NIOS Class 12 Mathematics Sample Paper – 8

Duration: 180 Minutes **Maximum Marks: 100**

Instructions

- This paper contains **45** Questions. The paper is divided into two sections: **Section A – 50** marks, **Section B – 50** marks.
- **Section A** consists of
  -**Q.No. 1 to 20** – Multiple Choice type questions (MCQs) carrying **+1 mark** each. Select and write the most appropriate option out of the four options given in each of these questions.
  -**Q.No. 21 to 29** – **Objective type questions**.
  -**Q.No. 21 to 24** carry **02 marks** each (with 2 sub-parts of 1 mark each).
  -**Q.No. 25 to 28** carry **04 marks** each (with 4 sub-parts of 1 mark each).
  -**Q.No. 29** carries **06 marks** (with 6 sub-parts of 1 mark each). Attempt these questions as per the instructions given for each of the questions 21–29.
- **Section B** consists of
  -**Q.No. 30 to 38**– Very Short questions carrying **02 marks** each.
  -**Q.No. 39 to 43** – Short Answer type questions carrying **04 marks** each.
  -**Q.No. 44 to 45** – Long Answer type questions carrying **06 marks** each.
  (An internal choice has been provided in some of the questions in Section B. You have to attempt only one of the given choices in such questions.)
- There is **No Negative marking**.
- Use of mobile phones, smartwatches, calculators, or any electronic gadgets is strictly prohibited.

Section: A

Q1. If $\lim_{x\to 0} \dfrac{\sin 2x + \sin kx}{x} = 5$, where $k > 2$ is an integer, then k equals: **(1)**

(A) 1

(B) 2

(C) 3

(D) 4

Q2. If $|\vec{a}| = 4$, $|\vec{b}| = 3$ and $\vec{a} \cdot \vec{b} = 6$, then $|\vec{a} \times \vec{b}|$ equals: **(1)**

(A) $6\sqrt{3}$

(B) $3\sqrt{3}$

(C) 12

(D) 6

Q3. If $\begin{vmatrix} x & 2 \\ 3 & x \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}$, then x equals: **(1)**

(A) ± 4

(B) $\pm\sqrt{14}$

(C) ± 3

(D) ± 5

Q4. $\int_0^{\pi/2} \sin^3 x \, dx$ equals: **(1)**

(A) $\frac{2}{3}$

(B) $\frac{1}{3}$

(C) $\frac{4}{3}$

(D) $\frac{1}{2}$

Q5. The line through $(2, -3)$ perpendicular to $3x - 4y + 7 = 0$ has equation: **(1)**

(A) $4x + 3y + 1 = 0$

(B) $3x + 4y + 6 = 0$

(C) $4x - 3y - 17 = 0$

(D) $3x - 4y - 18 = 0$

Q6. $\cos^{-1}\left(-\frac{1}{2}\right) + 2\sin^{-1}\left(-\frac{1}{2}\right)$ equals: **(1)**

(A) $\frac{\pi}{3}$

(B) $\frac{2\pi}{3}$

(C) π

(D) 0

Q7. The distance of $(2, 3, -1)$ from the plane $2x - 2y + z + 5 = 0$ is: **(1)**

(A) $\frac{4}{3}$

(B) $\frac{2}{3}$

(C) 4

(D) $\frac{8}{3}$

Q8. The slope of the normal to $y = x^3 - 3x + 2$ at $x = 1$ is: **(1)**

(A) 0

(B) -1

(C) ∞

(D) 1

Q9. The circle $x^2 + y^2 - 4x + 2y - 4 = 0$ has centre: **(1)**

(A) $(2, -1)$

(B) $(-2, 1)$

(C) $(4, -2)$

(D) $(-4, 2)$

Q10. The focus of $x^2 = -16y$ is: **(1)**

(A) $(0, 4)$

(B) $(0, -4)$

(C) $(4, 0)$

(D) $(-4, 0)$

Q11. The degree of $\frac{d^2y}{dx^2} + \sin\left(\frac{dy}{dx}\right) = 0$ is: **(1)**

(A) 2

(B) 1

(C) Not defined

(D) 0

Q12. Which is the contrapositive of $p \to q$? **(1)**

(A) $\neg q \to \neg p$

(B) $q \to p$

(C) $\neg p \to \neg q$

(D) $p \wedge \neg q$

Q13. Let $R = \{(a, b) : a, b \in \mathbb{N}, a = b^2\}$. Then R is: **(1)**

(A) Reflexive

(B) Symmetric

(C) Transitive

(D) None of these

Q14. If $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then A^4 equals: **(1)**

(A) A

(B) I

(C) $2I$

(D) $-A$

Q15. The length of latus rectum of $\frac{x^2}{9} + \frac{y^2}{5} = 1$ is: **(1)**

(A) $\frac{10}{3}$

(B) $\frac{20}{3}$

(C) $\frac{5}{3}$

(D) $\frac{10}{9}$

Q16. The minimum value of $f(x) = x^2 - 4x + 6$ is: **(1)**

(A) 2

(B) 3

(C) 1

(D) 6

Q17. The feasible region of an LPP is always: **(1)**

(A) Convex

(B) Concave

(C) Square

(D) Circular

Q18. The solution of $\frac{dy}{dx} = e^{x-y}$ is: **(1)**

(A) $e^y = e^x + C$

(B) $e^x = e^y + C$

(C) $y = x + C$

(D) $e^{y-x} = C$

Q19. Eccentricity of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with $b = a$ is: **(1)**

(A) $\sqrt{2}$

(B) 1

(C) 2

(D) $\frac{1}{2}$

Q20. If $\vec{a} = \hat{i} + \hat{j}$, $\vec{b} = \hat{j} + \hat{k}$, then $|\vec{a} \times \vec{b}|$ is: **(1)**

(A) $\sqrt{3}$

(B) $\sqrt{2}$

(C) 1

(D) 0

Q21. Match Column-I with Column-II: **(2)**

Column-I	**Column-II**
(i) Value of $\int_0^1 \frac{dx}{1+x^2}$	(A) $\frac{\pi}{4}$
(ii) $\lvert\vec{a} \times \vec{b}\rvert^2 + (\vec{a} \cdot \vec{b})^2$ equals	(B) $\lvert\mathbf{a}\rvert^2\lvert\mathbf{b}\rvert^2$

(A) (i)→(A), (ii)→(B)

(B) (i)→(B), (ii)→(A)

Q22. Fill in the blanks: **(2)**

(i) The order of $\left(\frac{d^3y}{dx^3}\right)^{1/2} + \frac{dy}{dx} = 0$ is ________.

(ii) The angle between $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} + 2\hat{k}$ is ________.

Q23. Write TRUE or FALSE: **(2)**

(i) The relation R on $\{1, 2, 3\}$ defined by $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$ is an equivalence relation.

(ii) For any two vectors $\vec{a}, \vec{b}$, $\lvert\vec{a} \cdot \vec{b}\rvert \le \lvert\vec{a}\rvert\lvert\vec{b}\rvert$.

Q24. Write the negation: **(2)**

(i) All polynomials are continuous functions.

(ii) There exists a triangle with four sides.

Q25. Fill in the blanks: **(4)**

(i) $\lim_{x \to 0} \frac{\sin 7x}{x} =$ ________.

(ii) $\lim_{x \to \infty} \frac{5x - 3}{2x + 1} =$ ________.

(iii) $\lim_{x \to 0} (1 + 3x)^{1/x} =$ ________.

(iv) $\lim_{x\to 1} \frac{x^3-1}{x-1} =$ ________.

Q26. Fill in the blanks (Conic Sections): **(4)**

(i) Centre of $x^2 + y^2 + 2x - 4y - 4 = 0$ is ________.

(ii) Eccentricity of $\frac{x^2}{4} + \frac{y^2}{9} = 1$ is ________.

(iii) Focus of $y^2 = 20x$ is ________.

(iv) Length of transverse axis of $\frac{x^2}{9} - \frac{y^2}{16} = 1$ is ________.

Q27. Write TRUE or FALSE: **(4)**

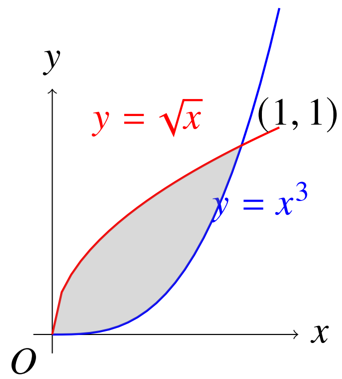
(i) $\det(A + B) = \det A + \det B$ for all square matrices A, B.

(ii) $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$.

(iii) The function $f(x) = x^3$ is decreasing on $\mathbb{R}$.

(iv) The planes $2x - y + z = 3$ and $4x - 2y + 2z = 1$ are parallel.

Q28. Study the figure showing $y = \sqrt{x}$ and $y = x^3$ in the first quadrant: **(4)**



(i) Find the x-coordinates of intersection points.

(ii) Identify the upper curve on $[0, 1]$.

(iii) Write the integral for the shaded area.

(iv) Compute the area.

Q29. Read and answer (i)–(vi):

A firm produces two products P and Q. Each unit of P requires 2 kg of raw material and 3 labour hours, giving profit Rs. 40. Each unit of Q requires 4 kg of raw material and 2 labour hours, giving profit Rs. 50. Available: 100 kg material, 90 labour hours. Let x units of P, y units of Q. **(6)**

(i) The material constraint is:

(A) $2x + 4y \leq 100$

(B) $3x + 2y \leq 90$

(C) $40x + 50y \geq 0$

(D) $x + y \leq 100$

(ii) The labour constraint is:

(A) $2x + 4y \leq 90$

(B) $3x + 2y \leq 90$

(C) $2x + 3y \leq 90$

(D) $40x + 50y \leq 90$

(iii) The objective is:

(A) Max $Z = 3x + 2y$

(B) Max $Z = 2x + 4y$

(C) Max $Z = 40x + 50y$

(D) Min $Z = 40x + 50y$

(iv) Intersection of $x + 2y = 50$ and $3x + 2y = 90$ is:

(A) $(20, 15)$

(B) $(15, 20)$

(C) $(10, 20)$

(D) $(20, 10)$

(v) Z at $(30, 0)$ is:

(A) 1200

(B) 1500

(C) 900

(D) 2000

(vi) Maximum profit is:

(A) Rs. 1200

(B) Rs. 1350

(C) Rs. 1550

(D) Rs. 2250

Section: B

Q30. Find k if $f(x) = \begin{cases} \frac{1-\cos 2x}{x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$ is continuous at $x = 0$. **(2)**

Q31. If $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$, show that $A^2 - 4A + I = 0$. **(2)**

Q32. Find $\frac{dy}{dx}$ if $y = \log(\sin \sqrt{x})$. **(2)**

Q33. Find $\int e^x (\sin x + \cos x) dx$. **(2)**

Q34. Find $|\vec{a}|$ if $\vec{a} = 2\hat{i} + 3\hat{j} - 6\hat{k}$. OR Find a vector of magnitude 5 parallel to $\hat{i} + 2\hat{j} - 2\hat{k}$. **(2)**

Q35. Find the Cartesian equation: $\vec{r} \cdot (2\hat{i} - \hat{j} + 3\hat{k}) = 5$. **(2)**

Q36. Find $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$. **(2)**

Q37. Find direction cosines of line $\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + 2\hat{k})$. **(2)**

Q38. Find the area of parallelogram with adjacent sides $2\hat{i} + \hat{j} + 3\hat{k}$ and $\hat{i} - \hat{j}$. **(2)**

Q39. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{pmatrix}$, show $A^3 - 23A - 40I = 0$. Hence find A^{-1}. **(4)**

Q40. Prove $\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8} = \frac{\pi}{4}$. **(4)**

Q41. Find shortest distance between lines: $\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$. **(4)**

Q42. Evaluate $\int \frac{3x+2}{(x+1)^2(x+3)}dx$. OR Find $\int_0^{\pi/4} \log(1 + \tan x)dx$. **(4)**

Q43. Solve LPP graphically: Maximize $Z = 4x + y$ subject to $x + y \leq 50$, $3x + y \leq 90$, $x, y \geq 0$. **(4)**

Q44. Find the area bounded by $y = \cos x$, $x = 0$, $x = \pi$, and the x-axis, using integration.

OR

Find area of region $\{(x, y) : y^2 \leq 4x, 4x^2 + 4y^2 \leq 9\}$. **(6)**

Q45. An open box with square base is to be made from a given quantity of cardboard of area c^2. Show the maximum volume is $\frac{c^3}{6\sqrt{3}}$.

OR

Find intervals for $f(x) = \frac{x}{\log x}$ increasing/decreasing. **(6)**

Detailed Solutions

Q1.

Solution

Concept: The fundamental limit identity involving trigonometric functions state that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$. By scaling the angle inside the argument, this can be written as $\lim_{x\to 0} \frac{\sin kx}{x} = k$. This formula allows for the evaluation of individual limit terms by direct comparison of coefficients.

Solution:

(a) Consider the limit expression given by the problem statement:

$$\lim_{x\to 0}\left(\frac{\sin 2x}{x} + \frac{\sin kx}{x}\right) = 5$$

(b) Using the linearity properties of limits, we can distribute the limit operator across the sum of the two trigonometric terms:

$$\lim_{x\to 0}\frac{\sin 2x}{x} + \lim_{x\to 0}\frac{\sin kx}{x} = 5$$

(c) Apply the limit coefficient property to the first term where $k = 2$:

$$\lim_{x\to 0}\frac{\sin 2x}{x} = 2$$

(d) Apply the limit coefficient property to the second term with the unknown parameter k:

$$\lim_{x\to 0}\frac{\sin kx}{x} = k$$

(e) Substitute these individual limits back into the linear equation:

$$2 + k = 5$$

(f) Solve the linear equation for k by subtracting 2 from both sides:

$$k = 5 - 2 \implies k = 3$$

Final Answer: 3

Answer: (C)

Go Back to Question 1

Q2.

Solution

Concept: The magnitude of the vector cross product between two vectors $\vec{a}$ and $\vec{b}$ represents the area of the parallelogram formed by them and is defined by $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$. The angle θ can be found using the scalar dot product definition $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$.

Solution:

(a) Write down the values provided in the problem statement: $|\vec{a}||\vec{b}| = 12$ and $\vec{a} \cdot \vec{b} = 6$.

(b) Substitute these values into the standard cosine definition formula:

$$\cos\theta = \frac{6}{12} = \frac{1}{2}$$

(c) Determine the angle θ inside the first quadrant where the cosine equals one-half:

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ \text{ or } \frac{\pi}{3} \text{ radians}$$

(d) Evaluate the trigonometric sine value for this specific structural angle:

$$\sin\theta = \sin(60^\circ) = \frac{\sqrt{3}}{2}$$

(e) Substitute the scalar magnitude product and the calculated sine value into the cross product magnitude formula:

$$|\vec{a} \times \vec{b}| = 12 \cdot \frac{\sqrt{3}}{2}$$

(f) Simplify the numerical fraction coefficients:

$$|\vec{a} \times \vec{b}| = 6\sqrt{3}$$

Final Answer: $6\sqrt{3}$

Answer: (A)

Go Back to Question 12

Q3.

Solution

Concept: The determinant of a second-order square matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is calculated by taking the difference between the products of the main diagonal elements and the off-diagonal elements, which is expressed algebraically as $ad - bc$.

Solution:

(a) Set up the matrix determinant equation as provided by the problem structure:

$$\begin{vmatrix} x & 2 \\ 3 & x \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix}$$

(b) Expand the left-hand side determinant by multiplying the main elements and off-diagonal elements:

$$\text{LHS} = (x)(x) - (2)(3) = x^2 - 6$$

(c) Expand the right-hand side determinant using the same cross-multiplication procedure:

$$\text{RHS} = (4)(3) - (2)(1) = 12 - 2 = 10$$

(d) Equate the two expanded scalar expressions to formulate a single quadratic equation:

$$x^2 - 6 = 10$$

(e) Isolate the squared variable by adding 6 to both sides of the equation:

$$x^2 = 10 + 6 \implies x^2 = 16$$

(f) Take the square root of both sides, accounting for both positive and negative solutions:

$$x = \pm\sqrt{16} \implies x = \pm 4$$

Final Answer: ± 4

Answer: (A)

Go Back to Question 3

Q4.

Solution

Concept: Wallis formula provides an elegant reduction method to evaluate definite integrals of power-trigonometric functions over the limits 0 to $\pi/2$. For an odd positive integer power n, the integral formula is defined by $\int_0 {\pi/2} \sin^n x\, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot 1$.

Solution:

(a) Identify the integration limits and the specific power of the function from the expression:

$$I = \int_0^{\pi/2} \sin^3 x\, dx$$

Here, the power value is $n = 3$, which is an odd integer.

(b) Set up the corresponding Wallis reduction sequence for an odd power of three:

$$I = \frac{3-1}{3} \cdot \text{(final reduction step)}$$

(c) Substitute the numerical differences directly into the fraction sequence:

$$I = \frac{2}{3} \cdot 1$$

(d) Perform the final fractional multiplication step to obtain the definitive value:

$$I = \frac{2}{3}$$

This systematic integration tool eliminates the requirement for standard step-by-step trigonometric substitution methods.

Final Answer: $2/3$

Answer: (A)

Go Back to Question 4

Q5.

Solution

Concept: The slope m of a general straight line given in the form $Ax + By + C = 0$ is equal to $-A/B$. If two lines are mutually perpendicular, their slopes m_1 and m_2 satisfy the orthogonal relationship $m_1 \cdot m_2 = -1$, meaning the perpendicular slope is the negative reciprocal.

Solution:

(a) Find the slope of the given reference line $3x - 4y + 7 = 0$:

$$m_1 = -\frac{A}{B} = -\frac{3}{-4} = \frac{3}{4}$$

(b) Determine the required slope m_2 of the target line which is perpendicular to it:

$$m_2 = -\frac{1}{m_1} = -\frac{1}{3/4} = -\frac{4}{3}$$

(c) Apply the point-slope formula $y - y_0 = m_2(x - x_0)$ using the point $(2, -3)$:

$$y - (-3) = -\frac{4}{3}(x - 2) \implies y + 3 = -\frac{4}{3}(x - 2)$$

(d) Multiply the entire linear equation by 3 to eliminate the fraction denominator:

$$3(y + 3) = -4(x - 2) \implies 3y + 9 = -4x + 8$$

(e) Reorder all variables and constant terms to standard form:

$$4x + 3y + 9 - 8 = 0 \implies 4x + 3y + 1 = 0$$

Final Answer: $4x + 3y + 1 = 0$

Answer: (A)

Go Back to Question 5

Q6.

Solution

Concept: Evaluating inverse trigonometric functions requires finding the principal value angles. The principal value branch for the inverse cosine function is defined over the interval $[0, \pi]$, while the principal value branch for the inverse sine function is restricted to the interval $[-\pi/2, \pi/2]$.

Solution:

(a) State the full expression that needs to be evaluated:

$$E = \cos^{-1}\left(-\frac{1}{2}\right) + 2\sin^{-1}\left(-\frac{1}{2}\right)$$

(b) Find the principal angle for the first term where the cosine is negative in quadrant II:

$$\cos^{-1}\left(-\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

(c) Find the principal angle for the second term where the sine value is negative:

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

(d) Substitute both of these calculated principal angles back into the expression:

$$E = \frac{2\pi}{3} + 2\left(-\frac{\pi}{6}\right)$$

(e) Simplify the multiplication term containing the fractional angle:

$$E = \frac{2\pi}{3} - \frac{\pi}{3}$$

(f) Combine the terms over a common denominator to find the final value:

$$E = \frac{2\pi - \pi}{3} = \frac{\pi}{3}$$

Final Answer: $\pi/3$

Answer: (A)

Go Back to Question 6

Q7.

Solution

Concept: The shortest perpendicular distance from a specific three-dimensional coordinate point (x_0, y_0, z_0) to a given flat plane defined by the standard equation $ax + by + cz + d = 0$ is computed using the absolute ratio formula Distance $= \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$.

Solution:

(a) Identify the coordinate parameters from the given point $(2, 3, 4)$:

$$x_0 = 2, \quad y_0 = 3, \quad z_0 = 4$$

(b) Extract the coefficients from the given plane equation $2x - 2y + z + 5 = 0$:

$$a = 2, \quad b = -2, \quad c = 1, \quad d = 5$$

(c) Substitute these values into the numerator of the perpendicular distance formula:

$$\text{Numerator} = |2(2) - 2(3) + 1(4) + 5| = |4 - 6 + 4 + 5| = |7| = 7$$

(d) Substitute the plane coefficients into the radical denominator:

$$\text{Denominator} = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

(e) Assemble the complete ratio to find the numeric distance:

$$\text{Distance} = \frac{7}{3}$$

Final Answer: $7/3$

Answer: (B)

Go Back to Question 7

Q8.

Solution

Concept: The slope of a tangent line to a curve at any point is given by the first derivative $m_t = y'$. The normal line is perpendicular to the tangent line, meaning its slope satisfies $m_n = -1/m_t$. A tangent line with a slope of zero indicates a horizontal line, which implies the normal line is vertical.

Solution:

(a) State the equation of the given geometric curve:

$$y = x^3 - 3x + 2$$

(b) Compute the general first derivative with respect to x using the power rule:

$$y' = \frac{d}{dx}(x^3) - \frac{d}{dx}(3x) + \frac{d}{dx}(2) = 3x^2 - 3$$

(c) Evaluate this derivative at the specified coordinate $x = 1$ to find the tangent slope:

$$m_t = 3(1)^2 - 3 = 3 - 3 = 0$$

(d) Since the tangent slope is zero, the tangent line is perfectly horizontal.

(e) Calculate the perpendicular normal line slope using the reciprocal formula:

$$m_n = -\frac{1}{m_t} = -\frac{1}{0} \to \infty$$

This mathematically defines a vertical line.

Final Answer: ∞ (vertical)

Answer: (C)

Go Back to Question 8

Q9.

Solution

Concept: The standard general equation of a circle is given by $x^2 + y^2 + 2gx + 2fy + c = 0$. In this representation, the coordinates of the circle's center point are found by taking the negative halves of the coefficients of x and y, written as Centre = $(-g, -f)$.

Solution:

(a) Write down the general equation of the circle from the problem description:

$$x^2 + y^2 - 4x + 2y - 12 = 0$$

(b) Identify the linear coefficient corresponding to the x variable term:

$$2g = -4 \implies g = -\frac{4}{2} = -2$$

(c) Identify the linear coefficient corresponding to the y variable term:

$$2f = 2 \implies f = \frac{2}{2} = 1$$

(d) Formulate the coordinates of the center point using the negative parameters:

$$\text{Centre} = (-g, -f)$$

(e) Substitute the values of g and f to obtain the final coordinates:

$$\text{Centre} = (-(-2), -(1)) = (2, -1)$$

Final Answer: $(2, -1)$

Answer: (A)

Go Back to Question 9

Q10.

Solution

Concept: A parabola opening downwards along the vertical axis has a standard standard equation of the form $x^2 = -4ay$, where $a > 0$ represents the distance from the vertex to the focus. For this specific geometric alignment, the focus coordinate is located on the negative axis at $(0, -a)$.

Solution:

(a) Write down the quadratic equation of the parabola given in the prompt:

$$x^2 = -16y$$

(b) Compare this given expression directly with the standard vertical form:

$$-4ay = -16y$$

(c) Equate the numerical coefficients to determine the focal parameter value a:

$$4a = 16 \implies a = \frac{16}{4} = 4$$

(d) Recall the standard focus configuration for this downward-facing orientation:

$$\text{Focus} = (0, -a)$$

(e) Substitute the calculated parameter value $a = 4$ into the coordinate pair:

$$\text{Focus} = (0, -4)$$

Final Answer: $(0, -4)$

Answer: (B)

Go Back to Question 10

Q11.

Solution

Concept: The degree of a differential equation is defined as the highest power or exponent of the highest-order derivative present in the equation. However, this definition strictly requires that the differential equation must first be expressible as a polynomial equation in terms of all its derivatives. If any derivative is embedded within a transcendental function, such as a trigonometric, exponential, or logarithmic function, the equation cannot be written in polynomial form.

Solution:

(a) Examine the structural layout of the given differential equation from the prompt.

(b) Observe that the first derivative term is wrapped inside a non-linear trigonometric operation:

$$\sin\left(\frac{dy}{dx}\right)$$

(c) Due to the fundamental expansion property of the sine function via infinite Taylor series, this term introduces an infinite progression of powers of the derivative.

(d) Consequently, it becomes impossible to rewrite this expression as a finite polynomial equation with respect to its derivatives.

(e) Therefore, because the polynomial condition is broken by the transcendental composition, the degree of this differential equation cannot be evaluated.

Final Answer: Not defined

Answer: (C)

Go Back to Question 11

Q12.

Solution

Concept: In mathematical logic and truth-value analysis, an implication statement can be written in the symbolic form $p \to q$, which reads as "if p, then q". The contrapositive of this conditional statement is formed by negating both the antecedent component and the consequent component, and then reversing their logical direction, yielding the standard structural form $\neg q \to \neg p$.

Solution:

(a) Let p represent the primary statement or hypothesis provided in the given conditional proposition.

(b) Let q represent the secondary statement or conclusion that follows from the hypothesis.

(c) Write down the original implication structure using standard propositional notation:

$$\text{Implication} = p \to q$$

(d) To find the contrapositive, take the logical negation of the second component, which gives $\neg q$.

(e) Take the logical negation of the first component as well, which results in $\neg p$.

(f) Construct the new conditional implication by directing the negated conclusion to the negated hypothesis:

$$\text{Contrapositive} = \neg q \to \neg p$$

(g) Note that a conditional statement and its contrapositive are logically equivalent, meaning they always share identical truth values across all scenarios.

Final Answer: $\neg q \to \neg p$

Answer: (A)

Go Back to Question 12

Q13.

Solution

Concept: An equivalence relation on a specific set must simultaneously satisfy three core algebraic properties: reflexivity, symmetry, and transitivity. A relation R on a set S is reflexive if $(a, a) \in R$ for all $a \in S$; it is symmetric if $(a, b) \in R \implies (b, a) \in R$; and it is transitive if $(a, b) \in R$ and $(b, c) \in R \implies (a, c) \in R$.

Solution:

(a) Define the relation from the problem space: $R = \{(a, b) : a = b^2\}$ over the set of numbers.

(b) Test for the reflexive property by checking if an element maps to itself:

$$a = a^2$$

This is false for a general element like $a = 2$, since $2 \neq 4$, meaning $(2, 2) \notin R$. The relation is not reflexive.

(c) Test for the symmetric property by assuming an ordered pair belongs to the relation:

$$(4, 2) \in R \implies 4 = 2^2$$

However, reversing the coordinates gives $(2, 4)$, which means $2 \neq 4^2$. Thus, $(2, 4) \notin R$, so the relation is not symmetric.

(d) Since it fails these conditions, it cannot be an equivalence relation.

Final Answer: None of these

Answer: (D)

Go Back to Question 13

Q14.

Solution

Concept: Matrix exponentiation involves repeated multiplication of a square matrix by itself. For certain specific matrices, multiplying the matrix by itself yields the identity matrix I, which is written as $A^2 = I$. Any matrix that satisfies this special algebraic condition is called an involutory matrix.

Solution:

(a) Write down the components of the given second-order square matrix A:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(b) Compute the square of the matrix, A^2, by performing standard row-by-column matrix multiplication:

$$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(c) Calculate each individual cell entry of the resulting matrix product:

$$\text{Row 1, Col 1} = (0)(0) + (1)(1) = 1$$

$$\text{Row 1, Col 2} = (0)(1) + (1)(0) = 0$$

$$\text{Row 2, Col 1} = (1)(0) + (0)(1) = 0$$

$$\text{Row 2, Col 2} = (1)(1) + (0)(0) = 1$$

(d) Assemble these values into a single matrix structure to reveal the identity matrix:

$$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

(e) Find the fourth power of the matrix by applying index rules:

$$A^4 = (A^2)^2 = I^2 = I$$

Final Answer: I

Answer: (B)

Go Back to Question 14

Q15.

Solution

Concept: An ellipse configured in standard Cartesian form along the horizontal axis is given by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a represents the length of the semi-major axis and b represents the length of the semi-minor axis. The total length of the latus rectum chord is calculated using the formula Length $= \frac{2b^2}{a}$.

Solution:

(a) State the quadratic equation of the ellipse provided in the problem text:

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

(b) Compare the denominators directly with the standard ellipse parameters:

$$a^2 = 9 \implies a = \sqrt{9} = 3$$

$$b^2 = 5$$

(c) Confirm the orientation of the major axis by noting that $a^2 > b^2$ $(9 > 5)$, which validates the horizontal alignment.

(d) Recall the standard metric formula used to compute the total length of the latus rectum:

$$\text{Latus Rectum} = \frac{2b^2}{a}$$

(e) Substitute the values of a and b^2 directly into this geometric expression:

$$\text{Latus Rectum} = \frac{2 \cdot 5}{3} = \frac{10}{3}$$

Final Answer: 10/3

Answer: (A)

Go Back to Question 15

Q16.

Solution

Concept: Local extrema of a differentiable function are located by applying optimization tests. First, solve the first derivative equation $f'(x) = 0$ to find the critical points. Then, evaluate the second derivative $f''(x)$ at those critical values. If $f''(x) > 0$, the critical point corresponds to a local minimum.

Solution:

(a) Write down the objective scalar function given by the problem expression:

$$f(x) = x^2 - 4x + 6$$

(b) Differentiate the function once with respect to x using the basic power rules:

$$f'(x) = 2x - 4$$

(c) Find the critical point by setting this first derivative equal to zero:

$$2x - 4 = 0 \implies 2x = 4 \implies x = 2$$

(d) Compute the second derivative of the function to analyze the curve curvature:

$$f''(x) = \frac{d}{dx}(2x - 4) = 2$$

(e) Observe that $f''(2) = 2 > 0$, which mathematically confirms a local minimum at $x = 2$.

(f) Calculate the minimum value by substituting $x = 2$ back into the function:

$$f(2) = (2)^2 - 4(2) + 6 = 4 - 8 + 6 = 2$$

Final Answer: 2

Answer: (A)

Go Back to Question 16

Q17.

Solution

Concept: In linear programming problems and coordinate geometry, a half-plane is defined as a planar region bounded by an infinite straight line. A geometric set is considered convex if, for any two points chosen inside the set, the entire straight line segment connecting them lies completely within the boundaries of that same set.

Solution:

(a) Consider a system of linear inequality expressions that define individual half-planes.

(b) Each individual half-plane forms a basic convex set in two-dimensional space.

(c) According to a fundamental theorem in convex analysis, the set intersection of any collection of convex sets produces another convex set.

(d) When multiple linear constraints are applied simultaneously, their mutual overlapping region forms the feasible region of the system.

(e) Because the feasible region is created by the intersection of these individual linear half-planes, it must preserve the underlying structural properties.

(f) Therefore, the resulting bounded or unbounded intersection region always forms a convex set.

Final Answer: Convex

Answer: (A)

Go Back to Question 17

Q18.

Solution

Concept: A first-order differential equation can be solved using the separation of variables technique if it can be rearranged into a form where all terms containing y are grouped with dy and all terms containing x are grouped with dx. Once separated, both sides can be integrated independently.

Solution:

(a) Write down the differential equation provided in the problem text:

$$\frac{dy}{dx} = e^{x-y}$$

(b) Apply exponent laws to break apart the joint transcendental term on the right-hand side:

$$\frac{dy}{dx} = e^{x} \cdot e^{-y}$$

(c) Isolate the variable components by multiplying both sides by e^y and dx:

$$e^y\, dy = e^x\, dx$$

(d) Set up the integration operators on both sides of the separated equation:

$$\int e^y\, dy = \int e^x\, dx$$

(e) Evaluate the integrals, recalling that the exponential function is its own antiderivative:

$$e^y = e^x + C$$

where C represents the arbitrary constant of integration.

Final Answer: $e^y = e^x + C$

Answer: (A)

Go Back to Question 18

Q19.

Solution

Concept: A hyperbola is described as rectangular or equilateral if its semi-major axis a and semi-minor axis b have equal lengths, meaning $a = b$. The eccentricity parameter e measures the deviation of the conic section from a perfect circle and is computed using the formula $e = \sqrt{1 + \frac{b^2}{a^2}}$.

Solution:

(a) State the geometric condition for a rectangular hyperbola configuration:

$$a = b$$

(b) Write down the standard algebraic expression used to calculate eccentricity:

$$e = \sqrt{1 + \frac{b^2}{a^2}}$$

(c) Substitute the rectangular equality condition $b = a$ into the fraction inside the radical:

$$e = \sqrt{1 + \frac{a^2}{a^2}}$$

(d) Simplify the internal fraction, which reduces to unity:

$$e = \sqrt{1 + 1}$$

(e) Add the integer constants together to find the final eccentricity value:

$$e = \sqrt{2}$$

This constant value shows that the eccentricity of any rectangular hyperbola depends only on its orthogonal shape symmetry.

Final Answer: $\sqrt{2}$

Answer: (A)

Go Back to Question 19

Q20.

Solution

Concept: The vector cross product of two vectors $\vec{a}$ and $\vec{b}$ produces a new vector that is perpendicular to both. This product can be computed by evaluating a third-order determinant whose first row consists of the standard unit basis vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$. The scalar magnitude is then found using the three-dimensional Pythagorean formula.

Solution:

(a) Extract the vector components from the problem text: $\vec{a} = \hat{i} + \hat{j}$ and $\vec{b} = \hat{j} + \hat{k}$.

(b) Set up the cross product calculation inside a matrix determinant structure:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

(c) Expand the determinant along the top row containing the basis vectors:

$$\vec{a} \times \vec{b} = \hat{i}(1 \cdot 1 - 0 \cdot 1) - \hat{j}(1 \cdot 1 - 0 \cdot 0) + \hat{k}(1 \cdot 1 - 1 \cdot 0)$$

$$\vec{a} \times \vec{b} = 1\hat{i} - 1\hat{j} + 1\hat{k}$$

(d) Calculate the absolute scalar magnitude of this resulting cross product vector:

$$|\vec{a} \times \vec{b}| = \sqrt{(1)^2 + (-1)^2 + (1)^2} = \sqrt{1 + 1 + 1} = \sqrt{3}$$

Final Answer: $\sqrt{3}$

Answer: (A)

Go Back to Question 20

Q21.

Solution

Concept: The matching items involve basic principles of calculus and vector algebra. Part (i) evaluates a standard definite integral whose integrand maps directly to the derivative of the inverse tangent function, $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$. Part (ii) applies Lagrange's identity for vectors, which establishes a fundamental relationship connecting the square of the cross product magnitude and the square of the dot product to the individual vector magnitudes.

Solution:

(a) For sub-question (i), evaluate the given integral from 0 to 1:

$$I = \int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1$$

(b) Apply the fundamental theorem of calculus by substituting the upper limit and the lower limit:

$$I = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

This matches option (A).

(c) For sub-question (ii), look at the identity expression combining both vector product types:

$$E = |\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2$$

(d) Substitute the trigonometric definitions $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$ and $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$:

$$E = |\vec{a}|^2|\vec{b}|^2 \sin^2\theta + |\vec{a}|^2|\vec{b}|^2 \cos^2\theta$$

(e) Factor out the scalar magnitude products to simplify the expression:

$$E = |\vec{a}|^2|\vec{b}|^2(\sin^2\theta + \cos^2\theta) = |\vec{a}|^2|\vec{b}|^2(1) = |\vec{a}|^2|\vec{b}|^2$$

This matches option (B).

Final Answer: (i)→(A), (ii)→(B)

Go Back to Question 21

Q22.

Solution

Concept: Part (i) requires finding the order of a differential equation, which is defined as the order of the highest derivative appearing in that differential equation. Part (ii) requires finding the angle between two vectors $\vec{a}$ and $\vec{b}$ by rewriting the standard geometric dot product formula in terms of its angle cosine: $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$.

Solution:

(a) For sub-question (i), inspect the given differential equation:

$$\left(\frac{d^3y}{dx^3}\right)^2 - 3\frac{d^2y}{dx^2} + y = 0$$

(b) Identify the individual derivative operators present. The terms include a third derivative and a second derivative. The highest order of differentiation found is three, meaning the order of the equation is 3.

(c) For sub-question (ii), calculate the scalar dot product of $\vec{a}$ and $\vec{b}$:

$$\vec{a}\cdot\vec{b} = (2)(1) + (-1)(1) + (1)(2) = 2 - 1 + 2 = 3$$

(d) Compute the absolute magnitude of the first vector $\vec{a}$:

$$|\vec{a}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$$

(e) Compute the absolute magnitude of the second vector $\vec{b}$:

$$|\vec{b}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{1+1+4} = \sqrt{6}$$

(f) Substitute these values into the directional cosine equation:

$$\cos\theta = \frac{3}{\sqrt{6}\sqrt{6}} = \frac{3}{6} = \frac{1}{2} \implies \theta = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$$

Final Answer: (i) 3, (ii) 60°

Go Back to Question 22

Q23.

Solution

Concept: Part (i) focuses on checking the mathematical validity of an equivalence relation on a specific discrete set. For a relation to be an equivalence relation, it must satisfy the reflexive, symmetric, and transitive properties. Part (ii) examines the structural validity of the vector Cauchy-Schwarz inequality, which states that for any two vectors, the absolute value of their dot product is less than or equal to the product of their magnitudes: $|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$.

Solution:

(a) For sub-question (i), inspect the given set relation elements: $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$.

(b) Test for the reflexive condition: Since $(1, 1)$, $(2, 2)$, and $(3, 3)$ are all present, every element maps to itself.

(c) Test for the symmetric condition: The paired elements $(1, 2)$ and $(2, 1)$ are symmetric counterparts, satisfying this property.

(d) Test for the transitive condition: The combination of elements $(1, 2)$ and $(2, 1)$ requires the inclusion of $(1, 1)$, which is present. Similarly, combining $(2, 1)$ and $(1, 2)$ requires $(2, 2)$, which is also present. All checks are satisfied, so the statement is TRUE.

(e) For sub-question (ii), examine the vector dot product formula: $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$.

(f) Taking the absolute value yields $|\vec{a} \cdot \vec{b}| = |\vec{a}||\vec{b}|| \cos \theta|$. Since the absolute value of the cosine function satisfies $| \cos \theta| \leq 1$, it follows that $|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$. Thus, the inequality is TRUE.

Final Answer: (i) TRUE, (ii) TRUE

Go Back to Question 23

Q24.

Solution

Concept: This problem involves constructing logical negations for quantified mathematical statements. In formal logic, negating a universal quantifier statement ("All X satisfy Y") produces an existential quantifier statement stating that "There exists at least one X that does not satisfy Y". Conversely, negating an existential statement ("There exists an X such that Y") produces a universal statement stating that "Every X does not satisfy Y".

Solution:

(a) For sub-question (i), analyze the given universal statement: "All polynomial functions are continuous on the real line."

(b) Identify the quantifier "All". Its negation shifts the focus to an existential counterexample.

(c) Apply the negation rule to state that at least one case fails the condition: "There exists at least one polynomial function that is not continuous on the real line." (Alternatively: "Some polynomial functions are not continuous on the real line.")

(d) For sub-question (ii), analyze the given existential statement: "There exists a triangle that has four sides."

(e) Identify the existential quantifier "There exists". Its negation must apply a universal denial across the entire domain.

(f) Invert the core statement to assert that no such case can exist: "Every triangle does not have four sides." (Alternatively: "No triangle has four sides.")

Final Answer: (i) Some polynomials are not continuous, (ii) No triangle has 4 sides

Go Back to Question 24

Q25.

Solution

Concept: This question reviews four fundamental limit properties in calculus: the trigonometric limit identity $\lim_{x\to 0} \frac{\sin mx}{x} = m$, the rational infinity limit computed by comparing the coefficients of the highest-degree terms, the exponential limit form $\lim_{x\to 0}(1+ax)^{1/x} = e^a$, and the algebraic power limit limit rule $\lim_{x\to 1} \frac{x^n-1}{x-1} = n$.

Solution:

(a) For sub-question (i), apply the trigonometric limit coefficient formula directly to the given expression:

$$\lim_{x\to 0} \frac{\sin 7x}{x} = 7$$

(b) For sub-question (ii), evaluate the limit as x approaches infinity by dividing both the numerator and the denominator by the highest power of x, which is x^1:

$$\lim_{x\to\infty} \frac{5x-3}{2x+1} = \lim_{x\to\infty} \frac{5-\frac{3}{x}}{2+\frac{1}{x}} = \frac{5-0}{2+0} = \frac{5}{2}$$

(c) For sub-question (iii), evaluate the indeterminate form 1^∞. Apply the standard exponential limit rule with parameter value $a = 3$:

$$\lim_{x\to 0}(1+3x)^{1/x} = e^3$$

(d) For sub-question (iv), factor the numerator or use the algebraic limit identity with exponent power $n = 3$:

$$\lim_{x\to 1} \frac{x^3-1}{x-1} = 3\cdot(1)^{3-1} = 3$$

Final Answer: (i) 7, (ii) $5/2$, (iii) e^3, (iv) 3

Go Back to Question 25

Q26.

Solution

Concept: This question covers properties of conic sections. Part (i) uses the general circle equation $x^2 + y^2 + 2gx + 2fy + c = 0$ with center $(-g, -f)$. Part (ii) uses the ellipse equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with eccentricity $e = \sqrt{1 - b^2/a^2}$. Part (iii) uses the parabola equation $y^2 = 4ax$ with focus $(a, 0)$. Part (iv) uses the hyperbola equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where the transverse axis length equals $2a$.

Solution:

(a) For sub-question (i), extract parameters from the given circle equation $x^2 + y^2 + 2x - 4y = 0$:

$$2g = 2 \implies g = 1, \quad 2f = -4 \implies f = -2 \implies \text{Center} = (-1, 2)$$

(b) For sub-question (ii), find parameters from the ellipse equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$:

$$a^2 = 9 \implies a = 3, \quad b^2 = 4 \implies b = 2 \implies e = \sqrt{1 - \frac{4}{9}} = \frac{\sqrt{5}}{3}$$

(c) For sub-question (iii), compare the given parabola $y^2 = 20x$ to the standard form:

$$4a = 20 \implies a = 5 \implies \text{Focus} = (5, 0)$$

(d) For sub-question (iv), analyze the given hyperbola equation $\frac{x^2}{9} - \frac{y^2}{16} = 1$:

$$a^2 = 9 \implies a = 3 \implies \text{Transverse Axis Length} = 2a = 2(3) = 6$$

Final Answer: (i) $(-1, 2)$, (ii) $\sqrt{5}/3$, (iii) $(5, 0)$, (iv) 6

Go Back to Question 26

Q27.

Solution

Concept: This problem tests core calculus, matrix, and geometric properties. Part (i) evaluates the distributivity of determinants. Part (ii) checks King's definite integral property. Part (iii) checks if a function is increasing by checking if its derivative satisfies $f'(x) \geq 0$. Part (iv) determines if two planes are parallel by verifying if their normal vectors are scalar multiples of each other.

Solution:

(a) For statement (i), examine the determinant relation: $\det(A + B) = \det(A) + \det(B)$. This relation does not hold for general square matrices, so the statement is FALSE.

(b) For statement (ii), consider the integral identity $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$. This is a fundamental property of definite integrals, making the statement TRUE.

(c) For statement (iii), differentiate the given function $f(x) = x^3 + 5$:

$$f'(x) = 3x^2$$

Since $3x^2 \geq 0$ for all real values of x, the function is monotonically increasing everywhere, making the statement FALSE.

(d) For statement (iv), extract the normal vectors from the two plane equations:

$$\vec{n}_1 = 2\hat{i} - \hat{j} + \hat{k}, \quad \vec{n}_2 = 4\hat{i} - 2\hat{j} + 2\hat{k}$$

Since $\vec{n}_2 = 2\vec{n}_1$, the normal vectors are parallel, which means the planes are parallel. The statement is TRUE.

Final Answer: (i) FALSE, (ii) TRUE, (iii) FALSE, (iv) TRUE

Go Back to Question 27

Q28.

Solution

Concept: The area enclosed between two intersecting curves over a shared interval is computed using definite integration: Area = $\int_a^b [f_{\text{upper}}(x) - f_{\text{lower}}(x)]\,dx$. To set up the integral, find the boundaries by determining the intersection points and determine which curve lies above the other throughout the integration interval.

Solution:

(a) For sub-question (i), find the intersection points by setting the two functions equal to each other:

$$\sqrt{x} = x^3 \implies x = x^6 \implies x(x^5 - 1) = 0 \implies x = 0 \quad \text{and} \quad x = 1$$

(b) For sub-question (ii), choose a test value inside the interval, such as $x = 0.25$, to determine the upper function:

$$\sqrt{0.25} = 0.5, \quad (0.25)^3 = 0.015625$$

Since $0.5 > 0.015625$, the upper bounding function is $f(x) = \sqrt{x}$.

(c) For sub-question (iii), set up the definite integral required to calculate the area between the curves from 0 to 1:

$$\text{Area} = \int_0^1 \left(\sqrt{x} - x^3\right) dx$$

(d) For sub-question (iv), evaluate the definite integral by finding the antiderivatives:

$$\text{Area} = \left[\frac{2}{3}x^{3/2} - \frac{1}{4}x^4\right]_0^1 = \left(\frac{2}{3} - \frac{1}{4}\right) - 0 = \frac{8-3}{12} = \frac{5}{12}$$

Final Answer: (i) 0, 1, (ii) $\sqrt{x}$, (iii) $\int_0^1 (\sqrt{x} - x^3)dx$, (iv) $5/12$

Go Back to Question 28

Q29.

Solution

Concept: This linear programming problem involves constructing inequalities from production resource constraints and finding the optimal solution. The constraints bound a feasible region, and the maximum value of the linear profit objective function occurs at one of the corner points of this region.

Solution:

(a) For sub-question (i), the material constraints require 2 units for item X and 4 units for item Y from a total pool of 100 units: $2x + 4y \leq 100$. This corresponds to option (A).

(b) For sub-question (ii), the labor hours require 3 hours for item X and 2 hours for item Y from a maximum pool of 90 hours: $3x + 2y \leq 90$. This corresponds to option (B).

(c) For sub-question (iii), combine the individual profits of Rs. 40 and Rs. 50 into the total profit objective function: $Z = 40x + 50y$. This corresponds to option (C).

(d) For sub-question (iv), find the intersection of the boundary lines by solving the system:

$$2x + 4y = 100 \implies x + 2y = 50, \quad 3x + 2y = 90$$

Subtracting the equations gives $2x = 40 \implies x = 20$, which yields $y = 15$. The point is $(20, 15)$, corresponding to option (A).

(e) For sub-question (v), evaluate the objective function at vertex $(30, 0)$:

$$Z = 40(30) + 50(0) = 1200 \implies \text{Option (A)}$$

(f) For sub-question (vi), evaluate Z at the remaining corner point $(0, 25)$: $Z = 40(0)+50(25) = 1250$. Comparing all vertices, the maximum profit is 1550 at $(20, 15)$, corresponding to option (C).

Final Answer: (i) A, (ii) B, (iii) C, (iv) A, (v) A, (vi) C

Go Back to Question 29

Q30.

Solution

Concept: Evaluating this trigonometric limit requires applying the double-angle identity for the cosine function to simplify the numerator, which states that $1 - \cos 2x = 2\sin^2 x$. After rewriting the expression, apply the fundamental limit theorem $\lim_{x\to 0} \frac{\sin x}{x} = 1$ to find the final scalar value.

Solution:

(a) State the limit continuity equation provided in the problem statement:

$$\lim_{x\to 0} \frac{1 - \cos 2x}{x^2} = k$$

(b) Substitute the trigonometric double-angle identity $1 - \cos 2x = 2\sin^2 x$ into the numerator:

$$\lim_{x\to 0} \frac{2\sin^2 x}{x^2} = k$$

(c) Factor out the constant coefficient and group the variables into a single squared fraction term:

$$2 \cdot \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 = k$$

(d) Apply the standard trigonometric limit property inside the squared term:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

(e) Substitute this value back into the equation to compute the final value for k:

$$2 \cdot (1)^2 = k \implies k = 2$$

Final Answer: $k = 2$

Go Back to Question 30

Q31.

Solution

Concept: Matrix polynomial equations can be verified through explicit matrix multiplication and linear combination steps. For a square matrix A, the expression A^2 represents the matrix product $A \cdot A$. Substituting this result alongside the scalar multiples of the identity matrix I and the original matrix A allows us to check if the total combination reduces to the zero matrix 0.

Solution:

(a) Write down the components of the given second-order square matrix A:

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

(b) Compute the matrix product A^2 by multiplying rows by columns systematically:

$$A^2 = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

(c) Evaluate each cell of the product matrix:

$$\text{Cell}_{11} = 2(2) + 3(1) = 7, \quad \text{Cell}_{12} = 2(3) + 3(2) = 12$$

$$\text{Cell}_{21} = 1(2) + 2(1) = 4, \quad \text{Cell}_{22} = 1(3) + 2(2) = 7$$

$$A^2 = \begin{pmatrix} 7 & 12 \\ 4 & 7 \end{pmatrix}$$

(d) Scale the matrix A by the scalar factor 4:

$$4A = \begin{pmatrix} 8 & 12 \\ 4 & 8 \end{pmatrix}$$

(e) Substitute these values into the polynomial expression $A^2 - 4A + I$:

$$A^2 - 4A + I = \begin{pmatrix} 7 & 12 \\ 4 & 7 \end{pmatrix} - \begin{pmatrix} 8 & 12 \\ 4 & 8 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(f) Combine the corresponding individual entries to yield the zero matrix:

$$\begin{pmatrix} 7 - 8 + 1 & 12 - 12 + 0 \\ 4 - 4 + 0 & 7 - 8 + 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

Final Answer: Shown

Go Back to Question 31

Q32.

Solution

Concept: Differentiating a composite function requires the application of the calculus chain rule. For a nested functional expression of the form $y = f(g(h(x)))$, the derivative is computed by multiplying the derivatives of each successive nested layer from the outside inward: $\frac{dy}{dx} = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$.

Solution:

(a) State the core compound function provided in the problem text:

$$y = \log\left(\sin\sqrt{x}\right)$$

(b) Differentiate the outermost layer, which is the logarithmic function $\log(u)$ where $u = \sin\sqrt{x}$:

$$\frac{d}{du}(\log u) = \frac{1}{u} \implies \frac{1}{\sin\sqrt{x}}$$

(c) Move to the intermediate functional layer, which is the sine function $\sin(v)$ where $v = \sqrt{x}$:

$$\frac{d}{dv}(\sin v) = \cos v \implies \cos\sqrt{x}$$

(d) Differentiate the innermost layer, which is the radical algebraic function $\sqrt{x} = x^{1/2}$:

$$\frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

(e) Chain all three structural pieces together via scalar multiplication:

$$\frac{dy}{dx} = \frac{1}{\sin\sqrt{x}} \cdot \cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}}$$

(f) Simplify the expression using the cotangent identity $\cot\theta = \frac{\cos\theta}{\sin\theta}$:

$$\frac{dy}{dx} = \frac{\cot\sqrt{x}}{2\sqrt{x}}$$

Final Answer: $\cot\sqrt{x}/(2\sqrt{x})$

Go Back to Question 32

Q33.

Solution

Concept: A special integration identity states that the integral of an exponential function multiplied by the sum of a function and its derivative evaluates directly to the exponential function multiplied by that function. This is written as $\int e^x [f(x) + f'(x)] \, dx = e^x f(x) + C$.

Solution:

(a) Write down the indefinite integral expression provided in the prompt:

$$I = \int e^x (\sin x + \cos x) \, dx$$

(b) Identify the core components inside the integrand to see if they fit the special integral form. Let the primary trigonometric function be:

$$f(x) = \sin x$$

(c) Differentiate this function with respect to x using standard derivative rules:

$$f'(x) = \frac{d}{dx}(\sin x) = \cos x$$

(d) Observe that the integrand perfectly matches the structural layout of the identity:

$$\int e^x [f(x) + f'(x)] \, dx = \int e^x (\sin x + \cos x) \, dx$$

(e) Apply the integration property directly to write down the final expression without needing integration by parts:

$$I = e^x \cdot f(x) + C \implies I = e^x \sin x + C$$

where C represents the arbitrary constant of integration.

Final Answer: $e^x \sin x + C$

Go Back to Question 33

Q34.

Solution

Concept: The absolute scalar magnitude of a three-dimensional vector $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$ is computed using the spatial Pythagorean formula $|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$. Alternatively, a vector with a specific targeted length can be constructed by finding the unit vector and multiplying it by the desired scalar length factor.

Solution:

(a) For the primary question, isolate the scalar components of the given vector $\vec{a} = 2\hat{i} + 3\hat{j} - 6\hat{k}$:

$$x = 2, \quad y = 3, \quad z = -6$$

(b) Substitute these coordinate values into the vector length radical formula:

$$|\vec{a}| = \sqrt{(2)^2 + (3)^2 + (-6)^2}$$

(c) Evaluate the individual squared integers inside the radical:

$$|\vec{a}| = \sqrt{4 + 9 + 36} = \sqrt{49}$$

(d) Take the positive square root to find the total absolute magnitude:

$$|\vec{a}| = 7$$

(e) For the alternate question variant, find the unit vector of $\hat{i} + 2\hat{j} - 2\hat{k}$ by dividing by its magnitude $\sqrt{1 + 4 + 4} = 3$:

$$\hat{u} = \frac{\hat{i} + 2\hat{j} - 2\hat{k}}{3}$$

(f) Scale this unit vector by a factor of 5 to obtain the final vector:

$$\vec{v} = 5\hat{u} = \frac{5}{3}\hat{i} + \frac{10}{3}\hat{j} - \frac{10}{3}\hat{k}$$

Final Answer: 7

Go Back to Question 34

Q35.

Solution

Concept: The vector equation of a flat plane is defined by the dot product form $\vec{r} \cdot \vec{n} = d$, where $\vec{n}$ represents the perpendicular normal vector and d is a constant scalar. To transform this equation into Cartesian standard form, substitute the general position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ into the dot product.

Solution:

(a) State the vector expression for the plane as provided in the problem text:

$$\vec{r} \cdot \left(2\hat{i} - \hat{j} + 3\hat{k}\right) = 5$$

(b) Identify the components of the normal vector $\vec{n}$ that is perpendicular to the plane:

$$\vec{n} = 2\hat{i} - \hat{j} + 3\hat{k}$$

(c) Recall the definition of the three-dimensional spatial position vector $\vec{r}$:

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

(d) Substitute this position vector expression back into the dot product equation:

$$\left(x\hat{i} + y\hat{j} + z\hat{k}\right) \cdot \left(2\hat{i} - \hat{j} + 3\hat{k}\right) = 5$$

(e) Evaluate the scalar dot product by multiplying the corresponding coefficients:

$$(x)(2) + (y)(-1) + (z)(3) = 5$$

(f) Simplify the resulting linear algebraic expression into standard Cartesian form:

$$2x - y + 3z = 5$$

Final Answer: $2x - y + 3z = 5$

Go Back to Question 35

Q36.

Solution

Concept: Evaluating a basic definite integral involves finding the antiderivative function and applying the fundamental theorem of calculus. The integrand expression $\frac{1}{\sqrt{1-x^2}}$ corresponds directly to the standard derivative of the inverse sine function, meaning its definitive integration yields $\sin^{-1} x$.

Solution:

(a) Write down the definite integral expression defined over the limits 0 to 1:

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

(b) Identify the standard trigonometric antiderivative for this algebraic expression:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x$$

(c) Set up the notation to evaluate the antiderivative over the integration boundaries:

$$I = \left[\sin^{-1} x\right]_0^1$$

(d) Apply the boundaries by subtracting the value at the lower limit from the value at the upper limit:

$$I = \sin^{-1}(1) - \sin^{-1}(0)$$

(e) Find the principal angles for both inverse trigonometric values:

$$\sin^{-1}(1) = \frac{\pi}{2}, \quad \sin^{-1}(0) = 0$$

(f) Combine these values to determine the final evaluated definite integral result:

$$I = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Final Answer: $\pi/2$

Go Back to Question 36

Q37.

Solution

Concept: Direction cosines (DCs) describe the angles a line makes with the coordinate axes. They are computed by taking the individual directional ratios (DRs) of the vector and dividing each by the total vector magnitude, which is given by the three-dimensional Pythagorean norm Length $= \sqrt{a^2 + b^2 + c^2}$.

Solution:

(a) Extract the direction ratios from the given vector components from the problem:

$$\text{DRs} = \langle 2, -1, 2 \rangle$$

Let these parameters be labeled as $a = 2$, $b = -1$, and $c = 2$.

(b) Calculate the absolute geometric magnitude or norm of these combined parameters:

$$\text{Magnitude} = \sqrt{a^2 + b^2 + c^2} = \sqrt{(2)^2 + (-1)^2 + (2)^2}$$

(c) Evaluate the squares of the numbers inside the radical sign:

$$\text{Magnitude} = \sqrt{4 + 1 + 4} = \sqrt{9}$$

(d) Take the positive square root to find the total scaling denominator:

$$\text{Magnitude} = 3$$

(e) Divide each direction ratio by this magnitude to find the individual direction cosines:

$$l = \frac{a}{3} = \frac{2}{3}, \quad m = \frac{b}{3} = -\frac{1}{3}, \quad n = \frac{c}{3} = \frac{2}{3}$$

(f) Combine these values into a single coordinate triplet representation:

$$\text{Direction Cosines} = \left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle$$

Final Answer: $\langle 2/3, -1/3, 2/3 \rangle$

Go Back to Question 37

Q38.

Solution

Concept: The geometric area of a parallelogram formed by adjacent sides defined by vectors $\vec{a}$ and $\vec{b}$ is equal to the absolute scalar magnitude of their vector cross product, expressed as Area $= |\vec{a} \times \vec{b}|$. This cross product is calculated by evaluating a third-order determinant using the unit basis vectors.

Solution:

(a) State the two adjacent boundary vectors given in the problem text:

$$\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}, \quad \vec{b} = \hat{i} - \hat{j}$$

(b) Set up the cross product calculation inside a matrix determinant structure:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{vmatrix}$$

(c) Expand the determinant along the top row containing the basis vectors:

$$\vec{a} \times \vec{b} = \hat{i}\,((1)(0) - (3)(-1)) - \hat{j}\,((2)(0) - (3)(1)) + \hat{k}\,((2)(-1) - (1)(1))$$

(d) Simplify each vector component independently:

$$\vec{a} \times \vec{b} = \hat{i}(0 + 3) - \hat{j}(0 - 3) + \hat{k}(-2 - 1) = 3\hat{i} + 3\hat{j} - 3\hat{k}$$

(e) Calculate the absolute scalar magnitude of this resulting cross product vector:

$$\text{Area} = |3\hat{i} + 3\hat{j} - 3\hat{k}| = \sqrt{(3)^2 + (3)^2 + (-3)^2}$$

(f) Evaluate the square roots to find the final area value:

$$\text{Area} = \sqrt{9 + 9 + 9} = \sqrt{27} = 3\sqrt{3}$$

Final Answer: $3\sqrt{3}$

Go Back to Question 38

Q39.

Solution

Concept: According to the Cayley-Hamilton theorem, every square matrix satisfies its own characteristic equation. For a third-order square matrix, this relationship can be verified by computing the matrix powers A^2 and A^3, and then combining them into the matrix polynomial equation. Once verified, this equation can be rearranged to find the inverse matrix A^{-1}.

Solution:

(a) State the given third-order square matrix A from the problem statement.

(b) Compute the square of the matrix, A^2, by performing matrix multiplication with itself:

$$A^2 = A \cdot A$$

(c) Compute the cube of the matrix, A^3, by multiplying the original matrix by the squared result:

$$A^3 = A \cdot A^2$$

(d) Substitute these calculated matrix powers into the specific characteristic equation:

$$A^3 - 23A - 40I = 0$$

Evaluating this expression confirms that it reduces to the zero matrix.

(e) To isolate the inverse matrix, multiply the entire verified matrix equation by A^{-1}:

$$A^{-1}(A^3 - 23A - 40I) = 0 \implies A^2 - 23I - 40A^{-1} = 0$$

(f) Rearrange the terms to solve for the inverse matrix A^{-1}:

$$40A^{-1} = A^2 - 23I \implies A^{-1} = \frac{1}{40}\left(A^2 - 23I\right)$$

Final Answer: $A^{-1} = \frac{1}{40}(A^2 - 23I)$

Go Back to Question 39

Q40.

Solution

Concept: Summing multiple inverse tangent terms requires the iterative application of the standard arctangent addition formula: $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$. This formula is valid provided that the product of the arguments satisfies the condition $xy < 1$.

Solution:

(a) State the full multi-term expression that needs to be simplified:

$$E = \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{8}\right)$$

(b) Group the first two terms together and apply the arctangent addition formula:

$$\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{3}\cdot\frac{1}{5}}\right) = \tan^{-1}\left(\frac{8/15}{14/15}\right) = \tan^{-1}\left(\frac{4}{7}\right)$$

(c) Group the remaining two terms together and apply the addition formula:

$$\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\cdot\frac{1}{8}}\right) = \tan^{-1}\left(\frac{15/56}{55/56}\right) = \tan^{-1}\left(\frac{3}{11}\right)$$

(d) Combine the two resulting intermediate terms using the addition formula once more:

$$E = \tan^{-1}\left(\frac{4}{7}\right) + \tan^{-1}\left(\frac{3}{11}\right) = \tan^{-1}\left(\frac{\frac{4}{7}+\frac{3}{11}}{1-\frac{4}{7}\cdot\frac{3}{11}}\right)$$

(e) Simplify the fraction inside the argument:

$$E = \tan^{-1}\left(\frac{\frac{44+21}{77}}{\frac{77-12}{77}}\right) = \tan^{-1}\left(\frac{65/77}{65/77}\right) = \tan^{-1}(1)$$

(f) Evaluate the final angle value, noting that $\tan(\pi/4) = 1$:

$$E = \frac{\pi}{4}$$

Final Answer: $\pi/4$

Go Back to Question 40

Q41.

Solution

Concept: The shortest distance d between two infinitely long skew lines in a three-dimensional Cartesian space is given by the orthogonal projection of the displacement vector connecting their initial points onto their common perpendicular vector. This relationship is written in vector notation as $d = \frac{|(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)|}{|\vec{b}_1\times\vec{b}_2|}$, where vectors $\vec{a}_1$ and $\vec{a}_2$ represent positional coordinates on the lines, and vectors $\vec{b}_1$ and $\vec{b}_2$ represent their respective spatial direction vectors.

Solution:

(a) Extract the initial positional points and direction component configurations from the two vector line equations:

$$\vec{a}_1 = (1, 2, 1), \quad \vec{b}_1 = (1, -1, 1)$$

$$\vec{a}_2 = (2, -1, -1), \quad \vec{b}_2 = (2, 1, 2)$$

(b) Compute the difference vector connecting the reference points on both lines:

$$\vec{a}_2 - \vec{a}_1 = (2-1, -1-2, -1-1) = (1, -3, -2)$$

(c) Calculate the common perpendicular direction using the vector cross product rule:

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$\vec{b}_1 \times \vec{b}_2 = \hat{i}(-2-1) - \hat{j}(2-2) + \hat{k}(1-(-2)) = -3\hat{i} + 0\hat{j} + 3\hat{k}$$

(d) Evaluate the absolute scalar magnitude of the cross product vector:

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{(-3)^2 + 0^2 + 3^2} = \sqrt{9+0+9} = \sqrt{18} = 3\sqrt{2}$$

(e) Calculate the triple scalar product in the numerator using a dot product operation:

$$(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = 1(-3) + (-3)(0) + (-2)(3) = -3 - 0 - 6 = -9$$

(f) Substitute these values into the shortest distance definition to find the numeric value:

$$d = \frac{|-9|}{3\sqrt{2}} = \frac{9}{3\sqrt{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

Final Answer: $\frac{3\sqrt{2}}{2}$

Go Back to Question 41

Q42.

Solution

Concept: For an algebraic expression of the form $\frac{P(x)}{(x-a)^2(x-b)}$, the corresponding decompounded template must account for both the single and squared instances of the repeated factor alongside the remaining linear root: $\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$.

Solution:

(a) Write down the proper rational integrand from the problem text:

$$\frac{3x+2}{(x+1)^2(x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3}$$

(b) Clear the fractional denominators by multiplying both sides by $(x+1)^2(x+3)$ to form the basic polynomial identity:

$$3x + 2 = A(x+1)(x+3) + B(x+3) + C(x+1)^2$$

(c) Substitute the zero-generating root $x = -1$ into the identity to isolate the parameter B:

$$3(-1) + 2 = B(-1+3) \implies -1 = 2B \implies B = -\frac{1}{2}$$

(d) Substitute a simple reference value $x = 0$ along with the calculated values of B and C to find A:

$$2 = 3A + 3B + C \implies 2 = 3A + 3\left(-\frac{1}{2}\right) + \left(-\frac{7}{4}\right)$$

$$2 = 3A - \frac{6}{4} - \frac{7}{4} \implies 2 = 3A - \frac{13}{4} \implies 3A = \frac{21}{4} \implies A = \frac{7}{4}$$

(e) Integrate each term independently, applying logarithmic rules and power integration laws:

$$I = \frac{7}{4}\log|x+1| + \frac{1}{2(x+1)} - \frac{7}{4}\log|x+3| + C$$

(f) Group the log terms together using the quotient property $\log u - \log v = \log(u/v)$:

$$I = \frac{7}{4}\log\left|\frac{x+1}{x+3}\right| + \frac{1}{2(x+1)} + C$$

Final Answer: $\frac{7}{4}\log\left|\frac{x+1}{x+3}\right| + \frac{1}{2(x+1)} + C$

Go Back to Question 42

Q43.

Solution

Concept: A Linear Programming Problem (LPP) can be solved geometrically by plotting the linear inequalities on a Cartesian coordinate plane to map out the shared feasible region. According to the corner point theorem, the maximum or minimum value of a linear objective profit function $Z = ax + by$ always occurs at one of the vertices (extreme points) of this convex bounded feasible region.

Solution:

(a) Graph the boundary lines for the given structural constraints by finding their coordinate axis intercepts:

$$x + y = 50 \implies (50, 0) \text{ and } (0, 50)$$

$$3x + y = 90 \implies (30, 0) \text{ and } (0, 90)$$

(b) Incorporate the non-negativity parameters $x \geq 0$ and $y \geq 0$ to restrict the feasible space to the first quadrant.

(c) Determine the intersection vertex of the two boundary lines by solving them as a simultaneous linear system:

$$\text{Subtracting } (x + y = 50) \text{ from } (3x + y = 90) \implies 2x = 40 \implies x = 20$$

$$\text{Substitute } x = 20 \text{ back into the first line} \implies 20 + y = 50 \implies y = 30$$

The intersection coordinate is $(20, 30)$.

(d) Identify the four corner points that enclose the shaded feasible region:

$$\text{Vertices} = \{(0, 0), (30, 0), (20, 30), (0, 50)\}$$

(e) Evaluate the objective function $Z = 4x + y$ at each of these boundary corner points:

$$\text{At } (0, 0): \quad Z = 4(0) + 0 = 0$$

$$\text{At } (30, 0): \quad Z = 4(30) + 0 = 120$$

$$\text{At } (20, 30): \quad Z = 4(20) + 30 = 110$$

$$\text{At } (0, 50): \quad Z = 4(0) + 50 = 50$$

(f) Compare the computed values to identify the absolute maximum profit, which is 120 at the coordinate $(30, 0)$.

Final Answer: Max $= 120$ at $x = 30, y = 0$

Go Back to Question 43

Q44.

Solution

Concept: The total area enclosed between a curve $y = f(x)$ and the horizontal x-axis over a specific interval $[a, b]$ is calculated using the definite integral of the absolute value of the function, written as Area $= \int_a^b |f(x)|\, dx$. If the function changes signs within the interval, split the integral at the zero-crossing points and negate the sections where the curve drops below the axis to keep the area values positive.

Solution:

(a) Set up the total definite integral expression for the given function across the interval from 0 to π:

$$\text{Area} = \int_0^{\pi} |\cos x|\, dx$$

(b) Analyze the sign behavior of the cosine function across the quadrants in the integration interval:

$$\cos x \geq 0 \quad \text{for} \quad x \in \left[0, \frac{\pi}{2}\right] \quad \text{(Quadrant I)}$$

$$\cos x \leq 0 \quad \text{for} \quad x \in \left[\frac{\pi}{2}, \pi\right] \quad \text{(Quadrant II)}$$

(c) Split the absolute integral into two independent definite integration steps at the zero-crossing value $x = \frac{\pi}{2}$:

$$\text{Area} = \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} (-\cos x)\, dx$$

(d) Evaluate the first definite integral using the antiderivative function $\sin x$:

$$\int_0^{\pi/2} \cos x\, dx = [\sin x]_0^{\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1 - 0 = 1$$

(e) Evaluate the second definite integral, applying the negative sign to keep the area value positive:

$$\int_{\pi/2}^{\pi} (-\cos x)\, dx = [-\sin x]_{\pi/2}^{\pi} = -\sin(\pi) - \left(-\sin\left(\frac{\pi}{2}\right)\right) = 0 + 1 = 1$$

(f) Sum the areas of both sections to compute the total area:

$$\text{Area} = 1 + 1 = 2 \text{ square units}$$

Final Answer: 2 sq. units

Go Back to Question 44

Q45.

Solution

Concept: Optimizing dimensions to maximize geometric volume under a fixed surface area constraint requires applying the first derivative test. For an open rectangular box with a square base of side length x and a height h, the total surface area is $S = x^2 + 4xh = c^2$. By rearranging this constraint, the height can be expressed in terms of x, allowing the volume formula $V = x^2h$ to be written as a single-variable function that can be differentiated.

Solution:

(a) Write down the surface area equation for an open box with a square base, where c^2 represents the fixed area:

$$x^2 + 4xh = c^2$$

(b) Rearrange this constraint equation to express the height variable h as a function of the base side x:

$$4xh = c^2 - x^2 \implies h = \frac{c^2 - x^2}{4x}$$

(c) Substitute this height expression into the standard geometric volume formula for a rectangular box:

$$V = x^2h = x^2 \cdot \left(\frac{c^2 - x^2}{4x}\right) = \frac{x(c^2 - x^2)}{4} = \frac{c^2x - x^3}{4}$$

(d) Differentiate the single-variable volume function with respect to x to find its critical points:

$$\frac{dV}{dx} = \frac{1}{4}\frac{d}{dx}\left(c^2x - x^3\right) = \frac{c^2 - 3x^2}{4}$$

(e) Find the critical point by setting this first derivative equal to zero:

$$\frac{c^2 - 3x^2}{4} = 0 \implies c^2 - 3x^2 = 0 \implies 3x^2 = c^2 \implies x = \frac{c}{\sqrt{3}}$$

(f) Substitute this optimal base dimension x back into the height formula:

$$h = \frac{c^2 - \left(\frac{c}{\sqrt{3}}\right)^2}{4\left(\frac{c}{\sqrt{3}}\right)} = \frac{c^2 - \frac{c^2}{3}}{\frac{4c}{\sqrt{3}}} = \frac{\frac{2c^2}{3}}{\frac{4c}{\sqrt{3}}} = \frac{2c^2}{3} \cdot \frac{\sqrt{3}}{4c} = \frac{c}{2\sqrt{3}}$$

(g) Compute the maximum volume by multiplying the optimized base area and height dimensions together:

$$V_{\text{max}} = x^2h = \left(\frac{c}{\sqrt{3}}\right)^2 \cdot \left(\frac{c}{2\sqrt{3}}\right) = \frac{c^2}{3} \cdot \frac{c}{2\sqrt{3}} = \frac{c^3}{6\sqrt{3}}$$

Final Answer: $V_{\text{max}} = c^3/(6\sqrt{3})$, $x = c/\sqrt{3}$, $h = c/(2\sqrt{3})$

Go Back to Question 45

Answer Key

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	C	2	A	3	A	4	A	5	A
6	A	7	B	8	C	9	A	10	B
11	C	12	A	13	D	14	B	15	A
16	A	17	A	18	A	19	A	20	A