{21\sqrt{2}}{10}$ (or plane $4x + 5y + z + 2 = 0$)
Go Back to Question 44

Q45.

Solution

Concept:
A function $f(x)$ is strictly increasing where $f'(x) > 0$ and strictly decreasing where $f'(x) < 0$. Critical points occur where $f'(x) = 0$ or $f'(x)$ does not exist. The sign of f' is analyzed using a number line (sign chart). Local extrema occur at points where f' changes sign.
Solution:

(a) $f(x) = \frac{x}{\log x}$, defined for $x > 0, x \neq 1$ (since $\log x \neq 0$).

(b) Compute $f'(x)$ using the quotient rule:

$$f'(x) = \frac{(1)(\log x) - x(\frac{1}{x})}{(\log x)^2} = \frac{\log x - 1}{(\log x)^2}$$

(c) Critical point: $f'(x) = 0 \implies \log x - 1 = 0 \implies \log x = 1 \implies x = e$.

(d) Sign analysis of $f'(x) = \frac{\log x - 1}{(\log x)^2}$:

- Denominator $(\log x)^2 > 0$ for all $x > 0, x \neq 1$. So the sign of f' depends only on $\log x - 1$.
- For $0 < x < 1$: $\log x < 0$, so $\log x - 1 < 0$. Hence $f'(x) < 0$. Function is decreasing on $(0, 1)$.
- For $1 < x < e$: $0 < \log x < 1$, so $\log x - 1 < 0$. Hence $f'(x) < 0$. Function is decreasing on $(1, e)$.
- For $x > e$: $\log x > 1$, so $\log x - 1 > 0$. Hence $f'(x) > 0$. Function is increasing on (e, ∞).

(e) Combine decreasing intervals: f is strictly decreasing on $(0, 1) \cup (1, e)$.

(f) f is strictly increasing on (e, ∞).

(g) At $x = e$, f' changes from negative to positive, so $x = e$ is a point of local minimum. $f(e) = \frac{e}{\log e} = \frac{e}{1} = e$.

(h) Note: $x = 1$ is a vertical asymptote (not in domain), not an extremum. The function has a local minimum at (e, e).

Final Answer: (a) Strictly increasing on (e, ∞). (b) Strictly decreasing on $(0, 1) \cup (1, e)$. Local minimum at $x = e$.

Go Back to Question 45

Answer Key

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	C	2	A	3	B	4	A	5	A
6	A	7	A	8	B	9	A	10	A
11	A	12	A	13	A	14	A	15	A
16	D	17	A	18	B	19	A	20	A