QA CAT 2025 Slot 3 Question Paper with Solution

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1. For a 4-digit number (greater than 1000), sum of the digits in the thousands, hundreds, and tens places is 15. Sum of the digits in the hundreds, tens, and units places is 16. Also, the digit in the tens place is 6 more than the digit in the units place. The difference between the largest and smallest possible value of the number is
(A) 40
(B) 78
(C) 811
(D) 735

Correct Answer: (3) 811

Solution:
Step 1: Define the digits.
Let the 4-digit number be $ABCD$, where A = thousands digit ($A \neq 0$), B = hundreds digit, C = tens digit, D = units digit.

Step 2: Form equations from the conditions.

$$A + B + C = 15 \tag{i}$$
$$B + C + D = 16 \tag{ii}$$
$$C = D + 6 \tag{iii}$$

Step 3: Determine possible values of C and D.
From (iii), $C = D + 6$. Since both are digits (0–9), possible values are:

$$D = 0 \Rightarrow C = 6,$$
$$D = 1 \Rightarrow C = 7,$$
$$D = 2 \Rightarrow C = 8,$$
$$D = 3 \Rightarrow C = 9.$$

Values $D \geq 4$ give $C \geq 10$, not allowed.

Step 4: Test all valid cases.
Case 1: $D = 0, C = 6$

$$B + 6 + 0 = 16 \Rightarrow B = 10 \quad \text{(invalid digit)}.$$

Case 2: $D = 1, C = 7$

$$B + 7 + 1 = 16 \Rightarrow B = 8.$$
$$A + 8 + 7 = 15 \Rightarrow A = 0 \quad \text{(invalid thousands digit)}.$$

Case 3: $D = 2, C = 8$

$$B = 16 - 8 - 2 = 6.$$
$$A = 15 - 6 - 8 = 1.$$

Valid number: 1682.
Case 4: $D = 3, C = 9$

$$B = 16 - 9 - 3 = 4.$$
$$A = 15 - 4 - 9 = 2.$$

Valid number: 2493.

Step 5: Compute required difference.

$$\text{Largest number} = 2493,$$
$$\text{Smallest number} = 1682,$$
$$\text{Difference} = 2493 - 1682 = 811.$$

Quick Tip

When solving digit-based problems, convert the wording into equations and restrict possibilities using the fact that digits are only 0–9. This usually reduces the problem to a few quick checks.

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2. ABCD is a trapezium in which AB is parallel to DC, AD is perpendicular to AB, and $AB = 3DC$. If a circle inscribed in the trapezium touching all the sides has a radius of 3 cm, then the area, in sq. cm, of the trapezium is
(A) 54
(B) $30\sqrt{3}$
(C) 48
(D) $36\sqrt{2}$

Correct Answer: (3) 48

Solution:
Step 1: Analyze the given information.
We have a trapezium $ABCD$ with:

- $AB \parallel DC$,
- $AD \perp AB$ (so AD is the height),
- $AB = 3DC$,
- A circle of radius $r = 3$ cm inscribed in the trapezium, touching all four sides.

Step 2: Determine the height AD.
Since the circle is tangent to both parallel sides AB and DC, the perpendicular distance between them equals the diameter of the circle.

$$\text{Height } h = AD = 2r = 2 \times 3 = 6 \text{ cm}.$$

Step 3: Use the property of a tangential quadrilateral.
A quadrilateral that circumscribes a circle (tangential quadrilateral) satisfies:

$$AB + DC = AD + BC.$$

Let $DC = x$. Then $AB = 3x$. Using the property:

$$3x + x = 6 + BC \Rightarrow 4x = 6 + BC \Rightarrow BC = 4x - 6. \quad (1)$$

Step 4: Apply Pythagoras theorem.
Drop a perpendicular from C to AB, meeting at E. Then:

- $CE = AD = 6$ cm (height),
- $EB = AB - DC = 3x - x = 2x$,
- $CB = BC$ is the hypotenuse.

In right triangle CEB:

$$BC^2 = EB^2 + CE^2 = (2x)^2 + 6^2 = 4x^2 + 36.$$

So,

$$BC = \sqrt{4x^2 + 36}. \tag{2}$$

Step 5: Solve for x.
From (1) and (2):

$$4x - 6 = \sqrt{4x^2 + 36}.$$

Square both sides:

$$\begin{aligned}(4x-6)^2 &= 4x^2 + 36\\ 16x^2 - 48x + 36 &= 4x^2 + 36\\ 16x^2 - 4x^2 - 48x &= 0\\ 12x^2 - 48x &= 0\\ 12x(x-4) &= 0.\end{aligned}$$

Since a side length cannot be zero, $x = 4$. Thus,

$$DC = x = 4 \text{ cm}, \quad AB = 3x = 12 \text{ cm}.$$

Step 6: Find the area of the trapezium.
Area of a trapezium:

$$\text{Area} = \frac{1}{2}(\text{sum of parallel sides}) \times \text{height}.$$

So,

$$\begin{aligned}\text{Area} &= \frac{1}{2}(AB + DC) \times AD\\ &= \frac{1}{2}(12 + 4) \times 6\\ &= \frac{1}{2} \times 16 \times 6\\ &= 8 \times 6 = 48 \text{ sq. cm.}\end{aligned}$$

Hence, the area of the trapezium is 48 sq. cm.

> **Quick Tip**
>
> For any quadrilateral that has an incircle (tangential quadrilateral), the sum of the lengths of opposite sides is equal. Also, if a circle is tangent to two parallel sides, the distance between them equals the diameter of the circle.

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3. Vessels A and B contain 60 litres of alcohol and 60 litres of water, respectively. A certain volume is taken out from A and poured into B. After stirring, the same volume is taken out from B and poured into A. If the resultant ratio of alcohol and water in A is 15 : 4, then the volume, in litres, initially taken out from A is

Solution:
Step 1: Initial state.
Vessel A: 60 L alcohol, 0 L water.
Vessel B: 0 L alcohol, 60 L water.

Step 2: First transfer (from A to B).
Let x litres be taken from A and poured into B.

- Vessel A now has $60 - x$ L alcohol, 0 L water.
- Vessel B now has x L alcohol and 60 L water.

Total volume in B is $60 + x$ L, with ratio of alcohol to water $= x : 60$.

Step 3: Second transfer (from B to A).
Again, x L is taken from B and poured into A.
Fraction of alcohol in B:

$$\frac{x}{60+x}, \quad \text{and fraction of water in B: } \frac{60}{60+x}.$$

So, in the x L taken from B:

$$\text{Alcohol taken} = x \cdot \frac{x}{60+x} = \frac{x^2}{60+x},$$
$$\text{Water taken} = x \cdot \frac{60}{60+x} = \frac{60x}{60+x}.$$

Step 4: Final contents of vessel A.
Alcohol in A:

$$\begin{aligned}\text{Alc}_A &= (60 - x) + \frac{x^2}{60+x} \\ &= \frac{(60-x)(60+x) + x^2}{60+x} \\ &= \frac{3600 - x^2 + x^2}{60+x} = \frac{3600}{60+x}.\end{aligned}$$

Water in A:

$$\text{Wat}_A = \frac{60x}{60+x}.$$

Step 5: Use the given ratio in vessel A.
Given final ratio of alcohol to water in A is 15 : 4:

$$\frac{\text{Alc}_A}{\text{Wat}_A} = \frac{15}{4}.$$

Substitute:

$$\frac{\frac{3600}{60+x}}{\frac{60x}{60+x}} = \frac{15}{4} \Rightarrow \frac{3600}{60x} = \frac{15}{4}.$$

Simplify:

$$\frac{60}{x} = \frac{15}{4} \Rightarrow 15x = 60 \times 4 = 240 \Rightarrow x = \frac{240}{15} = 16.$$

So, the volume initially taken out from A is $\boxed{16}$ litres.

Quick Tip

In transfer-and-mix problems, track the *fraction* of each component after mixing, then multiply by the transferred volume. Ratios often simplify nicely when common denominators cancel.

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4. In a class of 150 students, 75 students chose physics, 111 students chose mathematics and 40 students chose chemistry. All students chose at least one of the three subjects and at least one student chose all three subjects. The number of students who chose both physics and chemistry is equal to the number of students who chose both chemistry and mathematics, and this is half the

number of students who chose both physics and mathematics. The maximum possible number of students who chose physics but not mathematics, is
(A) 30
(B) 55
(C) 35
(D) 40

Correct Answer: (3) 35

Solution:
Step 1: Define the variables.
Let the sets of students who chose Physics, Mathematics, and Chemistry be P, M, and C respectively.

$$\begin{aligned} n(P \cup M \cup C) &= 150, \\ n(P) &= 75, \\ n(M) &= 111, \\ n(C) &= 40. \end{aligned}$$

Let

$$n(P \cap M \cap C) = x.$$

Let the number of students who chose both Physics and Chemistry be y. Then:

$$\begin{aligned} n(P \cap C) &= y, \\ n(C \cap M) &= y, \\ n(P \cap M) &= 2y \quad \text{(given: this is double the others).} \end{aligned}$$

Step 2: Use inclusion–exclusion principle.
For three sets:

$$n(P \cup M \cup C) = n(P) + n(M) + n(C) - [n(P \cap M) + n(M \cap C) + n(C \cap P)] + n(P \cap M \cap C).$$

Substitute known values:

$$\begin{aligned} 150 &= 75 + 111 + 40 - [2y + y + y] + x \\ 150 &= 226 - 4y + x. \end{aligned}$$

Rearrange to express x in terms of y:

$$\begin{aligned} 4y - x &= 226 - 150 = 76 \\ x &= 4y - 76. \end{aligned} \tag{1}$$

Step 3: Express the required quantity.
We want the maximum number of students who chose Physics but not Mathematics, i.e.:

$$n(P \text{ but not } M) = n(P) - n(P \cap M) = 75 - 2y.$$

To maximize $75 - 2y$, we need to *minimize* y, subject to constraints.

Step 4: Apply constraints to find minimum y.
Constraint 1: At least one student chose all three subjects:

$$x \geq 1.$$

Using (1):

$$4y - 76 \geq 1 \Rightarrow 4y \geq 77 \Rightarrow y \geq 19.25.$$

Since y is an integer, $y \geq 20$.

Constraint 2: No region can have negative students. Check "Chemistry only" students.

$$n(C \text{ only}) = n(C) - n(C \cap P) - n(C \cap M) + n(P \cap M \cap C).$$

Substitute:

$$\begin{aligned} n(C \text{ only}) &= 40 - y - y + x \\ &= 40 - 2y + x. \end{aligned}$$

Using $x = 4y - 76$:

$$\begin{aligned} n(C \text{ only}) &= 40 - 2y + (4y - 76) \\ &= 2y - 36. \end{aligned}$$

For this to be non-negative:

$$2y - 36 \geq 0 \Rightarrow y \geq 18.$$

Combining with earlier constraint $y \geq 20$, the minimum feasible y is $y = 20$.

Step 5: Compute the maximum number of students who chose Physics but not Mathematics.

$$n(P \text{ but not } M) = 75 - 2y = 75 - 2 \times 20 = 75 - 40 = 35.$$

Thus, the maximum possible number of students who chose Physics but not Mathematics is 35.

Quick Tip

In set and Venn diagram problems with three sets, translate all conditions into algebraic relations and use the inclusion–exclusion formula. Then apply non-negativity and "at least one" constraints to pin down valid integer values.

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5. In ΔABC, $AB = AC = 12$ cm and D is a point on side BC such that $AD = 8$ cm. If AD is extended to a point E such that $\angle ACB = \angle AEB$, then the length, in cm, of AE is
(A) 18
(B) 16
(C) 20
(D) 14

Correct Answer: (1) 18

Solution:
Step 1: Analyze the given information.
In $\triangle ABC$:

- $AB = AC = 12$ cm $\Rightarrow$ triangle ABC is isosceles.
- D is a point on BC such that $AD = 8$ cm.
- AD is extended to a point E such that $\angle ACB = \angle AEB$.

Step 2: Relate the angles.
Since $AB = AC$, the base angles are equal:

$$\angle ABC = \angle ACB.$$

Given that:

$$\angle ACB = \angle AEB.$$

Hence,

$$\angle ABC = \angle AEB.$$

Note that $\angle ABC$ is the same as $\angle ABD$ (point D lies on BC), so:

$$\angle ABD = \angle AEB.$$

Step 3: Prove triangle similarity.
Consider triangles $\triangle ABD$ and $\triangle AEB$:

- $\angle ABD = \angle AEB$ (from above),
- $\angle BAD = \angle EAB$ (common angle at A).

Thus, by AA similarity:

$$\triangle ABD \sim \triangle AEB.$$

Step 4: Use similarity ratios.
From $\triangle ABD \sim \triangle AEB$, corresponding sides are proportional. Match the vertices: $A \leftrightarrow A$, $B \leftrightarrow B$, $D \leftrightarrow E$. So:

$$\frac{AD}{AB} = \frac{AB}{AE}.$$

Step 5: Calculate AE.
Substitute $AD = 8$ cm and $AB = 12$ cm:

$$\frac{8}{12} = \frac{12}{AE}.$$

Cross-multiply:

$$8 \cdot AE = 12 \cdot 12 = 144,$$
$$AE = \frac{144}{8} = 18 \text{ cm}.$$

Therefore, the required length is $AE = 18$ cm.

Quick Tip

In geometry problems where an angle at an external point equals an internal angle, look for similar triangles formed by extending sides. AA similarity often gives a direct ratio to find unknown lengths.

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6. If $\left(x^2 + \frac{1}{x^2}\right) = 25$ and $x > 0$, then the value of $\left(x^7 + \frac{1}{x^7}\right)$ is
(A) $44859\sqrt{3}$
(B) $44853\sqrt{3}$
(C) $44850\sqrt{3}$
(D) $44856\sqrt{3}$

Correct Answer: (2) $44853\sqrt{3}$

Solution:
Step 1: Find the value of $\left(x + \frac{1}{x}\right)$.
We use the identity:

$$\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2.$$

Given:

$$x^2 + \frac{1}{x^2} = 25,$$

so:

$$\left(x+\frac{1}{x}\right)^2 = 25+2 = 27.$$

Since $x > 0$, the expression is positive:

$$x+\frac{1}{x} = \sqrt{27} = 3\sqrt{3}.$$

Step 2: Find $\left(x^3+\frac{1}{x^3}\right)$.
Use:

$$a^3+b^3 = (a+b)^3 - 3ab(a+b),$$

with $a = x$, $b = \frac{1}{x}$ and $ab = 1$:

$$x^3+\frac{1}{x^3} = \left(x+\frac{1}{x}\right)^3 - 3\left(x\cdot\frac{1}{x}\right)\left(x+\frac{1}{x}\right).$$

Substitute $x+\frac{1}{x} = 3\sqrt{3}$:

$$\begin{aligned} x^3+\frac{1}{x^3} &= (3\sqrt{3})^3 - 3(3\sqrt{3}) \\ &= 27(3\sqrt{3}) - 9\sqrt{3} \\ &= 81\sqrt{3} - 9\sqrt{3} = 72\sqrt{3}. \end{aligned}$$

Step 3: Find $\left(x^4+\frac{1}{x^4}\right)$.
Square $x^2+\frac{1}{x^2}$:

$$\left(x^2+\frac{1}{x^2}\right)^2 = x^4+\frac{1}{x^4}+2.$$

So:

$$\begin{aligned} 25^2 &= x^4+\frac{1}{x^4}+2 \\ 625 &= x^4+\frac{1}{x^4}+2 \\ x^4+\frac{1}{x^4} &= 623. \end{aligned}$$

Step 4: Compute $\left(x^7+\frac{1}{x^7}\right)$.
Consider the product:

$$\left(x^3+\frac{1}{x^3}\right)\left(x^4+\frac{1}{x^4}\right).$$

Expand:

$$\begin{aligned} \left(x^3+\frac{1}{x^3}\right)\left(x^4+\frac{1}{x^4}\right) &= x^7 + x^3\cdot\frac{1}{x^4} + \frac{1}{x^3}\cdot x^4 + \frac{1}{x^7} \\ &= x^7+\frac{1}{x}+x+\frac{1}{x^7}. \end{aligned}$$

Thus,

$$\left(x^3+\frac{1}{x^3}\right)\left(x^4+\frac{1}{x^4}\right) = \left(x^7+\frac{1}{x^7}\right)+\left(x+\frac{1}{x}\right).$$

So,

$$x^7+\frac{1}{x^7} = \left(x^3+\frac{1}{x^3}\right)\left(x^4+\frac{1}{x^4}\right) - \left(x+\frac{1}{x}\right).$$

Substitute known values:

$$\begin{aligned} x^7+\frac{1}{x^7} &= (72\sqrt{3})\cdot 623 - 3\sqrt{3} \\ &= 72\cdot 623\cdot\sqrt{3} - 3\sqrt{3}. \end{aligned}$$

Compute 72×623:

$$72 \times 623 = 44856.$$

Thus:

$$\begin{aligned} x^7 + \frac{1}{x^7} &= 44856\sqrt{3} - 3\sqrt{3} \\ &= (44856 - 3)\sqrt{3} \\ &= 44853\sqrt{3}. \end{aligned}$$

So, the value is $44853\sqrt{3}$.

Quick Tip

For expressions like $x^n + \dfrac{1}{x^n}$, first find $x + \dfrac{1}{x}$ or $x^2 + \dfrac{1}{x^2}$, then build higher powers using algebraic identities and products such as $\left(x^a + \dfrac{1}{x^a}\right)\left(x^b + \dfrac{1}{x^b}\right)$.

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7. Rahul starts on his journey at 5 pm at a constant speed so that he reaches his destination at 11 pm the same day. However, on his way, he stops for 20 minutes, and after that, increases his speed by 3 km per hour to reach on time. If he had stopped for 10 minutes more, he would have had to increase his speed by 5 km per hour to reach on time. His initial speed, in km per hour, was

(A) 20
(B) 15
(C) 12
(D) 18

Correct Answer: (2) 15

Solution:
Step 1: Set up the journey parameters.
Rahul travels from 5 pm to 11 pm, so the total scheduled time is:

$$T = 6 \text{ hours.}$$

Let his initial speed be v km/h. Let the total distance be D. Then:

$$D = v \times 6 = 6v.$$

Let t be the time (in hours) he travels at speed v before stopping.

Step 2: Scenario 1 (20-minute stop, speed increases by 3 km/h).
Stop duration = 20 minutes = $\frac{20}{60} = \frac{1}{3}$ hour. New speed = $v + 3$ km/h.
Time left for the second part:

$$6 - t - \frac{1}{3} = \frac{17}{3} - t.$$

Distance equation:

$$vt + (v+3)\left(\frac{17}{3} - t\right) = 6v.$$

Expand:

$$\begin{aligned} vt + \frac{17v}{3} - vt + 17 - 3t &= 6v \\ \frac{17v}{3} + 17 - 3t &= 6v. \end{aligned}$$

Multiply by 3:

$$\begin{aligned} 17v + 51 - 9t &= 18v \\ 51 - 9t &= v. \end{aligned} \tag{i}$$

Step 3: Scenario 2 (30-minute stop, speed increases by 5 km/h).
Now he stops 10 minutes more, so total stop:

$$30 \text{ minutes} = \frac{1}{2} \text{ hour}.$$

New speed $= v + 5$ km/h. Remaining time:

$$6 - t - \frac{1}{2} = 5.5 - t.$$

Distance equation:

$$vt + (v + 5)(5.5 - t) = 6v.$$

Expand:

$$\begin{aligned} vt + 5.5v - vt + 27.5 - 5t &= 6v \\ 5.5v + 27.5 - 5t &= 6v \\ 27.5 - 5t &= 0.5v. \end{aligned}$$

Multiply by 2:

$$55 - 10t = v. \tag{ii}$$

Step 4: Solve for v and t.
From (i) and (ii):

$$v = 51 - 9t, \quad v = 55 - 10t.$$

Equate:

$$\begin{aligned} 51 - 9t &= 55 - 10t \\ 10t - 9t &= 55 - 51 \\ t &= 4 \text{ hours}. \end{aligned}$$

Substitute $t = 4$ in (i):

$$v = 51 - 9 \cdot 4 = 51 - 36 = 15.$$

So, the initial speed was 15 km/h.

Step 5: Quick verification.
Total distance:

$$D = 6v = 6 \times 15 = 90 \text{ km}.$$

He covers in the first part:

$$4 \times 15 = 60 \text{ km, remaining 30 km}.$$

Scenario 1: Remaining time:

$$6 - 4 - \frac{1}{3} = \frac{5}{3} \text{ hours}.$$

Required speed:

$$\frac{30}{5/3} = 18 \text{ km/h} = 15 + 3 \quad \checkmark$$

Scenario 2: Remaining time:

$$6 - 4 - 0.5 = 1.5 \text{ hours}.$$

Required speed:

$$\frac{30}{1.5} = 20 \text{ km/h} = 15 + 5 \quad \checkmark$$

Everything is consistent.

Quick Tip

When a traveller must arrive "on time" under different stop durations and new speeds, keep the total distance the same and write a distance equation for each scenario. Equating the expressions for speed or distance usually gives a simple system of equations.

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8. The sum of all possible real values of x for which

$$\log_{x-3}(x^2 - 9) = \log_{x-3}(x+1) + 2,$$

is

(A) -3

(B) $\sqrt{33}$

(C) $\dfrac{3+\sqrt{33}}{2}$

(D) 3

Correct Answer: (3) $\dfrac{3+\sqrt{33}}{2}$

Solution:

Step 1: Domain of validity.

For logarithms, the base must be positive and not equal to 1, and the argument must be positive.

Base conditions:

$$x - 3 > 0 \Rightarrow x > 3, \qquad x - 3 \neq 1 \Rightarrow x \neq 4.$$

Argument conditions:

$$x^2 - 9 > 0 \Rightarrow (x-3)(x+3) > 0 \Rightarrow x > 3 \text{ or } x < -3,$$

$$x + 1 > 0 \Rightarrow x > -1.$$

Combining with $x > 3$, the domain is:

$$\boxed{x > 3,\ x \neq 4}.$$

Step 2: Rewrite the equation.

Given:

$$\log_{x-3}(x^2 - 9) = \log_{x-3}(x+1) + 2.$$

Write the constant 2 as a logarithm with the same base:

$$2 = \log_{x-3}((x-3)^2),$$

since

$$\log_{x-3}((x-3)^2) = 2.$$

So:

$$\log_{x-3}(x^2 - 9) = \log_{x-3}(x+1) + \log_{x-3}((x-3)^2).$$

Use $\log_b m + \log_b n = \log_b(mn)$:

$$\log_{x-3}(x^2 - 9) = \log_{x-3}\left((x+1)(x-3)^2\right).$$

Since the base is the same and valid, equate arguments:

$$x^2 - 9 = (x+1)(x-3)^2.$$

Step 3: Simplify the equation.
Factor the left-hand side:

$$(x-3)(x+3) = (x+1)(x-3)^2.$$

Since $x > 3$, we have $x - 3 \neq 0$, so divide both sides by $(x-3)$:

$$x + 3 = (x+1)(x-3).$$

Expand and rearrange:

$$\begin{aligned} x + 3 &= x^2 - 3x + x - 3 \\ x + 3 &= x^2 - 2x - 3 \\ 0 &= x^2 - 2x - 3 - x - 3 \\ 0 &= x^2 - 3x - 6. \end{aligned}$$

So we must solve:

$$x^2 - 3x - 6 = 0.$$

Step 4: Solve the quadratic.
Use the quadratic formula:

$$x = \frac{3 \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot (-6)}}{2} = \frac{3 \pm \sqrt{9+24}}{2} = \frac{3 \pm \sqrt{33}}{2}.$$

So,

$$x_1 = \frac{3+\sqrt{33}}{2}, \qquad x_2 = \frac{3-\sqrt{33}}{2}.$$

Step 5: Check against the domain.
Recall the domain: $x > 3,\ x \neq 4$.

- For $x_2 = \frac{3-\sqrt{33}}{2}$: since $\sqrt{33} > 5$, $3 - \sqrt{33} < 0$, hence $x_2 < 0$, not in the domain (invalid).
- For $x_1 = \frac{3+\sqrt{33}}{2}$: numerically $\sqrt{33} \approx 5.74$, so $x_1 \approx \frac{8.74}{2} \approx 4.37$, which satisfies $x > 3$ and $x \neq 4$ (valid).

Thus, there is only one valid solution:

$$x = \frac{3+\sqrt{33}}{2}.$$

Step 6: Sum of all possible real values.
Since there is only one valid value in the domain, the sum of all possible real values of x is:

$$\boxed{\frac{3+\sqrt{33}}{2}}.$$

Quick Tip

For equations with logarithms, always start by finding the domain (base and arguments). Then, if all logs share a base, use log properties to convert the equation into an algebraic one, and finally discard any roots that fall outside the domain.

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9. The ratio of the number of coins in boxes A and B was 17:7. After 108 coins were shifted from box A to box B, this ratio became 37:20. The number of coins that needs to be shifted further from A to B, to make this ratio 1:1, is

Solution:
Step 1: Define the initial quantities.
Let the number of coins in box A and box B be $17x$ and $7x$ respectively.

$$\text{Initial coins in A} = 17x, \quad \text{Initial coins in B} = 7x.$$

Step 2: After shifting 108 coins from A to B.
New number of coins:

$$\text{A: } 17x - 108, \quad \text{B: } 7x + 108.$$

Given that the new ratio is 37 : 20:

$$\frac{17x - 108}{7x + 108} = \frac{37}{20}.$$

Step 3: Solve for x.
Cross-multiply:

$$\begin{aligned} 20(17x - 108) &= 37(7x + 108) \\ 340x - 2160 &= 259x + 3996 \\ 340x - 259x &= 3996 + 2160 \\ 81x &= 6156 \\ x &= \frac{6156}{81} = 76. \end{aligned}$$

Step 4: Find the current number of coins in each box.
After the first shift:

$$\begin{aligned} \text{Coins in A} &= 17 \cdot 76 - 108 = 1292 - 108 = 1184, \\ \text{Coins in B} &= 7 \cdot 76 + 108 = 532 + 108 = 640. \end{aligned}$$

Total coins:

$$1184 + 640 = 1824.$$

Step 5: Make the ratio 1:1.
For a 1:1 ratio, both boxes must have:

$$\text{Target in each box} = \frac{1824}{2} = 912.$$

Currently, box A has 1184 coins, so coins to be shifted from A to B:

$$\text{Shift needed} = 1184 - 912 = 272.$$

(Alternatively, let the further shift be y, then

$$\frac{1184 - y}{640 + y} = 1 \Rightarrow 1184 - y = 640 + y \Rightarrow 2y = 544 \Rightarrow y = 272.$$

)
Thus, the number of coins that must be shifted further from A to B is 272.

Quick Tip

For ratio problems with transfers between two containers, first express initial quantities with a variable using the given ratio, then use the new ratio after transfer to form an equation. Finally, use total quantity (which stays constant) to handle any further equalisation like making the ratio 1:1.

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10. The rate of water flow through three pipes A, B and C are in the ratio 4 : 9 : 36. An empty tank can be filled up completely by pipe A in 15 hours. If all the three pipes are used simultaneously to fill up this empty tank, the time, in minutes, required to fill up the entire tank completely is nearest to
(A) 76
(B) 78
(C) 73
(D) 71

Correct Answer: (3) 73

Solution:
Step 1: Assume the flow rates.
The rates of flow of pipes A, B and C are in the ratio 4 : 9 : 36. Let:

$$R_A = 4 \text{ units/hour}, \quad R_B = 9 \text{ units/hour}, \quad R_C = 36 \text{ units/hour}.$$

Step 2: Find the capacity of the tank.
Pipe A alone fills the tank in 15 hours.
Using:

$$\text{Work} = \text{Rate} \times \text{Time},$$

we get:

$$\text{Total capacity} = R_A \times 15 = 4 \times 15 = 60 \text{ units}.$$

Step 3: Find the combined rate of all three pipes.

$$\text{Combined rate} = R_A + R_B + R_C = 4 + 9 + 36 = 49 \text{ units/hour}.$$

Step 4: Time taken to fill the tank together.

$$\text{Time} = \frac{\text{Total capacity}}{\text{Combined rate}} = \frac{60}{49} \text{ hours}.$$

Step 5: Convert the time into minutes.

$$\text{Time in minutes} = \frac{60}{49} \times 60 = \frac{3600}{49} \text{ minutes}.$$

Compute:

$$\frac{3600}{49} \approx 73.47 \text{ minutes}.$$

Rounding to the nearest integer, we get:

$$\boxed{73 \text{ minutes}}.$$

Quick Tip

When dealing with pipes and cisterns in ratios, assign simple proportional rates (like 4, 9, 36), use the given individual time to infer total capacity, then combine the rates and convert the final time into the required units.

---

11. Teams A, B, and C consist of five, eight, and ten members, respectively, such that every member within a team is equally productive. Working separately, teams A, B, and C can complete a certain job in 40 hours, 50 hours, and 4 hours, respectively. Two members from team A, three members from team B, and one member from team C together start the job, and the member

from team C leaves after 23 hours. The number of additional member(s) from team B, that would be required to replace the member from team C, to finish the job in the next one hour, is
(A) 1
(B) 2
(C) 3
(D) 4

Correct Answer: (2) 2

Solution:
Step 1: Calculate individual work rates.
Let the total work be W units.
Team A: 5 members finish the work in 40 hours:

$$5 \times 40 = 200 \text{ man-hours.}$$

Team B: 8 members finish the work in 50 hours:

$$8 \times 50 = 400 \text{ man-hours.}$$

Team C: 10 members finish the work in 4 hours:

$$10 \times 4 = 40 \text{ man-hours.}$$

Choose $W = 400$ units (LCM of 200, 400, 40).
Then the rate of one member from each team is:

$$r_A = \frac{400}{200} = 2 \text{ units/hour}, \quad r_B = \frac{400}{400} = 1 \text{ unit/hour}, \quad r_C = \frac{400}{40} = 10 \text{ units/hour.}$$

Step 2: Work done in the first 23 hours.
Initially working together:

- 2 members from A,
- 3 members from B,
- 1 member from C.

Combined hourly rate:

$$R_1 = 2r_A + 3r_B + 1r_C = 2 \cdot 2 + 3 \cdot 1 + 1 \cdot 10 = 4 + 3 + 10 = 17 \text{ units/hour.}$$

Work done in 23 hours:

$$W_1 = 17 \times 23 = 391 \text{ units.}$$

Step 3: Remaining work.

$$\text{Remaining work} = W - W_1 = 400 - 391 = 9 \text{ units.}$$

Step 4: Required rate for the last hour.
The job must be completed in the next 1 hour.
After the member from C leaves, we still have:

- 2 members from A,
- 3 members from B,
- plus x additional members from B (to be determined).

New combined rate R_2:

$$R_2 = 2r_A + 3r_B + xr_B = 2 \cdot 2 + 3 \cdot 1 + x \cdot 1 = 4 + 3 + x = 7 + x.$$

We need:

$$R_2 \times 1 = 9 \Rightarrow 7 + x = 9 \Rightarrow x = 2.$$

So, 2 additional members from Team B are required.

> **Quick Tip**
>
> In "teams and work" problems, convert everything into a common work unit using total work and individual rates. Once you know the work done in the first phase, the remaining work and time directly tell you the required combined rate for the final phase.

---

12. Ankita walks from A to C through B, and runs back through the same route at a speed that is 40% more than her walking speed. She takes exactly 3 hours 30 minutes to walk from B to C as well as to run from B to A. The total time, in minutes, she would take to walk from A to B and run from B to C, is

Solution:
Step 1: Define speeds.
Let Ankita's walking speed be w km/h. Her running speed is 40% more:

$$r = 1.4w.$$

Given:

$$3 \text{ hours } 30 \text{ minutes} = 3.5 \text{ hours} = 210 \text{ minutes}.$$

Step 2: Use the given times.
Walking from $B \to C$:

$$\frac{\text{Distance } BC}{w} = 3.5 \Rightarrow \text{Distance } BC = 3.5w.$$

Running from $B \to A$:

$$\frac{\text{Distance } BA}{r} = 3.5 \Rightarrow \text{Distance } BA = 3.5r = 3.5(1.4w) = 4.9w.$$

Step 3: Time to walk from A to B.
Walking from $A \to B$, using speed w:

$$\text{Time}(A \to B)_{\text{walk}} = \frac{\text{Distance } AB}{w} = \frac{4.9w}{w} = 4.9 \text{ hours}.$$

Convert to minutes:

$$4.9 \times 60 = 294 \text{ minutes}.$$

Step 4: Time to run from B to C.
Running from $B \to C$, using speed $r = 1.4w$:

$$\text{Time}(B \to C)_{\text{run}} = \frac{\text{Distance } BC}{r} = \frac{3.5w}{1.4w} = \frac{3.5}{1.4} = 2.5 \text{ hours}.$$

Convert to minutes:

$$2.5 \times 60 = 150 \text{ minutes}.$$

Step 5: Total required time.

$$\text{Total time} = 294 + 150 = 444 \text{ minutes.}$$

Quick Tip

When speed changes by a fixed percentage, keep everything in terms of one speed (like w), express all distances using that speed and given times, then recompute the required times with the new speed.

---

13. If $12^{12x} \times 4^{24x+12} \times 5^{2y} = 8^{4z} \times 20^{12x} \times 243^{3x-6}$, where x, y and z are natural numbers, then $x+y+z$ equals

Solution:
Step 1: Express all bases using prime factors.
Rewrite each base in terms of primes 2, 3 and 5:

$$12 = 2^2 \cdot 3, \quad 4 = 2^2, \quad 5 = 5, \quad 8 = 2^3, \quad 20 = 2^2 \cdot 5, \quad 243 = 3^5.$$

Substitute into the given equation:

$$(2^2 \cdot 3)^{12x} \times (2^2)^{24x+12} \times 5^{2y} = (2^3)^{4z} \times (2^2 \cdot 5)^{12x} \times (3^5)^{3x-6}.$$

Step 2: Simplify the exponents.
Left-hand side (LHS):

$$(2^2 \cdot 3)^{12x} = 2^{24x} \cdot 3^{12x},$$
$$(2^2)^{24x+12} = 2^{48x+24}.$$

So LHS becomes:

$$2^{24x} \cdot 3^{12x} \cdot 2^{48x+24} \cdot 5^{2y} = 2^{(24x+48x+24)} \cdot 3^{12x} \cdot 5^{2y} = 2^{72x+24} \cdot 3^{12x} \cdot 5^{2y}.$$

Right-hand side (RHS):

$$(2^3)^{4z} = 2^{12z},$$
$$(2^2 \cdot 5)^{12x} = 2^{24x} \cdot 5^{12x},$$
$$(3^5)^{3x-6} = 3^{5(3x-6)} = 3^{15x-30}.$$

So RHS becomes:

$$2^{12z} \cdot 2^{24x} \cdot 5^{12x} \cdot 3^{15x-30} = 2^{12z+24x} \cdot 3^{15x-30} \cdot 5^{12x}.$$

Step 3: Equate exponents of corresponding primes.
Since the expressions are equal and factorized into primes, equate exponents of 2, 3 and 5.
For base 3:

$$12x = 15x - 30 \Rightarrow 30 = 3x \Rightarrow x = 10.$$

For base 5:

$$2y = 12x.$$

Substitute $x = 10$:

$$2y = 12 \cdot 10 = 120 \Rightarrow y = 60.$$

For base 2:

$$72x + 24 = 12z + 24x.$$

Substitute $x = 10$:

$$72 \cdot 10 + 24 = 12z + 24 \cdot 10 \Rightarrow 720 + 24 = 12z + 240 \Rightarrow 744 = 12z + 240 \Rightarrow 12z = 504 \Rightarrow z = 42.$$

Step 4: Compute $x + y + z$.

$$x + y + z = 10 + 60 + 42 = 112.$$

Thus, $x + y + z = \boxed{112}$.

> Quick Tip
>
> Whenever you see an equation involving different bases with exponents, convert all bases to prime factors. Then compare the exponents of each prime on both sides — it turns the problem into a simple system of linear equations.

---

14. The sum of all the digits of the number $(10^{50} + 10^{25} - 123)$, is
(A) 21
(B) 221
(C) 324
(D) 255

Correct Answer: (2) 221

Solution:
Step 1: Rewrite the expression.
Let

$$N = 10^{50} + 10^{25} - 123.$$

Group as:

$$N = 10^{50} + (10^{25} - 123).$$

Step 2: Understand the form of $10^{25} - 123$.
The number 10^{25} is a 1 followed by 25 zeros:

$$10^{25} = 1\underbrace{00\ldots0}_{25 \text{ zeros}}.$$

Subtracting 123 affects only the last three digits, and the preceding zeros become 9's due to borrowing.
A pattern:

$$1000 - 123 = 877, \quad 10000 - 123 = 9877,$$

so in general:

$$10^n - 123 = \underbrace{99\ldots9}_{n-3 \text{ times}} 877.$$

Here $n = 25$, so:

$$10^{25} - 123 = \underbrace{99\ldots9}_{22 \text{ times}}877.$$

Step 3: Sum of digits of $10^{25} - 123$.
Digits: twenty-two 9's, then 8, 7, 7.
Sum of digits:

$$22 \times 9 + 8 + 7 + 7 = 198 + 22 = 220.$$

Step 4: Add 10^{50}.
The number 10^{50} is:

$$10^{50} = 1\underbrace{00\ldots0}_{50 \text{ zeros}}.$$

Adding $10^{25} - 123$, which has only 25 digits, affects only the last 25 positions. There is no carry up to the leading 1 of 10^{50}.
Thus, the final number N has:

- a leading digit 1 (from 10^{50}),
- some zeros in between,
- then the digits of $10^{25} - 123$, whose sum is 220.

So, the total sum of digits is:

$$1 + 220 = 221.$$

Quick Tip

When subtracting a small number from a large power of 10, think in terms of borrowing: most of the leading zeros turn into 9's, and only the last few digits change. This makes it much easier to find digit sums.

---

15. If $f(x) = (x^2 + 3x)(x^2 + 3x + 2)$, then the sum of all real roots of the equation $\sqrt{f(x) + 1} = 9701$, is
(A) 6
(B) -3
(C) -6
(D) 3

Correct Answer: (2) -3

Solution:
Step 1: Simplify the function $f(x)$.
Let

$$y = x^2 + 3x.$$

Then

$$f(x) = (x^2 + 3x)(x^2 + 3x + 2) = y(y + 2) = y^2 + 2y.$$

Step 2: Substitute into the given equation.
We are given:

$$\sqrt{f(x) + 1} = 9701.$$

Substitute $f(x) = y^2 + 2y$:

$$\sqrt{y^2 + 2y + 1} = 9701.$$

Note that:

$$y^2 + 2y + 1 = (y + 1)^2,$$

so:

$$\sqrt{(y + 1)^2} = 9701 \Rightarrow |y + 1| = 9701.$$

Step 3: Solve for y.
Two possible cases:

$$y + 1 = 9701 \Rightarrow y = 9700,$$

$$y + 1 = -9701 \Rightarrow y = -9702.$$

Recall $y = x^2 + 3x$.

Step 4: Find the corresponding x-values.
Case A: $y = 9700$:

$$x^2 + 3x = 9700 \Rightarrow x^2 + 3x - 9700 = 0.$$

This is a quadratic with real roots (discriminant $D = 3^2 - 4(1)(-9700) = 9 + 38800 = 38809 > 0$).

For $ax^2 + bx + c = 0$, sum of roots is $-\frac{b}{a}$. Here:

$$\text{Sum of roots} = -\frac{3}{1} = -3.$$

Case B: $y = -9702$:

$$x^2 + 3x = -9702 \Rightarrow x^2 + 3x + 9702 = 0.$$

Discriminant:

$$D = 3^2 - 4(1)(9702) = 9 - 38808 = -38799 < 0,$$

so there are *no real roots* from this case.

Step 5: Sum of all real roots.
Only Case A contributes real roots, whose sum is -3.
Hence, the sum of all real roots is:

$$\boxed{-3}.$$

Quick Tip

When you see an expression like $(x^2+3x)(x^2+3x+2)$, try substituting $y = x^2+3x$ to reduce the degree of the equation. Also, once you get quadratics, use the sum of roots formula $-\frac{b}{a}$ instead of solving for each root separately.

---

16. For real values of x, the range of the function $f(x) = \frac{2x-3}{2x^2+4x-6}$ is

(A) $(-\infty, \frac{1}{8}] \cup [1, \infty)$

(B) $(-\infty, \frac{1}{8}] \cup [\frac{1}{2}, \infty)$

(C) $(-\infty, \frac{1}{4}] \cup [\frac{1}{2}, \infty)$

Correct Answer: (2) $(-\infty, \frac{1}{8}] \cup [\frac{1}{2}, \infty)$

Solution:
Step 1: Let $y = f(x)$.

$$y = \frac{2x-3}{2x^2+4x-6}.$$

Step 2: Form a quadratic equation in x.
Cross-multiply:

$$y(2x^2 + 4x - 6) = 2x - 3.$$

Expand:

$$2yx^2 + 4yx - 6y = 2x - 3.$$

Bring all terms to one side:

$$2yx^2 + (4y-2)x + (3-6y) = 0.$$

This is a quadratic in x with:

$$A = 2y, \quad B = 4y - 2, \quad C = 3 - 6y.$$

Step 3: Condition for real x (discriminant ≥ 0).

$$D = B^2 - 4AC \geq 0.$$

Substitute:

$$(4y-2)^2 - 4(2y)(3-6y) \geq 0.$$

Expand:

$$16y^2 - 16y + 4 - 8y(3-6y) \geq 0,$$
$$16y^2 - 16y + 4 - 24y + 48y^2 \geq 0.$$

Combine like terms:

$$(16y^2 + 48y^2) + (-16y - 24y) + 4 \geq 0,$$
$$64y^2 - 40y + 4 \geq 0.$$

Step 4: Solve the quadratic inequality.
Divide by 4:

$$16y^2 - 10y + 1 \geq 0.$$

Solve $16y^2 - 10y + 1 = 0$:

$$y = \frac{10 \pm \sqrt{(-10)^2 - 4(16)(1)}}{2 \cdot 16} = \frac{10 \pm \sqrt{100-64}}{32} = \frac{10 \pm \sqrt{36}}{32} = \frac{10 \pm 6}{32}.$$

Thus:

$$y_1 = \frac{10-6}{32} = \frac{4}{32} = \frac{1}{8}, \quad y_2 = \frac{10+6}{32} = \frac{16}{32} = \frac{1}{2}.$$

Since the leading coefficient (16) is positive and the inequality is ≥ 0, the solution set lies *outside* the interval between the roots:

$$y \leq \frac{1}{8} \quad \text{or} \quad y \geq \frac{1}{2}.$$

Step 5: Write the range in interval form.

$$\text{Range of } f(x) = (-\infty, \tfrac{1}{8}] \cup [\tfrac{1}{2}, \infty).$$

Quick Tip

To find the range of a rational function $\dfrac{P(x)}{Q(x)}$, set $y = \dfrac{P(x)}{Q(x)}$, rearrange to a quadratic (or higher-degree) in x, and use the discriminant condition $D \geq 0$ to get an inequality in y. That inequality directly gives the range.

---

17. The monthly sales of a product from January to April were 120, 135, 150 and 165 units, respectively. The cost price of the product was Rs. 240 per unit, and a fixed marked price was used for the product in all the four months. Discounts of 20%, 10% and 5% were given on the marked price per unit in January, February and March, respectively, while no discounts were given in April. If the total profit from January to April was Rs. 138825, then the marked price per unit, in rupees, was
(A) 525
(B) 510
(C) 520
(D) 515

Correct Answer: (1) 525

Solution:
Step 1: Define variables and constants.
Let the marked price per unit be M (in rupees). Cost price (CP) per unit = 240 rupees. Total profit (January to April) = 138825 rupees.

Step 2: Compute total cost.
Total units sold:

$$120 + 135 + 150 + 165 = 570 \text{ units.}$$

Total cost:

$$\text{Total Cost} = 570 \times 240 = 136800 \text{ rupees.}$$

Step 3: Express total revenue in terms of M.

January: Discount $= 20\% \Rightarrow$ Selling price $= 0.8M$. Units sold $= 120$.

$$R_J = 120 \times 0.8M = 96M.$$

February: Discount $= 10\% \Rightarrow$ Selling price $= 0.9M$. Units sold $= 135$.

$$R_F = 135 \times 0.9M = 121.5M.$$

March: Discount $= 5\% \Rightarrow$ Selling price $= 0.95M$. Units sold $= 150$.

$$R_M = 150 \times 0.95M = 142.5M.$$

April: No discount $\Rightarrow$ Selling price $= M$. Units sold $= 165$.

$$R_A = 165 \times M = 165M.$$

Total revenue:

$$R = 96M + 121.5M + 142.5M + 165M = 525M.$$

Step 4: Use the profit equation.

$$\text{Profit} = \text{Total Revenue} - \text{Total Cost}.$$

Given profit $= 138825$:

$$138825 = 525M - 136800.$$

Solve for M:

$$525M = 138825 + 136800 = 275625,$$
$$M = \frac{275625}{525} = 525.$$

Thus, the marked price per unit is $\boxed{525}$ rupees.

Quick Tip

In multi-month profit questions, first find total cost using total units and cost price, then express each month's revenue in terms of the marked price and discount. Adding these gives a single-variable equation that's easy to solve.

---

18. A triangle ABC is formed with $AB = AC = 50$ cm and $BC = 80$ cm. Then, the sum of the lengths, in cm, of all three altitudes of the triangle ABC is

Solution:
Step 1: Identify the type of triangle.
Given:

$$AB = AC = 50 \text{ cm}, \quad BC = 80 \text{ cm}.$$

Since two sides are equal, $\triangle ABC$ is an isosceles triangle with base BC and equal sides AB and AC.

Step 2: Altitude from A to base BC (call it h_1).
Let AD be the altitude from vertex A to side BC.
In an isosceles triangle, the altitude from the vertex to the base bisects the base:

$$BD = DC = \frac{BC}{2} = \frac{80}{2} = 40 \text{ cm}.$$

Consider right triangle $\triangle ADC$:

$$AC = 50 \text{ cm (hypotenuse)}, \quad DC = 40 \text{ cm (base)}, \quad AD = h_1 \text{ (height)}.$$

Using Pythagoras' theorem:

$$\begin{aligned} h_1^2 + 40^2 &= 50^2 \\ h_1^2 + 1600 &= 2500 \\ h_1^2 &= 2500 - 1600 = 900 \\ h_1 &= 30 \text{ cm}. \end{aligned}$$

Step 3: Find the area of $\triangle ABC$.
Using base BC and altitude AD:

$$\begin{aligned} \text{Area} &= \frac{1}{2} \times \text{base} \times \text{height} \\ &= \frac{1}{2} \times 80 \times 30 \\ &= 40 \times 30 = 1200 \text{ cm}^2. \end{aligned}$$

Step 4: Altitudes to sides AB and AC (call them h_2 and h_3).
Let h_2 be the altitude to side AC and h_3 be the altitude to side AB.
Since $AB = AC$, the corresponding altitudes are equal:

$$h_2 = h_3.$$

Using the area formula with base $AC = 50$ cm and height h_2:

$$\begin{aligned} \text{Area} &= \frac{1}{2} \times AC \times h_2 \\ 1200 &= \frac{1}{2} \times 50 \times h_2 \\ 1200 &= 25h_2 \\ h_2 &= \frac{1200}{25} = 48 \text{ cm}. \end{aligned}$$

Thus:

$$h_3 = h_2 = 48 \text{ cm}.$$

Step 5: Sum of all three altitudes.

$$\begin{aligned} \text{Sum} &= h_1 + h_2 + h_3 \\ &= 30 + 48 + 48 \\ &= 126 \text{ cm}. \end{aligned}$$

Quick Tip

In an isosceles triangle, the altitude to the base not only gives the height but also splits the base into two equal parts. Once you know the area from one base–height pair, you can easily find the other altitudes using the same area with different bases.

19. Let p, q and r be three natural numbers such that their sum is 900, and r is a perfect square whose value lies between 150 and 500. If p is not less than $0.3q$ and not more than $0.7q$, then the sum of the maximum and minimum possible values of p is

Solution:
Step 1: Write down the conditions.

$$p + q + r = 900,$$

$$r \text{ is a perfect square with } 150 < r < 500,$$

$$0.3q \le p \le 0.7q.$$

Step 2: List possible values of r.
Perfect squares strictly between 150 and 500:

$13^2 = 169,\ 14^2 = 196,\ 15^2 = 225,\ 16^2 = 256,\ 17^2 = 289,\ 18^2 = 324,\ 19^2 = 361,\ 20^2 = 400,\ 21^2 = 441,\ 22^2 = 484.$

So, $169 \le r \le 484$.

Step 3: Express p in terms of r.
From the sum:

$$q = 900 - p - r.$$

The inequality

$$0.3q \le p \le 0.7q$$

becomes (substitute $q = 900 - p - r$):
Left inequality:

$$0.3(900 - p - r) \le p \Rightarrow 0.3(900 - r) - 0.3p \le p \Rightarrow 0.3(900 - r) \le 1.3p \Rightarrow p \ge \frac{0.3}{1.3}(900 - r) = \frac{3}{13}(900 - r).$$

Right inequality:

$$p \le 0.7(900 - p - r) \Rightarrow p \le 0.7(900 - r) - 0.7p \Rightarrow 1.7p \le 0.7(900 - r) \Rightarrow p \le \frac{0.7}{1.7}(900 - r) = \frac{7}{17}(900 - r).$$

Thus, for any admissible r,

$$\frac{3}{13}(900 - r) \le p \le \frac{7}{17}(900 - r).$$

Step 4: Find the minimum possible value of p.
The lower bound

$$p_{\min}(r) = \frac{3}{13}(900 - r)$$

decreases as r increases (since $900 - r$ gets smaller).
So the smallest possible p occurs at the largest r, i.e., $r = 484$:

$$p_{\min} = \frac{3}{13}(900 - 484) = \frac{3}{13} \cdot 416 = 3 \cdot 32 = 96.$$

Check corresponding q:

$$q = 900 - p - r = 900 - 96 - 484 = 320,$$

and

$$0.3q = 96, \quad 0.7q = 224,$$

so $0.3q \le p \le 0.7q$ holds. Hence, the minimum possible value of p is 96.

Step 5: Find the maximum possible value of p.
The upper bound

$$p_{\max}(r) = \frac{7}{17}(900 - r)$$

increases as r decreases (since $900 - r$ becomes larger).
So the largest possible p occurs at the smallest r, i.e., $r = 169$:

$$p_{\max} = \frac{7}{17}(900 - 169) = \frac{7}{17} \cdot 731 = 7 \cdot 43 = 301.$$

Check corresponding q:

$$q = 900 - p - r = 900 - 301 - 169 = 430,$$

and

$$0.3q = 129, \quad 0.7q = 301,$$

so $129 \le 301 \le 301$ holds. Hence, the maximum possible value of p is 301.

Step 6: Sum of maximum and minimum values of p.

$$\text{Required sum} = p_{\min} + p_{\max} = 96 + 301 = 397.$$

Quick Tip

When you have constraints like $ap \le q \le bp$ along with $p + q + r =$ constant, try expressing q in terms of p and r, then convert the inequalities into bounds for p in terms of r. After that, use monotonicity (increasing/decreasing behavior) to decide which extreme values of r give the extreme values of p.

---

20. The average salary of 5 managers and 25 engineers in a company is 60000 rupees. If each of the managers received 20% salary increase while the salary of the engineers remained unchanged, the average salary of all 30 employees would have increased by 5%. The average salary, in rupees, of the engineers is
(A) 40000
(B) 54000
(C) 50000
(D) 45000

Correct Answer: (2) 54000

Solution:
Step 1: Set up the initial equations.
Number of managers: $n_m = 5$
Number of engineers: $n_e = 25$
Total employees: $N = 30$
Initial overall average salary: Rs. 60000.
Let:

$$M = \text{average salary of a manager}, \quad E = \text{average salary of an engineer}.$$

Total salary of all 30 employees:

$$\text{Total salary} = 30 \times 60000 = 1800000.$$

So:

$$5M + 25E = 1800000. \tag{1}$$

Step 2: Use the salary increase information.
Each manager gets a 20% increase; engineers' salaries do not change.
Overall average salary increases by 5%:

$$\text{Increase in average} = 5\% \text{ of } 60000 = 0.05 \times 60000 = 3000.$$

Total increase in salary for all employees:

$$\text{Total increase} = 3000 \times 30 = 90000.$$

This total increase comes only from the managers (since engineers' salaries remain unchanged):

$$\text{Total increase from managers} = 5 \times (0.20M) = 5 \times 0.2M = 1.0M = M.$$

Equate:

$$M = 90000.$$

So the initial average salary of each manager is Rs. 90000.

Step 3: Find the average salary of the engineers.
Substitute $M = 90000$ into equation (1):

$$\begin{aligned} 5(90000) + 25E &= 1800000 \\ 450000 + 25E &= 1800000 \\ 25E &= 1800000 - 450000 = 1350000 \\ E &= \frac{1350000}{25}. \end{aligned}$$

Compute:

$$E = \frac{1350000}{25} = 54000.$$

Thus, the average salary of the engineers is $\boxed{54000}$ rupees.

> **Quick Tip**
>
> When only one group in a mixed-average problem gets a percentage raise, compute the total increase using the change in overall average, then equate it to the increase from that group alone. This often gives a direct equation for that group's average.

---

21. In a school with 1500 students, each student chooses any one of the streams out of science, arts, and commerce, by paying a fee of Rs 1100, Rs 1000, and Rs 800, respectively. The total fee paid by all the students is Rs 15,50,000. If the number of science students is not more than the number of arts students, then the maximum possible number of science students in the school is

Solution:
Step 1: Define variables.
Let:

$$s = \text{number of Science students}, \quad a = \text{number of Arts students}, \quad c = \text{number of Commerce students}.$$

Total students:

$$s + a + c = 1500. \tag{1}$$

Step 2: Use the fee information.
Fees per student:

$$\text{Science} = 1100, \quad \text{Arts} = 1000, \quad \text{Commerce} = 800.$$

Total fee collected:

$$1100s + 1000a + 800c = 1550000.$$

Divide the whole equation by 100 to simplify:

$$11s + 10a + 8c = 15500. \tag{2}$$

Step 3: Eliminate c.
From (1):

$$c = 1500 - (s + a).$$

Substitute into (2):

$$\begin{aligned} 11s + 10a + 8(1500 - s - a) &= 15500 \\ 11s + 10a + 12000 - 8s - 8a &= 15500 \\ (11s - 8s) + (10a - 8a) + 12000 &= 15500 \\ 3s + 2a &= 3500. \end{aligned} \tag{3}$$

Step 4: Express a in terms of s. From (3):

$$2a = 3500 - 3s \Rightarrow a = \frac{3500 - 3s}{2} = 1750 - 1.5s.$$

Step 5: Apply the condition $s \le a$.
Given that number of science students is not more than number of arts students:

$$s \le a.$$

Substitute $a = 1750 - 1.5s$:

$$\begin{aligned} s &\le 1750 - 1.5s \\ s + 1.5s &\le 1750 \\ 2.5s &\le 1750 \\ s &\le \frac{1750}{2.5} = 700. \end{aligned}$$

So, the maximum possible value of s is 700, if it is feasible.

Step 6: Check feasibility for $s = 700$.

$$a = 1750 - 1.5 \cdot 700 = 1750 - 1050 = 700,$$

$$c = 1500 - (s + a) = 1500 - (700 + 700) = 100.$$

All are non-negative integers, and $s \le a$ holds $(700 = 700)$. So this distribution is valid.
Hence, the maximum possible number of science students is $\boxed{700}$.

Quick Tip

In word problems with headcount and revenue constraints, first set up two equations: one for the total number of people and another for total money. Then eliminate one variable to get a simple linear relation and apply given inequalities to find extrema.

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22. In an arithmetic progression, if the sum of fourth, seventh and tenth terms is 99, and the sum of the first fourteen terms is 497, then the sum of first five terms is

Solution:
Step 1: Define variables.
Let the first term of the AP be a and the common difference be d.
The n-th term:

$$T_n = a + (n - 1)d$$

The sum of the first n terms:

$$S_n = \frac{n}{2}[2a + (n - 1)d].$$

Step 2: Use the condition on the 4th, 7th and 10th terms.
Given:

$$T_4 + T_7 + T_{10} = 99.$$

Now:

$$T_4 = a + 3d, \quad T_7 = a + 6d, \quad T_{10} = a + 9d.$$

So:

$$\begin{aligned} (a + 3d) + (a + 6d) + (a + 9d) &= 99 \\ 3a + 18d &= 99 \\ a + 6d &= 33. \end{aligned} \tag{1}$$

Step 3: Use the condition on the sum of first 14 terms.
Given:

$$S_{14} = 497.$$

Using the sum formula:

$$\begin{aligned} S_{14} &= \frac{14}{2}[2a + (14 - 1)d] \\ 497 &= 7[2a + 13d] \\ 2a + 13d &= 71. \end{aligned} \tag{2}$$

Step 4: Solve the system of equations for a and d.
From (1):

$$a = 33 - 6d.$$

Substitute into (2):

$$\begin{aligned} 2(33 - 6d) + 13d &= 71 \\ 66 - 12d + 13d &= 71 \\ 66 + d &= 71 \\ d &= 71 - 66 = 5. \end{aligned}$$

Now back-substitute into (1):

$$\begin{aligned} a + 6(5) &= 33 \\ a + 30 &= 33 \\ a &= 3. \end{aligned}$$

So the AP begins:

$$3, \ 8, \ 13, \ 18, \ \ldots$$

Step 5: Find the sum of the first five terms.
Use the sum formula with $n = 5$:

$$\begin{aligned} S_5 &= \frac{5}{2}[2a + (5 - 1)d] \\ &= \frac{5}{2}[2(3) + 4(5)] \\ &= \frac{5}{2}[6 + 20] \\ &= \frac{5}{2} \cdot 26 \\ &= 5 \cdot 13 = 65. \end{aligned}$$

Thus, the sum of the first five terms is 65.

Quick Tip

When given conditions on specific terms and on the sum of terms in an AP, convert them into equations using $T_n = a + (n-1)d$ and $S_n = \frac{n}{2}[2a + (n-1)d]$. Two independent conditions will usually give you two linear equations in a and d.
