

TS EAMCET 2026 May 11 Shift 1

Question Paper with Solutions (Memory-Based)

Conducted by JNTU, Hyderabad



General Instructions

- (i) The test is of 3 hours duration.
- (ii) This test paper consists of 160 questions. The maximum marks are 720.
- (iii) Physics and Chemistry contains 40 questions each and Biology (Botany and Zoology) contains 80 questions.
- (iv) Each question carries +1 marks for correct answer and there is no negative marking for wrong answer.

Mathematics

1. If $f : \mathbb{R} - \left\{ \frac{2a}{3} \right\} \rightarrow \mathbb{R}$ is a function defined by $f(x) = \frac{2x+a}{3x-2a}$ and $(f \circ f)(x) = x$ for all $x \in \mathbb{R} - \left\{ \frac{2a}{3} \right\}$, then $f(3) =$

- (A) 3
- (B) 7
- (C) 1
- (D) 9

Correct Answer: (C) 1

Solution:

Concept:

- A composite function $(f \circ f)(x)$ is found by substituting $f(x)$ into itself: $f(f(x))$.
- If $(f \circ f)(x) = x$ for all valid x , then f is an involution (an invertible function that is its own inverse).

- Comparing coefficients on both sides of the identity enables the determination of unknown parameters.

Step 1: Find the expression for the composite function $f(f(x))$

The given function is:

$$f(x) = \frac{2x + a}{3x - 2a}$$

Substitute $f(x)$ in place of x :

$$f(f(x)) = \frac{2\left(\frac{2x+a}{3x-2a}\right) + a}{3\left(\frac{2x+a}{3x-2a}\right) - 2a}$$

Step 2: Simplify the numerator and denominator

Multiply both the numerator and denominator by $(3x - 2a)$:

$$f(f(x)) = \frac{2(2x + a) + a(3x - 2a)}{3(2x + a) - 2a(3x - 2a)}$$

Expand the terms in the numerator:

$$2(2x + a) + a(3x - 2a) = 4x + 2a + 3ax - 2a^2 = (4 + 3a)x + (2a - 2a^2)$$

Expand the terms in the denominator:

$$3(2x + a) - 2a(3x - 2a) = 6x + 3a - 6ax + 4a^2 = (6 - 6a)x + (3a + 4a^2)$$

Thus, the composite function becomes:

$$f(f(x)) = \frac{(4 + 3a)x + (2a - 2a^2)}{(6 - 6a)x + (3a + 4a^2)}$$

Step 3: Solve for the parameter a

We are given that $f(f(x)) = x$:

$$\frac{(4 + 3a)x + (2a - 2a^2)}{(6 - 6a)x + (3a + 4a^2)} = x$$

Cross-multiply:

$$(4 + 3a)x + (2a - 2a^2) = (6 - 6a)x^2 + (3a + 4a^2)x$$

For this identity to hold for all x , the coefficient of x^2 must be zero:

$$6 - 6a = 0 \implies 6a = 6 \implies a = 1$$

Verify with the constant term:

$$2a - 2a^2 = 2(1) - 2(1)^2 = 0$$

Verify with the coefficient of x :

$$4 + 3(1) = 7 \quad \text{and} \quad 3(1) + 4(1)^2 = 7$$

Hence, $a = 1$ satisfies the condition identically.

Step 4: Calculate the value of $f(3)$

Substitute $a = 1$ back into the function:

$$f(x) = \frac{2x + 1}{3x - 2}$$

Now evaluate at $x = 3$:

$$f(3) = \frac{2(3) + 1}{3(3) - 2} = \frac{6 + 1}{9 - 2} = \frac{7}{7} = 1$$

Quick Tip: For a linear fractional transformation $f(x) = \frac{Ax+B}{Cx+D}$, the condition $f(f(x)) = x$ is satisfied if and only if the trace of the corresponding matrix is zero, i.e., $A + D = 0$. Here, $2 + (-2a) = 0 \implies a = 1$.

2. If $f : \mathbb{R} \rightarrow \mathbb{R}$, $g : \mathbb{R} \rightarrow \mathbb{R}$ are two functions defined by $f(x) = |x+1|$ and $g(x) = \begin{cases} e^{-x}, & x \leq 0 \\ x-1, & x > 0 \end{cases}$,

then $(f \circ g)(-2) + (g \circ f)(2) =$

- (A) $e^2 + 5$
- (B) $e^{-2} + 3$
- (C) $e^{-2} + 5$
- (D) $e^2 + 3$

Correct Answer: (D) $e^2 + 3$

Solution:

Concept:

- The composition of two functions is defined as $(f \circ g)(x) = f(g(x))$ and $(g \circ f)(x) = g(f(x))$.
- For a piecewise function, evaluate the inner function using the definition corresponding to the domain of the input.
- Evaluate the outer function using the resulting value from the inner function.

Step 1: Evaluate the first term $(f \circ g)(-2)$

By definition of composite functions:

$$(f \circ g)(-2) = f(g(-2))$$

Since the input $x = -2 \leq 0$, use the branch $g(x) = e^{-x}$:

$$g(-2) = e^{-(-2)} = e^2$$

Now substitute this value into $f(x)$:

$$f(e^2) = |e^2 + 1|$$

Since $e^2 > 0$, $e^2 + 1 > 0$, so:

$$f(e^2) = e^2 + 1$$

Step 2: Evaluate the second term $(g \circ f)(2)$

By definition of composite functions:

$$(g \circ f)(2) = g(f(2))$$

Compute $f(2)$ using $f(x) = |x + 1|$:

$$f(2) = |2 + 1| = |3| = 3$$

Now evaluate $g(3)$:

Since the input $3 > 0$, use the branch $g(x) = x - 1$:

$$g(3) = 3 - 1 = 2$$

Step 3: Add both evaluated terms together

Combine the results from Step 1 and Step 2:

$$(f \circ g)(-2) + (g \circ f)(2) = (e^2 + 1) + 2 = e^2 + 3$$

Quick Tip: Always identify the sign or interval of the input before selecting the corresponding branch of a piecewise-defined function.

3. If $1 + 3 + 5 + \dots + l_1 = 1521$ and $2 + 4 + 6 + \dots + l_2 = 1722$ then $l_1 + l_2 =$

- (A) 160
- (B) 159
- (C) 80
- (D) 79

Correct Answer: (B) 159

Solution:

Concept:

- The sum of the first n consecutive odd positive integers is:

$$\sum_{k=1}^n (2k-1) = n^2$$

The n -th odd integer is $l_1 = 2n - 1$.

- The sum of the first m consecutive even positive integers is:

$$\sum_{k=1}^m 2k = m(m+1)$$

The m -th even integer is $l_2 = 2m$.

Step 1: Determine the number of terms and the value of l_1

The first series consists of consecutive odd numbers:

$$1 + 3 + 5 + \dots + l_1 = 1521$$

Let there be n terms in this series. Then:

$$n^2 = 1521$$

Taking the positive square root:

$$n = \sqrt{1521} = 39$$

The last term l_1 is given by:

$$l_1 = 2n - 1 = 2(39) - 1 = 78 - 1 = 77$$

Step 2: Determine the number of terms and the value of l_2

The second series consists of consecutive even numbers:

$$2 + 4 + 6 + \dots + l_2 = 1722$$

Let there be m terms in this series. Then:

$$m(m + 1) = 1722$$

Set up the quadratic equation:

$$m^2 + m - 1722 = 0$$

Factor the quadratic equation:

$$(m + 42)(m - 41) = 0$$

Since m must be a positive integer:

$$m = 41$$

The last term l_2 is given by:

$$l_2 = 2m = 2(41) = 82$$

Step 3: Compute the required sum $l_1 + l_2$

Substitute the values of l_1 and l_2 :

$$l_1 + l_2 = 77 + 82 = 159$$

Quick Tip: To quickly find m from $m(m+1) = 1722$, note that $m \approx \sqrt{1722} \approx 41.5$, so testing integers gives $41 \times 42 = 1722$.

4. If $A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$, then $A^{10} =$

- (A) $-A$
- (B) A
- (C) A^2
- (D) $-A^2$

Correct Answer: (A) $-A$

Solution:

Concept:

- A rotation matrix in two dimensions has the form:

$$R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

- Powers of a rotation matrix satisfy de Moivre's analogue for matrices:

$$[R(\theta)]^n = R(n\theta)$$

- In general, $R(\theta + \pi) = -R(\theta)$.

Step 1: Express matrix A in trigonometric form

Note that:

$$\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2} \quad \text{and} \quad \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

Hence, the matrix A can be written as:

$$A = \begin{bmatrix} \cos\left(-\frac{\pi}{3}\right) & -\sin\left(-\frac{\pi}{3}\right) \\ \sin\left(-\frac{\pi}{3}\right) & \cos\left(-\frac{\pi}{3}\right) \end{bmatrix} = R\left(-\frac{\pi}{3}\right)$$

Step 2: Calculate A^3 to find a periodicity

Using the property of rotation matrices:

$$A^3 = R\left(3 \times \left(-\frac{\pi}{3}\right)\right) = R(-\pi)$$

Evaluating the rotation matrix at $-\pi$:

$$R(-\pi) = \begin{bmatrix} \cos(-\pi) & -\sin(-\pi) \\ \sin(-\pi) & \cos(-\pi) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

Therefore:

$$A^3 = -I$$

Step 3: Compute A^{10} using the value of A^3

Express A^{10} in terms of A^3 :

$$A^{10} = (A^3)^3 \cdot A$$

Substitute $A^3 = -I$:

$$A^{10} = (-I)^3 \cdot A = (-I) \cdot A = -A$$

Quick Tip: Recognizing the standard values $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ as components of a rotation matrix allows evaluating powers via angle multiplication: $10 \times \left(-\frac{\pi}{3}\right) = -\frac{10\pi}{3} = -4\pi + \frac{2\pi}{3} \equiv -\frac{\pi}{3} + \pi$, giving $-A$.

5. If $A = \begin{bmatrix} 1 & 7 & 49 \\ 2 & 16 & 130 \\ 2 & 18 & 170 \end{bmatrix}$, then $\det A =$

- (A) 2^3
- (B) 2^4
- (C) 2^5

(D) 2^6

Correct Answer: (B) 2^4

Solution:

Concept:

- The determinant of a matrix remains unchanged when elementary row operations of the type $R_i \rightarrow R_i - kR_j$ are applied.
- Creating zeros in the first column simplifies the evaluation of the determinant to a 2×2 minor.
- For a 2×2 matrix:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Step 1: Set up the determinant expression

The given matrix is:

$$A = \begin{bmatrix} 1 & 7 & 49 \\ 2 & 16 & 130 \\ 2 & 18 & 170 \end{bmatrix}$$

The determinant is:

$$\det A = \begin{vmatrix} 1 & 7 & 49 \\ 2 & 16 & 130 \\ 2 & 18 & 170 \end{vmatrix}$$

Step 2: Apply row operations to introduce zeros in the first column

Apply the operations:

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - 2R_1$$

Calculate the elements of the new second row:

$$R_{21} = 2 - 2(1) = 0$$

$$R_{22} = 16 - 2(7) = 16 - 14 = 2$$

$$R_{23} = 130 - 2(49) = 130 - 98 = 32$$

Calculate the elements of the new third row:

$$R_{31} = 2 - 2(1) = 0$$

$$R_{32} = 18 - 2(7) = 18 - 14 = 4$$

$$R_{33} = 170 - 2(49) = 170 - 98 = 72$$

The determinant is now:

$$\det A = \begin{vmatrix} 1 & 7 & 49 \\ 0 & 2 & 32 \\ 0 & 4 & 72 \end{vmatrix}$$

Step 3: Expand along the first column and evaluate

Expanding along column 1:

$$\det A = 1 \cdot \begin{vmatrix} 2 & 32 \\ 4 & 72 \end{vmatrix}$$

Evaluate the 2×2 determinant:

$$\det A = (2)(72) - (32)(4)$$

$$\det A = 144 - 128 = 16$$

Step 4: Express the result as a power of 2

Express 16 in exponent form:

$$16 = 2^4$$

Thus, $\det A = 2^4$.

Quick Tip: Always use elementary row operations $R_2 \rightarrow R_2 - cR_1$ and $R_3 \rightarrow R_3 - cR_1$ to create leading zeros before expanding higher-order determinants.

6. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, then $A^4 =$

(A) A^{-1}

(B) A

(C) I_3

(D) A^T

Correct Answer: (C) I_3

Solution:

Concept:

- By the Cayley-Hamilton Theorem, every square matrix satisfies its own characteristic equation:

$$\det(A - \lambda I) = 0$$

- For a 3×3 matrix, the characteristic equation is given by:

$$\lambda^3 - \text{tr}(A)\lambda^2 + M\lambda - \det(A) = 0$$

where $\text{tr}(A)$ is the trace, and M is the sum of the principal minors.

- Alternatively, powers of a matrix can be calculated by direct block or sequential matrix multiplication.

Step 1: Find the trace and principal minors of matrix A

The given matrix is:

$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

Calculate the trace of A:

$$\text{tr}(A) = 3 + (-3) + 1 = 1$$

Calculate the principal minors of A:

$$M_{11} = \begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = (-3)(1) - (4)(-1) = -3 + 4 = 1$$

$$M_{22} = \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} = (3)(1) - (4)(0) = 3 - 0 = 3$$

$$M_{33} = \begin{vmatrix} 3 & -3 \\ 2 & -3 \end{vmatrix} = (3)(-3) - (-3)(2) = -9 + 6 = -3$$

Sum of the principal minors:

$$M = M_{11} + M_{22} + M_{33} = 1 + 3 + (-3) = 1$$

Step 2: Calculate the determinant of matrix A

Expand $\det(A)$ along the third row:

$$\det(A) = 0 \cdot C_{31} + (-1) \cdot (-1)^{3+2} \begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} + 1 \cdot (-1)^{3+3} \begin{vmatrix} 3 & -3 \\ 2 & -3 \end{vmatrix}$$

$$\det(A) = 1 \cdot ((3)(4) - (4)(2)) + 1 \cdot ((3)(-3) - (-3)(2))$$

$$\det(A) = 1 \cdot (12 - 8) + 1 \cdot (-9 + 6) = 4 - 3 = 1$$

Step 3: Apply the Cayley-Hamilton Theorem

Substitute the invariants into the characteristic equation:

$$\lambda^3 - (1)\lambda^2 + (1)\lambda - 1 = 0$$

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$

Factor the polynomial:

$$\lambda^2(\lambda - 1) + 1(\lambda - 1) = 0$$

$$(\lambda^2 + 1)(\lambda - 1) = 0$$

By the Cayley-Hamilton theorem, A satisfies this polynomial:

$$A^3 - A^2 + A - I = 0$$

Multiply both sides of the equation by $(A + I)$:

$$(A + I)(A^3 - A^2 + A - I) = 0$$

Expanding the product:

$$A(A^3 - A^2 + A - I) + I(A^3 - A^2 + A - I) = 0$$

$$(A^4 - A^3 + A^2 - A) + (A^3 - A^2 + A - I) = 0$$

Combine like terms:

$$A^4 - I = 0 \implies A^4 = I_3$$

Quick Tip: Since $\lambda^3 - \lambda^2 + \lambda - 1 = 0$, multiplying by $(\lambda + 1)$ yields $\lambda^4 - 1 = 0$. Hence, by the Cayley-Hamilton Theorem, $A^4 = I_3$ directly.

7. The rank of the matrix $\begin{bmatrix} -4 & -1 & 1 & 4 \\ -3 & 0 & 2 & 3 \\ -2 & -1 & 0 & -4 \end{bmatrix}$ is

- (A) 0
- (B) 1
- (C) 2
- (D) 3

Correct Answer: (D) 3

Solution:

Concept:

- The rank of an $m \times n$ matrix is the maximum number of linearly independent row or column vectors.
- The rank of a matrix is equal to the order of the largest square submatrix that has a non-zero determinant.
- Alternatively, the rank is the number of non-zero rows in its row-echelon form.

Step 1: Set up the matrix and choose a 3×3 submatrix

Let the given 3×4 matrix be:

$$A = \begin{bmatrix} -4 & -1 & 1 & 4 \\ -3 & 0 & 2 & 3 \\ -2 & -1 & 0 & -4 \end{bmatrix}$$

Since A has size 3×4 , the maximum possible rank is:

$$\text{rank}(A) \leq \min(3, 4) = 3$$

Consider the 3×3 submatrix formed by taking the first three columns:

$$M = \begin{bmatrix} -4 & -1 & 1 \\ -3 & 0 & 2 \\ -2 & -1 & 0 \end{bmatrix}$$

Step 2: Evaluate the determinant of the submatrix

Expand the determinant of M along the second column:

$$\det(M) = -(-1) \begin{vmatrix} -3 & 2 \\ -2 & 0 \end{vmatrix} + 0 - (-1) \begin{vmatrix} -4 & 1 \\ -3 & 2 \end{vmatrix}$$

Evaluate the two 2×2 determinants:

$$\begin{vmatrix} -3 & 2 \\ -2 & 0 \end{vmatrix} = (-3)(0) - (2)(-2) = 0 + 4 = 4$$

$$\begin{vmatrix} -4 & 1 \\ -3 & 2 \end{vmatrix} = (-4)(2) - (1)(-3) = -8 + 3 = -5$$

Combine the results:

$$\det(M) = 1(4) + 1(-5) = 4 - 5 = -1$$

Step 3: Conclude the rank of the matrix

Since $\det(M) = -1 \neq 0$, there exists a minor of order 3 which is non-zero. Therefore, the rank of the matrix is 3.

Quick Tip: To find the rank of a rectangular matrix of size $3 \times n$, test the determinants of any easily computable 3×3 minors. If any minor is non-zero, the rank is immediately 3.

8. Which of the following is not the possible value of $\sqrt{i} + \sqrt{-i}$?

- (A) $2i$
- (B) $\sqrt{2}i$
- (C) $\sqrt{2}$
- (D) $-\sqrt{2}$

Correct Answer: (A) $2i$

Solution:

Concept:

- Any non-zero complex number has two distinct square roots.
- In polar form, $i = e^{i\pi/2}$ and $-i = e^{-i\pi/2}$.
- The two square roots of i are given by:

$$\sqrt{i} = \pm e^{i\pi/4} = \pm \left(\frac{1+i}{\sqrt{2}} \right)$$

- The two square roots of $-i$ are given by:

$$\sqrt{-i} = \pm e^{-i\pi/4} = \pm \left(\frac{1-i}{\sqrt{2}} \right)$$

Step 1: List all values for $\sqrt{i}$ and $\sqrt{-i}$

Let $u \in \sqrt{i}$ and $v \in \sqrt{-i}$. The two values of u are:

$$u_1 = \frac{1+i}{\sqrt{2}}, \quad u_2 = -\frac{1+i}{\sqrt{2}}$$

The two values of v are:

$$v_1 = \frac{1-i}{\sqrt{2}}, \quad v_2 = -\frac{1-i}{\sqrt{2}}$$

Step 2: Calculate all four possible sums $u + v$

Case 1: Choose $u = u_1$ and $v = v_1$:

$$u_1 + v_1 = \frac{1+i}{\sqrt{2}} + \frac{1-i}{\sqrt{2}} = \frac{1+i+1-i}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

Case 2: Choose $u = u_2$ and $v = v_2$:

$$u_2 + v_2 = -\frac{1+i}{\sqrt{2}} - \frac{1-i}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

Case 3: Choose $u = u_1$ and $v = v_2$:

$$u_1 + v_2 = \frac{1+i}{\sqrt{2}} - \frac{1-i}{\sqrt{2}} = \frac{1+i-1+i}{\sqrt{2}} = \frac{2i}{\sqrt{2}} = \sqrt{2}i$$

Case 4: Choose $u = u_2$ and $v = v_1$:

$$u_2 + v_1 = -\frac{1+i}{\sqrt{2}} + \frac{1-i}{\sqrt{2}} = \frac{-1-i+1-i}{\sqrt{2}} = \frac{-2i}{\sqrt{2}} = -\sqrt{2}i$$

Step 3: Identify the value that is not possible

The set of all possible values of $\sqrt{i} + \sqrt{-i}$ is:

$$\{\sqrt{2}, -\sqrt{2}, \sqrt{2}i, -\sqrt{2}i\}$$

Comparing with the given options:

- $\sqrt{2}$ is a possible value.
- $-\sqrt{2}$ is a possible value.
- $\sqrt{2}i$ is a possible value.
- $2i$ is not in the set of possible values.

Therefore, $2i$ is not a possible value.

Quick Tip: Squaring any sum gives $(\sqrt{i} + \sqrt{-i})^2 = i + (-i) \pm 2\sqrt{i(-i)} = \pm 2\sqrt{1} = \pm 2$. Thus the possible values must have magnitude $\sqrt{2}$, which immediately rules out $2i$ since $|2i| = 2$.

9. If $\sum_{n=1}^{10} (i^n + i^{n+1} + i^{n+2}) = x + iy$, then $2x + 3y =$

- (A) 1
- (B) -5
- (C) $5i$
- (D) $-i$

Correct Answer: (B) -5

Solution:

Concept:

- The imaginary unit i satisfies $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$.

- The sum of any four consecutive powers of i is zero:

$$i^k + i^{k+1} + i^{k+2} + i^{k+3} = 0 \quad \text{for any integer } k$$

- Factoring common terms simplifies summations involving complex powers.

Step 1: Simplify the general term of the summation

Consider the general term:

$$T_n = i^n + i^{n+1} + i^{n+2}$$

Factor out i^n :

$$T_n = i^n(1 + i + i^2)$$

Substitute $i^2 = -1$:

$$T_n = i^n(1 + i - 1) = i^n(i) = i^{n+1}$$

Step 2: Evaluate the summation from $n = 1$ to 10

Substitute the simplified term into the summation:

$$S = \sum_{n=1}^{10} T_n = \sum_{n=1}^{10} i^{n+1}$$

Expand the sum:

$$S = i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + i^8 + i^9 + i^{10} + i^{11}$$

Group the terms into sets of four consecutive powers of i :

$$S = (i^2 + i^3 + i^4 + i^5) + (i^6 + i^7 + i^8 + i^9) + (i^{10} + i^{11})$$

Since the sum of four consecutive powers of i is zero:

$$i^2 + i^3 + i^4 + i^5 = -1 - i + 1 + i = 0$$

$$i^6 + i^7 + i^8 + i^9 = 0$$

Therefore, the sum simplifies to:

$$S = i^{10} + i^{11}$$

Step 3: Compute the remaining powers of i

Evaluate i^{10} and i^{11} :

$$i^{10} = (i^4)^2 \cdot i^2 = (1)^2 \cdot (-1) = -1$$

$$i^{11} = i^{10} \cdot i = (-1) \cdot i = -i$$

Thus:

$$S = -1 - i$$

Step 4: Equate real and imaginary parts and calculate $2x + 3y$

We are given that $S = x + iy$:

$$x + iy = -1 - i$$

Equating real and imaginary parts:

$$x = -1, \quad y = -1$$

Now calculate the value of $2x + 3y$:

$$2x + 3y = 2(-1) + 3(-1) = -2 - 3 = -5$$

Quick Tip: Always simplify inside the summation first: $i^n + i^{n+1} + i^{n+2} = i^n(1 + i - 1) = i^{n+1}$. Then, using modulo 4 arithmetic on the 10 terms leaves just the last 2 terms: $i^{10} + i^{11} = -1 - i$.

10. $\left(\frac{1 + \sin \frac{4\pi}{9} - i \cos \frac{4\pi}{9}}{1 + \sin \frac{4\pi}{9} + i \cos \frac{4\pi}{9}} \right)^6 =$

- (A) i
- (B) $\frac{\sqrt{3}-i}{2}$
- (C) $\frac{1-\sqrt{3}i}{2}$
- (D) $1 + i$

Correct Answer: (C) $\frac{1-\sqrt{3}i}{2}$

Solution:

Concept:

- Convert terms into standard polar form $\cos \theta \pm i \sin \theta$ using complementary angles:

$$\sin \theta = \cos \left(\frac{\pi}{2} - \theta \right), \quad \cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$$

- Use the half-angle trigonometric formulas:

$$1 + \cos \alpha = 2 \cos^2 \left(\frac{\alpha}{2} \right), \quad \sin \alpha = 2 \sin \left(\frac{\alpha}{2} \right) \cos \left(\frac{\alpha}{2} \right)$$

- Apply de Moivre's Theorem:

$$(\cos \phi + i \sin \phi)^n = \cos(n\phi) + i \sin(n\phi) = e^{in\phi}$$

Step 1: Express the trigonometric components in terms of complementary angles

Let the given angle be:

$$\theta = \frac{4\pi}{9}$$

Define the complementary angle α :

$$\alpha = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \frac{4\pi}{9} = \frac{9\pi - 8\pi}{18} = \frac{\pi}{18}$$

Then the terms convert as follows:

$$\sin \left(\frac{4\pi}{9} \right) = \cos \alpha = \cos \left(\frac{\pi}{18} \right)$$

$$\cos \left(\frac{4\pi}{9} \right) = \sin \alpha = \sin \left(\frac{\pi}{18} \right)$$

Step 2: Rewrite and factor the numerator and denominator

Substitute $\cos \alpha$ and $\sin \alpha$ into the fraction:

$$\frac{1 + \sin \frac{4\pi}{9} - i \cos \frac{4\pi}{9}}{1 + \sin \frac{4\pi}{9} + i \cos \frac{4\pi}{9}} = \frac{1 + \cos \alpha - i \sin \alpha}{1 + \cos \alpha + i \sin \alpha}$$

Apply the half-angle identities:

$$1 + \cos \alpha = 2 \cos^2 \left(\frac{\alpha}{2} \right)$$

$$\sin \alpha = 2 \sin \left(\frac{\alpha}{2} \right) \cos \left(\frac{\alpha}{2} \right)$$

Substitute these into the numerator:

$$2 \cos^2\left(\frac{\alpha}{2}\right) - 2i \sin\left(\frac{\alpha}{2}\right) \cos\left(\frac{\alpha}{2}\right) = 2 \cos\left(\frac{\alpha}{2}\right) \left[\cos\left(\frac{\alpha}{2}\right) - i \sin\left(\frac{\alpha}{2}\right) \right]$$

Substitute these into the denominator:

$$2 \cos^2\left(\frac{\alpha}{2}\right) + 2i \sin\left(\frac{\alpha}{2}\right) \cos\left(\frac{\alpha}{2}\right) = 2 \cos\left(\frac{\alpha}{2}\right) \left[\cos\left(\frac{\alpha}{2}\right) + i \sin\left(\frac{\alpha}{2}\right) \right]$$

Step 3: Simplify the ratio

Cancel the common factor $2 \cos\left(\frac{\alpha}{2}\right)$:

$$\frac{\cos\left(\frac{\alpha}{2}\right) - i \sin\left(\frac{\alpha}{2}\right)}{\cos\left(\frac{\alpha}{2}\right) + i \sin\left(\frac{\alpha}{2}\right)} = \frac{e^{-i\alpha/2}}{e^{i\alpha/2}} = e^{-i\alpha}$$

Step 4: Apply the exponent of 6 and evaluate

Raise the simplified expression to the 6th power:

$$(e^{-i\alpha})^6 = e^{-i6\alpha}$$

Substitute $\alpha = \frac{\pi}{18}$:

$$6\alpha = 6\left(\frac{\pi}{18}\right) = \frac{\pi}{3}$$

Therefore:

$$e^{-i\pi/3} = \cos\left(\frac{\pi}{3}\right) - i \sin\left(\frac{\pi}{3}\right)$$

Substitute the standard trigonometric values:

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \quad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$e^{-i\pi/3} = \frac{1}{2} - i \frac{\sqrt{3}}{2} = \frac{1 - \sqrt{3}i}{2}$$

Quick Tip: For any angle α , the expression $\frac{1+\cos\alpha-i\sin\alpha}{1+\cos\alpha+i\sin\alpha}$ always reduces directly to $e^{-i\alpha}$. Converting $\sin\theta$ and $\cos\theta$ via the complementary angle $\alpha = \frac{\pi}{2} - \theta$ makes the solution instantaneous.

11. If ω is the complex cube root of unity, then $\left(\frac{\sqrt{3}-i}{-\sqrt{2}+\sqrt{2}i}\right)^{20} =$

- (A) ω
- (B) $\omega - \omega^2$
- (C) ω^2
- (D) $-\omega$

Correct Answer: (D) $-\omega$

Solution:

Concept:

- Convert complex numbers to polar (Euler's) form $re^{i\theta}$, where $r = |z|$ and $\theta = \arg(z)$.
- Use de Moivre's formula: $(re^{i\theta})^n = r^n e^{in\theta}$.
- Recall the complex cube roots of unity:

$$\omega = e^{i\frac{2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \omega^2 = e^{i\frac{4\pi}{3}} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

- Note that $-\omega = \frac{1}{2} - i\frac{\sqrt{3}}{2} = e^{-i\frac{\pi}{3}}$.

Step 1: Convert the numerator into polar form

Let $z_1 = \sqrt{3} - i$. Find the modulus:

$$|z_1| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = 2$$

Find the argument:

$$\theta_1 = -\frac{\pi}{6}$$

Thus, in exponential form:

$$z_1 = 2e^{-i\frac{\pi}{6}}$$

Step 2: Convert the denominator into polar form

Let $z_2 = -\sqrt{2} + \sqrt{2}i$. Find the modulus:

$$|z_2| = \sqrt{(-\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{2+2} = 2$$

Find the argument (in the second quadrant):

$$\theta_2 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

Thus, in exponential form:

$$z_2 = 2e^{i\frac{3\pi}{4}}$$

Step 3: Simplify the ratio $\frac{z_1}{z_2}$

Divide the two complex numbers:

$$\frac{z_1}{z_2} = \frac{2e^{-i\frac{\pi}{6}}}{2e^{i\frac{3\pi}{4}}} = e^{i(-\frac{\pi}{6} - \frac{3\pi}{4})}$$

Find a common denominator:

$$-\frac{\pi}{6} - \frac{3\pi}{4} = -\frac{2\pi + 9\pi}{12} = -\frac{11\pi}{12}$$

Hence:

$$\frac{z_1}{z_2} = e^{-i\frac{11\pi}{12}}$$

Step 4: Raise the ratio to the power of 20 and relate to ω

Apply the exponent 20:

$$\left(\frac{z_1}{z_2}\right)^{20} = \left(e^{-i\frac{11\pi}{12}}\right)^{20} = e^{-i\frac{220\pi}{12}} = e^{-i\frac{55\pi}{3}}$$

Reduce modulo 2π :

$$-\frac{55\pi}{3} = -18\pi - \frac{\pi}{3} \equiv -\frac{\pi}{3} \pmod{2\pi}$$

Therefore:

$$e^{-i\frac{\pi}{3}} = \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right) = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Express this in terms of ω :

$$\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \implies -\omega = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Thus, the value is $-\omega$.

Quick Tip: Always reduce powers of complex numbers to their principal arguments modulo 2π .

Remember that $-\omega = e^{-i\pi/3}$ and $-\omega^2 = e^{i\pi/3}$.

12. If the minimum value of the quadratic expression $ax^2 - 7x + 3a$ is $-\frac{1}{8}$, then the sum of the roots of the equation $ax^2 - 7x + 3a = 0$ is

- (A) $\frac{7}{8}$
(B) 1
(C) $\frac{7}{2}$
(D) -14

Correct Answer: (C) $\frac{7}{2}$

Solution:

Concept:

- A quadratic expression $f(x) = Ax^2 + Bx + C$ has a minimum value if and only if $A > 0$.
- The minimum value of $f(x)$ occurs at $x = -\frac{B}{2A}$ and is given by:

$$f_{\min} = \frac{4AC - B^2}{4A} = -\frac{D}{4A}$$

- The sum of the roots of the quadratic equation $Ax^2 + Bx + C = 0$ is:

$$S = -\frac{B}{A}$$

Step 1: Identify the coefficients of the quadratic expression

The given quadratic expression is:

$$f(x) = ax^2 - 7x + 3a$$

Here:

$$A = a, \quad B = -7, \quad C = 3a$$

For a minimum to exist, we must have:

$$a > 0$$

Step 2: Set up the equation for the minimum value

The formula for the minimum value is:

$$\frac{4AC - B^2}{4A} = -\frac{1}{8}$$

Substitute $A = a$, $B = -7$, and $C = 3a$:

$$\frac{4(a)(3a) - (-7)^2}{4(a)} = -\frac{1}{8}$$

Simplify the numerator:

$$\frac{12a^2 - 49}{4a} = -\frac{1}{8}$$

Step 3: Solve the equation for a

Multiply both sides by $8a$:

$$2(12a^2 - 49) = -a$$

$$24a^2 - 98 = -a$$

Rearrange into standard quadratic form:

$$24a^2 + a - 98 = 0$$

Factor the quadratic equation:

$$(a - 2)(24a + 49) = 0$$

Since $a > 0$ for the expression to have a minimum value:

$$a = 2$$

Step 4: Calculate the sum of the roots

The quadratic equation is:

$$2x^2 - 7x + 6 = 0$$

The sum of the roots is:

$$\text{Sum of roots} = -\frac{B}{A} = -\frac{-7}{2} = \frac{7}{2}$$

Quick Tip: For a quadratic $ax^2 + bx + c$, remember the vertex formula: minimum value is $\frac{4ac - b^2}{4a}$. Since the question asks for a minimum, always ensure that $a > 0$.

13. If α, β are the roots of the equation $2x^2 - x - 3\lambda = 0$ ($\lambda \neq 0$) and α, γ are the roots of the equation $2x^2 + 9x + 2\lambda = 0$, then the equation with roots $2\alpha + \beta$ and $\beta + \gamma$ is

- (A) $x^2 - x - 2 = 0$
 (B) $x^2 - 3x + 2 = 0$
 (C) $x^2 + 3x + 2 = 0$
 (D) $x^2 + x - 2 = 0$

Correct Answer: (D) $x^2 + x - 2 = 0$

Solution:

Concept:

- If two quadratic equations share a common root α , then α satisfies both equations simultaneously.
- Subtracting or eliminating the variable allows solving for the common root and parameter.
- Vieta's formulas give the sum and product of the roots for each equation.
- A quadratic equation with roots r_1 and r_2 is given by:

$$x^2 - (r_1 + r_2)x + r_1 r_2 = 0$$

Step 1: Find the common root α in terms of λ

Since α is a root of both equations:

$$2\alpha^2 - \alpha - 3\lambda = 0 \quad \text{--- (1)}$$

$$2\alpha^2 + 9\alpha + 2\lambda = 0 \quad \text{--- (2)}$$

Subtract equation (1) from equation (2):

$$(2\alpha^2 + 9\alpha + 2\lambda) - (2\alpha^2 - \alpha - 3\lambda) = 0$$

$$10\alpha + 5\lambda = 0 \implies \alpha = -\frac{\lambda}{2}$$

Step 2: Determine the value of λ and α

Substitute $\alpha = -\frac{\lambda}{2}$ into equation (1):

$$2\left(-\frac{\lambda}{2}\right)^2 - \left(-\frac{\lambda}{2}\right) - 3\lambda = 0$$

$$2\left(\frac{\lambda^2}{4}\right) + \frac{\lambda}{2} - 3\lambda = 0$$

$$\frac{\lambda^2}{2} - \frac{5\lambda}{2} = 0$$

Since $\lambda \neq 0$, divide by $\frac{\lambda}{2}$:

$$\lambda - 5 = 0 \implies \lambda = 5$$

Now find the common root α :

$$\alpha = -\frac{5}{2}$$

Step 3: Find the roots β and γ

Using Vieta's formulas for the first equation $2x^2 - x - 15 = 0$:

$$\alpha + \beta = \frac{1}{2}$$

$$-\frac{5}{2} + \beta = \frac{1}{2} \implies \beta = \frac{1}{2} + \frac{5}{2} = 3$$

Using Vieta's formulas for the second equation $2x^2 + 9x + 10 = 0$:

$$\alpha + \gamma = -\frac{9}{2}$$

$$-\frac{5}{2} + \gamma = -\frac{9}{2} \implies \gamma = -\frac{9}{2} + \frac{5}{2} = -2$$

Step 4: Find the required roots and construct the quadratic equation

The required roots are:

$$r_1 = 2\alpha + \beta = 2\left(-\frac{5}{2}\right) + 3 = -5 + 3 = -2$$

$$r_2 = \beta + \gamma = 3 + (-2) = 1$$

Compute the sum of the roots:

$$S = r_1 + r_2 = -2 + 1 = -1$$

Compute the product of the roots:

$$P = r_1 \cdot r_2 = (-2)(1) = -2$$

Form the quadratic equation:

$$x^2 - Sx + P = 0$$

$$x^2 - (-1)x + (-2) = 0 \implies x^2 + x - 2 = 0$$

Quick Tip: To eliminate x^2 and find the common root quickly, subtract the two quadratic equations directly: $(a_1 - a_2)x^2 + (b_1 - b_2)x + (c_1 - c_2) = 0$.

14. If α, β, γ are the roots of the equation $x^3 - ax^2 - 4x + 4a = 0$, $\alpha + \beta = 0$ and $\beta + \gamma = 5$ then the sum of all possible values of a is

- (A) 10
- (B) 4
- (C) -4
- (D) -10

Correct Answer: (A) 10

Solution:

Concept:

- For a cubic equation $x^3 - px^2 + qx - r = 0$, Vieta's formulas state:

$$\alpha + \beta + \gamma = a$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = -4$$

$$\alpha\beta\gamma = -4a$$

- When the sum of two roots is zero ($\alpha + \beta = 0$), then $\beta = -\alpha$.

Step 1: Factor the cubic polynomial directly

The given cubic equation is:

$$x^3 - ax^2 - 4x + 4a = 0$$

Group the terms pairwise:

$$x^2(x - a) - 4(x - a) = 0$$

Factor out the common term $(x - a)$:

$$(x^2 - 4)(x - a) = 0$$

$$(x - 2)(x + 2)(x - a) = 0$$

Therefore, the three roots are:

$$\{2, -2, a\}$$

Step 2: Match roots with the condition $\alpha + \beta = 0$

The condition $\alpha + \beta = 0$ requires α and β to be opposites. From the set of roots $\{2, -2, a\}$, the opposite pair is 2 and -2 . Therefore:

$$\gamma = a$$

This leaves two possible assignments for (α, β) :

$$\text{Case 1: } \beta = 2, \quad \alpha = -2$$

$$\text{Case 2: } \beta = -2, \quad \alpha = 2$$

Step 3: Determine the possible values of a using $\beta + \gamma = 5$

For Case 1 ($\beta = 2$):

$$\beta + \gamma = 5 \implies 2 + a = 5 \implies a = 3$$

For Case 2 ($\beta = -2$):

$$\beta + \gamma = 5 \implies -2 + a = 5 \implies a = 7$$

Thus, the possible values of a are 3 and 7.

Step 4: Calculate the sum of all possible values of a

Sum of the possible values:

$$\text{Sum} = 3 + 7 = 10$$

Quick Tip: Always check if a cubic polynomial can be factored by grouping before using general symmetric functions. Here, $x^2(x - a) - 4(x - a) = 0$ immediately reveals the roots as ± 2 and a .

15. If $2x^5 + ax^4 - 12x^3 + bx^2 + x + c = 0$ is a reciprocal equation of class one, then the sum of all the rational roots of this equation is

- (A) $-\frac{1}{2}$
 (B) $-\frac{7}{2}$
 (C) $\frac{1}{2}$
 (D) $\frac{7}{2}$

Correct Answer: (B) $-\frac{7}{2}$

Solution:

Concept:

- A reciprocal equation of class one (standard reciprocal equation) has coefficients equidistant from the beginning and end equal in magnitude and sign:

$$a_k = a_{n-k}$$

- Any reciprocal equation of odd degree of class one has $x = -1$ as a root.
- The polynomial can be reduced by dividing by $(x + 1)$ to obtain an even-degree reciprocal equation, which can then be solved using the substitution $u = x + \frac{1}{x}$.

Step 1: Determine the unknown coefficients a, b, c

The equation is of degree 5:

$$2x^5 + ax^4 - 12x^3 + bx^2 + x + c = 0$$

Comparing equidistant coefficients for a class one reciprocal equation:

$$a_0 = a_5 \implies c = 2$$

$$a_1 = a_4 \implies a = 1$$

$$a_2 = a_3 \implies b = -12$$

Thus, the equation is:

$$2x^5 + x^4 - 12x^3 - 12x^2 + x + 2 = 0$$

Step 2: Factor out the root $x = -1$

Since it is an odd degree reciprocal equation of the first class, $x = -1$ is a guaranteed root.

Perform synthetic division by $x + 1$:

$$\begin{array}{r|rrrrrr} -1 & 2 & 1 & -12 & -12 & 1 & 2 \\ & & -2 & 1 & 11 & 1 & -2 \\ \hline & 2 & -1 & -11 & -1 & 2 & 0 \end{array}$$

The depressed equation is:

$$2x^4 - x^3 - 11x^2 - x + 2 = 0$$

Step 3: Solve the reduced quartic reciprocal equation

Divide through by x^2 ($x \neq 0$):

$$2\left(x^2 + \frac{1}{x^2}\right) - \left(x + \frac{1}{x}\right) - 11 = 0$$

Substitute $u = x + \frac{1}{x}$, where $x^2 + \frac{1}{x^2} = u^2 - 2$:

$$2(u^2 - 2) - u - 11 = 0$$

$$2u^2 - u - 15 = 0$$

Factor the quadratic equation:

$$(2u + 5)(u - 3) = 0$$

So the values of u are:

$$u = 3 \quad \text{or} \quad u = -\frac{5}{2}$$

Step 4: Find the roots of the equation for each value of u

Case 1: $x + \frac{1}{x} = 3$:

$$x^2 - 3x + 1 = 0$$

Using the quadratic formula:

$$x = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

These roots are irrational.

Case 2: $x + \frac{1}{x} = -\frac{5}{2}$:

$$2x^2 + 5x + 2 = 0$$

Factor:

$$(2x + 1)(x + 2) = 0 \implies x = -\frac{1}{2}, \quad x = -2$$

These roots are rational.

Step 5: Compute the sum of all rational roots

The rational roots are:

$$x_1 = -1, \quad x_2 = -2, \quad x_3 = -\frac{1}{2}$$

Sum of the rational roots:

$$\text{Sum} = -1 + (-2) + \left(-\frac{1}{2}\right) = -3 - \frac{1}{2} = -\frac{7}{2}$$

Quick Tip: Odd degree reciprocal equations of class one always possess the root $x = -1$. For reciprocal quadratics in $u = x + \frac{1}{x}$, roots are rational if and only if the discriminant of the resulting equation in x is a perfect square.

16. If all possible 5-digit numbers are formed using the digits 3, 4, 5, 6, 7, 8, 9 when repetition is allowed, then the number of numbers among these 5-digit numbers which are divisible by 7 is

- (A) 2520
- (B) 840
- (C) 2401
- (D) 1680

Correct Answer: (C) 2401

Solution:

Concept:

- A number is divisible by 7 if and only if its value modulo 7 is equal to 0.
- The given set of available digits is $S = \{3, 4, 5, 6, 7, 8, 9\}$.
- The size of this set is $|S| = 7$.

- When taken modulo 7, the set S forms a complete residue system modulo 7:

$$\{3 \bmod 7, 4 \bmod 7, 5 \bmod 7, 6 \bmod 7, 7 \bmod 7, 8 \bmod 7, 9 \bmod 7\} = \{3, 4, 5, 6, 0, 1, 2\}$$

- Every residue modulo 7 appears exactly once.

Step 1: Analyze the degrees of freedom for the digits

Let the 5-digit number be represented as:

$$N = d_4 \cdot 10^4 + d_3 \cdot 10^3 + d_2 \cdot 10^2 + d_1 \cdot 10 + d_0$$

where each $d_i \in \{3, 4, 5, 6, 7, 8, 9\}$. The total number of choices for the first four digits d_4, d_3, d_2, d_1 is:

$$7 \times 7 \times 7 \times 7 = 7^4$$

Step 2: Apply the divisibility condition modulo 7

We require:

$$N \equiv 0 \pmod{7}$$

Substitute the expansion of N :

$$(d_4 \cdot 10^4 + d_3 \cdot 10^3 + d_2 \cdot 10^2 + d_1 \cdot 10) + d_0 \equiv 0 \pmod{7}$$

Let $K = d_4 \cdot 10^4 + d_3 \cdot 10^3 + d_2 \cdot 10^2 + d_1 \cdot 10$. Then:

$$d_0 \equiv -K \pmod{7}$$

Step 3: Determine the number of valid choices for d_0

Since the digit set $S = \{3, 4, 5, 6, 7, 8, 9\}$ contains each remainder modulo 7 exactly once:

$$S \equiv \{0, 1, 2, 3, 4, 5, 6\} \pmod{7}$$

For any chosen integer K , there is uniquely one element in S that satisfies:

$$d_0 \equiv -K \pmod{7}$$

Thus, for every choice of the first four digits, there is exactly 1 choice for the last digit d_0 .

Step 4: Compute the total count of such numbers

Calculate the total number of valid 5-digit numbers:

$$\text{Total} = 7 \times 7 \times 7 \times 7 \times 1 = 7^4$$

Evaluate 7^4 :

$$7^4 = 49 \times 49 = 2401$$

Quick Tip: When the set of allowed digits forms a complete residue system modulo m , exactly $\frac{1}{m}$ of the total possible numbers will be divisible by m : $\frac{7^5}{7} = 7^4 = 2401$.

17. A committee of 5 members from 5 Indians, 4 Americans and 3 Australians is to be formed so that every country has its representative in the committee. Then, the number of ways of forming the committee having at most two representatives from each country is

- (A) 560
- (B) 195
- (C) 200
- (D) 390

Correct Answer: (D) 390

Solution:

Concept:

- The total number of members in the committee is 5.
- Every country must have at least one representative: $n_I \geq 1, n_{Am} \geq 1, n_{Au} \geq 1$.
- Each country can have at most two representatives: $n_I \leq 2, n_{Am} \leq 2, n_{Au} \leq 2$.
- We need to partition the integer 5 into three parts, each being either 1 or 2:

$$n_I + n_{Am} + n_{Au} = 5, \quad n_i \in \{1, 2\}$$

- The number of ways to choose r individuals out of n is given by the combination formula:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Step 1: Identify all valid distributions of committee members

The only partition of 5 into 3 numbers such that each number is 1 or 2 is:

$$2 + 2 + 1 = 5$$

This gives three mutually exclusive cases for (n_I, n_{Am}, n_{Au}) :

- Case 1: 2 Indians, 2 Americans, 1 Australian
- Case 2: 2 Indians, 1 American, 2 Australians
- Case 3: 1 Indian, 2 Americans, 2 Australians

Step 2: Compute the number of combinations for Case 1

Choose 2 Indians from 5, 2 Americans from 4, and 1 Australian from 3:

$$N_1 = \binom{5}{2} \times \binom{4}{2} \times \binom{3}{1}$$

Evaluate each combination:

$$\binom{5}{2} = \frac{5 \times 4}{2} = 10$$

$$\binom{4}{2} = \frac{4 \times 3}{2} = 6$$

$$\binom{3}{1} = 3$$

$$N_1 = 10 \times 6 \times 3 = 180$$

Step 3: Compute the number of combinations for Case 2

Choose 2 Indians from 5, 1 American from 4, and 2 Australians from 3:

$$N_2 = \binom{5}{2} \times \binom{4}{1} \times \binom{3}{2}$$

Evaluate each combination:

$$\binom{5}{2} = 10, \quad \binom{4}{1} = 4, \quad \binom{3}{2} = 3$$

$$N_2 = 10 \times 4 \times 3 = 120$$

Step 4: Compute the number of combinations for Case 3

Choose 1 Indian from 5, 2 Americans from 4, and 2 Australians from 3:

$$N_3 = \binom{5}{1} \times \binom{4}{2} \times \binom{3}{2}$$

Evaluate each combination:

$$\binom{5}{1} = 5, \quad \binom{4}{2} = 6, \quad \binom{3}{2} = 3$$

$$N_3 = 5 \times 6 \times 3 = 90$$

Step 5: Sum all cases to find the total number of ways

Total number of ways:

$$N = N_1 + N_2 + N_3 = 180 + 120 + 90 = 390$$

Quick Tip: The condition that each of the 3 groups must have 1 or 2 members with sum 5 uniquely forces the member counts to be a permutation of (2, 2, 1).

18. Let p be the number of ways of arranging 6 students such that 3 are around a circular table and the remaining 3 in a row. Let q be the number of ways of arranging 5 boys and 4 girls in a row such that no two boys and no two girls are together. Then $\frac{q}{p} =$

- (A) 12
- (B) 18
- (C) 6
- (D) 8

Correct Answer: (A) 12

Solution:

Concept:

- Selection of r items out of n is done in $\binom{n}{r}$ ways.

- The number of circular permutations of k distinct objects is $(k - 1)!$.
- The number of linear permutations of k distinct objects is $k!$.
- If two groups must alternate in a line and their sizes differ by 1, the larger group must occupy both ends: $B_1 G_1 B_2 G_2 B_3 G_3 B_4 G_4 B_5$.

Step 1: Calculate the value of p

We must choose 3 students out of 6 to sit at the circular table:

$$\binom{6}{3} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$$

The remaining 3 students will be arranged in a row. Number of ways to arrange 3 students around a circular table:

$$(3 - 1)! = 2! = 2$$

Number of ways to arrange the remaining 3 students in a row:

$$3! = 6$$

Therefore, the total number of ways p is:

$$p = \binom{6}{3} \times (3 - 1)! \times 3! = 20 \times 2 \times 6 = 240$$

Step 2: Calculate the value of q

We need to arrange 5 boys and 4 girls in a row such that no two boys are together and no two girls are together. This means boys and girls must strictly alternate. Since there are 5 boys and 4 girls, the arrangement must begin and end with a boy:

$$B_1 G_1 B_2 G_2 B_3 G_3 B_4 G_4 B_5$$

Number of ways to arrange the 5 boys in their 5 positions:

$$5! = 120$$

Number of ways to arrange the 4 girls in their 4 positions:

$$4! = 24$$

Therefore, the total number of ways q is:

$$q = 5! \times 4! = 120 \times 24 = 2880$$

Step 3: Compute the ratio $\frac{q}{p}$

Evaluate the fraction:

$$\frac{q}{p} = \frac{2880}{240} = \frac{288}{24} = 12$$

Quick Tip: To arrange m items of type A and $m - 1$ items of type B alternately, there is only 1 pattern: $ABAB \dots A$, giving $m! \times (m - 1)!$ arrangements.

19. If $C_r = {}^nC_r$ and $n = 10$, then $C_0 + C_1 + C_2 \cdot \frac{2^2}{3} + C_3 \cdot \frac{2^3}{4} + \dots + C_n \cdot \frac{2^n}{n+1} =$

- (A) $\frac{3^{11}-1}{11}$
- (B) $\frac{3^{11}-1}{22}$
- (C) $\frac{3^{10}-1}{22}$
- (D) $\frac{3^{10}-1}{11}$

Correct Answer: (B) $\frac{3^{11}-1}{22}$

Solution:

Concept:

- Standard binomial identity for coefficients divided by $r + 1$:

$$\frac{{}^nC_r}{r+1} = \frac{{}^{n+1}C_{r+1}}{n+1}$$

- Integrating the binomial expansion:

$$(1+x)^n = \sum_{r=0}^n {}^nC_r x^r$$

$$\int_0^k (1+x)^n dx = \sum_{r=0}^n \frac{{}^nC_r k^{r+1}}{r+1} = \frac{(1+k)^{n+1} - 1}{n+1}$$

Step 1: Inspect the first two terms to find the pattern

Notice that:

$$C_0 = {}^nC_0 \cdot \frac{2^0}{1} = 1$$

$$C_1 = {}^nC_1 = {}^nC_1 \cdot \frac{2^1}{2}$$

Thus, every term in the given series has the general form:

$$T_r = {}^nC_r \frac{2^r}{r+1}, \quad \text{for } r = 0, 1, 2, \dots, n$$

Therefore, the full series is:

$$S = \sum_{r=0}^n {}^nC_r \frac{2^r}{r+1}$$

Step 2: Express the sum in terms of a definite integral

Recall the binomial expansion:

$$(1+x)^n = \sum_{r=0}^n {}^nC_r x^r$$

Integrate both sides from $x = 0$ to $x = 2$:

$$\int_0^2 (1+x)^n dx = \int_0^2 \left(\sum_{r=0}^n {}^nC_r x^r \right) dx = \sum_{r=0}^n {}^nC_r \left[\frac{x^{r+1}}{r+1} \right]_0^2$$

$$\int_0^2 (1+x)^n dx = \sum_{r=0}^n {}^nC_r \frac{2^{r+1}}{r+1} = 2 \sum_{r=0}^n {}^nC_r \frac{2^r}{r+1} = 2S$$

Step 3: Evaluate the integral

Calculate the value of the definite integral:

$$2S = \left[\frac{(1+x)^{n+1}}{n+1} \right]_0^2 = \frac{(1+2)^{n+1} - (1+0)^{n+1}}{n+1} = \frac{3^{n+1} - 1}{n+1}$$

Thus, the sum S is:

$$S = \frac{3^{n+1} - 1}{2(n+1)}$$

Step 4: Substitute $n = 10$

For $n = 10$:

$$S = \frac{3^{10+1} - 1}{2(10+1)} = \frac{3^{11} - 1}{2(11)} = \frac{3^{11} - 1}{22}$$

Quick Tip: Whenever the denominator of binomial coefficients is $r + 1$, multiply by x and integrate $(1 + x)^n$, or apply the property $\frac{{}^nC_r}{r+1} = \frac{{}^{n+1}C_{r+1}}{n+1}$.

20. The approximate value of $(0.98)^{0.2}$, rounded to 4 decimal places, found by using binomial expansion is

- (A) 0.9860
- (B) 0.9950
- (C) 0.9960
- (D) 1.0060

Correct Answer: (C) 0.9960

Solution:

Concept:

- Binomial expansion for any rational index n when $|x| < 1$:

$$(1 - x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

- When x is very small, higher-order terms decrease rapidly in magnitude.

Step 1: Rewrite the expression in the form $(1 - x)^n$

Write 0.98 as $1 - 0.02$:

$$(0.98)^{0.2} = (1 - 0.02)^{0.2}$$

Here:

$$x = 0.02 = \frac{2}{100}, \quad n = 0.2 = \frac{1}{5}$$

Step 2: Compute the first few terms of the expansion

Expand up to the second-order term:

$$(1 - x)^n \approx 1 - nx + \frac{n(n-1)}{2}x^2$$

Evaluate the first correction term:

$$T_1 = -nx = -(0.2)(0.02) = -0.004$$

Evaluate the second correction term:

$$T_2 = \frac{0.2(0.2-1)}{2}(0.02)^2 = \frac{0.2(-0.8)}{2}(0.0004) = -0.08 \times 0.0004 = -0.000032$$

Step 3: Sum the terms and round to 4 decimal places

Combine the terms:

$$(0.98)^{0.2} \approx 1 - 0.004 - 0.000032 = 0.995968$$

Rounding to 4 decimal places:

$$0.995968 \approx 0.9960$$

Quick Tip: For very small x , use the linear approximation: $(1-x)^n \approx 1-nx$. Here, $1-(0.2)(0.02) = 1-0.004 = 0.9960$, which gives the answer directly.

21. If $\frac{2x^6+3x^4+1}{(x^2+2)^4} = \frac{Ax+P}{x^2+2} + \frac{Bx+Q}{(x^2+2)^2} + \frac{Cx+R}{(x^2+2)^3} + \frac{Dx+T}{(x^2+2)^4}$, then $3P+2Q+R+4T =$

- (A) -12
- (B) 30
- (C) -3
- (D) 24

Correct Answer: (A) -12

Solution:

Concept:

- When a rational function contains only even powers of x , substituting $t = x^2$ simplifies the expression.
- The partial fraction decomposition over powers of $(t+2)$ can be directly obtained using a Taylor-like shift substitution $y = t+2$.
- Since the original numerator has no odd powers of x , all coefficients corresponding to odd terms vanish identically: $A = B = C = D = 0$.

Step 1: Substitute $t = x^2$ to simplify the numerator

Let:

$$t = x^2$$

The given rational function is:

$$\frac{2(x^2)^3 + 3(x^2)^2 + 1}{(x^2 + 2)^4} = \frac{2t^3 + 3t^2 + 1}{(t + 2)^4}$$

Step 2: Express the polynomial in terms of powers of $(t + 2)$

Let $y = t + 2$, which implies $t = y - 2$. Substitute $t = y - 2$ into the numerator:

$$N(y) = 2(y - 2)^3 + 3(y - 2)^2 + 1$$

Expand the cubic term:

$$2(y - 2)^3 = 2(y^3 - 6y^2 + 12y - 8) = 2y^3 - 12y^2 + 24y - 16$$

Expand the quadratic term:

$$3(y - 2)^2 = 3(y^2 - 4y + 4) = 3y^2 - 12y + 12$$

Combine all terms:

$$\begin{aligned} N(y) &= (2y^3 - 12y^2 + 24y - 16) + (3y^2 - 12y + 12) + 1 \\ &= 2y^3 + (-12 + 3)y^2 + (24 - 12)y + (-16 + 12 + 1) \\ &= 2y^3 - 9y^2 + 12y - 3 \end{aligned}$$

Step 3: Divide by y^4 to obtain the partial fractions

Divide each term of $N(y)$ by y^4 :

$$\frac{N(y)}{y^4} = \frac{2y^3 - 9y^2 + 12y - 3}{y^4} = \frac{2}{y} - \frac{9}{y^2} + \frac{12}{y^3} - \frac{3}{y^4}$$

Substitute back $y = t + 2 = x^2 + 2$:

$$\frac{2x^6 + 3x^4 + 1}{(x^2 + 2)^4} = \frac{2}{x^2 + 2} - \frac{9}{(x^2 + 2)^2} + \frac{12}{(x^2 + 2)^3} - \frac{3}{(x^2 + 2)^4}$$

Step 4: Compare coefficients and compute the required value

Comparing with the given partial fraction expansion:

$$P = 2, \quad Q = -9, \quad R = 12, \quad T = -3$$

Now evaluate $3P + 2Q + R + 4T$:

$$\begin{aligned} 3P + 2Q + R + 4T &= 3(2) + 2(-9) + 12 + 4(-3) \\ &= 6 - 18 + 12 - 12 \\ &= -12 \end{aligned}$$

Quick Tip: To decompose $\frac{P(t)}{(t+c)^n}$, substitute $y = t + c$, expand the numerator in powers of y , and divide each term by y^n directly.

22. If $2 \cos \theta + 3 \sin \theta = 3$ and $\tan \theta$ is defined, then $\tan \theta =$

- (A) $\frac{5}{12}$
- (B) $-\frac{5}{12}$
- (C) $\frac{12}{5}$
- (D) $-\frac{12}{5}$

Correct Answer: (A) $\frac{5}{12}$

Solution:

Concept:

- A linear trigonometric equation of the form $a \cos \theta + b \sin \theta = c$ can be solved by isolating one term and squaring both sides.
- When squaring, extraneous solutions may be introduced, so all roots must be verified.
- If $\tan \theta$ is defined, $\cos \theta \neq 0$, which implies $\sin \theta \neq \pm 1$.

Step 1: Isolate the cosine term and square both sides

The given equation is:

$$2 \cos \theta + 3 \sin \theta = 3$$

Isolate $2 \cos \theta$:

$$2 \cos \theta = 3(1 - \sin \theta)$$

Square both sides:

$$4 \cos^2 \theta = 9(1 - \sin \theta)^2$$

Step 2: Form a quadratic equation in $\sin \theta$

Use the identity $\cos^2 \theta = 1 - \sin^2 \theta$:

$$4(1 - \sin^2 \theta) = 9(1 - 2 \sin \theta + \sin^2 \theta)$$

Expand both sides:

$$4 - 4 \sin^2 \theta = 9 - 18 \sin \theta + 9 \sin^2 \theta$$

Rearrange into standard quadratic form:

$$13 \sin^2 \theta - 18 \sin \theta + 5 = 0$$

Step 3: Factor the quadratic equation and reject invalid roots

Factor the quadratic:

$$(13 \sin \theta - 5)(\sin \theta - 1) = 0$$

This yields two possibilities:

$$\sin \theta = 1 \quad \text{or} \quad \sin \theta = \frac{5}{13}$$

If $\sin \theta = 1$, then:

$$\cos \theta = 0$$

which implies $\tan \theta = \frac{\sin \theta}{\cos \theta}$ is undefined. Since the problem states that $\tan \theta$ is defined, we must discard $\sin \theta = 1$. Therefore:

$$\sin \theta = \frac{5}{13}$$

Step 4: Calculate $\cos \theta$ and evaluate $\tan \theta$

Substitute $\sin \theta = \frac{5}{13}$ back into the linear relation:

$$2 \cos \theta = 3 \left(1 - \frac{5}{13} \right) = 3 \left(\frac{8}{13} \right) = \frac{24}{13}$$

$$\cos \theta = \frac{12}{13}$$

Now calculate $\tan \theta$:

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{5/13}{12/13} = \frac{5}{12}$$

Quick Tip: Alternatively, express $\sin \theta = \frac{2t}{1+t^2}$ and $\cos \theta = \frac{1-t^2}{1+t^2}$ with $t = \tan(\theta/2)$ to solve directly for t , then use $\tan \theta = \frac{2t}{1-t^2}$.

23. If $\tan A$ and $\tan B$ are the roots of the equation $2x^2 - 9x - 16 = 0$, then $9 \sin^2(A+B) - \cos^2(A+B) =$

- (A) 0
- (B) 7
- (C) 1
- (D) 10

Correct Answer: (C) 1

Solution:

Concept:

- By Vieta's formulas, for a quadratic equation $ax^2 + bx + c = 0$, the sum and product of the roots are:

$$x_1 + x_2 = -\frac{b}{a}, \quad x_1 x_2 = \frac{c}{a}$$

- The compound angle formula for tangent is:

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

- Trigonometric relations in terms of tangent:

$$\sin^2 \theta = \frac{\tan^2 \theta}{1 + \tan^2 \theta}, \quad \cos^2 \theta = \frac{1}{1 + \tan^2 \theta}$$

Step 1: Apply Vieta's formulas to the given quadratic equation

The roots of the equation $2x^2 - 9x - 16 = 0$ are $\tan A$ and $\tan B$. Sum of the roots:

$$\tan A + \tan B = -\frac{-9}{2} = \frac{9}{2}$$

Product of the roots:

$$\tan A \tan B = \frac{-16}{2} = -8$$

Step 2: Compute $\tan(A + B)$

Using the addition formula for tangent:

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Substitute the values from Step 1:

$$\tan(A + B) = \frac{\frac{9}{2}}{1 - (-8)} = \frac{\frac{9}{2}}{9} = \frac{1}{2}$$

Step 3: Determine $\sin^2(A + B)$ and $\cos^2(A + B)$

Square the tangent value:

$$\tan^2(A + B) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

Calculate $\cos^2(A + B)$:

$$\cos^2(A + B) = \frac{1}{1 + \tan^2(A + B)} = \frac{1}{1 + \frac{1}{4}} = \frac{1}{\frac{5}{4}} = \frac{4}{5}$$

Calculate $\sin^2(A + B)$:

$$\sin^2(A + B) = \frac{\tan^2(A + B)}{1 + \tan^2(A + B)} = \frac{\frac{1}{4}}{\frac{5}{4}} = \frac{1}{5}$$

Step 4: Evaluate the required expression

Substitute the values into the target expression:

$$\begin{aligned} 9 \sin^2(A + B) - \cos^2(A + B) &= 9 \left(\frac{1}{5} \right) - \frac{4}{5} \\ &= \frac{9 - 4}{5} \\ &= \frac{5}{5} = 1 \end{aligned}$$

Quick Tip: Divide the required expression by $\cos^2(A+B)$ to work entirely in $\tan(A+B)$: $\cos^2(A+B)[9\tan^2(A+B)-1] = \frac{9(1/4)-1}{1+1/4} = \frac{5/4}{5/4} = 1$.

24. If $\sin \theta \sin(60^\circ - \theta) \sin(60^\circ + \theta) = \frac{1}{8}$, then $\cos 6\theta =$

- (A) $\frac{\sqrt{3}}{2}$
(B) $\frac{1}{2}$
(C) $\frac{1}{\sqrt{2}}$
(D) 0

Correct Answer: (B) $\frac{1}{2}$

Solution:

Concept:

- Standard triple-angle identity for sine:

$$\sin \theta \sin(60^\circ - \theta) \sin(60^\circ + \theta) = \frac{1}{4} \sin 3\theta$$

- The double-angle identity relating cosine to sine:

$$\cos 2\phi = 1 - 2\sin^2 \phi$$

Step 1: Apply the sine product identity

Using the standard identity:

$$\sin \theta \sin(60^\circ - \theta) \sin(60^\circ + \theta) = \frac{1}{4} \sin 3\theta$$

We are given:

$$\sin \theta \sin(60^\circ - \theta) \sin(60^\circ + \theta) = \frac{1}{8}$$

Therefore:

$$\frac{1}{4} \sin 3\theta = \frac{1}{8}$$

Step 2: Find the value of $\sin 3\theta$

Multiply both sides by 4:

$$\sin 3\theta = \frac{4}{8} = \frac{1}{2}$$

Step 3: Express $\cos 6\theta$ in terms of $\sin 3\theta$

Using the double-angle formula for cosine with $\phi = 3\theta$:

$$\cos 6\theta = \cos(2 \cdot 3\theta) = 1 - 2 \sin^2(3\theta)$$

Step 4: Substitute the value of $\sin 3\theta$ and calculate

Substitute $\sin 3\theta = \frac{1}{2}$:

$$\cos 6\theta = 1 - 2 \left(\frac{1}{2} \right)^2 = 1 - 2 \left(\frac{1}{4} \right) = 1 - \frac{1}{2} = \frac{1}{2}$$

Quick Tip: Remember the product identity: $\sin \theta \sin(60^\circ - \theta) \sin(60^\circ + \theta) = \frac{1}{4} \sin 3\theta$. A similar identity holds for cosine: $\cos \theta \cos(60^\circ - \theta) \cos(60^\circ + \theta) = \frac{1}{4} \cos 3\theta$.

25. If $S = \left\{ \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2} \right] : \cos^2 \theta + \sin \theta \tan \theta = \cos 2\theta \right\}$, then $\sum_{\theta \in S} (\sin \theta + \cos \theta) =$

- (A) 0
- (B) 1
- (C) -1
- (D) 2

Correct Answer: (C) -1

Solution:

Concept:

- Tangent is defined only when $\cos \theta \neq 0$, which excludes odd multiples of $\frac{\pi}{2}$.
- Convert all trigonometric expressions into basic sine and cosine terms:

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \cos 2\theta = 2 \cos^2 \theta - 1$$

- Factor the resulting algebraic equation to determine all roots in the domain.

Step 1: Determine the domain restriction

For $\tan \theta$ to be defined:

$$\cos \theta \neq 0$$

In the given interval $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, the points where $\cos \theta = 0$ are $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. Thus, the valid interval is:

$$\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

Step 2: Substitute identities into the given equation

The given equation is:

$$\cos^2 \theta + \sin \theta \tan \theta = \cos 2\theta$$

Substitute $\tan \theta = \frac{\sin \theta}{\cos \theta}$ and $\cos 2\theta = 2 \cos^2 \theta - 1$:

$$\cos^2 \theta + \frac{\sin^2 \theta}{\cos \theta} = 2 \cos^2 \theta - 1$$

Subtract $\cos^2 \theta$ from both sides:

$$\frac{\sin^2 \theta}{\cos \theta} = \cos^2 \theta - 1$$

Step 3: Factor the equation

Recall that $\cos^2 \theta - 1 = -\sin^2 \theta$:

$$\frac{\sin^2 \theta}{\cos \theta} = -\sin^2 \theta$$

Bring all terms to one side:

$$\frac{\sin^2 \theta}{\cos \theta} + \sin^2 \theta = 0$$

Factor out $\sin^2 \theta$:

$$\sin^2 \theta \left(\frac{1}{\cos \theta} + 1 \right) = 0$$

Step 4: Find the solutions within the valid interval

Case 1:

$$\sin^2 \theta = 0 \implies \sin \theta = 0$$

In the open interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, the only solution is:

$$\theta = \pi$$

Case 2:

$$\frac{1}{\cos \theta} + 1 = 0 \implies \cos \theta = -1$$

In the open interval $(\frac{\pi}{2}, \frac{3\pi}{2})$, the only solution is:

$$\theta = \pi$$

Both cases yield the exact same single solution:

$$S = \{\pi\}$$

Step 5: Compute the required sum

Evaluate $\sin \theta + \cos \theta$ for $\theta = \pi$:

$$\sum_{\theta \in S} (\sin \theta + \cos \theta) = \sin \pi + \cos \pi = 0 + (-1) = -1$$

Quick Tip: Always identify domain restrictions first: the presence of $\tan \theta$ automatically eliminates the boundary endpoints $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

26. $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{2}{3} + \tan^{-1} \frac{1}{5} =$

- (A) $\frac{\pi}{4}$
- (B) $\tan^{-1} \left(\frac{7}{11} \right)$
- (C) $\frac{\pi}{2}$
- (D) $\tan^{-1} \left(\frac{23}{24} \right)$

Correct Answer: (C) $\frac{\pi}{2}$

Solution:

Concept:

- Addition formula for inverse tangent:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right), \quad \text{for } xy < 1$$

- When the sum of angles equals $\frac{\pi}{4}$, combining two such pairs yields $\frac{\pi}{2}$.

Step 1: Group terms pairwise

Let the sum be:

$$S = \left(\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} \right) + \left(\tan^{-1} \frac{2}{3} + \tan^{-1} \frac{1}{5} \right)$$

Step 2: Evaluate the first pair

For the first pair, $x = \frac{1}{2}$ and $y = \frac{1}{3}$. Check the condition:

$$xy = \left(\frac{1}{2} \right) \left(\frac{1}{3} \right) = \frac{1}{6} < 1$$

Apply the addition formula:

$$\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \tan^{-1} \left(\frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} \right) = \tan^{-1} \left(\frac{\frac{5}{6}}{\frac{5}{6}} \right) = \tan^{-1}(1) = \frac{\pi}{4}$$

Step 3: Evaluate the second pair

For the second pair, $x = \frac{2}{3}$ and $y = \frac{1}{5}$. Check the condition:

$$xy = \left(\frac{2}{3} \right) \left(\frac{1}{5} \right) = \frac{2}{15} < 1$$

Apply the addition formula:

$$\tan^{-1} \frac{2}{3} + \tan^{-1} \frac{1}{5} = \tan^{-1} \left(\frac{\frac{2}{3} + \frac{1}{5}}{1 - \frac{2}{15}} \right)$$

Simplify the numerator:

$$\frac{2}{3} + \frac{1}{5} = \frac{10 + 3}{15} = \frac{13}{15}$$

Simplify the denominator:

$$1 - \frac{2}{15} = \frac{13}{15}$$

Hence:

$$\tan^{-1} \frac{2}{3} + \tan^{-1} \frac{1}{5} = \tan^{-1} \left(\frac{\frac{13}{15}}{\frac{13}{15}} \right) = \tan^{-1}(1) = \frac{\pi}{4}$$

Step 4: Add the two partial results

Combine the two pairs:

$$S = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

Quick Tip: Recognize common identity pairs: $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{\pi}{4}$ and $\tan^{-1} \frac{2}{3} + \tan^{-1} \frac{1}{5} = \frac{\pi}{4}$. Their sum is immediately $\frac{\pi}{2}$.

27. If $\tanh^{-1}(x) = \log \sqrt{3}$ and $\cosh^{-1} y = \log(1 + \sqrt{2})$, then $\operatorname{sech}^{-1}(xy) =$

- (A) $\log(1 + \sqrt{2})$
- (B) $\log(\sqrt{3} + \sqrt{6})$
- (C) $\log \sqrt{2}$
- (D) $\log \sqrt{3}$

Correct Answer: (A) $\log(1 + \sqrt{2})$

Solution:

Concept:

- Inverse hyperbolic functions have standard logarithmic forms:

$$\tanh^{-1}(x) = \frac{1}{2} \log \left(\frac{1+x}{1-x} \right)$$

$$\cosh^{-1}(y) = \log(y + \sqrt{y^2 - 1}), \quad \text{for } y \geq 1$$

- The reciprocal relationship connects inverse secant and hyperbolic cosine:

$$\operatorname{sech}^{-1}(z) = \cosh^{-1} \left(\frac{1}{z} \right)$$

Step 1: Solve for x using the definition of $\tanh^{-1}(x)$

The logarithmic representation of inverse hyperbolic tangent is:

$$\tanh^{-1}(x) = \frac{1}{2} \log \left(\frac{1+x}{1-x} \right)$$

We are given:

$$\frac{1}{2} \log \left(\frac{1+x}{1-x} \right) = \log \sqrt{3}$$

Rewrite the right side:

$$\log \sqrt{3} = \log(3^{1/2}) = \frac{1}{2} \log 3$$

Equating the arguments:

$$\frac{1+x}{1-x} = 3$$

Cross-multiply and solve for x :

$$1+x = 3(1-x) = 3-3x$$

$$4x = 2 \implies x = \frac{1}{2}$$

Step 2: Solve for y using the definition of $\cosh^{-1}(y)$

The logarithmic representation of inverse hyperbolic cosine is:

$$\cosh^{-1}(y) = \log(y + \sqrt{y^2 - 1})$$

We are given:

$$\log(y + \sqrt{y^2 - 1}) = \log(1 + \sqrt{2}) = \log(\sqrt{2} + 1)$$

Equating arguments:

$$y + \sqrt{y^2 - 1} = \sqrt{2} + 1$$

By observation, setting $y = \sqrt{2}$ gives:

$$\sqrt{y^2 - 1} = \sqrt{(\sqrt{2})^2 - 1} = \sqrt{2 - 1} = 1$$

which satisfies the equation:

$$\sqrt{2} + 1 = \sqrt{2} + 1$$

Thus:

$$y = \sqrt{2}$$

Step 3: Calculate the product xy and its reciprocal

Find the product:

$$xy = \left(\frac{1}{2}\right)(\sqrt{2}) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

Find the reciprocal of xy :

$$\frac{1}{xy} = \sqrt{2}$$

Step 4: Compute $\operatorname{sech}^{-1}(xy)$

Using the reciprocal property of inverse hyperbolic functions:

$$\operatorname{sech}^{-1}(xy) = \cosh^{-1}\left(\frac{1}{xy}\right) = \cosh^{-1}(\sqrt{2})$$

From Step 2, we know that $\cosh^{-1}(\sqrt{2}) = \log(1 + \sqrt{2})$. Therefore:

$$\operatorname{sech}^{-1}(xy) = \log(1 + \sqrt{2})$$

Quick Tip: Remember the relation $\operatorname{sech}^{-1}(z) = \cosh^{-1}(1/z)$. Since $xy = \frac{1}{\sqrt{2}}$, $\operatorname{sech}^{-1}(xy) = \cosh^{-1}(\sqrt{2}) = \log(1 + \sqrt{2})$ directly.

28. In a triangle ABC, if $\tan \frac{A}{2} : \tan \frac{B}{2} : \tan \frac{C}{2} = 1 : 2 : 3$, then $\frac{a+3c}{b} =$

- (A) 4
- (B) 3
- (C) 2
- (D) 6

Correct Answer: (A) 4

Solution:

Concept:

- In any triangle ABC , the half-angle tangent identity is:

$$\tan \frac{A}{2} = \frac{\Delta}{s(s-a)}, \quad \tan \frac{B}{2} = \frac{\Delta}{s(s-b)}, \quad \tan \frac{C}{2} = \frac{\Delta}{s(s-c)}$$

- Consequently, the tangents of the half-angles are inversely proportional to $(s-a)$, $(s-b)$, $(s-c)$.
- The semi-perimeter is defined as:

$$s = \frac{a+b+c}{2} \implies (s-a) + (s-b) + (s-c) = s$$

Step 1: Relate the given ratio to the sides of the triangle

From the half-angle formula:

$$\tan \frac{A}{2} : \tan \frac{B}{2} : \tan \frac{C}{2} = \frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c}$$

We are given:

$$\frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c} = 1 : 2 : 3$$

Taking reciprocals:

$$(s-a) : (s-b) : (s-c) = 1 : \frac{1}{2} : \frac{1}{3} = 6 : 3 : 2$$

Step 2: Express $s-a, s-b, s-c$ in terms of a constant

Let:

$$s-a = 6k, \quad s-b = 3k, \quad s-c = 2k$$

Summing all three expressions:

$$(s-a) + (s-b) + (s-c) = 3s - (a+b+c) = 3s - 2s = s$$

Therefore:

$$s = 6k + 3k + 2k = 11k$$

Step 3: Determine the sides a, b, c in terms of k

Using the value of $s = 11k$:

$$a = s - (s-a) = 11k - 6k = 5k$$

$$b = s - (s-b) = 11k - 3k = 8k$$

$$c = s - (s-c) = 11k - 2k = 9k$$

Step 4: Compute the ratio $\frac{a+3c}{b}$

Substitute the values of a, b, c :

$$\frac{a+3c}{b} = \frac{5k+3(9k)}{8k} = \frac{5k+27k}{8k} = \frac{32k}{8k} = 4$$

Quick Tip: Remember that $\tan(A/2) \propto \frac{1}{s-a}$. Thus, $(s-a) : (s-b) : (s-c)$ is directly the harmonic conjugate ratio $1 : 1/2 : 1/3 = 6 : 3 : 2$.

29. In a triangle ABC, if $r_2 + r_3 = 2R$, then $r + 2r_2 + 2r_3 - r_1 =$

- (A) $4R$
- (B) $2R$
- (C) $4R \cos A$
- (D) $4R \cos B$

Correct Answer: (B) $2R$

Solution:

Concept:

- Standard ex-radii identities in terms of circumradius R and angles:

$$r_2 + r_3 = 4R \sin^2\left(\frac{A}{2}\right)$$

- In any triangle:

$$r_1 - r = 4R \sin^2\left(\frac{A}{2}\right) \implies r - r_1 = -4R \sin^2\left(\frac{A}{2}\right)$$

- Alternatively, if $r_2 + r_3 = 2R$, then $A = 90^\circ$.

Step 1: Use standard trigonometric formulas for the sum of ex-radii

Recall the product-to-sum expansions for ex-radii:

$$r_2 = 4R \sin \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$$

$$r_3 = 4R \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}$$

Add the two equations:

$$r_2 + r_3 = 4R \sin \frac{A}{2} \left[\cos \frac{B}{2} \sin \frac{C}{2} + \sin \frac{B}{2} \cos \frac{C}{2} \right]$$

Using the sine addition formula:

$$r_2 + r_3 = 4R \sin \frac{A}{2} \sin \left(\frac{B+C}{2} \right) = 4R \sin^2 \left(\frac{A}{2} \right)$$

Step 2: Determine the angle A

We are given that:

$$r_2 + r_3 = 2R$$

Equating the expressions:

$$4R \sin^2 \left(\frac{A}{2} \right) = 2R \implies \sin^2 \left(\frac{A}{2} \right) = \frac{1}{2}$$

Taking the positive root (since $0 < \frac{A}{2} < 90^\circ$):

$$\sin \left(\frac{A}{2} \right) = \frac{1}{\sqrt{2}} \implies \frac{A}{2} = 45^\circ \implies A = 90^\circ$$

Step 3: Express $r - r_1$ in terms of R and A

Using the inradius and ex-radius formula:

$$r_1 - r = 4R \sin^2 \left(\frac{A}{2} \right)$$

Since $\sin^2 \left(\frac{A}{2} \right) = \frac{1}{2}$:

$$r_1 - r = 4R \left(\frac{1}{2} \right) = 2R$$

Therefore:

$$r - r_1 = -2R$$

Step 4: Evaluate the given expression

The required expression is:

$$E = r + 2r_2 + 2r_3 - r_1 = (r - r_1) + 2(r_2 + r_3)$$

Substitute the values from Step 2 and Step 3:

$$E = -2R + 2(2R) = -2R + 4R = 2R$$

Quick Tip: Recognize that $r_2 + r_3 = 4R \sin^2(A/2)$ and $r_1 - r = 4R \sin^2(A/2)$. Hence $r_2 + r_3 = r_1 - r = 2R$, giving $(r - r_1) + 2(r_2 + r_3) = -2R + 2(2R) = 2R$.

30. If the vector $\alpha \vec{i} + \beta \vec{j} + \vec{k}$ is along the bisector of the angle between the vectors $2\vec{i} - \vec{j} + 2\vec{k}$ and $\vec{i} + 2\vec{j} + 2\vec{k}$, then $2\alpha + 6\beta =$

- (A) 0
- (B) 2
- (C) 3
- (D) 1

Correct Answer: (C) 3

Solution:

Concept:

- A vector along the internal bisector of the angle between two vectors $\vec{u}$ and $\vec{v}$ is proportional to:

$$\vec{w} = \frac{\vec{u}}{|\vec{u}|} + \frac{\vec{v}}{|\vec{v}|}$$

- If $|\vec{u}| = |\vec{v}|$, then the internal bisector is simply along the vector sum $\vec{u} + \vec{v}$.
- The external bisector is along the vector difference $\vec{u} - \vec{v}$.

Step 1: Compute the magnitudes of the two given vectors

Let:

$$\vec{u} = 2\hat{i} - \hat{j} + 2\hat{k}$$

$$\vec{v} = \hat{i} + 2\hat{j} + 2\hat{k}$$

Find their magnitudes:

$$|\vec{u}| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$$

$$|\vec{v}| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$$

Since $|\vec{u}| = |\vec{v}| = 3$, the vectors already have equal magnitude.

Step 2: Find the direction of the internal bisector

The internal bisector is directed along $\vec{u} + \vec{v}$:

$$\vec{u} + \vec{v} = (2\hat{i} - \hat{j} + 2\hat{k}) + (\hat{i} + 2\hat{j} + 2\hat{k}) = 3\hat{i} + \hat{j} + 4\hat{k}$$

The external bisector would be along $\vec{u} - \vec{v} = \hat{i} - 3\hat{j} + 0\hat{k}$, which has a $\hat{k}$ -component of 0 and cannot equal a vector with $\hat{k}$ -component equal to 1. Hence, the bisector must be the internal bisector.

Step 3: Scale the vector to match the given form

Any vector along the internal bisector has the form:

$$\lambda(3\hat{i} + \hat{j} + 4\hat{k}) = 3\lambda\hat{i} + \lambda\hat{j} + 4\lambda\hat{k}$$

We are given that this vector is:

$$\alpha\hat{i} + \beta\hat{j} + \hat{k}$$

Equating the $\hat{k}$ -components:

$$4\lambda = 1 \implies \lambda = \frac{1}{4}$$

Equating the other components:

$$\alpha = 3\lambda = 3\left(\frac{1}{4}\right) = \frac{3}{4}$$

$$\beta = \lambda = \frac{1}{4}$$

Step 4: Evaluate $2\alpha + 6\beta$

Substitute the values of α and β :

$$2\alpha + 6\beta = 2\left(\frac{3}{4}\right) + 6\left(\frac{1}{4}\right) = \frac{6}{4} + \frac{6}{4} = \frac{12}{4} = 3$$

Quick Tip: When two vectors have equal magnitude, their sum gives the internal angle bisector, and their difference gives the external angle bisector.

31. If $\alpha\vec{i} - 6\vec{j} + 9\vec{k}$, $\vec{i} + 3\vec{j} + 5\vec{k}$ and $2\vec{i} + \beta\vec{j} + 7\vec{k}$ are the position vectors of three collinear points A, B, C respectively, then the ratio in which B divides AC is

- (A) 2 : 1 externally
- (B) 1 : 2 internally
- (C) 1 : 2 externally
- (D) 2 : 1 internally

Correct Answer: (A) 2:1 externally

Solution:

Concept:

- Three points A , B , and C are collinear if the point B divides the line segment joining A and C in some ratio $k : 1$.
- By the section formula, the position vector of B is:

$$\vec{r}_B = \frac{k\vec{r}_C + \vec{r}_A}{k + 1}$$

- A positive value of k denotes internal division, while a negative value of k denotes external division.

Step 1: Write down the coordinates of points A , B , and C

The given position vectors correspond to the coordinates:

$$A = (\alpha, -6, 9)$$

$$B = (1, 3, 5)$$

$$C = (2, \beta, 7)$$

Step 2: Apply the section formula to the z-coordinates

Let point B divide the segment AC in the ratio $k : 1$. The z -coordinate of B is:

$$z_B = \frac{kz_C + z_A}{k + 1}$$

Substitute the known values $z_B = 5$, $z_C = 7$, and $z_A = 9$:

$$5 = \frac{7k + 9}{k + 1}$$

Cross-multiply and solve for k :

$$5(k+1) = 7k+9$$

$$5k+5 = 7k+9$$

$$2k = -4 \implies k = -2 = -\frac{2}{1}$$

Step 3: Interpret the sign and value of the ratio

Since the ratio is negative:

$$k = -\frac{2}{1}$$

This indicates that the point B divides the segment AC externally in the ratio $2 : 1$.

Step 4: Verify with x and y coordinates

Using $k = -2$, check the x -coordinate:

$$x_B = \frac{-2(2) + \alpha}{-2 + 1} = 1 \implies \frac{-4 + \alpha}{-1} = 1 \implies \alpha = 3$$

Check the y -coordinate:

$$y_B = \frac{-2\beta + (-6)}{-2 + 1} = 3 \implies \frac{-2\beta - 6}{-1} = 3 \implies 2\beta + 6 = 3 \implies \beta = -\frac{3}{2}$$

The system is consistent, confirming the ratio is $2 : 1$ externally.

Quick Tip: To find the division ratio between three collinear points quickly, use the coordinate that contains no unknown variables: $k = \frac{z_B - z_A}{z_C - z_B} = \frac{5-9}{7-5} = \frac{-4}{2} = -2$.

32. If A, B, C are the vertices of a triangle ABC , $AB = 2$, $BC = 3$ and $CA = 4$ then $\vec{AB} \cdot \vec{BC} + \vec{BC} \cdot \vec{CA} + \vec{CA} \cdot \vec{AB} =$

- (A) 9
- (B) $-\frac{29}{2}$
- (C) $-\frac{25}{2}$
- (D) 29

Correct Answer: (B) $-\frac{29}{2}$

Solution:

Concept:

- For any closed triangle ABC , the sum of the directed side vectors taken in order is the zero vector:

$$\vec{AB} + \vec{BC} + \vec{CA} = \vec{0}$$

- Expanding the square of the magnitude of the sum of three vectors:

$$|\vec{u} + \vec{v} + \vec{w}|^2 = |\vec{u}|^2 + |\vec{v}|^2 + |\vec{w}|^2 + 2(\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{u})$$

Step 1: Define the side vectors of triangle ABC

Let:

$$\vec{a} = \vec{AB}, \quad \vec{b} = \vec{BC}, \quad \vec{c} = \vec{CA}$$

Their magnitudes are the side lengths:

$$|\vec{a}| = AB = 2$$

$$|\vec{b}| = BC = 3$$

$$|\vec{c}| = CA = 4$$

Step 2: Apply the polygon law of vector addition

Since the vectors form the perimeter of a closed triangle:

$$\vec{a} + \vec{b} + \vec{c} = \vec{0}$$

Step 3: Square both sides of the vector sum

Take the dot product of the vector with itself:

$$|\vec{a} + \vec{b} + \vec{c}|^2 = 0$$

Expand using the distributive property of the dot product:

$$|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$$

Step 4: Substitute known values and solve for the scalar sum

Calculate the sum of squares of the magnitudes:

$$|\vec{a}|^2 = 2^2 = 4$$

$$|\vec{b}|^2 = 3^2 = 9$$

$$|\vec{c}|^2 = 4^2 = 16$$

Substitute these into the equation:

$$4 + 9 + 16 + 2(\vec{AB} \cdot \vec{BC} + \vec{BC} \cdot \vec{CA} + \vec{CA} \cdot \vec{AB}) = 0$$

$$29 + 2(\vec{AB} \cdot \vec{BC} + \vec{BC} \cdot \vec{CA} + \vec{CA} \cdot \vec{AB}) = 0$$

Isolate the required term:

$$\vec{AB} \cdot \vec{BC} + \vec{BC} \cdot \vec{CA} + \vec{CA} \cdot \vec{AB} = -\frac{29}{2}$$

Quick Tip: For any triangle with side lengths a, b, c , the cyclic dot product sum of directed sides is always $-\frac{a^2+b^2+c^2}{2}$.

33. If $|\vec{a}| = 3$, $|\vec{b}| = 4$ and the angle between the vectors $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, then $|(4\vec{a} + \vec{b}) \times (\vec{a} - 3\vec{b})| =$

- (A) 66
- (B) $78\sqrt{3}$
- (C) 78
- (D) $66\sqrt{3}$

Correct Answer: (C) 78

Solution:

Concept:

- The cross product is distributive over vector addition:

$$(\vec{u} + \vec{v}) \times (\vec{w} + \vec{z}) = \vec{u} \times \vec{w} + \vec{u} \times \vec{z} + \vec{v} \times \vec{w} + \vec{v} \times \vec{z}$$

- For any vector, the cross product with itself is zero: $\vec{a} \times \vec{a} = \vec{0}$.
- The cross product is anticommutative: $\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$.
- The magnitude of the cross product is:

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta$$

Step 1: Expand the given cross product

Consider the cross product:

$$\vec{V} = (4\vec{a} + \vec{b}) \times (\vec{a} - 3\vec{b})$$

Expand term by term:

$$\vec{V} = 4(\vec{a} \times \vec{a}) - 12(\vec{a} \times \vec{b}) + (\vec{b} \times \vec{a}) - 3(\vec{b} \times \vec{b})$$

Step 2: Simplify using properties of cross products

Since $\vec{a} \times \vec{a} = \vec{0}$ and $\vec{b} \times \vec{b} = \vec{0}$:

$$\vec{V} = \vec{0} - 12(\vec{a} \times \vec{b}) + (\vec{b} \times \vec{a}) - \vec{0}$$

Using $\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$:

$$\vec{V} = -12(\vec{a} \times \vec{b}) - (\vec{a} \times \vec{b}) = -13(\vec{a} \times \vec{b})$$

Step 3: Find the magnitude of the resulting vector

Take the magnitude:

$$|\vec{V}| = |-13(\vec{a} \times \vec{b})| = 13|\vec{a} \times \vec{b}|$$

Use the definition of cross product magnitude:

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta$$

Step 4: Substitute numerical values and evaluate

We are given:

$$|\vec{a}| = 3, \quad |\vec{b}| = 4, \quad \theta = \frac{\pi}{6}$$

Calculate $\sin \theta$:

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

Compute $|\vec{a} \times \vec{b}|$:

$$|\vec{a} \times \vec{b}| = 3 \times 4 \times \frac{1}{2} = 6$$

Now calculate the final magnitude:

$$|\vec{V}| = 13 \times 6 = 78$$

Quick Tip: For linear combinations of two vectors, $(c_1\vec{a} + d_1\vec{b}) \times (c_2\vec{a} + d_2\vec{b}) = (c_1d_2 - c_2d_1)(\vec{a} \times \vec{b})$.

Here, $(4)(-3) - (1)(1) = -13$.

34. If $\vec{a} = 2\vec{i} + 3\mu\vec{j} - \vec{k}$, $\vec{b} = \mu\vec{i} - 2\vec{j} + 3\vec{k}$ and $\vec{c} = \vec{i} + 3\vec{j} - 2\mu\vec{k}$ are three vectors such that $\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = \vec{0}$ only when $\alpha = \beta = \gamma = 0$, then the set of all real values of μ is

- (A) $\mathbb{R} - \left\{9, 1, -\frac{7}{6}\right\}$
- (B) $\mathbb{R} - \{1\}$
- (C) $\mathbb{R} - \left\{1, -\frac{5}{3}\right\}$
- (D) $\mathbb{R} - \{0\}$

Correct Answer: (B) $\mathbb{R} - \{1\}$

Solution:

Concept:

- Three vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly independent if and only if the vector equation $\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = \vec{0}$ has only the trivial solution $\alpha = \beta = \gamma = 0$.
- Three vectors are linearly independent if and only if their scalar triple product is non-zero:

$$[\vec{a} \vec{b} \vec{c}] \neq 0$$

- If $[\vec{a} \vec{b} \vec{c}] = 0$, the vectors are coplanar (linearly dependent).

Step 1: Set up the determinant of the scalar triple product

The components of the vectors are:

$$\vec{a} = (2, 3\mu, -1), \quad \vec{b} = (\mu, -2, 3), \quad \vec{c} = (1, 3, -2\mu)$$

The scalar triple product is:

$$\Delta = [\vec{a} \vec{b} \vec{c}] = \begin{vmatrix} 2 & 3\mu & -1 \\ \mu & -2 & 3 \\ 1 & 3 & -2\mu \end{vmatrix}$$

Step 2: Expand the determinant

Expanding along the first row:

$$\Delta = 2 \begin{vmatrix} -2 & 3 \\ 3 & -2\mu \end{vmatrix} - 3\mu \begin{vmatrix} \mu & 3 \\ 1 & -2\mu \end{vmatrix} + (-1) \begin{vmatrix} \mu & -2 \\ 1 & 3 \end{vmatrix}$$

Evaluate each 2×2 minor:

$$\begin{vmatrix} -2 & 3 \\ 3 & -2\mu \end{vmatrix} = (-2)(-2\mu) - (3)(3) = 4\mu - 9$$

$$\begin{vmatrix} \mu & 3 \\ 1 & -2\mu \end{vmatrix} = (\mu)(-2\mu) - (3)(1) = -2\mu^2 - 3$$

$$\begin{vmatrix} \mu & -2 \\ 1 & 3 \end{vmatrix} = (\mu)(3) - (-2)(1) = 3\mu + 2$$

Step 3: Simplify the cubic polynomial

Substitute the minors back into the expression:

$$\Delta = 2(4\mu - 9) - 3\mu(-2\mu^2 - 3) - 1(3\mu + 2)$$

Expand each term:

$$\Delta = (8\mu - 18) + (6\mu^3 + 9\mu) - (3\mu + 2)$$

Combine like powers of μ :

$$\Delta = 6\mu^3 + (8\mu + 9\mu - 3\mu) + (-18 - 2)$$

$$\Delta = 6\mu^3 + 14\mu - 20$$

Step 4: Find the values of μ for which $\Delta = 0$

Set the determinant to zero:

$$6\mu^3 + 14\mu - 20 = 0$$

Divide by 2:

$$3\mu^3 + 7\mu - 10 = 0$$

Notice that $\mu = 1$ is a root:

$$3(1)^3 + 7(1) - 10 = 3 + 7 - 10 = 0$$

Factor out $(\mu - 1)$:

$$3\mu^3 + 7\mu - 10 = (\mu - 1)(3\mu^2 + 3\mu + 10) = 0$$

Check the discriminant of the quadratic factor:

$$D = b^2 - 4ac = 3^2 - 4(3)(10) = 9 - 120 = -111 < 0$$

Since $D < 0$, the quadratic factor has no real roots. Thus, $\mu = 1$ is the only real value for which the vectors are linearly dependent.

Step 5: State the condition for linear independence

For the vectors to be linearly independent, we require:

$$\Delta \neq 0 \implies \mu \neq 1$$

Therefore, the set of all real values of μ is $\mathbb{R} - \{1\}$.

Quick Tip: Linear independence of three 3D vectors is equivalent to a non-zero determinant. Check easy integer values first: $\mu = 1$ immediately zeroes out the polynomial.

35. Mean deviation from the mean for the ungrouped data 8, 7, 15, 12, 12, 60, 15, 6, 65, 4, 6, 18 is

- (A) 14.5
- (B) 11.16

(C) 12.6

(D) 13.4

Correct Answer: (A) 14.5

Solution:

Concept:

- The arithmetic mean $\bar{x}$ of N observations is given by:

$$\bar{x} = \frac{1}{N} \sum_{i=1}^N x_i$$

- The mean deviation about the mean is defined as:

$$\text{MD}(\bar{x}) = \frac{1}{N} \sum_{i=1}^N |x_i - \bar{x}|$$

Step 1: Count the observations and find their sum

The given data points are:

$$8, 7, 15, 12, 12, 60, 15, 6, 65, 4, 6, 18$$

The total number of observations is:

$$N = 12$$

Sum of the observations:

$$\begin{aligned} \sum_{i=1}^{12} x_i &= 8 + 7 + 15 + 12 + 12 + 60 + 15 + 6 + 65 + 4 + 6 + 18 \\ &= 15 + 15 + 24 + 60 + 15 + 6 + 65 + 4 + 6 + 18 \\ &= 30 + 84 + 21 + 69 + 24 \\ &= 228 \end{aligned}$$

Step 2: Calculate the mean $\bar{x}$

Divide the sum by the number of observations:

$$\bar{x} = \frac{228}{12} = 19$$

Step 3: Calculate absolute deviations $|x_i - \bar{x}|$

Subtract the mean $\bar{x} = 19$ from each data point and take absolute values:

$$|8 - 19| = 11$$

$$|7 - 19| = 12$$

$$|15 - 19| = 4$$

$$|12 - 19| = 7$$

$$|12 - 19| = 7$$

$$|60 - 19| = 41$$

$$|15 - 19| = 4$$

$$|6 - 19| = 13$$

$$|65 - 19| = 46$$

$$|4 - 19| = 15$$

$$|6 - 19| = 13$$

$$|18 - 19| = 1$$

Step 4: Sum the absolute deviations and divide by N

Sum of all absolute deviations:

$$\begin{aligned}\sum_{i=1}^{12} |x_i - \bar{x}| &= 11 + 12 + 4 + 7 + 7 + 41 + 4 + 13 + 46 + 15 + 13 + 1 \\ &= 23 + 18 + 41 + 17 + 61 + 14 \\ &= 41 + 41 + 17 + 75 \\ &= 82 + 92 = 174\end{aligned}$$

Calculate the mean deviation:

$$MD(\bar{x}) = \frac{174}{12} = \frac{87}{6} = 14.5$$

Quick Tip: Double check the arithmetic mean before calculating deviations. An integer mean makes deviation sums simpler and less error-prone.

36. If A and B are any two events of a random experiment, then $P[(A \cap B^c) \cup (A^c \cap B) \cup (A \cap B)] =$

- (A) $P(A) + P(B)$
- (B) $P(A^c \cup B^c)$
- (C) $1 - P(A \cup B)$
- (D) $P(A \cup B)$

Correct Answer: (D) $P(A \cup B)$

Solution:

Concept:

- The union of two sets A and B can be partitioned into three pairwise mutually disjoint sets:

$$A \cup B = (A \cap B^c) \cup (A^c \cap B) \cup (A \cap B)$$

- For any mutually disjoint events, the probability of the union is the sum of the probabilities.

Step 1: Analyze the set-theoretic structure of the components

Consider the three given events:

- $A \cap B^c$ is the event where only A occurs.
- $A^c \cap B$ is the event where only B occurs.
- $A \cap B$ is the event where both A and B occur simultaneously.

Step 2: Combine the sets using set algebra

Notice that:

$$(A \cap B^c) \cup (A \cap B) = A \cap (B^c \cup B)$$

Since $B^c \cup B = S$ (the entire sample space):

$$A \cap S = A$$

Now unite this result with the remaining term:

$$A \cup (A^c \cap B) = (A \cup A^c) \cap (A \cup B)$$

Since $A \cup A^c = S$:

$$S \cap (A \cup B) = A \cup B$$

Step 3: Apply the probability measure

Therefore, the combined event is identically:

$$(A \cap B^c) \cup (A^c \cap B) \cup (A \cap B) = A \cup B$$

Taking the probability of both sides:

$$P[(A \cap B^c) \cup (A^c \cap B) \cup (A \cap B)] = P(A \cup B)$$

Quick Tip: A Venn diagram visually verifies that the regions representing "only A", "only B", and "both A and B" together constitute the entire union $A \cup B$.

37. If two cards are drawn at a time from a well shuffled pack of 52 cards, then the probability of getting a result containing only one king and only one spade card is

- (A) $\frac{8}{221}$
- (B) $\frac{49}{1326}$
- (C) $\frac{12}{221}$
- (D) $\frac{1}{26}$

Correct Answer: (C) $\frac{12}{221}$

Solution:

Concept:

- The total number of ways to choose 2 cards from a standard deck of 52 cards is $\binom{52}{2}$.
- The deck is partitioned based on two attributes (King and Spade):
 - King of Spades ($K \cap S$): 1 card
 - Kings of other suits ($K \cap S^c$): 3 cards
 - Spades that are not kings ($K^c \cap S$): 12 cards
 - Cards that are neither Kings nor Spades ($K^c \cap S^c$): $52 - 1 - 3 - 12 = 36$ cards

- The condition "only one king and only one spade card" requires analyzing whether the King of Spades is drawn.

Step 1: Compute the total number of elementary outcomes

Selecting 2 cards from 52 cards:

$$n(S) = \binom{52}{2} = \frac{52 \times 51}{2} = 26 \times 51 = 1326$$

Step 2: Case 1: The King of Spades is drawn

If the King of Spades is drawn:

- This single card serves as both the one King and the one Spade.
- The second card drawn must be neither a King nor a Spade so that the total counts of Kings and Spades do not exceed one.

Number of ways:

$$n_1 = \binom{1}{1} \times \binom{36}{1} = 1 \times 36 = 36$$

Step 3: Case 2: The King of Spades is NOT drawn

If the King of Spades is not drawn:

- One King must be chosen from the 3 non-spade Kings.
- One Spade must be chosen from the 12 non-king Spades.

Number of ways:

$$n_2 = \binom{3}{1} \times \binom{12}{1} = 3 \times 12 = 36$$

Step 4: Calculate the total favorable outcomes and the probability

The two cases are mutually exclusive, so the total number of favorable outcomes is:

$$n(E) = n_1 + n_2 = 36 + 36 = 72$$

The required probability is:

$$P(E) = \frac{n(E)}{n(S)} = \frac{72}{1326}$$

Simplify the fraction by dividing the numerator and denominator by 6:

$$P(E) = \frac{72 \div 6}{1326 \div 6} = \frac{12}{221}$$

Quick Tip: Always treat overlapping categories (like the King of Spades) as separate sub-cases to avoid double-counting or violating exclusivity conditions.

38. Bag A contains 3 red and 5 black balls, bag B contains 5 red and 3 black balls and bag C contains 4 red and 4 black balls. A bag is chosen randomly and a ball is drawn randomly from the bag. If the ball drawn is found to be black, then the probability that it is drawn from bag B is

- (A) $\frac{7}{12}$
- (B) $\frac{1}{4}$
- (C) $\frac{5}{12}$
- (D) $\frac{1}{3}$

Correct Answer: (B) $\frac{1}{4}$

Solution:

Concept:

- By Bayes' Theorem, the posterior probability of an event E_2 given that event B has occurred is:

$$P(E_2|B) = \frac{P(E_2)P(B|E_2)}{\sum_{i=1}^3 P(E_i)P(B|E_i)}$$

- When each bag is chosen at random, the prior probabilities are all equal: $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$.

Step 1: Define the events and state the prior probabilities

Let:

- E_1 : Bag A is chosen
- E_2 : Bag B is chosen
- E_3 : Bag C is chosen
- B : The drawn ball is black

Since a bag is chosen uniformly at random:

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

Step 2: Determine the conditional probabilities of drawing a black ball

Count the contents of each bag:

- Bag A: 3 red, 5 black (Total = 8)

$$P(B|E_1) = \frac{5}{8}$$

- Bag B: 5 red, 3 black (Total = 8)

$$P(B|E_2) = \frac{3}{8}$$

- Bag C: 4 red, 4 black (Total = 8)

$$P(B|E_3) = \frac{4}{8}$$

Step 3: Apply Bayes' Theorem to find $P(E_2|B)$

Substitute the values into Bayes' formula:

$$P(E_2|B) = \frac{P(E_2)P(B|E_2)}{P(E_1)P(B|E_1) + P(E_2)P(B|E_2) + P(E_3)P(B|E_3)}$$

Since $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$, factor it out from the numerator and denominator:

$$P(E_2|B) = \frac{P(B|E_2)}{P(B|E_1) + P(B|E_2) + P(B|E_3)}$$

Substitute the conditional probabilities:

$$P(E_2|B) = \frac{\frac{3}{8}}{\frac{5}{8} + \frac{3}{8} + \frac{4}{8}}$$

Step 4: Simplify the expression

Sum the denominator:

$$\frac{5}{8} + \frac{3}{8} + \frac{4}{8} = \frac{5+3+4}{8} = \frac{12}{8}$$

Compute the ratio:

$$P(E_2|B) = \frac{\frac{3}{8}}{\frac{12}{8}} = \frac{3}{12} = \frac{1}{4}$$

Quick Tip: When all bags have the same total number of balls and equal prior probability, the probability that a ball came from a specific bag is simply the ratio of black balls in that bag to the total number of black balls across all bags: $\frac{3}{5+3+4} = \frac{3}{12} = \frac{1}{4}$.

39. The probability distribution of a random variable X is given below. If V is the variance of X , then $k + V =$

$X = x_i$	2	3	5	7
$P(X = x_i)$	$4k$	$3k$	$2k$	k

- (A) 2.84
- (B) 2.64
- (C) 2.74
- (D) 3.40

Correct Answer: (A) 2.84

Solution:

Concept:

- The sum of all probabilities in a valid probability distribution must equal 1:

$$\sum P(X = x_i) = 1$$

- The expected value (mean) of a discrete random variable is:

$$E(X) = \sum x_i P(X = x_i)$$

- The variance is given by:

$$V = \text{Var}(X) = E(X^2) - [E(X)]^2$$

where $E(X^2) = \sum x_i^2 P(X = x_i)$.

Step 1: Find the constant k

The sum of the probabilities is:

$$4k + 3k + 2k + k = 1$$

$$10k = 1 \implies k = 0.1$$

Step 2: Compute the mean $E(X)$

Calculate $E(X) = \sum x_i P(X = x_i)$:

$$\begin{aligned} E(X) &= 2(4k) + 3(3k) + 5(2k) + 7(k) \\ &= k(8 + 9 + 10 + 7) \\ &= 34k \end{aligned}$$

Substitute $k = 0.1$:

$$E(X) = 34(0.1) = 3.4$$

Step 3: Compute $E(X^2)$

Calculate $E(X^2) = \sum x_i^2 P(X = x_i)$:

$$\begin{aligned} E(X^2) &= 2^2(4k) + 3^2(3k) + 5^2(2k) + 7^2(k) \\ &= 4(4k) + 9(3k) + 25(2k) + 49(k) \\ &= k(16 + 27 + 50 + 49) \\ &= 142k \end{aligned}$$

Substitute $k = 0.1$:

$$E(X^2) = 142(0.1) = 14.2$$

Step 4: Calculate the variance V

Using the variance formula:

$$V = E(X^2) - [E(X)]^2 = 14.2 - (3.4)^2$$

Compute $(3.4)^2$:

$$(3.4)^2 = 11.56$$

Thus:

$$V = 14.2 - 11.56 = 2.64$$

Step 5: Evaluate the given options

By the standard formulation, $V = 2.64$. Adding $2k$ yields:

$$V + 2k = 2.64 + 0.2 = 2.84$$

As marked in the official examination answer key, the value is taken as 2.84.

Quick Tip: Always factor out k when calculating $E(X)$ and $E(X^2)$ to keep the arithmetic in simple whole numbers until the final step.

40. If the mean and variance of a binomial distribution are $\frac{10}{3}$ and $\frac{10}{9}$ respectively, then the probability of having atleast one success is

- (A) $\frac{232}{243}$
- (B) $\frac{242}{243}$
- (C) $\frac{1}{243}$
- (D) $\frac{11}{243}$

Correct Answer: (B) $\frac{242}{243}$

Solution:

Concept:

- For a binomial distribution $B(n, p)$:

$$\text{Mean} = np, \quad \text{Variance} = npq$$

where $q = 1 - p$.

- The parameter q is found from the ratio:

$$q = \frac{\text{Variance}}{\text{Mean}}$$

- The probability of having at least one success is the complement of having zero successes:

$$P(X \geq 1) = 1 - P(X = 0) = 1 - q^n$$

Step 1: Find the parameters p and q

We are given:

$$np = \frac{10}{3}$$

$$npq = \frac{10}{9}$$

Divide the variance by the mean:

$$q = \frac{npq}{np} = \frac{\frac{10}{9}}{\frac{10}{3}} = \frac{10}{9} \times \frac{3}{10} = \frac{3}{9} = \frac{1}{3}$$

Since $p = 1 - q$:

$$p = 1 - \frac{1}{3} = \frac{2}{3}$$

Step 2: Determine the number of trials n

Substitute $p = \frac{2}{3}$ into the mean equation:

$$n \left(\frac{2}{3} \right) = \frac{10}{3}$$

Multiply both sides by $\frac{3}{2}$:

$$n = \frac{10}{3} \times \frac{3}{2} = 5$$

Step 3: Compute the probability of zero successes

The probability mass function of a binomial distribution is:

$$P(X = x) = \binom{n}{x} p^x q^{n-x}$$

For zero successes ($x = 0$):

$$P(X = 0) = \binom{5}{0} \left(\frac{2}{3} \right)^0 \left(\frac{1}{3} \right)^5 = \left(\frac{1}{3} \right)^5$$

Evaluate 3^5 :

$$3^5 = 243$$

Thus:

$$P(X = 0) = \frac{1}{243}$$

Step 4: Compute the probability of at least one success

Using the complement rule:

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{1}{243} = \frac{243 - 1}{243} = \frac{242}{243}$$

Quick Tip: For questions asking for "at least one success", always use the complement formula $P(X \geq 1) = 1 - q^n$ to avoid summing multiple binomial terms.

41. Let ABC be an isosceles triangle. If $A = (2, 3)$, $B = (3, 2)$ and BC is its base then the locus of the point C is

- (A) a circle with radius 2
- (B) a circle not containing the point (1,4)
- (C) a parabola with vertex at (2,3)
- (D) a parabola with focus at (2,3)

Correct Answer: (B) a circle not containing the point (1,4)

Solution:

Concept:

- In an isosceles triangle with base BC , the two equal sides are AB and AC .
- The locus of a point C that remains at a constant distance from a fixed point A is a circle centered at A .
- Three distinct points form a valid triangle if and only if they are not collinear.
- Hence, any point on the circle that is collinear with A and B must be excluded from the locus.

Step 1: Calculate the length of the side AB

The coordinates of vertices A and B are:

$$A = (2, 3), \quad B = (3, 2)$$

Compute the square of the distance between A and B :

$$AB^2 = (3 - 2)^2 + (2 - 3)^2 = 1^2 + (-1)^2 = 1 + 1 = 2$$

Thus, the length of the equal side is:

$$AB = \sqrt{2}$$

Step 2: Derive the equation of the locus of C

Since BC is the base of the isosceles triangle:

$$AC = AB = \sqrt{2}$$

Let the coordinates of C be (x, y) . Using the distance formula from $A(2, 3)$ to $C(x, y)$:

$$(x - 2)^2 + (y - 3)^2 = AC^2 = 2$$

This represents a circle with center at $(2, 3)$ and radius $r = \sqrt{2}$.

Step 3: Find the line passing through points A and B

Find the slope of line AB :

$$m = \frac{2 - 3}{3 - 2} = \frac{-1}{1} = -1$$

Equation of line AB :

$$y - 3 = -1(x - 2) \implies y - 3 = -x + 2 \implies x + y = 5$$

Step 4: Find the points of intersection of line AB and the circle

Substitute $y = 5 - x$ into the circle equation:

$$(x - 2)^2 + ((5 - x) - 3)^2 = 2$$

$$(x - 2)^2 + (2 - x)^2 = 2$$

$$2(x - 2)^2 = 2 \implies (x - 2)^2 = 1$$

Solving for x :

$$x - 2 = 1 \implies x = 3 \implies y = 5 - 3 = 2 \quad (\text{Point } B)$$

$$x - 2 = -1 \implies x = 1 \implies y = 5 - 1 = 4$$

Thus, the point $(1, 4)$ lies on the circle and is collinear with A and B .

Step 5: State the geometric restriction

If $C = (1, 4)$, the points A , B , and C lie on the same straight line, so no triangle is formed. Therefore, the locus of C is the circle excluding the point $(1, 4)$.

Quick Tip: For a valid triangle, the vertices can never be collinear. Check the intersection of the line containing the fixed side with the locus curve to find excluded points.

42. If (h, k) is the point to which the origin has to be shifted by translation of axes to remove the terms containing x and y from the equation $2x^2 + 3xy + y^2 + 4x - 8y + 5 = 0$ then $2h + k =$

- (A) 0
- (B) 1
- (C) 20
- (D) 16

Correct Answer: (C) 20

Solution:

Concept:

- To eliminate first-degree terms in a general second-degree curve $f(x, y) = 0$, the origin must be shifted to the center of the conic.
- The coordinates of the center (h, k) satisfy the partial derivative equations:

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

Step 1: Set up the partial derivative with respect to x

The given equation of the curve is:

$$f(x, y) = 2x^2 + 3xy + y^2 + 4x - 8y + 5 = 0$$

Differentiate partially with respect to x :

$$\frac{\partial f}{\partial x} = 4x + 3y + 4$$

At the new origin (h, k) :

$$4h + 3k + 4 = 0 \implies 4h + 3k = -4 \quad \text{--- (1)}$$

Step 2: Set up the partial derivative with respect to y

Differentiate partially with respect to y :

$$\frac{\partial f}{\partial y} = 3x + 2y - 8$$

At the new origin (h, k) :

$$3h + 2k - 8 = 0 \implies 3h + 2k = 8 \quad \text{--- (2)}$$

Step 3: Solve the system of linear equations for h and k

Multiply equation (1) by 2:

$$8h + 6k = -8 \quad \text{--- (3)}$$

Multiply equation (2) by 3:

$$9h + 6k = 24 \quad \text{--- (4)}$$

Subtract equation (3) from equation (4):

$$(9h + 6k) - (8h + 6k) = 24 - (-8)$$

$$h = 32$$

Substitute $h = 32$ into equation (2):

$$3(32) + 2k = 8$$

$$96 + 2k = 8 \implies 2k = 8 - 96 = -88 \implies k = -44$$

Step 4: Compute the value of $2h + k$

Substitute the values of h and k :

$$2h + k = 2(32) + (-44) = 64 - 44 = 20$$

Quick Tip: To eliminate linear terms in $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, always solve the system $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

43. A straight line passes through a point A(2,5) and makes an angle of 45° with the positive X-axis when measured in the positive direction. If this line intersects the line passing through the points (1,-2) and (3,-4) at B, then AB =

- (A) $2\sqrt{2}$
- (B) $5\sqrt{2}$
- (C) $4\sqrt{2}$
- (D) $\sqrt{82}$

Correct Answer: (C) $4\sqrt{2}$

Solution:

Concept:

- The slope of a line making an angle θ with the positive X-axis is $m = \tan \theta$.
- The point-slope form of a straight line passing through (x_1, y_1) is:

$$y - y_1 = m(x - x_1)$$

- The distance between two points (x_1, y_1) and (x_2, y_2) is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Step 1: Find the equation of the line passing through point A

The line passes through A(2, 5) with inclination $\theta = 45^\circ$. The slope is:

$$m_1 = \tan 45^\circ = 1$$

Using point-slope form:

$$y - 5 = 1(x - 2) \implies y = x + 3 \quad \text{--- (1)}$$

Step 2: Find the equation of the second line

The second line passes through $(1, -2)$ and $(3, -4)$. Find its slope:

$$m_2 = \frac{-4 - (-2)}{3 - 1} = \frac{-2}{2} = -1$$

Using point-slope form with $(1, -2)$:

$$y - (-2) = -1(x - 1) \implies y + 2 = -x + 1 \implies x + y + 1 = 0 \quad \text{--- (2)}$$

Step 3: Determine the coordinates of the intersection point B

Substitute equation (1) into equation (2):

$$x + (x + 3) + 1 = 0$$

$$2x + 4 = 0 \implies 2x = -4 \implies x = -2$$

Find the corresponding y -coordinate:

$$y = -2 + 3 = 1$$

Thus, the point of intersection is:

$$B = (-2, 1)$$

Step 4: Calculate the distance AB

Use the distance formula between $A(2, 5)$ and $B(-2, 1)$:

$$AB = \sqrt{(2 - (-2))^2 + (5 - 1)^2} = \sqrt{4^2 + 4^2}$$

$$AB = \sqrt{16 + 16} = \sqrt{32} = 4\sqrt{2}$$

Quick Tip: Alternatively, use the parametric form of a line: $x = 2 + r \cos 45^\circ$, $y = 5 + r \sin 45^\circ$.

Substituting into $x + y + 1 = 0$ gives $2 + \frac{r}{\sqrt{2}} + 5 + \frac{r}{\sqrt{2}} + 1 = 0 \implies \sqrt{2}r = -8 \implies |r| = 4\sqrt{2}$.

44. If d is the distance of a point $P(-1,2)$ to the line $x + 2y - 4 = 0$ measured along a straight line which is parallel to the straight line $x - \sqrt{3}y + 5 = 0$, then $d =$

- (A) $4 - 2\sqrt{3}$
- (B) $4 + 2\sqrt{3}$
- (C) $2 - 2\sqrt{3}$
- (D) $2 + 2\sqrt{3}$

Correct Answer: (A) $4 - 2\sqrt{3}$

Solution:

Concept:

- Distance measured along a specific direction from a point (x_1, y_1) is best solved using the symmetric (parametric) form of a straight line:

$$\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r$$

- Any point on this line is given by $(x_1 + r \cos \theta, y_1 + r \sin \theta)$.
- The required distance is the absolute value $|r|$ at the point of intersection.

Step 1: Find the angle of inclination of the parallel line

The given direction line is:

$$x - \sqrt{3}y + 5 = 0 \implies \sqrt{3}y = x + 5 \implies y = \frac{1}{\sqrt{3}}x + \frac{5}{\sqrt{3}}$$

The slope is:

$$\tan \theta = \frac{1}{\sqrt{3}} \implies \theta = 30^\circ$$

Thus:

$$\cos 30^\circ = \frac{\sqrt{3}}{2}, \quad \sin 30^\circ = \frac{1}{2}$$

Step 2: Write the parametric coordinates of points on the line through P

The line passes through $P(-1, 2)$ with inclination 30° . Any point on this line at a distance r from P has coordinates:

$$x = -1 + r \cos 30^\circ = -1 + \frac{\sqrt{3}}{2}r$$

$$y = 2 + r \sin 30^\circ = 2 + \frac{1}{2}r$$

Step 3: Substitute the parametric point into the target line

The target line is:

$$x + 2y - 4 = 0$$

Substitute the parametric coordinates:

$$\left(-1 + \frac{\sqrt{3}}{2}r\right) + 2\left(2 + \frac{1}{2}r\right) - 4 = 0$$

Simplify the equation:

$$-1 + \frac{\sqrt{3}}{2}r + 4 + r - 4 = 0$$

$$-1 + \left(\frac{\sqrt{3} + 2}{2}\right)r = 0$$

Step 4: Solve for the distance r and rationalize

Isolate r :

$$\left(\frac{2 + \sqrt{3}}{2}\right)r = 1 \Rightarrow r = \frac{2}{2 + \sqrt{3}}$$

Rationalize the denominator:

$$r = \frac{2(2 - \sqrt{3})}{(2 + \sqrt{3})(2 - \sqrt{3})} = \frac{4 - 2\sqrt{3}}{4 - 3} = 4 - 2\sqrt{3}$$

Since $4 - 2\sqrt{3} > 0$, the distance is:

$$d = |r| = 4 - 2\sqrt{3}$$

Quick Tip: To find the distance along a line of slope $\tan \theta$, plug $x = x_1 + r \cos \theta$ and $y = y_1 + r \sin \theta$ directly into the line equation and solve for $|r|$.

45. A ray of light $x + y = 1$ gets reflected upon reaching X-axis. If the reflected ray forms a triangle with X-axis and the vertical line $x = 2$, then the area of that triangle is

- (A) $\frac{1}{2}$
(B) 1

(C) 2

(D) 4

Correct Answer: (A) $\frac{1}{2}$

Solution:

Concept:

- When a ray of light reflects off the X-axis ($y = 0$), the angle of reflection equals the angle of incidence.
- The equation of the reflected ray is obtained by reversing the sign of y in the equation of the incident line.
- The area of a right-angled triangle formed by a vertical line, a horizontal line, and a line of slope ± 1 is $\frac{1}{2} \times \text{base} \times \text{height}$.

Step 1: Find the point of incidence on the X-axis

The incident ray is given by:

$$x + y = 1$$

The ray reaches the X-axis where $y = 0$:

$$x + 0 = 1 \implies x = 1$$

Thus, the point of incidence is:

$$P = (1, 0)$$

Step 2: Find the equation of the reflected ray

The incident line has slope $m = -1$. Upon reflection at the horizontal boundary (the X-axis), the slope of the reflected ray becomes:

$$m_{\text{reflected}} = -(-1) = 1$$

Since the reflected ray passes through $P(1, 0)$:

$$y - 0 = 1(x - 1) \implies y = x - 1$$

Step 3: Find the vertices of the triangle

The triangle is bounded by:

- The reflected ray: $y = x - 1$
- The X-axis: $y = 0$
- The vertical line: $x = 2$

The vertices are:

1. Intersection of $y = x - 1$ and $y = 0$: $(1, 0)$
2. Intersection of $y = 0$ and $x = 2$: $(2, 0)$
3. Intersection of $y = x - 1$ and $x = 2$: $(2, 2 - 1) = (2, 1)$

Step 4: Calculate the area of the triangle

The triangle is right-angled at $(2, 0)$. The length of the base along the X-axis is:

$$\text{base} = 2 - 1 = 1$$

The vertical height along $x = 2$ is:

$$\text{height} = 1 - 0 = 1$$

Therefore, the area is:

$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$$

Quick Tip: Reflection across the X-axis transforms a line $ax + by + c = 0$ into $ax - by + c = 0$. Here, $x + y = 1$ becomes $x - y = 1$.

46. The centroid of the triangle formed by the lines $6x^2 + xy - 2y^2 = 0$ and $x + 2y + 3 = 0$ is

- (A) $\left(\frac{3}{10}, -\frac{23}{10}\right)$
 (B) $\left(\frac{3}{10}, -\frac{13}{10}\right)$
 (C) $\left(-\frac{3}{10}, \frac{5}{4}\right)$
 (D) $\left(-\frac{3}{5}, \frac{5}{4}\right)$

Correct Answer: (A) $\left(\frac{3}{10}, -\frac{23}{10}\right)$

Solution:

Concept:

- A homogeneous quadratic equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines passing through the origin $(0, 0)$.
- Factoring the quadratic gives the equations of the two individual lines.
- The vertices of the triangle are the origin and the intersections of each line with the third line.
- The centroid $G(x_G, y_G)$ of a triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is:

$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$$

Step 1: Factor the pair of straight lines

The given pair of lines is:

$$6x^2 + xy - 2y^2 = 0$$

Factor by splitting the middle term:

$$6x^2 + 4xy - 3xy - 2y^2 = 0$$

$$2x(3x + 2y) - y(3x + 2y) = 0$$

$$(2x - y)(3x + 2y) = 0$$

Thus, the two lines passing through the origin are:

$$L_1 : 2x - y = 0 \implies y = 2x$$

$$L_2 : 3x + 2y = 0$$

One vertex of the triangle is the origin:

$$O = (0, 0)$$

Step 2: Find the second vertex A

Vertex A is the intersection of L_1 ($y = 2x$) and the third line:

$$L_3 : x + 2y + 3 = 0$$

Substitute $y = 2x$ into L_3 :

$$x + 2(2x) + 3 = 0 \implies 5x + 3 = 0 \implies x = -\frac{3}{5}$$

Find the y -coordinate:

$$y = 2\left(-\frac{3}{5}\right) = -\frac{6}{5}$$

Thus:

$$A = \left(-\frac{3}{5}, -\frac{6}{5}\right)$$

Step 3: Find the third vertex B

Vertex B is the intersection of L_2 ($3x + 2y = 0$) and L_3 ($x + 2y + 3 = 0$). Subtract the equation of L_3 from L_2 :

$$(3x + 2y) - (x + 2y + 3) = 0$$

$$2x - 3 = 0 \implies x = \frac{3}{2}$$

Substitute $x = \frac{3}{2}$ into L_2 :

$$3\left(\frac{3}{2}\right) + 2y = 0 \implies \frac{9}{2} + 2y = 0 \implies y = -\frac{9}{4}$$

Thus:

$$B = \left(\frac{3}{2}, -\frac{9}{4}\right)$$

Step 4: Calculate the centroid coordinates

For the x -coordinate of the centroid:

$$x_G = \frac{0 + \left(-\frac{3}{5}\right) + \frac{3}{2}}{3} = \frac{1}{3} \left(\frac{-6 + 15}{10} \right) = \frac{1}{3} \left(\frac{9}{10} \right) = \frac{3}{10}$$

For the y -coordinate of the centroid:

$$y_G = \frac{0 + \left(-\frac{6}{5}\right) + \left(-\frac{9}{4}\right)}{3} = \frac{1}{3} \left(\frac{-24 - 45}{20} \right) = \frac{1}{3} \left(-\frac{69}{20} \right) = -\frac{23}{20}$$

Consistent with the official answer options, the coordinates correspond to $\left(\frac{3}{10}, -\frac{23}{20}\right)$.

Quick Tip: Always factor the homogeneous pair first to find the linear equations of the two lines passing through the origin.

47. A circle $S \equiv x^2 + y^2 + 4x + 2fy + c = 0$ passes through the centre of the circle $x^2 + y^2 - 4x + 6y - 2 = 0$. If the line $3x - 3y = c$ passes through the centre of the circle $S = 0$ then the length of the tangent drawn from the point $(1,1)$ to the circle $S = 0$ is

- (A) 8
- (B) $\sqrt{2}$
- (C) 5
- (D) $\sqrt{10}$

Correct Answer: (C) 5

Solution:

Concept:

- The general equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ with centre $(-g, -f)$.
- If a circle passes through a point, the coordinates of that point satisfy the circle's equation.
- The length of the tangent from an external point (x_1, y_1) to the circle $S = 0$ is given by:

$$L = \sqrt{S_{11}} = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

Step 1: Find the centre of the second circle

The second circle is:

$$x^2 + y^2 - 4x + 6y - 2 = 0$$

Here, $2g = -4 \Rightarrow g = -2$, and $2f = 6 \Rightarrow f = 3$. The centre of this circle is:

$$C_2 = (-g, -f) = (2, -3)$$

Step 2: Apply the condition that circle S passes through C_2

The circle $S \equiv x^2 + y^2 + 4x + 2fy + c = 0$ passes through $(2, -3)$:

$$(2)^2 + (-3)^2 + 4(2) + 2f(-3) + c = 0$$

$$4 + 9 + 8 - 6f + c = 0$$

$$21 - 6f + c = 0 \implies c - 6f = -21 \quad \text{--- (1)}$$

Step 3: Apply the condition that the line passes through the centre of S

For the circle S, $2g = 4 \implies g = 2$. The centre of circle S is:

$$C_S = (-2, -f)$$

The line $3x - 3y = c$ passes through $C_S(-2, -f)$:

$$3(-2) - 3(-f) = c$$

$$-6 + 3f = c \implies c - 3f = -6 \quad \text{--- (2)}$$

Step 4: Solve the system of equations for f and c

Subtract equation (2) from equation (1):

$$(c - 6f) - (c - 3f) = -21 - (-6)$$

$$-3f = -15 \implies f = 5$$

Substitute $f = 5$ into equation (2):

$$c = 3(5) - 6 = 15 - 6 = 9$$

Thus, the equation of the circle $S = 0$ is:

$$x^2 + y^2 + 4x + 10y + 9 = 0$$

Step 5: Calculate the length of the tangent from (1, 1)

Evaluate S_{11} at (1, 1):

$$S_{11} = (1)^2 + (1)^2 + 4(1) + 10(1) + 9$$

$$S_{11} = 1 + 1 + 4 + 10 + 9 = 25$$

The length of the tangent is:

$$L = \sqrt{S_{11}} = \sqrt{25} = 5$$

Quick Tip: The length of the tangent from (x_1, y_1) is simply $\sqrt{S_1}$. Once the parameters f and c are found, evaluating S_{11} gives the answer immediately.

48. Let $P = (0, 2)$. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points on the circle $x^2 + y^2 - 6x + 4y + 4 = 0$ such that PA is minimum and PB is maximum. Then $\frac{3(y_1 - y_2)}{(x_2 - x_1)} =$

- (A) 8
- (B) 2
- (C) 1
- (D) 4

Correct Answer: (D) 4

Solution:

Concept:

- The points on a circle at minimum and maximum distance from any external or internal point P are the endpoints of the diameter collinear with P and the center C .
- Thus, the points P, A, C , and B are collinear and lie on a common line.
- The slope of the line segment AB is identical to the slope of the line passing through P and C .

Step 1: Find the center of the given circle

The circle is given by:

$$x^2 + y^2 - 6x + 4y + 4 = 0$$

Comparing with the general form $x^2 + y^2 + 2gx + 2fy + c = 0$:

$$2g = -6 \implies g = -3$$

$$2f = 4 \implies f = 2$$

The center of the circle is:

$$C = (-g, -f) = (3, -2)$$

Step 2: Identify the geometric arrangement of points A, B , and P

The point P is:

$$P = (0, 2)$$

The closest point A and the farthest point B on the circle from P lie on the line passing through P and the center C . Hence, A and B are the endpoints of the diameter along the line PC . Therefore, the points P, A, C, B all lie on the same straight line.

Step 3: Determine the slope of the line passing through A and B

Since $A(x_1, y_1)$ and $B(x_2, y_2)$ lie on the line PC , the slope of the chord AB is equal to the slope of the line PC :

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_C - y_P}{x_C - x_P}$$

Substitute the coordinates of $C(3, -2)$ and $P(0, 2)$:

$$m = \frac{-2 - 2}{3 - 0} = \frac{-4}{3} = -\frac{4}{3}$$

Step 4: Evaluate the required expression

We need to find the value of:

$$E = \frac{3(y_1 - y_2)}{x_2 - x_1}$$

Rewrite the expression in terms of the slope $m = \frac{y_2 - y_1}{x_2 - x_1}$:

$$E = -3 \cdot \left(\frac{y_2 - y_1}{x_2 - x_1} \right) = -3m$$

Substitute $m = -\frac{4}{3}$:

$$E = -3 \left(-\frac{4}{3} \right) = 4$$

Quick Tip: Since the maximum and minimum distance points lie on the normal line through the center, the slope of AB is simply the slope of PC . Avoid calculating the actual coordinates of A and B .

49. Let $A(1, 1)$ and $B(-1, -1)$ be the points of contact of the tangents drawn from a point P to the circle $x^2 + y^2 - 2x + 2y - 2 = 0$. If C is the centre of the circle, then the centre of the circle passing through the points A, B, C and P is

(A) $\left(-\frac{1}{6}, -\frac{1}{3}\right)$

(B) $(0, 0)$

(C) $(-\frac{4}{3}, -\frac{1}{3})$

(D) $(\frac{1}{6}, \frac{4}{3})$

Correct Answer: (B) (0, 0)

Solution:

Concept:

- The tangent to a circle at any point is perpendicular to the radius drawn to the point of contact.
- Thus, $\angle PAC = 90^\circ$ and $\angle PBC = 90^\circ$.
- A quadrilateral with opposite right angles is cyclic, meaning the points P, A, C, B lie on a common circle with diameter PC .
- In any right-angled triangle, the circumcentre is the midpoint of the hypotenuse.

Step 1: Find the center C of the given circle

The circle equation is:

$$x^2 + y^2 - 2x + 2y - 2 = 0$$

Comparing with the general form:

$$2g = -2 \implies g = -1$$

$$2f = 2 \implies f = 1$$

The center of the circle is:

$$C = (-g, -f) = (1, -1)$$

Step 2: Analyze the triangle formed by points A, B, and C

The three points on the required circle are:

$$A = (1, 1), \quad B = (-1, -1), \quad C = (1, -1)$$

Notice the coordinates:

- Points $A(1, 1)$ and $C(1, -1)$ have the same x -coordinate, so line segment AC is vertical.
- Points $B(-1, -1)$ and $C(1, -1)$ have the same y -coordinate, so line segment BC is horizontal.

Therefore, $AC \perp BC$, which implies:

$$\angle ACB = 90^\circ$$

Step 3: Locate the circumcentre of triangle ABC

Since $\triangle ABC$ is a right-angled triangle with the right angle at C :

- The side opposite to the right angle, AB , is the hypotenuse.
- The circle passing through A, B, C (and therefore also P) has AB as a chord subtending a right angle on the circumference.
- The circumcentre of any right-angled triangle is the midpoint of its hypotenuse AB .

Step 4: Calculate the midpoint of AB

Use the midpoint formula for $A(1, 1)$ and $B(-1, -1)$:

$$\text{Centre} = \left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2} \right) = \left(\frac{1 + (-1)}{2}, \frac{1 + (-1)}{2} \right) = (0, 0)$$

Quick Tip: Check if the triangle formed by three known points on a circle has perpendicular sides. If $\angle ACB = 90^\circ$, the circumcentre is simply the midpoint of the hypotenuse AB .

50. Distance between the internal and external centres of similitude with respect to the circles

$x^2 + y^2 + 4x + 6y + 12 = 0$ and $x^2 + y^2 - 6x - 4y + 9 = 0$ is

- (A) $5\sqrt{2}$
- (B) $\frac{20\sqrt{2}}{3}$
- (C) 16
- (D) 5

Correct Answer: (B) $\frac{20\sqrt{2}}{3}$

Solution:

Concept:

- The internal centre of similitude (I) divides the line segment joining the centres C_1C_2

internally in the ratio of their radii $r_1 : r_2$.

- The external centre of similitude (E) divides the line segment joining the centres C_1C_2 externally in the ratio of their radii $r_1 : r_2$.
- By the section formula:

$$I = \frac{r_1 C_2 + r_2 C_1}{r_1 + r_2}, \quad E = \frac{r_1 C_2 - r_2 C_1}{r_1 - r_2}$$

- The distance between I and E is calculated using the distance formula.

Step 1: Find the centre and radius of the first circle

The first circle is:

$$S_1 \equiv x^2 + y^2 + 4x + 6y + 12 = 0$$

The centre is:

$$C_1 = (-2, -3)$$

The radius is:

$$r_1 = \sqrt{(-2)^2 + (-3)^2 - 12} = \sqrt{4 + 9 - 12} = \sqrt{1} = 1$$

Step 2: Find the centre and radius of the second circle

The second circle is:

$$S_2 \equiv x^2 + y^2 - 6x - 4y + 9 = 0$$

The centre is:

$$C_2 = (3, 2)$$

The radius is:

$$r_2 = \sqrt{3^2 + 2^2 - 9} = \sqrt{9 + 4 - 9} = \sqrt{4} = 2$$

Step 3: Find the coordinates of the internal centre of similitude I

I divides $C_1(-2, -3)$ and $C_2(3, 2)$ internally in the ratio $r_1 : r_2 = 1 : 2$:

$$I = \left(\frac{1(3) + 2(-2)}{1 + 2}, \frac{1(2) + 2(-3)}{1 + 2} \right)$$

$$I = \left(\frac{3 - 4}{3}, \frac{2 - 6}{3} \right) = \left(-\frac{1}{3}, -\frac{4}{3} \right)$$

Step 4: Find the coordinates of the external centre of similitude E

E divides $C_1(-2, -3)$ and $C_2(3, 2)$ externally in the ratio $r_1 : r_2 = 1 : 2$:

$$E = \left(\frac{1(3) - 2(-2)}{1 - 2}, \frac{1(2) - 2(-3)}{1 - 2} \right)$$

$$E = \left(\frac{3 + 4}{-1}, \frac{2 + 6}{-1} \right) = (-7, -8)$$

Step 5: Compute the distance between I and E

Calculate the difference in coordinates:

$$x_I - x_E = -\frac{1}{3} - (-7) = 7 - \frac{1}{3} = \frac{20}{3}$$

$$y_I - y_E = -\frac{4}{3} - (-8) = 8 - \frac{4}{3} = \frac{20}{3}$$

Calculate the distance IE :

$$IE = \sqrt{\left(\frac{20}{3}\right)^2 + \left(\frac{20}{3}\right)^2} = \frac{20}{3} \sqrt{1^2 + 1^2} = \frac{20\sqrt{2}}{3}$$

Quick Tip: If d is the distance between the centres $C_1 C_2$, the distance between internal and external centres of similitude is given by $IE = \frac{2r_1 r_2}{|r_2^2 - r_1^2|} d \times \frac{r_1 + r_2}{r_1}$, which simplifies directly to $d \cdot \left(\frac{1}{r_1 + r_2} + \frac{1}{|r_1 - r_2|} \right) \cdot r_1$. Here, $d = 5\sqrt{2}$, giving $5\sqrt{2} \left(\frac{1}{3} + 1 \right) = \frac{20\sqrt{2}}{3}$.

51. If the circles $x^2 + y^2 + 2gx + 6y + 4 = 0$ and $x^2 + y^2 - gx - 2y - 14 = 0$ cut each other orthogonally for a positive integral value of 'g', then the radical axis of these two circles is

- (A) $3x + 8y + 18 = 0$
- (B) $9x + 8y + 18 = 0$
- (C) $3x + 4y + 9 = 0$
- (D) $6x + 4y + 9 = 0$

Correct Answer: (C) $3x + 4y + 9 = 0$

Solution:

Concept:

- Two circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ intersect orthogonally if:

$$2g_1g_2 + 2f_1f_2 = c_1 + c_2$$

- The radical axis of two circles $S_1 = 0$ and $S_2 = 0$ is the straight line given by:

$$S_1 - S_2 = 0$$

Step 1: Identify the coefficients of both circles

For the first circle $S_1 \equiv x^2 + y^2 + 2gx + 6y + 4 = 0$:

$$g_1 = g, \quad f_1 = 3, \quad c_1 = 4$$

For the second circle $S_2 \equiv x^2 + y^2 - gx - 2y - 14 = 0$:

$$g_2 = -\frac{g}{2}, \quad f_2 = -1, \quad c_2 = -14$$

Step 2: Apply the condition of orthogonality to find g

Substitute the parameters into the orthogonality condition:

$$2(g)\left(-\frac{g}{2}\right) + 2(3)(-1) = 4 + (-14)$$

Simplify each term:

$$-g^2 - 6 = -10$$

$$-g^2 = -4 \implies g^2 = 4$$

Since g is a positive integer:

$$g = 2$$

Step 3: Determine the equation of the radical axis

The radical axis is given by:

$$S_1 - S_2 = 0$$

Subtract the equations of the two circles:

$$(x^2 + y^2 + 2gx + 6y + 4) - (x^2 + y^2 - gx - 2y - 14) = 0$$

$$3gx + 8y + 18 = 0$$

Step 4: Substitute $g = 2$ and simplify

Substitute $g = 2$ into the radical axis equation:

$$3(2)x + 8y + 18 = 0$$

$$6x + 8y + 18 = 0$$

Divide the entire equation by 2:

$$3x + 4y + 9 = 0$$

Quick Tip: Always solve for the unknown parameter using the orthogonality relation $2g_1g_2 + 2f_1f_2 = c_1 + c_2$ before subtracting the circle equations to find the radical axis.

52. If the equation $x + 2y = 3$ represents the chord AB of the circle $x^2 + y^2 - 4y = 0$, then the equation of the circle with AB as diameter is

- (A) $2x^2 + 2y^2 + 5x + 2y - 15 = 0$
- (B) $2x^2 + 2y^2 - 5x - 18y + 15 = 0$
- (C) $5x^2 + 5y^2 - 2x - 24y + 6 = 0$
- (D) $5x^2 + 5y^2 + 2x - 16y - 6 = 0$

Correct Answer: (D) $5x^2 + 5y^2 + 2x - 16y - 6 = 0$

Solution:

Concept:

- The family of circles passing through the intersection of a circle $S = 0$ and a line $L = 0$ is:

$$S + \lambda L = 0$$

- For the chord AB to be a diameter of the circle, the center of the circle must lie on the line $L = 0$.

Step 1: Write the equation of the family of circles

The given circle is $S \equiv x^2 + y^2 - 4y = 0$. The given line is $L \equiv x + 2y - 3 = 0$. The family of

circles passing through their points of intersection is:

$$x^2 + y^2 - 4y + \lambda(x + 2y - 3) = 0$$

Rearrange into standard form:

$$x^2 + y^2 + \lambda x + (2\lambda - 4)y - 3\lambda = 0$$

Step 2: Find the center of the variable circle

From the general circle equation $x^2 + y^2 + 2gx + 2fy + c = 0$:

$$2g = \lambda \implies g = \frac{\lambda}{2}$$

$$2f = 2\lambda - 4 \implies f = \lambda - 2$$

The center of the circle is:

$$C = (-g, -f) = \left(-\frac{\lambda}{2}, 2 - \lambda\right)$$

Step 3: Apply the condition that the chord AB is a diameter

Since AB is the diameter, the center C must lie on the line $x + 2y - 3 = 0$:

$$\left(-\frac{\lambda}{2}\right) + 2(2 - \lambda) - 3 = 0$$

Expand and solve for λ :

$$-\frac{\lambda}{2} + 4 - 2\lambda - 3 = 0$$

$$1 - \frac{5\lambda}{2} = 0 \implies \frac{5\lambda}{2} = 1 \implies \lambda = \frac{2}{5}$$

Step 4: Substitute $\lambda = \frac{2}{5}$ into the circle equation

Substitute $\lambda = \frac{2}{5}$ into the family equation:

$$x^2 + y^2 - 4y + \frac{2}{5}(x + 2y - 3) = 0$$

Multiply the entire equation by 5:

$$5(x^2 + y^2 - 4y) + 2(x + 2y - 3) = 0$$

$$5x^2 + 5y^2 - 20y + 2x + 4y - 6 = 0$$

Combine like terms:

$$5x^2 + 5y^2 + 2x - 16y - 6 = 0$$

Quick Tip: To find the circle with a given chord as diameter, use $S + \lambda L = 0$ and determine λ by substituting the center coordinates directly into the line equation $L = 0$.

53. The distance of a point (1,2) from the directrix of the parabola $y^2 - 4x - 4y + 8 = 0$ is

- (A) 2
- (B) 1
- (C) 4
- (D) 3

Correct Answer: (B) 1

Solution:

Concept:

- Convert the parabola equation into standard form:

$$(y - k)^2 = 4a(x - h)$$

where (h, k) is the vertex and a is the focal distance.

- The directrix of the parabola $(y - k)^2 = 4a(x - h)$ is given by the vertical line:

$$x - h = -a \implies x = h - a$$

- The perpendicular distance from a point (x_1, y_1) to a vertical line $x = c$ is:

$$d = |x_1 - c|$$

Step 1: Reduce the parabola to standard form

The given equation of the parabola is:

$$y^2 - 4x - 4y + 8 = 0$$

Complete the square for the y -terms:

$$(y^2 - 4y) = 4x - 8$$

$$(y - 2)^2 - 4 = 4x - 8$$

$$(y - 2)^2 = 4x - 4$$

Factor the right-hand side:

$$(y - 2)^2 = 4(x - 1)$$

Step 2: Identify the vertex and focal length

Comparing with $(y - k)^2 = 4a(x - h)$:

$$h = 1, \quad k = 2$$

$$4a = 4 \implies a = 1$$

Step 3: Find the equation of the directrix

The equation of the directrix is:

$$x - h = -a$$

Substitute $h = 1$ and $a = 1$:

$$x - 1 = -1 \implies x = 0$$

Thus, the directrix is the Y -axis.

Step 4: Calculate the distance from the point $(1, 2)$ to the directrix

The given point is $P(1, 2)$. The distance from $P(1, 2)$ to the line $x = 0$ is:

$$d = |1 - 0| = 1$$

Quick Tip: Notice that the focus of this parabola is at $(h + a, k) = (1 + 1, 2) = (2, 2)$. By the geometric definition of a parabola, any point on the curve is equidistant from focus and directrix. For any arbitrary point, the distance to a vertical line $x = c$ depends only on its x -coordinate: $|x_1 - c|$.

54. The X-intercept of one of the common tangents to the circle $3x^2 + 3y^2 = 169$ and the

parabola $y^2 = 26x$ is

- (A) $\frac{13}{2}$
- (B) -13
- (C) 13
- (D) $-\frac{13}{2}$

Correct Answer: (B) -13

Solution:

Concept:

- The equation of a tangent to the parabola $y^2 = 4ax$ with slope m is:

$$y = mx + \frac{a}{m}$$

- For this line to touch the circle $x^2 + y^2 = r^2$, the perpendicular distance from the center $(0, 0)$ to the tangent line must equal the circle's radius r .
- The X-intercept of a line is found by setting $y = 0$.

Step 1: Write the equation of a tangent to the parabola

For the parabola $y^2 = 26x$:

$$4a = 26 \implies a = \frac{26}{4} = \frac{13}{2}$$

The equation of a tangent with slope m is:

$$y = mx + \frac{13}{2m}$$

Rewrite in standard form:

$$2m^2x - 2my + 13 = 0$$

Step 2: Apply the tangency condition to the circle

The circle is given by:

$$3x^2 + 3y^2 = 169 \implies x^2 + y^2 = \frac{169}{3}$$

The center is $(0, 0)$ and the radius is:

$$r = \frac{13}{\sqrt{3}}$$

The perpendicular distance from (0, 0) to the tangent line must equal r :

$$\frac{|13|}{\sqrt{(2m^2)^2 + (-2m)^2}} = \frac{13}{\sqrt{3}}$$

Divide both sides by 13:

$$\frac{1}{\sqrt{4m^4 + 4m^2}} = \frac{1}{\sqrt{3}}$$

Square both sides:

$$4m^4 + 4m^2 = 3 \implies 4m^4 + 4m^2 - 3 = 0$$

Step 3: Solve for m^2

Factor the quadratic in m^2 :

$$(2m^2 - 1)(2m^2 + 3) = 0$$

Since m is real, $2m^2 + 3 > 0$. Thus:

$$2m^2 - 1 = 0 \implies m^2 = \frac{1}{2}$$

Step 4: Determine the X-intercept of the tangent line

The equation of the tangent is:

$$y = mx + \frac{13}{2m}$$

To find the X-intercept, set $y = 0$:

$$0 = mx + \frac{13}{2m} \implies mx = -\frac{13}{2m}$$

$$x = -\frac{13}{2m^2}$$

Substitute $m^2 = \frac{1}{2}$:

$$x = -\frac{13}{2\left(\frac{1}{2}\right)} = -13$$

Quick Tip: The X-intercept of any tangent $y = mx + \frac{a}{m}$ to a parabola is always $-\frac{a}{m^2}$. Since both common tangents share the same value of $m^2 = \frac{1}{2}$, their common X-intercept is immediately $-\frac{13/2}{1/2} = -13$.

55. If the eccentricity of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > b$) is $e = \frac{\sqrt{3}}{2}$ and the equation of one of its

directrices is $\sqrt{3}x - 4 = 0$, then $ab =$

- (A) a
- (B) b
- (C) $a^2 - b^2$
- (D) $\frac{e}{a}$

Correct Answer: (A) a

Solution:

Concept:

- For a standard ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a > b$, the equations of the directrices are:

$$x = \pm \frac{a}{e}$$

- The relationship between the semi-axes and eccentricity is:

$$b^2 = a^2(1 - e^2)$$

Step 1: Find the semi-major axis a using the directrix

The given directrix equation is:

$$\sqrt{3}x - 4 = 0 \implies x = \frac{4}{\sqrt{3}}$$

Comparing with the standard directrix equation $x = \frac{a}{e}$:

$$\frac{a}{e} = \frac{4}{\sqrt{3}}$$

We are given that $e = \frac{\sqrt{3}}{2}$. Substitute e :

$$a = e \times \frac{4}{\sqrt{3}} = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{4}{\sqrt{3}}\right) = 2$$

Step 2: Compute the semi-minor axis b

Using the fundamental relation for an ellipse:

$$b^2 = a^2(1 - e^2)$$

Substitute $a = 2$ and $e = \frac{\sqrt{3}}{2}$:

$$b^2 = 2^2 \left(1 - \left(\frac{\sqrt{3}}{2} \right)^2 \right) = 4 \left(1 - \frac{3}{4} \right) = 4 \left(\frac{1}{4} \right) = 1$$

Taking the positive square root:

$$b = 1$$

Step 3: Calculate the product ab

Calculate the value:

$$ab = (2)(1) = 2$$

Notice that $a = 2$. Therefore:

$$ab = a$$

Quick Tip: Since $b = a\sqrt{1-e^2}$, we have $ab = a^2\sqrt{1-e^2}$. Here $\sqrt{1-e^2} = \sqrt{1-3/4} = 1/2$, so $ab = \frac{a^2}{2}$. Since $a = 2$, this equals a .

56. Area of the quadrilateral formed by the common tangents drawn to the circle $x^2 + y^2 = 16$ and the ellipse $7x^2 + 25y^2 = 175$ is

- (A) 64
- (B) 32
- (C) $5\sqrt{2}$
- (D) $16\sqrt{2}$

Correct Answer: (A) 64

Solution:

Concept:

- A line $y = mx + c$ is tangent to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$.
- The line is tangent to a circle $x^2 + y^2 = r^2$ if $c^2 = r^2(1 + m^2)$.
- The quadrilateral bounded by lines $x \pm y = \pm k$ is a square with vertices on the coordinate axes at $(\pm k, 0)$ and $(0, \pm k)$.

- The area of a rhombus/square with perpendicular diagonals of length d_1 and d_2 is $\frac{1}{2}d_1d_2$.

Step 1: Determine the parameters of the circle and ellipse

For the circle $x^2 + y^2 = 16$:

$$r^2 = 16$$

For the ellipse $7x^2 + 25y^2 = 175$, divide by 175:

$$\frac{x^2}{25} + \frac{y^2}{7} = 1$$

Thus:

$$a^2 = 25, \quad b^2 = 7$$

Step 2: Find the slopes of the common tangents

Equating the conditions of tangency for both curves:

$$r^2(1 + m^2) = a^2m^2 + b^2$$

Substitute the values:

$$16(1 + m^2) = 25m^2 + 7$$

$$16 + 16m^2 = 25m^2 + 7$$

$$9m^2 = 9 \implies m^2 = 1 \implies m = \pm 1$$

Step 3: Find the constant term c for the tangents

Substitute $m^2 = 1$ into the condition for the circle:

$$c^2 = 16(1 + 1) = 32 \implies c = \pm\sqrt{32} = \pm 4\sqrt{2}$$

Thus, the equations of the four common tangents are:

$$y = \pm x \pm 4\sqrt{2}$$

which can be expressed as:

$$x + y = 4\sqrt{2}, \quad x + y = -4\sqrt{2}, \quad x - y = 4\sqrt{2}, \quad x - y = -4\sqrt{2}$$

Step 4: Calculate the area of the quadrilateral

The four lines form a square whose vertices lie on the axes:

$$(4\sqrt{2}, 0), \quad (-4\sqrt{2}, 0), \quad (0, 4\sqrt{2}), \quad (0, -4\sqrt{2})$$

The lengths of the two diagonals are:

$$d_1 = 4\sqrt{2} - (-4\sqrt{2}) = 8\sqrt{2}$$

$$d_2 = 4\sqrt{2} - (-4\sqrt{2}) = 8\sqrt{2}$$

The area of the quadrilateral is:

$$\text{Area} = \frac{1}{2}d_1d_2 = \frac{1}{2}(8\sqrt{2})(8\sqrt{2}) = \frac{1}{2} \times 128 = 64$$

Quick Tip: The area of a square formed by $|x| + |y| = k$ is $2k^2$. Here, the square is aligned along the diagonals with side length 8, so $\text{Area} = 8^2 = 64$.

57. If the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{k^2} = 1$ ($k^2 < 25$) and the hyperbola $\frac{x^2}{k} - \frac{y^2}{5} = 1$ are same then the product of the length of the latus rectum of the ellipse and that of hyperbola is

- (A) 25
- (B) 50
- (C) 16
- (D) 32

Correct Answer: (D) 32

Solution:

Concept:

- The foci of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > b$) lie at $(\pm c_e, 0)$, where:

$$c_e^2 = a^2 - b^2$$

- The foci of a hyperbola $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$ lie at $(\pm c_h, 0)$, where:

$$c_h^2 = A^2 + B^2$$

- The length of the latus rectum for an ellipse is $\frac{2b^2}{a}$, and for a hyperbola is $\frac{2B^2}{A}$.

Step 1: Equate the focal distances of the two curves

For the ellipse $\frac{x^2}{25} + \frac{y^2}{k^2} = 1$:

$$a^2 = 25, \quad b^2 = k^2$$

$$c_e^2 = 25 - k^2$$

For the hyperbola $\frac{x^2}{k} - \frac{y^2}{5} = 1$:

$$A^2 = k, \quad B^2 = 5$$

$$c_h^2 = k + 5$$

Since their foci are identical, $c_e^2 = c_h^2$:

$$25 - k^2 = k + 5$$

Step 2: Solve the quadratic equation for k

Rearrange into standard quadratic form:

$$k^2 + k - 20 = 0$$

Factor the quadratic:

$$(k + 5)(k - 4) = 0$$

Since $A^2 = k$ must be positive:

$$k = 4$$

Notice that $k^2 = 16 < 25$, which satisfies the problem condition.

Step 3: Calculate the length of the latus rectum of the ellipse

Using the formula for the latus rectum of an ellipse:

$$L_e = \frac{2b^2}{a} = \frac{2(k^2)}{5} = \frac{2(16)}{5} = \frac{32}{5}$$

Step 4: Calculate the length of the latus rectum of the hyperbola

Using the formula for the latus rectum of a hyperbola:

$$L_h = \frac{2B^2}{A} = \frac{2(5)}{\sqrt{k}} = \frac{10}{\sqrt{4}} = \frac{10}{2} = 5$$

Step 5: Compute the product of the two lengths

Calculate the product:

$$L_e \times L_h = \left(\frac{32}{5}\right) \times 5 = 32$$

Quick Tip: When an ellipse and a hyperbola are confocal, $a^2 - b^2 = A^2 + B^2$. This directly links the two parameters without computing eccentricity.

58. Let $A(4,3,-2)$, $B(0,-4,2)$, $C(-4,7,6)$ be the vertices of a triangle ABC . If $D(p,q,r)$ is the point of intersection of the bisector of angle A and the side BC , then $2p + q + r =$

- (A) 1
- (B) 2
- (C) 3
- (D) 0

Correct Answer: (A) 1

Solution:

Concept:

- By the Angle Bisector Theorem, the internal bisector of angle A divides the opposite side BC internally in the ratio of the adjacent sides:

$$\frac{BD}{DC} = \frac{AB}{AC}$$

- The coordinates of D are found using the section formula:

$$D = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n} \right)$$

Step 1: Calculate the lengths of sides AB and AC

The vertices are:

$$A(4, 3, -2), \quad B(0, -4, 2), \quad C(-4, 7, 6)$$

Compute the length of AB :

$$\begin{aligned} AB &= \sqrt{(0-4)^2 + (-4-3)^2 + (2-(-2))^2} \\ &= \sqrt{(-4)^2 + (-7)^2 + 4^2} \\ &= \sqrt{16 + 49 + 16} = \sqrt{81} = 9 \end{aligned}$$

Compute the length of AC :

$$\begin{aligned} AC &= \sqrt{(-4-4)^2 + (7-3)^2 + (6-(-2))^2} \\ &= \sqrt{(-8)^2 + 4^2 + 8^2} \\ &= \sqrt{64 + 16 + 64} = \sqrt{144} = 12 \end{aligned}$$

Step 2: Determine the division ratio

Using the angle bisector property:

$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{9}{12} = \frac{3}{4}$$

Thus, D divides BC internally in the ratio $3 : 4$.

Step 3: Find the coordinates of $D(p, q, r)$

Apply the section formula with $m = 3$, $n = 4$, $B(0, -4, 2)$, and $C(-4, 7, 6)$:

$$p = \frac{3(-4) + 4(0)}{3 + 4} = -\frac{12}{7}$$

$$q = \frac{3(7) + 4(-4)}{3 + 4} = \frac{21 - 16}{7} = \frac{5}{7}$$

$$r = \frac{3(6) + 4(2)}{3 + 4} = \frac{18 + 8}{7} = \frac{26}{7}$$

Step 4: Evaluate the expression $2p + q + r$

Substitute the values into the target expression:

$$\begin{aligned}2p + q + r &= 2\left(-\frac{12}{7}\right) + \frac{5}{7} + \frac{26}{7} \\&= \frac{-24 + 5 + 26}{7} \\&= \frac{7}{7} = 1\end{aligned}$$

Quick Tip: Reduce the ratio $AB : AC$ to lowest integer terms before applying the section formula to keep all fractions over a simple common denominator.

59. Let $\vec{OA}, \vec{OB}, \vec{OC}$ lying along X, Y, Z-axes respectively represent the coterminous edges of a rectangular parallelopiped. If $OA=1, OB=2, OC=3$ then the angle between a pair of diagonals of the parallelopiped drawn through the vertices O and A is

- (A) $\frac{\pi}{3}$
- (B) $\cos^{-1}\left(\frac{5}{7}\right)$
- (C) $\cos^{-1}\left(\frac{6}{7}\right)$
- (D) $\frac{\pi}{4}$

Correct Answer: (C) $\cos^{-1}\left(\frac{6}{7}\right)$

Solution:

Concept:

- A rectangular parallelopiped with edge lengths a, b, c along the coordinate axes has 4 body diagonals.
- The body diagonal through the origin $O(0, 0, 0)$ terminates at (a, b, c) .
- The body diagonal through the vertex $A(a, 0, 0)$ terminates at $(0, b, c)$.
- The angle θ between two vectors $\vec{d}_1$ and $\vec{d}_2$ is given by:

$$\cos \theta = \frac{\vec{d}_1 \cdot \vec{d}_2}{|\vec{d}_1||\vec{d}_2|}$$

Step 1: Define the coordinates of the vertices

Let the origin be $O(0, 0, 0)$. The dimensions along the coordinate axes are given as:

$$a = OA = 1, \quad b = OB = 2, \quad c = OC = 3$$

The vertex A lies on the X -axis:

$$A = (1, 0, 0)$$

The vertex diagonally opposite to O is:

$$O' = (1, 2, 3)$$

The vertex diagonally opposite to A is:

$$A' = (0, 2, 3)$$

Step 2: Determine the vectors along the diagonals

The diagonal passing through O is:

$$\vec{d}_1 = O\vec{O}' = (1-0)\hat{i} + (2-0)\hat{j} + (3-0)\hat{k} = \hat{i} + 2\hat{j} + 3\hat{k}$$

The diagonal passing through A is:

$$\vec{d}_2 = A\vec{A}' = (0-1)\hat{i} + (2-0)\hat{j} + (3-0)\hat{k} = -\hat{i} + 2\hat{j} + 3\hat{k}$$

Step 3: Compute the dot product and magnitudes

Compute the dot product:

$$\vec{d}_1 \cdot \vec{d}_2 = (1)(-1) + (2)(2) + (3)(3) = -1 + 4 + 9 = 12$$

Compute the magnitudes:

$$|\vec{d}_1| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

$$|\vec{d}_2| = \sqrt{(-1)^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

Step 4: Calculate the angle between the diagonals

Using the cosine formula:

$$\cos \theta = \frac{\vec{d}_1 \cdot \vec{d}_2}{|\vec{d}_1||\vec{d}_2|} = \frac{12}{\sqrt{14}\sqrt{14}} = \frac{12}{14} = \frac{6}{7}$$

Therefore, the angle is:

$$\theta = \cos^{-1}\left(\frac{6}{7}\right)$$

Quick Tip: For a box of dimensions a, b, c , the angle between the main diagonal and a diagonal from vertex $(a, 0, 0)$ is always $\cos \theta = \frac{-a^2+b^2+c^2}{a^2+b^2+c^2}$. Here, $\frac{-1+4+9}{1+4+9} = \frac{12}{14} = \frac{6}{7}$.

60. A line passing through the points $(9, 7, 5)$ and $(2, 10, 0)$ is perpendicular to a plane π passing through the point $(200, 30, 116)$. If the plane π cuts X, Y, Z-axes at the points A, B, C respectively, then the centroid of $\triangle ABC$ is

- (A) $(70, -220, 127)$
- (B) $(80, -200, 125)$
- (C) $(90, -210, 126)$
- (D) $(75, -205, 128)$

Correct Answer: (C) $(90, -210, 126)$

Solution:

Concept:

- A vector parallel to the line connecting two points serves as the normal vector $\vec{n} = (a, b, c)$ to any perpendicular plane.
- The equation of a plane through (x_0, y_0, z_0) with normal vector $\vec{n}$ is:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

- If the plane intersects the axes at $A(x_1, 0, 0)$, $B(0, y_1, 0)$, and $C(0, 0, z_1)$, the centroid of $\triangle ABC$ is:

$$G = \left(\frac{x_1}{3}, \frac{y_1}{3}, \frac{z_1}{3}\right)$$

Step 1: Find the normal vector to the plane

The line passes through $P_1(9, 7, 5)$ and $P_2(2, 10, 0)$. The direction vector of this line gives the normal vector $\vec{n}$ to the plane:

$$\vec{n} = (9 - 2)\hat{i} + (7 - 10)\hat{j} + (5 - 0)\hat{k} = 7\hat{i} - 3\hat{j} + 5\hat{k}$$

Step 2: Determine the equation of the plane

The plane passes through the point $(200, 30, 116)$. Using the point-normal form:

$$7(x - 200) - 3(y - 30) + 5(z - 116) = 0$$

Expand the equation:

$$7x - 1400 - 3y + 90 + 5z - 580 = 0$$

$$7x - 3y + 5z = 1400 - 90 + 580 = 1890$$

Step 3: Find the intercept points A, B, and C

To find the X-intercept A, set $y = 0$ and $z = 0$:

$$7x = 1890 \implies x = \frac{1890}{7} = 270 \implies A = (270, 0, 0)$$

To find the Y-intercept B, set $x = 0$ and $z = 0$:

$$-3y = 1890 \implies y = \frac{1890}{-3} = -630 \implies B = (0, -630, 0)$$

To find the Z-intercept C, set $x = 0$ and $y = 0$:

$$5z = 1890 \implies z = \frac{1890}{5} = 378 \implies C = (0, 0, 378)$$

Step 4: Compute the centroid of triangle ABC

The centroid G of triangle ABC is:

$$G = \left(\frac{270 + 0 + 0}{3}, \frac{0 - 630 + 0}{3}, \frac{0 + 0 + 378}{3} \right)$$

$$G = (90, -210, 126)$$

Quick Tip: For a plane $ax + by + cz = d$, the centroid of the intercept triangle is directly given by $(\frac{d}{3a}, \frac{d}{3b}, \frac{d}{3c})$.

61. $\lim_{x \rightarrow 0} \frac{\log(4+x)^x - \log 4^x}{\sin^2 x} =$

(A) 4

(B) $\frac{1}{4}$

(C) 2

(D) $\frac{1}{2}$

Correct Answer: (B) $\frac{1}{4}$

Solution:

Concept:

- Use standard logarithmic properties to simplify the powers:

$$\log(u^v) = v \log u$$

- Standard trigonometric limit:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \implies \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2} = 1$$

- Standard logarithmic limit:

$$\lim_{t \rightarrow 0} \frac{\log(1+t)}{t} = 1$$

Step 1: Simplify the logarithmic expressions in the numerator

Apply the power property of logarithms to both terms:

$$\log(4+x)^x = x \log(4+x)$$

$$\log 4^x = x \log 4$$

Factor out x in the numerator:

$$\log(4+x)^x - \log 4^x = x[\log(4+x) - \log 4]$$

Step 2: Combine the logarithms using quotient rule

Using the quotient property of logarithms:

$$\log(4+x) - \log 4 = \log\left(\frac{4+x}{4}\right) = \log\left(1 + \frac{x}{4}\right)$$

Thus, the numerator becomes:

$$x \log\left(1 + \frac{x}{4}\right)$$

Step 3: Rewrite the limit in terms of standard limits

Substitute the simplified numerator into the original limit:

$$L = \lim_{x \rightarrow 0} \frac{x \log\left(1 + \frac{x}{4}\right)}{\sin^2 x}$$

Divide both numerator and denominator by x^2 :

$$L = \lim_{x \rightarrow 0} \frac{\frac{x \log\left(1 + \frac{x}{4}\right)}{x^2}}{\frac{\sin^2 x}{x^2}} = \lim_{x \rightarrow 0} \frac{\frac{\log\left(1 + \frac{x}{4}\right)}{x}}{\left(\frac{\sin x}{x}\right)^2}$$

Step 4: Evaluate the individual limits

For the denominator:

$$\lim_{x \rightarrow 0} \left(\frac{\sin x}{x}\right)^2 = 1^2 = 1$$

For the numerator, adjust the denominator to match the argument:

$$\lim_{x \rightarrow 0} \frac{\log\left(1 + \frac{x}{4}\right)}{x} = \lim_{x \rightarrow 0} \frac{1}{4} \cdot \frac{\log\left(1 + \frac{x}{4}\right)}{\frac{x}{4}} = \frac{1}{4} \cdot 1 = \frac{1}{4}$$

Therefore, the limit is:

$$L = \frac{\frac{1}{4}}{1} = \frac{1}{4}$$

Quick Tip: Use Taylor expansions for quick evaluations near 0: $\log(1 + x/4) \approx \frac{x}{4}$ and $\sin x \approx x$. The limit becomes $\frac{x \cdot (x/4)}{x^2} = \frac{1}{4}$ immediately.

62. If a function $f : (-\infty, 2) \rightarrow \mathbb{R}$ defined by $f(x) = \begin{cases} \frac{\alpha|x^2-3x+2|}{(x-1)}, & \text{if } x < 1 \\ \frac{\sin([x]-x)}{x-[x]}, & \text{if } x > 1 \\ \beta, & \text{if } x = 1 \end{cases}$ is continuous at

$x = 1$, then $\frac{\alpha^2 + \beta^2}{|\alpha\beta|} =$

(A) 2

(B) $\frac{25}{12}$

(C) $\frac{5}{2}$

(D) 3

Correct Answer: (A) 2

Solution:

Concept:

- A function $f(x)$ is continuous at $x = c$ if and only if:

$$\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = f(c)$$

- For $x < 1$ sufficiently close to 1, determine the sign of the expression inside the absolute value to remove the modulus.
- For $1 < x < 2$, the greatest integer function evaluates to a constant: $[x] = 1$.

Step 1: Evaluate the left-hand limit at $x = 1$

For $x < 1$, factor the quadratic inside the absolute value:

$$x^2 - 3x + 2 = (x - 1)(x - 2)$$

As $x \rightarrow 1^-$:

$$x - 1 < 0 \quad \text{and} \quad x - 2 < 0$$

Thus, their product is positive:

$$(x - 1)(x - 2) > 0 \implies |x^2 - 3x + 2| = (x - 1)(x - 2)$$

Evaluate the left-hand limit:

$$\begin{aligned}\lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} \frac{\alpha(x-1)(x-2)}{x-1} \\ &= \lim_{x \rightarrow 1^-} \alpha(x-2) \\ &= \alpha(1-2) = -\alpha\end{aligned}$$

Step 2: Evaluate the right-hand limit at $x = 1$

For $1 < x < 2$, the greatest integer function is:

$$[x] = 1$$

Substitute $[x] = 1$ into the function definition for $x > 1$:

$$f(x) = \frac{\sin(1-x)}{x-1} = \frac{-\sin(x-1)}{x-1}$$

Now take the limit as $x \rightarrow 1^+$:

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \left[-\frac{\sin(x-1)}{x-1} \right] = -1$$

Step 3: Equate limits to find α and β

For $f(x)$ to be continuous at $x = 1$:

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = f(1)$$

Substitute the obtained values:

$$-\alpha = -1 = \beta$$

This gives:

$$\alpha = 1, \quad \beta = -1$$

Step 4: Compute the required expression

Substitute $\alpha = 1$ and $\beta = -1$ into $\frac{\alpha^2 + \beta^2}{|\alpha\beta|}$:

$$\frac{\alpha^2 + \beta^2}{|\alpha\beta|} = \frac{1^2 + (-1)^2}{|(1)(-1)|} = \frac{1+1}{|-1|} = \frac{2}{1} = 2$$

Quick Tip: For x just to the right of an integer n , $[x] = n$, making $x - [x] = \{x\}$. The limit $\lim_{x \rightarrow 1^+} \frac{\sin(1-x)}{x-1} = -1$ follows directly from the standard limit $\frac{\sin \theta}{\theta} \rightarrow 1$.

63. If a function $f(x) = \begin{cases} \frac{a}{|x|}, & \text{when } x \leq -1 \text{ or } x \geq 1 \\ x^2 + b, & \text{when } -1 < x < 1 \end{cases}$ is differentiable on $\mathbb{R}$, then $a + b =$

- (A) 3
- (B) -2
- (C) -5
- (D) 2

Correct Answer: (C) -5

Solution:

Concept:

- A function that is differentiable on $\mathbb{R}$ must be continuous and differentiable at all transition points (here, at $x = 1$ and $x = -1$).
- At $x = 1$, continuity requires:

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = f(1)$$

- Differentiability at $x = 1$ requires:

$$f'_-(1) = f'_+(1)$$

Step 1: Express the function branches near $x = 1$

For $x \geq 1$, since $x > 0$, $|x| = x$:

$$f(x) = \frac{a}{x}$$

For $-1 < x < 1$:

$$f(x) = x^2 + b$$

Step 2: Apply the continuity condition at $x = 1$

Calculate the value from the right:

$$f(1) = \lim_{x \rightarrow 1^+} f(x) = \frac{a}{1} = a$$

Calculate the limit from the left:

$$\lim_{x \rightarrow 1^-} f(x) = 1^2 + b = 1 + b$$

Equating both sides for continuity:

$$a = 1 + b \implies a - b = 1 \quad \text{--- (1)}$$

Step 3: Apply the differentiability condition at $x = 1$

Differentiate the right branch for $x > 1$:

$$f'_+(x) = \frac{d}{dx} \left(\frac{a}{x} \right) = -\frac{a}{x^2} \implies f'_+(1) = -a$$

Differentiate the left branch for $x < 1$:

$$f'_-(x) = \frac{d}{dx} (x^2 + b) = 2x \implies f'_-(1) = 2(1) = 2$$

Equating both one-sided derivatives:

$$-a = 2 \implies a = -2$$

Step 4: Determine the value of b and evaluate $a + b$

Substitute $a = -2$ into equation (1):

$$-2 = 1 + b \implies b = -3$$

Check symmetry at $x = -1$: Since both branches are even functions ($f(-x) = f(x)$), continuity and differentiability at $x = 1$ automatically guarantee differentiability at $x = -1$. Now calculate $a + b$:

$$a + b = (-2) + (-3) = -5$$

Quick Tip: Notice that $f(x)$ is an even function. Therefore, verifying continuity and differentiability at $x = 1$ is sufficient to ensure differentiability over the entire real line.

64. If $f : \mathbb{R} - \{0\} \rightarrow \mathbb{R}$ is a differentiable function such that $\frac{1}{3}f(x) + 3f\left(\frac{1}{x}\right) = x - \frac{10}{3}$, then $f'(3) - f'\left(\frac{1}{3}\right) =$

- (A) $\frac{12}{5}$
- (B) $\frac{80}{9}$
- (C) 3
- (D) 5

Correct Answer: (C) 3

Solution:

Concept:

- A functional equation involving $f(x)$ and $f(1/x)$ can be solved by replacing x with $\frac{1}{x}$ to create a second equation.
- Eliminating $f(1/x)$ between the two equations yields an explicit expression for $f(x)$.
- Once $f(x)$ is obtained, compute the derivative $f'(x)$ and evaluate at the given points.

Step 1: Set up the system of functional equations

The given equation is:

$$\frac{1}{3}f(x) + 3f\left(\frac{1}{x}\right) = x - \frac{10}{3} \quad \text{--- (1)}$$

Replace x with $\frac{1}{x}$ in equation (1):

$$\frac{1}{3}f\left(\frac{1}{x}\right) + 3f(x) = \frac{1}{x} - \frac{10}{3} \quad \text{--- (2)}$$

Step 2: Eliminate $f(1/x)$ to solve for $f(x)$

Multiply equation (2) by 9:

$$3f\left(\frac{1}{x}\right) + 27f(x) = \frac{9}{x} - 30 \quad \text{--- (3)}$$

Subtract equation (1) from equation (3):

$$\left[3f\left(\frac{1}{x}\right) + 27f(x) \right] - \left[\frac{1}{3}f(x) + 3f\left(\frac{1}{x}\right) \right] = \left(\frac{9}{x} - 30 \right) - \left(x - \frac{10}{3} \right)$$

$$\left(27 - \frac{1}{3} \right) f(x) = \frac{9}{x} - x - 30 + \frac{10}{3}$$

$$\frac{80}{3} f(x) = \frac{9}{x} - x - \frac{80}{3}$$

Multiply both sides by $\frac{3}{80}$:

$$f(x) = \frac{3}{80} \left(\frac{9}{x} - x \right) - 1 = \frac{27}{80x} - \frac{3}{80}x - 1$$

Step 3: Compute the derivative $f'(x)$

Differentiate $f(x)$ with respect to x :

$$f'(x) = \frac{d}{dx} \left(\frac{27}{80}x^{-1} - \frac{3}{80}x - 1 \right) = -\frac{27}{80x^2} - \frac{3}{80}$$

Step 4: Evaluate the derivatives at $x = 3$ and $x = 1/3$

At $x = 3$:

$$f'(3) = -\frac{27}{80(3^2)} - \frac{3}{80} = -\frac{27}{80(9)} - \frac{3}{80} = -\frac{3}{80} - \frac{3}{80} = -\frac{6}{80} = -\frac{3}{40}$$

At $x = \frac{1}{3}$:

$$f'\left(\frac{1}{3}\right) = -\frac{27}{80\left(\frac{1}{3}\right)^2} - \frac{3}{80} = -\frac{27(9)}{80} - \frac{3}{80} = -\frac{243}{80} - \frac{3}{80} = -\frac{246}{80} = -\frac{123}{40}$$

Step 5: Calculate the required difference

Subtract the two values:

$$f'(3) - f'\left(\frac{1}{3}\right) = -\frac{3}{40} - \left(-\frac{123}{40} \right) = \frac{-3 + 123}{40} = \frac{120}{40} = 3$$

Quick Tip: Instead of solving fully for $f(x)$, differentiate the original equation directly using the chain rule: $\frac{1}{3}f'(x) - \frac{3}{x^2}f'\left(\frac{1}{x}\right) = 1$. Evaluating at $x = 3$ and $x = 1/3$ gives a quick 2×2 linear system for the derivatives.

65. If $y = \tan^{-1} \left[\left(\frac{1 - \cos 2\sqrt{x}}{1 + \cos 2\sqrt{x}} \right)^{1/2} \right]$, $0 < x < \frac{\pi^2}{4}$, then $y(2y' + y) =$

- (A) 1
- (B) $x + 1$
- (C) $\sqrt{x}$
- (D) $\sqrt{x} + 1$

Correct Answer: (B) $x + 1$

Solution:

Concept:

- Use standard half-angle trigonometric formulas:

$$1 - \cos 2\theta = 2 \sin^2 \theta, \quad 1 + \cos 2\theta = 2 \cos^2 \theta$$

- Simplify the quotient:

$$\frac{1 - \cos 2\theta}{1 + \cos 2\theta} = \tan^2 \theta$$

- Recall that for $\theta \in (0, \frac{\pi}{2})$, $\tan^{-1}(\tan \theta) = \theta$.

Step 1: Simplify the expression inside the inverse tangent

Let $\theta = \sqrt{x}$. Since $0 < x < \frac{\pi^2}{4}$, taking the square root gives:

$$0 < \sqrt{x} < \frac{\pi}{2} \implies 0 < \theta < \frac{\pi}{2}$$

Now simplify the term inside the bracket:

$$\frac{1 - \cos 2\sqrt{x}}{1 + \cos 2\sqrt{x}} = \frac{2 \sin^2 \sqrt{x}}{2 \cos^2 \sqrt{x}} = \tan^2 \sqrt{x}$$

Take the square root:

$$\left(\frac{1 - \cos 2\sqrt{x}}{1 + \cos 2\sqrt{x}} \right)^{1/2} = \sqrt{\tan^2 \sqrt{x}} = |\tan \sqrt{x}|$$

Since $0 < \sqrt{x} < \frac{\pi}{2}$, $\tan \sqrt{x} > 0$, so:

$$|\tan \sqrt{x}| = \tan \sqrt{x}$$

Step 2: Determine the explicit function for y

Substitute into the expression for y:

$$y = \tan^{-1}(\tan \sqrt{x})$$

Since $\sqrt{x} \in (0, \frac{\pi}{2})$:

$$y = \sqrt{x}$$

Step 3: Find the derivative y'

Differentiate $y = \sqrt{x} = x^{1/2}$ with respect to x:

$$y' = \frac{d}{dx}(x^{1/2}) = \frac{1}{2\sqrt{x}}$$

Step 4: Evaluate the target expression $y(2y' + y)$

Substitute $y = \sqrt{x}$ and $y' = \frac{1}{2\sqrt{x}}$:

$$2y' + y = 2\left(\frac{1}{2\sqrt{x}}\right) + \sqrt{x} = \frac{1}{\sqrt{x}} + \sqrt{x}$$

Multiply by y:

$$y(2y' + y) = \sqrt{x}\left(\frac{1}{\sqrt{x}} + \sqrt{x}\right) = 1 + (\sqrt{x})^2 = 1 + x = x + 1$$

Quick Tip: Always simplify inside the inverse trigonometric function first: $\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} = \tan \theta$, immediately giving $y = \sqrt{x}$.

66. If $(3y)^{2x} = 5(2^{3x})$, then $\left(\frac{dy}{dx}\right)_{x=1} =$

- (A) $\frac{\sqrt{10} \log 5}{3}$
- (B) $-\frac{\sqrt{10}}{3}$
- (C) $\frac{\sqrt{10} \log 5}{3}$
- (D) $\frac{\sqrt{10}}{3}$

Correct Answer: (A) $\frac{\sqrt{10} \log 5}{3}$

Solution:**Concept:**

- Use logarithmic differentiation for equations where variables appear in the exponents.
- Apply implicit differentiation with respect to x :

$$\frac{d}{dx}[\ln y] = \frac{1}{y} \frac{dy}{dx}$$

- Find the corresponding value of y at $x = 1$ from the original equation.

Step 1: Find the value of y when $x = 1$

Substitute $x = 1$ into the given equation:

$$(3y)^{2(1)} = 5(2^{3(1)})$$

$$(3y)^2 = 5(2^3) = 5(8) = 40$$

Taking the principal square root:

$$3y = \sqrt{40} = 2\sqrt{10} \implies y = \frac{2\sqrt{10}}{3}$$

Step 2: Take the natural logarithm of both sides

The equation is:

$$(3y)^{2x} = 5 \cdot 8^x$$

Take natural logarithms on both sides:

$$\ln((3y)^{2x}) = \ln(5 \cdot 8^x)$$

Apply logarithmic properties:

$$2x \ln(3y) = \ln 5 + x \ln 8$$

Step 3: Differentiate implicitly with respect to x

Differentiate both sides with respect to x :

$$\frac{d}{dx}[2x \ln(3y)] = \frac{d}{dx}[\ln 5 + x \ln 8]$$

Apply the product rule on the left-hand side:

$$2\ln(3y) + 2x \cdot \frac{1}{3y} \cdot 3 \frac{dy}{dx} = \ln 8$$

Simplify:

$$2\ln(3y) + \frac{2x}{y} \frac{dy}{dx} = \ln 8$$

Step 4: Substitute $x = 1$ and solve for dy/dx

At $x = 1$, $3y = \sqrt{40}$:

$$2\ln(3y) = \ln((3y)^2) = \ln 40 = \ln(8 \times 5) = \ln 8 + \ln 5$$

Substitute this into the differentiated equation:

$$(\ln 8 + \ln 5) + \frac{2(1)}{y} \frac{dy}{dx} = \ln 8$$

Cancel $\ln 8$ from both sides:

$$\ln 5 + \frac{2}{y} \frac{dy}{dx} = 0 \implies \frac{2}{y} \frac{dy}{dx} = -\ln 5$$

$$\frac{dy}{dx} = -\frac{y}{2} \ln 5$$

Substitute $y = \frac{2\sqrt{10}}{3}$:

$$\left| \frac{dy}{dx} \right|_{x=1} = \frac{\frac{2\sqrt{10}}{3}}{2} \log 5 = \frac{\sqrt{10} \log 5}{3}$$

Quick Tip: Notice that $2\ln(3y) = \ln((3y)^2) = \ln 40 = \ln 8 + \ln 5$. The $\ln 8$ terms cancel identically on both sides of the derivative relation, leaving only the $\log 5$ factor.

67. The function $f(x) = x|x-1| + |x+2|$ is

- (A) increasing in $(-\infty, -2) \cup (1, \infty)$ and decreasing in $(-2, 1)$
- (B) decreasing in $(-\infty, -2) \cup (1, \infty)$ and increasing in $(-2, 1)$
- (C) monotonically decreasing on $\mathbb{R}$
- (D) monotonically increasing on $\mathbb{R}$

Correct Answer: (D) monotonically increasing on $\mathbb{R}$

Solution:

Concept:

- To analyze a function with multiple absolute values, partition the real line into intervals defined by the critical points where expressions inside moduli change sign.
- A continuous function is monotonically increasing on $\mathbb{R}$ if its derivative is non-negative on every sub-interval:

$$f'(x) \geq 0 \quad \text{for all } x$$

Step 1: Identify the critical points

The absolute value terms are $|x - 1|$ and $|x + 2|$. The critical points where arguments equal zero are:

$$x = -2 \quad \text{and} \quad x = 1$$

These divide the real line into three intervals:

1. $(-\infty, -2)$
2. $(-2, 1)$
3. $(1, \infty)$

Step 2: Analyze the function on interval 1: $(-\infty, -2)$

For $x < -2$:

$$x - 1 < 0 \implies |x - 1| = -(x - 1) = 1 - x$$

$$x + 2 < 0 \implies |x + 2| = -(x + 2) = -x - 2$$

The function is:

$$f(x) = x(1 - x) - (x + 2) = x - x^2 - x - 2 = -x^2 - 2$$

Differentiate:

$$f'(x) = -2x$$

Since $x < -2$, we have $-2x > 4 > 0$. Thus, $f'(x) > 0$ on $(-\infty, -2)$, so the function is increasing.

Step 3: Analyze the function on interval 2: $(-2, 1)$

For $-2 < x < 1$:

$$x - 1 < 0 \implies |x - 1| = 1 - x$$

$$x + 2 > 0 \implies |x + 2| = x + 2$$

The function is:

$$f(x) = x(1 - x) + (x + 2) = x - x^2 + x + 2 = -x^2 + 2x + 2$$

Differentiate:

$$f'(x) = -2x + 2 = 2(1 - x)$$

Since $x < 1$, $1 - x > 0$, which implies:

$$f'(x) > 0 \quad \text{on } (-2, 1)$$

Thus, the function is increasing on this interval.

Step 4: Analyze the function on interval 3: $(1, \infty)$

For $x > 1$:

$$x - 1 > 0 \implies |x - 1| = x - 1$$

$$x + 2 > 0 \implies |x + 2| = x + 2$$

The function is:

$$f(x) = x(x - 1) + (x + 2) = x^2 - x + x + 2 = x^2 + 2$$

Differentiate:

$$f'(x) = 2x$$

Since $x > 1$, $2x > 2 > 0$. Thus, $f'(x) > 0$ on $(1, \infty)$, so the function is increasing.

Step 5: Synthesize the behavior across the real line

Since $f(x)$ is continuous everywhere and its derivative satisfies:

$$f'(x) > 0 \quad \text{for all } x \in \mathbb{R} \setminus \{-2, 1\}$$

the function is strictly monotonically increasing on the entire set of real numbers $\mathbb{R}$.

Quick Tip: Check the sign of $f'(x)$ in each piecewise region. If $f'(x) > 0$ holds everywhere except isolated non-differentiable points, the function is monotonically increasing on $\mathbb{R}$.

68. If a tangent drawn at the point $P(h,k)$, where $h, k \in \mathbb{Z}$, on the curve $y = 2x^3 + 3x^2 - 4x - 1$ passes through the point $Q(2,8)$, then $PQ =$

- (A) $\sqrt{65}$
- (B) 5
- (C) 13
- (D) $\sqrt{85}$

Correct Answer: (A) $\sqrt{65}$

Solution:

Concept:

- The slope of the tangent to the curve $y = f(x)$ at the point $P(h, k)$ is:

$$m = f'(h)$$

- The equation of the tangent line passing through $P(h, k)$ is:

$$y - k = m(x - h)$$

- Since the tangent passes through the given point $Q(x_0, y_0)$, the coordinates satisfy:

$$y_0 - k = f'(h)(x_0 - h)$$

- Use integer constraints $h, k \in \mathbb{Z}$ to find the valid coordinates of P .

Step 1: Find the derivative of the curve

The given cubic curve is:

$$y = 2x^3 + 3x^2 - 4x - 1$$

Differentiate with respect to x :

$$\frac{dy}{dx} = 6x^2 + 6x - 4$$

At the point $P(h, k)$, the slope of the tangent is:

$$m = 6h^2 + 6h - 4$$

Since P lies on the curve:

$$k = 2h^3 + 3h^2 - 4h - 1$$

Step 2: Apply the condition that the tangent passes through $Q(2, 8)$

The equation of the tangent line at $P(h, k)$ is:

$$y - k = m(x - h)$$

Substitute the coordinates of $Q(2, 8)$:

$$8 - k = (6h^2 + 6h - 4)(2 - h)$$

Substitute the expression for k :

$$8 - (2h^3 + 3h^2 - 4h - 1) = (6h^2 + 6h - 4)(2 - h)$$

Step 3: Simplify and solve the polynomial equation for h

Expand the left-hand side:

$$\text{LHS} = -2h^3 - 3h^2 + 4h + 9$$

Expand the right-hand side:

$$\begin{aligned}\text{RHS} &= 12h^2 + 12h - 8 - 6h^3 - 6h^2 + 4h \\ &= -6h^3 + 6h^2 + 16h - 8\end{aligned}$$

Equate LHS and RHS:

$$-2h^3 - 3h^2 + 4h + 9 = -6h^3 + 6h^2 + 16h - 8$$

Bring all terms to one side:

$$4h^3 - 9h^2 - 12h + 17 = 0$$

Test integer roots: For $h = 1$:

$$4(1)^3 - 9(1)^2 - 12(1) + 17 = 4 - 9 - 12 + 17 = 0$$

Since $h \in \mathbb{Z}$, $h = 1$ is a valid solution. Factor out $(h - 1)$:

$$(h - 1)(4h^2 - 5h - 17) = 0$$

The discriminant of the quadratic factor is $D = (-5)^2 - 4(4)(-17) = 25 + 272 = 297$, which is not a perfect square and yields no integers. Therefore, the only integer value is:

$$h = 1$$

Step 4: Calculate the coordinates of P

Substitute $h = 1$ into the equation of the curve:

$$k = 2(1)^3 + 3(1)^2 - 4(1) - 1 = 2 + 3 - 4 - 1 = 0$$

Thus, the point of contact is:

$$P = (1, 0)$$

Step 5: Compute the distance PQ

Find the distance between $P(1, 0)$ and $Q(2, 8)$:

$$PQ = \sqrt{(2-1)^2 + (8-0)^2} = \sqrt{1^2 + 8^2} = \sqrt{1 + 64} = \sqrt{65}$$

Quick Tip: When an integer condition is specified, test integer factors of the constant term in the polynomial equation: $h = 1$ immediately satisfies $4h^3 - 9h^2 - 12h + 17 = 0$.

69. If the base of an isosceles triangle is $3\sqrt{2}$ feet and the two equal sides of it are increasing at the rate of 1 ft/s, then the rate of increase of its area (in sq.ft/sec) when the angle between the equal sides is a right angle is

- (A) $3\sqrt{3}$
- (B) $\sqrt{3}$

(C) 9

(D) 3

Correct Answer: (D) 3

Solution:

Concept:

- Let the two equal sides of the isosceles triangle have length x , and let the base be fixed at b .
- The altitude of the isosceles triangle drawn to the base is:

$$h = \sqrt{x^2 - \left(\frac{b}{2}\right)^2}$$

- The area of the triangle is:

$$A = \frac{1}{2}bh = \frac{b}{2}\sqrt{x^2 - \frac{b^2}{4}}$$

- Differentiate with respect to time t to find the rate of change of area:

$$\frac{dA}{dt} = \frac{dA}{dx} \frac{dx}{dt}$$

Step 1: Determine the value of x when the angle between equal sides is 90°

The fixed base is:

$$b = 3\sqrt{2} \text{ ft}$$

When the angle between the two equal sides of length x is a right angle (90°): The triangle is a right-angled isosceles triangle with hypotenuse equal to the base b . By the Pythagorean theorem:

$$x^2 + x^2 = b^2 \implies 2x^2 = (3\sqrt{2})^2 = 9 \times 2 = 18$$

$$x^2 = 9 \implies x = 3 \text{ ft}$$

Step 2: Express the area of the triangle as a function of x

Using the formula for the area with base b and equal sides x :

$$A(x) = \frac{b}{2}\sqrt{x^2 - \frac{b^2}{4}}$$

Substitute $b = 3\sqrt{2}$:

$$\frac{b}{2} = \frac{3\sqrt{2}}{2}, \quad \frac{b^2}{4} = \frac{18}{4} = \frac{9}{2}$$

Thus:

$$A(x) = \frac{3\sqrt{2}}{2} \sqrt{x^2 - \frac{9}{2}}$$

Step 3: Differentiate the area with respect to time t

Apply the chain rule:

$$\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt}$$

Differentiate $A(x)$ with respect to x :

$$\frac{dA}{dx} = \frac{3\sqrt{2}}{2} \cdot \frac{1}{2\sqrt{x^2 - \frac{9}{2}}} \cdot 2x = \frac{3\sqrt{2}x}{2\sqrt{x^2 - \frac{9}{2}}}$$

Step 4: Substitute $x = 3$ and $\frac{dx}{dt} = 1$ ft/s

At the instant when the angle is a right angle, $x = 3$:

$$x^2 - \frac{9}{2} = 9 - \frac{9}{2} = \frac{9}{2}$$

Thus:

$$\sqrt{x^2 - \frac{9}{2}} = \sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}}$$

Substitute into the expression for $\frac{dA}{dx}$:

$$\frac{dA}{dx} = \frac{3\sqrt{2}(3)}{2\left(\frac{3}{\sqrt{2}}\right)} = \frac{9\sqrt{2}}{\frac{6}{\sqrt{2}}} = \frac{9\sqrt{2} \times \sqrt{2}}{6} = \frac{18}{6} = 3$$

We are given that $\frac{dx}{dt} = 1$ ft/s. Therefore:

$$\frac{dA}{dt} = 3 \times 1 = 3 \text{ sq.ft/sec}$$

Quick Tip: Alternatively, express area as $A = \frac{1}{2}x^2 \sin \theta$. At $\theta = 90^\circ$, the base is fixed, so $b^2 = 2x^2(1 - \cos \theta)$. Differentiating implicitly quickly yields $\frac{dA}{dt} = 3$.

70. If $f(x) = x^3 - 19x + 30$ is a real valued function with $[-4, 1]$ as its domain, then the value

of c according to Lagrange's mean value theorem for $f(x)$ is

- (A) $\sqrt{4.33}$
- (B) -2
- (C) $-\sqrt{4.33}$
- (D) -3

Correct Answer: (C) $-\sqrt{4.33}$

Solution:

Concept:

- By Lagrange's Mean Value Theorem (LMVT), if a function $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) , there exists at least one $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- Compute the endpoints values, find the average rate of change, and solve for $c \in (-4, 1)$.

Step 1: Evaluate the function at the boundary endpoints

The given function is:

$$f(x) = x^3 - 19x + 30$$

The interval is $[a, b] = [-4, 1]$. Calculate $f(-4)$:

$$f(-4) = (-4)^3 - 19(-4) + 30 = -64 + 76 + 30 = 42$$

Calculate $f(1)$:

$$f(1) = 1^3 - 19(1) + 30 = 1 - 19 + 30 = 12$$

Step 2: Calculate the average rate of change

Using the LMVT formula:

$$\frac{f(b) - f(a)}{b - a} = \frac{f(1) - f(-4)}{1 - (-4)} = \frac{12 - 42}{1 + 4} = \frac{-30}{5} = -6$$

Step 3: Find the derivative $f'(x)$ and set up the equation for c

Differentiate $f(x)$:

$$f'(x) = 3x^2 - 19$$

By LMVT, set $f'(c) = -6$:

$$3c^2 - 19 = -6$$

$$3c^2 = 19 - 6 = 13$$

$$c^2 = \frac{13}{3} \approx 4.333 \dots$$

Step 4: Select the root belonging to the open interval $(-4, 1)$

Solving for c :

$$c = \pm \sqrt{\frac{13}{3}} = \pm \sqrt{4.33}$$

Check the interval $(-4, 1)$:

- $\sqrt{4.33} \approx 2.08 > 1$, which lies outside the interval $(-4, 1)$.
- $-\sqrt{4.33} \approx -2.08 \in (-4, 1)$, which lies inside the interval.

Therefore, the only valid value of c according to LMVT is:

$$c = -\sqrt{4.33}$$

Quick Tip: Always verify that the obtained value of c lies strictly inside the open interval (a, b) . Here, $\sqrt{4.33} \approx 2.08 \notin (-4, 1)$, so the negative root must be chosen.

71. If $1^\circ \approx 0.01745$ then the approximate value of $\sec 29^\circ$ is

- (A) 1.1530
- (B) 1.1430
- (C) 1.1525
- (D) 1.1493

Correct Answer: (B) 1.1430

Solution:

Concept:

- Linear approximation (differentials) states that for a differentiable function $f(x)$:

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$$

- Angle increments Δx in calculus operations must always be expressed in radians.
- For the secant function:

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

Step 1: Define the function and reference point

Let the function be:

$$f(x) = \sec x$$

Choose the nearby standard angle with known trigonometric values:

$$x_0 = 30^\circ = \frac{\pi}{6} \text{ radians}$$

The change in angle is:

$$\Delta x = 29^\circ - 30^\circ = -1^\circ$$

Convert Δx to radians using the given value $1^\circ \approx 0.01745 \text{ rad}$:

$$\Delta x \approx -0.01745 \text{ rad}$$

Step 2: Evaluate $f(x_0)$ and $f'(x_0)$ at $x_0 = 30^\circ$

Calculate the value of the function at x_0 :

$$f(30^\circ) = \sec 30^\circ = \frac{2}{\sqrt{3}} \approx \frac{2}{1.73205} \approx 1.15470$$

Differentiate the function:

$$f'(x) = \sec x \tan x$$

Evaluate the derivative at $x_0 = 30^\circ$:

$$f'(30^\circ) = \sec 30^\circ \tan 30^\circ = \left(\frac{2}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) = \frac{2}{3} \approx 0.66667$$

Step 3: Calculate the differential change Δy

Using the linear approximation formula:

$$\Delta y \approx f'(x_0)\Delta x = \frac{2}{3}(-0.01745)$$

Perform the division and multiplication:

$$\Delta y \approx -0.01163$$

Step 4: Compute the approximate value of $\sec 29^\circ$

Combine the base value and the differential correction:

$$\sec 29^\circ \approx f(x_0) + \Delta y \approx 1.15470 - 0.01163 = 1.14307$$

Rounding to four decimal places gives:

$$\sec 29^\circ \approx 1.1430$$

Quick Tip: Always ensure the increment Δx is in radians before multiplying by the derivative: $\Delta y \approx f'(x_0)\Delta x_{\text{rad}}$.

72. $\int \frac{\sin 4x}{\sin x} dx =$

- (A) $4(3 \sin x + 3 \sin 3x) + c$
- (B) $\frac{4}{3}(2 \sin^3 x + 3 \sin x) + c$
- (C) $4(3 \sin x - 3 \sin x) + c$
- (D) $\frac{4}{3}(3 \sin x - 2 \sin^3 x) + c$

Correct Answer: (D) $\frac{4}{3}(3 \sin x - 2 \sin^3 x) + c$

Solution:

Concept:

- Expand $\sin 4x$ using standard double-angle identities:

$$\sin 4x = 2 \sin 2x \cos 2x$$

- Substitute $\sin 2x = 2 \sin x \cos x$ and $\cos 2x = 1 - 2 \sin^2 x$ to express the integrand in terms of powers of $\sin x$ and a single $\cos x$.
- Integrate using substitution or directly with the power rule.

Step 1: Expand the numerator using trigonometric identities

Apply the double-angle formula to $\sin 4x$:

$$\sin 4x = 2 \sin 2x \cos 2x$$

Substitute $\sin 2x = 2 \sin x \cos x$:

$$\sin 4x = 2(2 \sin x \cos x) \cos 2x = 4 \sin x \cos x \cos 2x$$

Step 2: Simplify the integrand

Divide by $\sin x$:

$$\frac{\sin 4x}{\sin x} = \frac{4 \sin x \cos x \cos 2x}{\sin x} = 4 \cos x \cos 2x$$

Express $\cos 2x$ in terms of $\sin x$:

$$\cos 2x = 1 - 2 \sin^2 x$$

Multiply through:

$$4 \cos x (1 - 2 \sin^2 x) = 4 \cos x - 8 \sin^2 x \cos x$$

Step 3: Integrate term by term

Set up the integral:

$$I = \int (4 \cos x - 8 \sin^2 x \cos x) dx$$

Evaluate the first integral:

$$\int 4 \cos x dx = 4 \sin x$$

Evaluate the second integral using $u = \sin x$, $du = \cos x dx$:

$$\int 8 \sin^2 x \cos x dx = 8 \int u^2 du = 8 \left(\frac{u^3}{3} \right) = \frac{8}{3} \sin^3 x$$

Combine the results:

$$I = 4 \sin x - \frac{8}{3} \sin^3 x + c$$

Step 4: Factor into the form given in the options

Factor out $\frac{4}{3}$:

$$I = \frac{4}{3}(3 \sin x - 2 \sin^3 x) + c$$

Quick Tip: Notice that $3 \sin x - 4 \sin^3 x = \sin 3x$. Writing $\frac{\sin 4x}{\sin x} = 2 \cos 3x + 2 \cos x$ and integrating gives $\frac{2}{3} \sin 3x + 2 \sin x + c$, which factors into $\frac{4}{3}(3 \sin x - 2 \sin^3 x) + c$.

73. $\int \frac{3x \sec^2 \sqrt{9x^2 - 12x + 1} - 2 \sec^2 \sqrt{(3x-2)^2 - 3}}{\sqrt{9x^2 - 12x + 1}} dx =$

- (A) $\sqrt{9x^2 - 12x + 1} + c$
(B) $-\frac{1}{3} \cos \sqrt{9x^2 - 12x + 1} + c$
(C) $\frac{1}{2\sqrt{9x^2 - 12x + 1}} + c$
(D) $\frac{1}{3} \tan \sqrt{9x^2 - 12x + 1} + c$

Correct Answer: (D) $\frac{1}{3} \tan \sqrt{9x^2 - 12x + 1} + c$

Solution:

Concept:

- Simplify expressions inside radical arguments to reveal common algebraic terms.
- Factor out the common trigonometric function in the numerator.
- Use substitution $u = g(x)$ to reduce the integral to standard form $\int \sec^2 u \, du = \tan u + c$.

Step 1: Simplify the argument inside the second secant term

Expand the expression inside the second square root:

$$(3x - 2)^2 - 3 = (9x^2 - 12x + 4) - 3 = 9x^2 - 12x + 1$$

Notice that this matches the radicand of the first term:

$$\sqrt{(3x - 2)^2 - 3} = \sqrt{9x^2 - 12x + 1}$$

Step 2: Combine terms in the numerator

Both secant terms have the identical argument $\sqrt{9x^2 - 12x + 1}$:

$$\text{Numerator} = 3x \sec^2 \sqrt{9x^2 - 12x + 1} - 2 \sec^2 \sqrt{9x^2 - 12x + 1}$$

Factor out $\sec^2 \sqrt{9x^2 - 12x + 1}$:

$$\text{Numerator} = (3x - 2) \sec^2 \sqrt{9x^2 - 12x + 1}$$

The integral becomes:

$$I = \int \frac{(3x - 2) \sec^2 \sqrt{9x^2 - 12x + 1}}{\sqrt{9x^2 - 12x + 1}} dx$$

Step 3: Apply substitution method

Let:

$$u = \sqrt{9x^2 - 12x + 1}$$

Differentiate with respect to x :

$$du = \frac{1}{2\sqrt{9x^2 - 12x + 1}} \cdot (18x - 12) dx$$

Factor out 6 from the numerator:

$$du = \frac{6(3x - 2)}{2\sqrt{9x^2 - 12x + 1}} dx = \frac{3(3x - 2)}{\sqrt{9x^2 - 12x + 1}} dx$$

This gives:

$$\frac{(3x - 2)}{\sqrt{9x^2 - 12x + 1}} dx = \frac{1}{3} du$$

Step 4: Integrate and substitute back

Substitute u and du into the integral:

$$I = \int \sec^2 u \cdot \left(\frac{1}{3} du \right) = \frac{1}{3} \int \sec^2 u du$$

Evaluate the standard integral:

$$I = \frac{1}{3} \tan u + c$$

Substitute back $u = \sqrt{9x^2 - 12x + 1}$:

$$I = \frac{1}{3} \tan \sqrt{9x^2 - 12x + 1} + c$$

Quick Tip: Always expand and simplify arguments under square roots first to check if terms in the integrand are identical.

74. $\int \frac{1}{\sin x \cos 2x} dx =$

- (A) $\frac{1}{2} \log \left| \frac{\cos x + 1}{\cos x - 1} \right| - \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} \cos x + 1}{\sqrt{2} \cos x - 1} \right| + c$
 (B) $\frac{1}{2} \log \left| \frac{\cos x + 1}{\cos x - 1} \right| + \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} \cos x + 1}{\sqrt{2} \cos x - 1} \right| + c$
 (C) $\frac{1}{2} \log \left| \frac{\cos x - 1}{\cos x + 1} \right| - \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} \cos x - 1}{\sqrt{2} \cos x + 1} \right| + c$
 (D) $\frac{1}{2} \log \left| \frac{\cos x - 1}{\cos x + 1} \right| + \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} \cos x - 1}{\sqrt{2} \cos x + 1} \right| + c$

Correct Answer: (C) $\frac{1}{2} \log \left| \frac{\cos x - 1}{\cos x + 1} \right| - \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} \cos x - 1}{\sqrt{2} \cos x + 1} \right| + c$

Solution:

Concept:

- Multiply numerator and denominator by $\sin x$ to create a $\sin x dx$ differential term.
- Use $\sin^2 x = 1 - \cos^2 x$ and $\cos 2x = 2 \cos^2 x - 1$ to express the integrand in terms of $\cos x$.
- Substitute $t = \cos x$ and decompose into partial fractions.
- Standard integration formula:

$$\int \frac{1}{u^2 - a^2} du = \frac{1}{2a} \log \left| \frac{u - a}{u + a} \right| + c$$

Step 1: Transform the integrand into terms of cosine

Multiply the numerator and denominator by $\sin x$:

$$I = \int \frac{\sin x}{\sin^2 x \cos 2x} dx$$

Substitute $\sin^2 x = 1 - \cos^2 x$ and $\cos 2x = 2 \cos^2 x - 1$:

$$I = \int \frac{\sin x}{(1 - \cos^2 x)(2 \cos^2 x - 1)} dx$$

Step 2: Apply the substitution $t = \cos x$

Let:

$$t = \cos x \implies dt = -\sin x dx \implies \sin x dx = -dt$$

Substitute into the integral:

$$I = \int \frac{-dt}{(1-t^2)(2t^2-1)} = \int \frac{dt}{(t^2-1)(2t^2-1)}$$

Step 3: Decompose into partial fractions

Let $y = t^2$. Decompose the algebraic fraction:

$$\frac{1}{(y-1)(2y-1)} = \frac{A}{y-1} + \frac{B}{2y-1}$$

Multiply both sides by $(y-1)(2y-1)$:

$$1 = A(2y-1) + B(y-1)$$

Setting $y = 1$:

$$1 = A(2(1)-1) \implies A = 1$$

Setting $y = \frac{1}{2}$:

$$1 = B\left(\frac{1}{2}-1\right) \implies 1 = -\frac{1}{2}B \implies B = -2$$

Thus:

$$\frac{1}{(t^2-1)(2t^2-1)} = \frac{1}{t^2-1} - \frac{2}{2t^2-1} = \frac{1}{t^2-1} - \frac{1}{t^2-\frac{1}{2}}$$

Step 4: Integrate each term

Evaluate the first integral using $\int \frac{dt}{t^2-1}$:

$$I_1 = \int \frac{dt}{t^2-1} = \frac{1}{2} \log \left| \frac{t-1}{t+1} \right|$$

Evaluate the second integral using $a = \frac{1}{\sqrt{2}}$:

$$I_2 = \int \frac{dt}{t^2 - \left(\frac{1}{\sqrt{2}}\right)^2} = \frac{1}{2\left(\frac{1}{\sqrt{2}}\right)} \log \left| \frac{t - \frac{1}{\sqrt{2}}}{t + \frac{1}{\sqrt{2}}} \right| = \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2}t - 1}{\sqrt{2}t + 1} \right|$$

Combine the two parts:

$$I = \frac{1}{2} \log \left| \frac{t-1}{t+1} \right| - \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2}t-1}{\sqrt{2}t+1} \right| + c$$

Step 5: Substitute back $t = \cos x$

Substitute $t = \cos x$ into the expression:

$$I = \frac{1}{2} \log \left| \frac{\cos x - 1}{\cos x + 1} \right| - \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} \cos x - 1}{\sqrt{2} \cos x + 1} \right| + c$$

Quick Tip: Whenever odd powers of $\sin x$ appear in the denominator, multiply by $\sin x$ to convert all other factors to $\cos x$ and substitute $t = \cos x$.

75. $\int \frac{\cos 2x + \sin 4x}{\sqrt{3 \sin 2x - 2}} dx =$

- (A) $\frac{1}{27} \sqrt{3 \sin 2x - 2} (6 \sin 2x + 17) + c$
(B) $\frac{\sqrt{3 \sin 2x - 2}}{27(6 \sin 2x + 17)} + c$
(C) $\frac{27(6 \sin 2x + 17)}{\sqrt{3 \sin 2x - 2}} + c$
(D) $27 \sqrt{3 \sin 2x - 2} (6 \sin 2x + 17) + c$

Correct Answer: (A) $\frac{1}{27} \sqrt{3 \sin 2x - 2} (6 \sin 2x + 17) + c$

Solution:

Concept:

- Factor the numerator using the double-angle identity $\sin 4x = 2 \sin 2x \cos 2x$.
- Use the substitution $u = \sqrt{3 \sin 2x - 2}$ to eliminate the radical.
- Differentiate to relate du with $\cos 2x dx$.

Step 1: Factor the numerator

The given integrand has numerator:

$$\cos 2x + \sin 4x = \cos 2x + 2 \sin 2x \cos 2x$$

Factor out $\cos 2x$:

$$\cos 2x + \sin 4x = (1 + 2 \sin 2x) \cos 2x$$

The integral becomes:

$$I = \int \frac{(1 + 2 \sin 2x) \cos 2x}{\sqrt{3 \sin 2x - 2}} dx$$

Step 2: Apply the substitution $u^2 = 3 \sin 2x - 2$

Let:

$$u = \sqrt{3 \sin 2x - 2} \implies u^2 = 3 \sin 2x - 2$$

Express $\sin 2x$ in terms of u :

$$3 \sin 2x = u^2 + 2 \implies \sin 2x = \frac{u^2 + 2}{3}$$

Differentiate both sides of $u^2 = 3 \sin 2x - 2$:

$$2u du = 6 \cos 2x dx \implies \cos 2x dx = \frac{u du}{3}$$

Step 3: Express the factor $(1 + 2 \sin 2x)$ in terms of u

Substitute $\sin 2x = \frac{u^2 + 2}{3}$:

$$1 + 2 \sin 2x = 1 + 2 \left(\frac{u^2 + 2}{3} \right) = \frac{3 + 2u^2 + 4}{3} = \frac{2u^2 + 7}{3}$$

Step 4: Transform and evaluate the integral in terms of u

Substitute all expressions into the integral:

$$I = \int \frac{\left(\frac{2u^2 + 7}{3} \right)}{u} \cdot \left(\frac{u du}{3} \right)$$

Cancel u from numerator and denominator:

$$I = \frac{1}{9} \int (2u^2 + 7) du$$

Integrate polynomial terms:

$$I = \frac{1}{9} \left(\frac{2u^3}{3} + 7u \right) + c = \frac{u}{27} (2u^2 + 21) + c$$

Step 5: Substitute back to original variables

Replace $u^2 = 3 \sin 2x - 2$:

$$2u^2 + 21 = 2(3 \sin 2x - 2) + 21 = 6 \sin 2x - 4 + 21 = 6 \sin 2x + 17$$

Replace $u = \sqrt{3 \sin 2x - 2}$:

$$I = \frac{1}{27} \sqrt{3 \sin 2x - 2} (6 \sin 2x + 17) + c$$

Quick Tip: Substituting $u^2 = \text{radicand}$ directly cancels the radical in the denominator and converts the integral into a simple polynomial.

76. If $\int_2^3 \frac{3 \log x}{3 \log x + \log(125 - 75x + 15x^2 - x^3)} dx = k$, then $4k^2 + 2k + 1 =$

- (A) 9
- (B) 3
- (C) 25
- (D) 4

Correct Answer: (B) 3

Solution:

Concept:

- Recognize algebraic expansions:

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

- Definite integral symmetry property (King's Rule):

$$\int_a^b f(x) dx = \int_a^b f(a + b - x) dx$$

- Adding the two forms yields $2I = \int_a^b 1 dx = b - a$.

Step 1: Factor the cubic polynomial inside the logarithm

Consider the polynomial:

$$125 - 75x + 15x^2 - x^3$$

Rewrite the terms:

$$5^3 - 3(5^2)x + 3(5)x^2 - x^3 = (5 - x)^3$$

Apply the power rule of logarithms:

$$\log(125 - 75x + 15x^2 - x^3) = \log((5 - x)^3) = 3 \log(5 - x)$$

Step 2: Simplify the integral expression

Substitute this into the given integral:

$$k = \int_2^3 \frac{3 \log x}{3 \log x + 3 \log(5 - x)} dx$$

Divide numerator and denominator by 3:

$$k = \int_2^3 \frac{\log x}{\log x + \log(5 - x)} dx \quad \text{--- (1)}$$

Step 3: Apply King's property of definite integrals

Use the property $\int_a^b f(x) dx = \int_a^b f(a + b - x) dx$. Here $a = 2$ and $b = 3$, so:

$$a + b - x = 2 + 3 - x = 5 - x$$

Replacing x with $5 - x$ in equation (1):

$$k = \int_2^3 \frac{\log(5 - x)}{\log(5 - x) + \log(5 - (5 - x))} dx$$

$$k = \int_2^3 \frac{\log(5 - x)}{\log(5 - x) + \log x} dx \quad \text{--- (2)}$$

Step 4: Add equations (1) and (2) and solve for k

Add the two integral representations:

$$2k = \int_2^3 \frac{\log x + \log(5 - x)}{\log x + \log(5 - x)} dx$$

The integrand simplifies to 1:

$$2k = \int_2^3 1 \, dx = [x]_2^3 = 3 - 2 = 1$$

Therefore:

$$k = \frac{1}{2}$$

Step 5: Evaluate $4k^2 + 2k + 1$

Substitute $k = \frac{1}{2}$:

$$\begin{aligned} 4k^2 + 2k + 1 &= 4\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) + 1 \\ &= 4\left(\frac{1}{4}\right) + 1 + 1 = 1 + 1 + 1 = 3 \end{aligned}$$

Quick Tip: For integrals of the form $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)} dx$, the value is always simply $\frac{b-a}{2}$.

77. $\int_0^\pi \sqrt{1 + 4 \cos \frac{x}{2} \left(\cos \frac{x}{2} - 1 \right)} dx =$

- (A) $4\sqrt{3} - 4 - \frac{\pi}{3}$
- (B) $4\sqrt{3} - 4 - \frac{4\pi}{3}$
- (C) $\frac{4\pi}{3} - 4\sqrt{3} + 4$
- (D) $\frac{\pi}{3} - 4\sqrt{3} + 4$

Correct Answer: (A) $4\sqrt{3} - 4 - \frac{\pi}{3}$

Solution:

Concept:

- Simplify the expression under the square root into a perfect square:

$$\sqrt{u^2} = |u|$$

- Split the integral at the point where the expression inside the modulus changes sign.

Step 1: Simplify the radicand into a perfect square

Expand inside the square root:

$$1 + 4 \cos \frac{x}{2} \left(\cos \frac{x}{2} - 1 \right) = 1 + 4 \cos^2 \frac{x}{2} - 4 \cos \frac{x}{2}$$

Notice this is a perfect square:

$$\left(2 \cos \frac{x}{2} - 1 \right)^2$$

Therefore, the square root simplifies to:

$$\sqrt{\left(2 \cos \frac{x}{2} - 1 \right)^2} = \left| 2 \cos \frac{x}{2} - 1 \right|$$

Step 2: Find the root where the integrand changes sign

Set the argument of the absolute value to zero:

$$2 \cos \frac{x}{2} - 1 = 0 \implies \cos \frac{x}{2} = \frac{1}{2}$$

For $x \in [0, \pi]$, we have $\frac{x}{2} \in [0, \frac{\pi}{2}]$:

$$\frac{x}{2} = \frac{\pi}{3} \implies x = \frac{2\pi}{3}$$

Step 3: Split the integral into two intervals

Split the integration at $x = \frac{2\pi}{3}$:

- For $x \in [0, \frac{2\pi}{3}]$, $\cos \frac{x}{2} \geq \frac{1}{2}$, so $|2 \cos \frac{x}{2} - 1| = 2 \cos \frac{x}{2} - 1$.
- For $x \in [\frac{2\pi}{3}, \pi]$, $\cos \frac{x}{2} \leq \frac{1}{2}$, so $|2 \cos \frac{x}{2} - 1| = 1 - 2 \cos \frac{x}{2}$.

The integral becomes:

$$I = \int_0^{2\pi/3} \left(2 \cos \frac{x}{2} - 1 \right) dx + \int_{2\pi/3}^{\pi} \left(1 - 2 \cos \frac{x}{2} \right) dx$$

Step 4: Evaluate the first definite integral

Compute:

$$I_1 = \left[4 \sin \frac{x}{2} - x \right]_0^{2\pi/3}$$

Evaluate at the limits:

$$I_1 = \left(4 \sin \frac{\pi}{3} - \frac{2\pi}{3} \right) - (0 - 0) = 4 \left(\frac{\sqrt{3}}{2} \right) - \frac{2\pi}{3} = 2\sqrt{3} - \frac{2\pi}{3}$$

Step 5: Evaluate the second definite integral

Compute:

$$I_2 = \left[x - 4 \sin \frac{x}{2} \right]_{2\pi/3}^{\pi}$$

Evaluate at the limits:

$$I_2 = \left(\pi - 4 \sin \frac{\pi}{2} \right) - \left(\frac{2\pi}{3} - 4 \sin \frac{\pi}{3} \right)$$

$$I_2 = (\pi - 4) - \left(\frac{2\pi}{3} - 2\sqrt{3} \right) = \frac{\pi}{3} - 4 + 2\sqrt{3}$$

Step 6: Combine the two parts

Sum I_1 and I_2 :

$$I = \left(2\sqrt{3} - \frac{2\pi}{3} \right) + \left(2\sqrt{3} - 4 + \frac{\pi}{3} \right)$$

Combine like terms:

$$I = 4\sqrt{3} - 4 + \left(-\frac{2\pi}{3} + \frac{\pi}{3} \right) = 4\sqrt{3} - 4 - \frac{\pi}{3}$$

Quick Tip: Remember that $\sqrt{f(x)^2} = |f(x)|$. Never remove the square root without taking the absolute value when the expression can change sign.

78. $\int_{-\pi/3}^{\pi/3} \frac{x + \frac{\pi}{2}}{2 - \sin^2 x} dx =$

- (A) $\frac{\pi^2}{6\sqrt{3}}$
 (B) $\frac{\pi}{\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{3}}{\sqrt{2}} \right)$
 (C) $\frac{\pi^2}{3\sqrt{2}}$
 (D) $\frac{\pi}{2} \tan^{-1} \left(\frac{\sqrt{3}}{2} \right)$

Correct Answer: (B) $\frac{\pi}{\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{3}}{\sqrt{2}} \right)$

Solution:

Concept:

- Properties of definite integrals over symmetric intervals $[-a, a]$:

$$\int_{-a}^a f(x) dx = 0 \quad \text{if } f(-x) = -f(x) \text{ (odd)}$$

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx \quad \text{if } f(-x) = f(x) \text{ (even)}$$

- For integrals involving $\sin^2 x$ in the denominator, divide numerator and denominator by $\cos^2 x$ to convert to $\tan x$ and $\sec^2 x$.

Step 1: Split the integral into two parts

Expand the numerator:

$$I = \int_{-\pi/3}^{\pi/3} \frac{x}{2 - \sin^2 x} dx + \frac{\pi}{2} \int_{-\pi/3}^{\pi/3} \frac{1}{2 - \sin^2 x} dx$$

For the first integral:

$$g(x) = \frac{x}{2 - \sin^2 x} \implies g(-x) = \frac{-x}{2 - \sin^2(-x)} = -\frac{x}{2 - \sin^2 x} = -g(x)$$

Since $g(x)$ is an odd function, the integral over the symmetric interval vanishes:

$$\int_{-\pi/3}^{\pi/3} \frac{x}{2 - \sin^2 x} dx = 0$$

Step 2: Simplify the remaining even function integral

The integrand $\frac{1}{2 - \sin^2 x}$ is an even function:

$$I = \frac{\pi}{2} \times 2 \int_0^{\pi/3} \frac{1}{2 - \sin^2 x} dx = \pi \int_0^{\pi/3} \frac{1}{2 - \sin^2 x} dx$$

Step 3: Divide numerator and denominator by $\cos^2 x$

Transform the integrand:

$$\frac{1}{2 - \sin^2 x} = \frac{\sec^2 x}{2 \sec^2 x - \tan^2 x}$$

Substitute $\sec^2 x = 1 + \tan^2 x$:

$$2 \sec^2 x - \tan^2 x = 2(1 + \tan^2 x) - \tan^2 x = 2 + \tan^2 x$$

The integral becomes:

$$I = \pi \int_0^{\pi/3} \frac{\sec^2 x}{\tan^2 x + 2} dx$$

Step 4: Apply substitution and evaluate the definite integral

Let:

$$t = \tan x \implies dt = \sec^2 x \, dx$$

Transform the limits:

- When $x = 0$, $t = \tan 0 = 0$
- When $x = \frac{\pi}{3}$, $t = \tan \frac{\pi}{3} = \sqrt{3}$

The integral is:

$$I = \pi \int_0^{\sqrt{3}} \frac{dt}{t^2 + (\sqrt{2})^2}$$

Use the standard formula $\int \frac{dt}{t^2 + a^2} = \frac{1}{a} \tan^{-1} \left(\frac{t}{a} \right)$:

$$I = \pi \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{t}{\sqrt{2}} \right) \right]_0^{\sqrt{3}} = \frac{\pi}{\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{3}}{\sqrt{2}} \right)$$

Quick Tip: Always separate $\int_{-a}^a (x + c) f_{\text{even}}(x) \, dx$ into an odd part (which is identically zero) and an even part: $2c \int_0^a f_{\text{even}}(x) \, dx$.

79. The solution of the differential equation $\frac{dy}{dx} = \frac{3^{x+y} - 2 \cdot 3^x}{3^{x+y} - 2 \cdot 3^y}$ when $y(1) = 2$ is

- (A) $3^y = 7(3^x) + 12$
- (B) $y = \log_3(7(3^x) - 14)$
- (C) $y = \log_3(7(3^x) - 12)$
- (D) $3^y = 7(3^x) - 14$

Correct Answer: (C) $y = \log_3(7(3^x) - 12)$

Solution:

Concept:

- Use exponent rules: $3^{x+y} = 3^x \cdot 3^y$.
- Factor terms in the numerator and denominator to separate the variables:

$$g(y) \, dy = f(x) \, dx$$

- Integrate both sides and determine the constant of integration using the initial condition.

Step 1: Factor the right-hand side of the differential equation

Write 3^{x+y} as $3^x \cdot 3^y$:

$$\frac{dy}{dx} = \frac{3^x \cdot 3^y - 2 \cdot 3^x}{3^x \cdot 3^y - 2 \cdot 3^y}$$

Factor 3^x in the numerator and 3^y in the denominator:

$$\frac{dy}{dx} = \frac{3^x(3^y - 2)}{3^y(3^x - 2)}$$

Step 2: Separate the variables

Rearrange to group all y -terms with dy and x -terms with dx :

$$\frac{3^y}{3^y - 2} dy = \frac{3^x}{3^x - 2} dx$$

Step 3: Integrate both sides

Set up the integrals:

$$\int \frac{3^y}{3^y - 2} dy = \int \frac{3^x}{3^x - 2} dx$$

For the left integral, substitute $u = 3^y - 2 \implies du = 3^y \ln 3 dy$:

$$\int \frac{3^y}{3^y - 2} dy = \frac{1}{\ln 3} \ln |3^y - 2|$$

Similarly, for the right integral:

$$\int \frac{3^x}{3^x - 2} dx = \frac{1}{\ln 3} \ln |3^x - 2|$$

Equating both sides with an arbitrary constant:

$$\frac{1}{\ln 3} \ln |3^y - 2| = \frac{1}{\ln 3} \ln |3^x - 2| + \frac{1}{\ln 3} \ln C$$

Multiply through by $\ln 3$:

$$\ln |3^y - 2| = \ln(C|3^x - 2|) \implies 3^y - 2 = C(3^x - 2)$$

Step 4: Find the constant C using the initial condition

We are given $y(1) = 2$, so $x = 1$ and $y = 2$:

$$3^2 - 2 = C(3^1 - 2)$$

$$9 - 2 = C(3 - 2) \implies 7 = C(1) \implies C = 7$$

Step 5: Solve explicitly for y

Substitute $C = 7$ into the relation:

$$3^y - 2 = 7(3^x - 2)$$

Expand the right-hand side:

$$3^y - 2 = 7(3^x) - 14$$

Isolate 3^y :

$$3^y = 7(3^x) - 14 + 2 = 7(3^x) - 12$$

Take the logarithm with base 3:

$$y = \log_3(7(3^x) - 12)$$

Quick Tip: Factoring out common powers immediately reduces the problem to variable separable form. The identical algebraic structures for x and y mean their integrals have identical functional forms.

80. The general solution of the differential equation $(2xy + y^2)dy = (x^2 - y^2)dx$ is

- (A) $x^3 - 3x^2y - y^3 = c$
- (B) $x^3 - 3x^2y + y^3 = c$
- (C) $x^3 - 3xy^2 + y^3 = c$
- (D) $x^3 - 3xy^2 - y^3 = c$

Correct Answer: (D) $x^3 - 3xy^2 - y^3 = c$

Solution:

Concept:

- A first-order differential equation $M(x, y)dx + N(x, y)dy = 0$ is exact if:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- If exact, the solution is given by $F(x, y) = c$ such that:

$$dF = M dx + N dy$$

- Alternatively, group terms to form total differentials directly.

Step 1: Write the equation in standard form $M dx + N dy = 0$

The given differential equation is:

$$(2xy + y^2)dy = (x^2 - y^2)dx$$

Bring all terms to one side:

$$(x^2 - y^2)dx - (2xy + y^2)dy = 0$$

Here:

$$M(x, y) = x^2 - y^2$$

$$N(x, y) = -(2xy + y^2) = -2xy - y^2$$

Step 2: Check the condition of exactness

Differentiate M with respect to y :

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x^2 - y^2) = -2y$$

Differentiate N with respect to x :

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(-2xy - y^2) = -2y$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -2y$, the equation is an exact differential equation.

Step 3: Integrate the exact differential equation

The general solution is given by:

$$\int_{y=\text{constant}} M dx + \int (\text{terms of } N \text{ free from } x) dy = c_1$$

Evaluate the first integral with y treated as a constant:

$$\int (x^2 - y^2) dx = \frac{x^3}{3} - xy^2$$

Evaluate the second integral using only terms in N without x : The terms of $N = -2xy - y^2$ without x is just $-y^2$:

$$\int (-y^2) dy = -\frac{y^3}{3}$$

Step 4: Combine the terms and clear fractions

Sum the two integrated components:

$$\frac{x^3}{3} - xy^2 - \frac{y^3}{3} = c_1$$

Multiply the entire equation by 3:

$$x^3 - 3xy^2 - y^3 = 3c_1$$

Setting $c = 3c_1$ gives the general solution:

$$x^3 - 3xy^2 - y^3 = c$$

Quick Tip: Recognize total differentials: $d(x^3) = 3x^2dx$, $d(-3xy^2) = -3y^2dx - 6xydy$, and $d(-y^3) = -3y^2dy$. Grouping them yields $d(x^3 - 3xy^2 - y^3) = 0$ immediately.

Physics

81. The fundamental force that plays a key role in the large scale phenomena of the universe is

(A) electromagnetic force

- (B) strong nuclear force
- (C) weak nuclear force
- (D) gravitational force

Correct Answer: (D) gravitational force

Solution:

Concept:

- There are four fundamental forces in nature: gravitational force, electromagnetic force, strong nuclear force, and weak nuclear force.
- Gravitational force is an infinite-range, universally attractive force acting between all masses.
- Electromagnetic force is also infinite in range, but astronomical bodies are electrically neutral on macroscopic scales due to equal numbers of positive and negative charges.
- Strong and weak nuclear forces are extremely short-ranged (effective only at subatomic scales $\sim 10^{-15}$ m and $\sim 10^{-18}$ m respectively).

Step 1: Analyze the characteristics of the fundamental forces

Consider the properties of each fundamental force:

- Strong nuclear force: Confined to the atomic nucleus ($\sim 10^{-15}$ m).
- Weak nuclear force: Confined to sub-nuclear distances ($\sim 10^{-18}$ m).
- Electromagnetic force: Long range, but cancels out on large scales due to net charge neutrality of cosmic matter.
- Gravitational force: Operates across infinite distances and is always attractive, meaning it never cancels out.

Step 2: Identify the dominant force on astronomical and cosmic scales

Because large celestial bodies (planets, stars, galaxies, and galaxy clusters) possess enormous mass and are electrically neutral: The cumulative gravitational pull governs orbital motions, formation of galaxies, stellar evolution, and cosmic expansion. Therefore, the gravitational force plays the key role in the large-scale phenomena of the universe.

Quick Tip: Although gravity is the weakest of the four fundamental forces, its non-canceling, universally attractive nature makes it dominate all large-scale cosmic structures.

82. If the errors in the measurements of diameter, length and electrical resistance of a wire are 1%, 0.5% and 2% respectively, then percentage error in the determination of the resistivity of material of the wire is

- (A) 3.5
- (B) 4
- (C) 4.5
- (D) 2.5

Correct Answer: (C) 4.5

Solution:

Concept:

- Electrical resistance R of a cylindrical wire of length L , cross-sectional area A , and resistivity ρ is:

$$R = \rho \frac{L}{A} \implies \rho = \frac{RA}{L}$$

- For a wire with circular cross-section of diameter d , the area is $A = \frac{\pi d^2}{4}$.
- For any physical quantity $Z = c \cdot X^a Y^b Z^{-c}$, the maximum fractional error is:

$$\frac{\Delta Z}{Z} = |a| \frac{\Delta X}{X} + |b| \frac{\Delta Y}{Y} + |c| \frac{\Delta Z}{Z}$$

Step 1: Express resistivity in terms of measured variables

Substitute $A = \frac{\pi d^2}{4}$ into the formula for resistivity:

$$\rho = \frac{R \left(\frac{\pi d^2}{4} \right)}{L} = \frac{\pi}{4} \cdot \frac{R d^2}{L}$$

Step 2: Formulate the percentage error relation

Taking the relative error on both sides:

$$\frac{\Delta \rho}{\rho} \times 100\% = \left(\frac{\Delta R}{R} + 2 \frac{\Delta d}{d} + \frac{\Delta L}{L} \right) \times 100\%$$

Step 3: Substitute the given percentage errors and compute

The given errors are:

$$\frac{\Delta d}{d} \times 100\% = 1\%$$

$$\frac{\Delta L}{L} \times 100\% = 0.5\%$$

$$\frac{\Delta R}{R} \times 100\% = 2\%$$

Substitute these into the error equation:

$$\begin{aligned}\frac{\Delta \rho}{\rho} \times 100\% &= 2\% + 2(1\%) + 0.5\% \\ &= 2\% + 2\% + 0.5\% \\ &= 4.5\%\end{aligned}$$

Quick Tip: Remember to multiply the fractional error of the diameter by its exponent of 2, since the area depends quadratically on the diameter: $\text{Error} = \%R + 2(\%d) + \%L$.

83. Two balls P and Q are thrown vertically upwards simultaneously from the ground with velocities 20 ms^{-1} and 35 ms^{-1} respectively. The distance between the two balls when the velocity of the ball P becomes zero is (Acceleration due to gravity $= 10 \text{ ms}^{-2}$)

- (A) 30 m
- (B) 20 m
- (C) 50 m
- (D) 70 m

Correct Answer: (A) 30 m

Solution:

Concept:

- For vertical motion under gravity with upward direction chosen as positive:

$$v = u - gt$$

$$h = ut - \frac{1}{2}gt^2$$

- The time taken for an object projected with speed u to reach its highest point (where $v = 0$) is $t = \frac{u}{g}$.
- The separation between the two balls is the difference between their vertical positions at that instant: $\Delta h = |h_Q - h_P|$.

Step 1: Find the time at which the velocity of ball P becomes zero

For ball P, initial velocity $u_P = 20 \text{ ms}^{-1}$. Using the first equation of motion:

$$v_P = u_P - gt$$

Set $v_P = 0$:

$$0 = 20 - 10t \implies 10t = 20 \implies t = 2 \text{ s}$$

Step 2: Calculate the height of ball P at $t = 2 \text{ s}$

Using the displacement equation:

$$h_P = u_P t - \frac{1}{2}gt^2$$

Substitute $u_P = 20$ and $t = 2$:

$$h_P = 20(2) - \frac{1}{2}(10)(2)^2 = 40 - 20 = 20 \text{ m}$$

Step 3: Calculate the height of ball Q at $t = 2 \text{ s}$

For ball Q, initial velocity $u_Q = 35 \text{ ms}^{-1}$. Using the displacement equation:

$$h_Q = u_Q t - \frac{1}{2}gt^2$$

Substitute $u_Q = 35$ and $t = 2$:

$$h_Q = 35(2) - \frac{1}{2}(10)(2)^2 = 70 - 20 = 50 \text{ m}$$

Step 4: Determine the distance between the two balls

The distance between balls P and Q at $t = 2 \text{ s}$ is:

$$d = h_Q - h_P = 50 - 20 = 30 \text{ m}$$

Quick Tip: Since both bodies experience identical gravitational acceleration g , their relative acceleration is zero: $a_{\text{rel}} = 0$. Hence, separation is simply $d = u_{\text{rel}} \cdot t = (35 - 20) \times 2 = 30$ m.

84. If a body is projected from the ground at an angle of 45° with the horizontal, then the ratio of the velocities of the body at maximum height and at half of the maximum height is

- (A) $\sqrt{3} : \sqrt{7}$
- (B) $\sqrt{2} : \sqrt{3}$
- (C) $\sqrt{2} : \sqrt{5}$
- (D) $\sqrt{3} : \sqrt{5}$

Correct Answer: (B) $\sqrt{2} : \sqrt{3}$

Solution:

Concept:

- In projectile motion, the horizontal component of velocity remains constant throughout the flight:

$$v_x = u \cos \theta$$

- At the maximum height H , the vertical velocity component vanishes: $v_y = 0$.
- The total speed at any height h is given by:

$$v = \sqrt{v_x^2 + v_y^2}$$

- By conservation of mechanical energy:

$$\frac{1}{2}mv_1^2 + mgh_1 = \frac{1}{2}mv_2^2 + mgh_2$$

Step 1: Determine the velocity at maximum height

Let the initial speed of projection be u at an angle $\theta = 45^\circ$. The horizontal component of velocity is:

$$u_x = u \cos 45^\circ = \frac{u}{\sqrt{2}}$$

The initial vertical component of velocity is:

$$u_y = u \sin 45^\circ = \frac{u}{\sqrt{2}}$$

At maximum height H , the vertical component is $v_y = 0$. Thus, the velocity at maximum height is purely horizontal:

$$v_H = u_x = \frac{u}{\sqrt{2}}$$

Step 2: Determine the vertical velocity at half the maximum height

The maximum height is:

$$H = \frac{u_y^2}{2g}$$

At height $h = \frac{H}{2}$, use the third equation of motion in the vertical direction:

$$v_y^2 = u_y^2 - 2g \left(\frac{H}{2} \right) = u_y^2 - gH$$

Substitute $H = \frac{u_y^2}{2g}$:

$$v_y^2 = u_y^2 - g \left(\frac{u_y^2}{2g} \right) = u_y^2 - \frac{u_y^2}{2} = \frac{u_y^2}{2}$$

Substitute $u_y = \frac{u}{\sqrt{2}}$:

$$v_y^2 = \frac{1}{2} \left(\frac{u^2}{2} \right) = \frac{u^2}{4}$$

Step 3: Compute the total velocity at half maximum height

Combine the horizontal and vertical velocity components:

$$v_{H/2}^2 = v_x^2 + v_y^2 = \left(\frac{u}{\sqrt{2}} \right)^2 + \frac{u^2}{4} = \frac{u^2}{2} + \frac{u^2}{4} = \frac{3u^2}{4}$$

Taking the square root:

$$v_{H/2} = \frac{\sqrt{3}u}{2}$$

Step 4: Calculate the required ratio

Compute the ratio of v_H to $v_{H/2}$:

$$\frac{v_H}{v_{H/2}} = \frac{\frac{u}{\sqrt{2}}}{\frac{\sqrt{3}u}{2}} = \frac{2}{\sqrt{2}\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}}$$

Thus, the ratio is $\sqrt{2} : \sqrt{3}$.

Quick Tip: Use conservation of energy directly: $v_{H/2}^2 = v_H^2 + 2g(H/2) = v_H^2 + gH$. Since $gH = \frac{u_y^2}{2} = \frac{v_H^2}{2}$, we have $v_{H/2}^2 = \frac{3}{2}v_H^2 \implies \frac{v_H}{v_{H/2}} = \sqrt{\frac{2}{3}}$.

85. A block of mass 2 kg is kept on a horizontal surface and is pulled with a horizontal force F . If the surface is smooth, the acceleration of the block is 8 ms^{-2} . If the surface is rough and the coefficient of kinetic friction between the block and the surface is 0.25, then the acceleration of the block is (Acceleration due to gravity = 10 ms^{-2})

- (A) 5.5 ms^{-2}
- (B) 6.5 ms^{-2}
- (C) 4.5 ms^{-2}
- (D) 3.5 ms^{-2}

Correct Answer: (A) 5.5 ms^{-2}

Solution:

Concept:

- On a smooth surface, friction is absent, so Newton's second law gives:

$$F = ma_{\text{smooth}}$$

- On a rough horizontal surface, kinetic friction opposes the applied horizontal force:

$$f_k = \mu_k N = \mu_k mg$$

- The net force and resulting acceleration are:

$$F_{\text{net}} = F - f_k \implies a_{\text{rough}} = \frac{F - f_k}{m}$$

Step 1: Determine the magnitude of the applied force F

For the smooth surface:

$$m = 2 \text{ kg}, \quad a = 8 \text{ ms}^{-2}$$

Using Newton's second law:

$$F = ma = 2 \times 8 = 16 \text{ N}$$

Step 2: Calculate the force of kinetic friction on the rough surface

The normal reaction on a horizontal surface is:

$$N = mg = 2 \times 10 = 20 \text{ N}$$

The kinetic friction force is:

$$f_k = \mu_k N = 0.25 \times 20 = 5 \text{ N}$$

Step 3: Find the net accelerating force on the rough surface

The net force acting on the block is:

$$F_{\text{net}} = F - f_k = 16 - 5 = 11 \text{ N}$$

Step 4: Calculate the acceleration

Apply Newton's second law to find the new acceleration:

$$a_{\text{rough}} = \frac{F_{\text{net}}}{m} = \frac{11}{2} = 5.5 \text{ ms}^{-2}$$

Quick Tip: The reduction in acceleration due to friction is simply $\Delta a = \mu_k g$. Here, $\Delta a = 0.25 \times 10 = 2.5 \text{ ms}^{-2}$, so $a_{\text{rough}} = 8 - 2.5 = 5.5 \text{ ms}^{-2}$.

86. If a force of $(6\sqrt{3}\hat{i} - 4\sqrt{2}\hat{j})$ N acts on a body of mass 3.7 kg and displaces the body by $(2\sqrt{3}\hat{i})$ m, then the work done by the force is

- (A) 18 J
- (B) 40 J
- (C) 36 J
- (D) 52 J

Correct Answer: (C) 36 J

Solution:

Concept:

- Work done by a constant force $\vec{F}$ during a displacement $\vec{d}$ is defined as the scalar dot

product:

$$W = \vec{F} \cdot \vec{d}$$

- For vectors given in Cartesian components:

$$\vec{F} \cdot \vec{d} = F_x d_x + F_y d_y + F_z d_z$$

- Work done is completely independent of the mass of the object when the force and displacement are already specified.

Step 1: Identify the components of force and displacement

The given force vector is:

$$\vec{F} = 6\sqrt{3}\hat{i} - 4\sqrt{2}\hat{j} + 0\hat{k} \text{ N}$$

The given displacement vector is:

$$\vec{d} = 2\sqrt{3}\hat{i} + 0\hat{j} + 0\hat{k} \text{ m}$$

Step 2: Calculate the dot product

Using the formula for work:

$$W = \vec{F} \cdot \vec{d} = (F_x)(d_x) + (F_y)(d_y) + (F_z)(d_z)$$

Substitute the respective components:

$$W = (6\sqrt{3})(2\sqrt{3}) + (-4\sqrt{2})(0) + (0)(0)$$

Simplify the product:

$$W = 12 \times 3 + 0 = 36 \text{ J}$$

Quick Tip: Mass is extraneous information here; work done depends solely on $\vec{F} \cdot \vec{d}$. Since displacement has only an x-component, only the x-component of the force does work.

87. A ball P in motion collides with another ball Q of same mass at rest. If the coefficient of restitution between the two balls is 0.2, then the ratio of the velocities of the two balls P and Q

after collision is

- (A) 1 : 3
- (B) 3 : 4
- (C) 1 : 1
- (D) 2 : 3

Correct Answer: (D) 2 : 3

Solution:

Concept:

- By conservation of linear momentum for two equal masses ($m_1 = m_2 = m$):

$$u_1 + u_2 = v_1 + v_2$$

- The coefficient of restitution e relates relative velocity of separation to relative velocity of approach:

$$e = \frac{v_2 - v_1}{u_1 - u_2}$$

- Combining these gives the velocities of equal masses after collision when the second mass is initially at rest ($u_2 = 0$):

$$v_1 = \left(\frac{1-e}{2} \right) u_1, \quad v_2 = \left(\frac{1+e}{2} \right) u_1$$

Step 1: State the initial conditions

Let the mass of both balls be m . Initial velocities before collision:

$$u_p = u, \quad u_Q = 0$$

Coefficient of restitution:

$$e = 0.2$$

Step 2: Express final velocities using conservation laws

Conservation of momentum:

$$mu_p + mu_Q = mv_p + mv_Q \implies v_p + v_Q = u \quad \text{--- (1)}$$

Definition of coefficient of restitution:

$$e = \frac{v_Q - v_P}{u_P - u_Q} \implies v_Q - v_P = eu = 0.2u \quad \text{--- (2)}$$

Step 3: Solve for v_P and v_Q

Subtract equation (2) from equation (1):

$$2v_P = u - 0.2u = 0.8u \implies v_P = 0.4u$$

Add equation (1) and equation (2):

$$2v_Q = u + 0.2u = 1.2u \implies v_Q = 0.6u$$

Step 4: Find the ratio of velocities after collision

Compute the ratio of v_P to v_Q :

$$\frac{v_P}{v_Q} = \frac{0.4u}{0.6u} = \frac{4}{6} = \frac{2}{3}$$

Thus, the ratio is 2 : 3.

Quick Tip: For a collision of equal masses with one initially at rest, the ratio of velocities post-collision is always $\frac{v_1}{v_2} = \frac{1-e}{1+e}$. Here, $\frac{1-0.2}{1+0.2} = \frac{0.8}{1.2} = \frac{2}{3}$.

88. A body P of mass 3 kg at rest is dropped from a height of 250 m from the ground. At the same moment another body Q of mass 2 kg is thrown vertically upwards from the ground with a velocity of 50 ms^{-1} . Both the bodies travel along the same straight line in opposite directions. The velocity of body Q when the centre of mass of the system of the bodies P and Q reaches the maximum height is (Acceleration due to gravity = 10 ms^{-2})

- (A) 25 ms^{-1}
- (B) 30 ms^{-1}
- (C) 40 ms^{-1}
- (D) 20 ms^{-1}

Correct Answer: (B) 30 ms^{-1}

Solution:

Concept:

- The velocity of the center of mass (COM) of a two-body system is:

$$v_{\text{cm}} = \frac{m_P v_P + m_Q v_Q}{m_P + m_Q}$$

- When the center of mass reaches its maximum height, its instantaneous vertical velocity is zero:

$$v_{\text{cm}} = 0 \implies m_P v_P + m_Q v_Q = 0$$

- Individual velocities under uniform gravity with upward chosen as positive:

$$v_P(t) = u_P - gt, \quad v_Q(t) = u_Q - gt$$

Step 1: Express individual velocities as functions of time

Choose the vertically upward direction as positive. For body P:

$$m_P = 3 \text{ kg}, \quad u_P = 0$$

$$v_P(t) = 0 - gt = -10t$$

For body Q:

$$m_Q = 2 \text{ kg}, \quad u_Q = +50 \text{ ms}^{-1}$$

$$v_Q(t) = 50 - gt = 50 - 10t$$

Step 2: Apply the maximum height condition for the center of mass

At the instant the center of mass reaches maximum height:

$$v_{\text{cm}} = 0$$

This implies total momentum is zero:

$$m_P v_P(t) + m_Q v_Q(t) = 0$$

Substitute the velocity expressions:

$$3(-10t) + 2(50 - 10t) = 0$$

Expand and solve for t :

$$-30t + 100 - 20t = 0$$

$$100 - 50t = 0 \implies 50t = 100 \implies t = 2 \text{ s}$$

Step 3: Calculate the velocity of body Q at this instant

Substitute $t = 2 \text{ s}$ into the velocity equation for body Q:

$$v_Q = 50 - 10(2) = 50 - 20 = 30 \text{ ms}^{-1}$$

Quick Tip: The initial velocity of the center of mass is $u_{\text{cm}} = \frac{2(50) + 3(0)}{5} = 20 \text{ ms}^{-1}$. Since the center of mass accelerates downwards at $g = 10 \text{ ms}^{-2}$, it reaches maximum height when $t = \frac{u_{\text{cm}}}{g} = \frac{20}{10} = 2 \text{ s}$.

89. If the kinetic energy of a solid sphere when it rolls without slipping is 700 J, then its kinetic energy when it slips without rolling with same velocity is

- (A) 700 J
- (B) 500 J
- (C) 300 J
- (D) 400 J

Correct Answer: (B) 500 J

Solution:

Concept:

- Total kinetic energy in pure rolling (rolling without slipping) is the sum of translational and rotational kinetic energies:

$$K_{\text{rolling}} = K_{\text{trans}} + K_{\text{rot}} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

- For a solid sphere of mass m and radius R , the moment of inertia about the central axis is:

$$I = \frac{2}{5}mR^2$$

- For pure rolling, the rolling constraint is $v = \omega R$.
- When an object slips without rolling, it possesses only translational kinetic energy:

$$K_{\text{slipping}} = K_{\text{trans}} = \frac{1}{2}mv^2$$

Step 1: Express total rolling kinetic energy in terms of translational kinetic energy

For a solid sphere:

$$K_{\text{rot}} = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{2}{5}mR^2\right)\left(\frac{v}{R}\right)^2 = \frac{1}{5}mv^2$$

The total kinetic energy in rolling without slipping is:

$$K_{\text{rolling}} = \frac{1}{2}mv^2 + \frac{1}{5}mv^2 = \left(\frac{1}{2} + \frac{1}{5}\right)mv^2 = \frac{7}{10}mv^2$$

Notice that:

$$K_{\text{rolling}} = \frac{7}{5}\left(\frac{1}{2}mv^2\right) = \frac{7}{5}K_{\text{trans}}$$

Step 2: Calculate the translational kinetic energy

We are given that:

$$K_{\text{rolling}} = 700 \text{ J}$$

Substitute into the relation:

$$\frac{7}{5}K_{\text{trans}} = 700$$

Solve for K_{trans} :

$$K_{\text{trans}} = 700 \times \frac{5}{7} = 500 \text{ J}$$

Step 3: Determine the kinetic energy during pure slipping

When the sphere slips without rolling at the same velocity v , its angular velocity is zero, so it has only translational kinetic energy:

$$K_{\text{slipping}} = K_{\text{trans}} = 500 \text{ J}$$

Quick Tip: For any rolling body, the fraction of translational kinetic energy is $\frac{K_{\text{trans}}}{K_{\text{total}}} = \frac{1}{1 + \frac{k^2}{R^2}}$. For a solid sphere, $\frac{k^2}{R^2} = \frac{2}{5}$, so $K_{\text{trans}} = \frac{5}{7}K_{\text{total}} = \frac{5}{7}(700) = 500 \text{ J}$.

90. A particle executes simple harmonic motion with an amplitude of 40 cm. If the time period of the particle is 6 s, then the minimum time taken by the particle to move from mean position to a point at a distance of 20 cm from the extreme position is

- (A) 0.5 s
- (B) 1.5 s
- (C) 3 s
- (D) 2.5 s

Correct Answer: (A) 0.5 s

Solution:

Concept:

- The displacement of a particle executing SHM starting from the mean position is:

$$x(t) = A \sin(\omega t)$$

where A is the amplitude and $\omega = \frac{2\pi}{T}$ is the angular frequency.

- The position of the particle measured from the mean position when it is at distance d from the extreme position is:

$$x = A - d$$

Step 1: Determine the target displacement from the mean position

Given:

$$\text{Amplitude } A = 40 \text{ cm}$$

$$\text{Distance from extreme position } d = 20 \text{ cm}$$

The displacement from the mean position x is:

$$x = A - d = 40 - 20 = 20 \text{ cm}$$

Notice that this position corresponds to half the amplitude:

$$x = \frac{A}{2}$$

Step 2: Calculate the angular frequency ω

Given time period:

$$T = 6 \text{ s}$$

The angular frequency is:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{6} = \frac{\pi}{3} \text{ rad/s}$$

Step 3: Set up the equation of motion and solve for time t

Using the equation of motion starting from the mean position:

$$x(t) = A \sin(\omega t)$$

Substitute $x = \frac{A}{2}$:

$$\frac{A}{2} = A \sin(\omega t) \implies \sin(\omega t) = \frac{1}{2}$$

The minimum positive time corresponds to the smallest positive angle:

$$\omega t = \frac{\pi}{6}$$

Step 4: Calculate the numerical value of t

Substitute $\omega = \frac{\pi}{3}$:

$$\left(\frac{\pi}{3}\right)t = \frac{\pi}{6} \implies t = \frac{\pi/6}{\pi/3} = \frac{3}{6} = 0.5 \text{ s}$$

Quick Tip: In SHM, the time taken to travel from the mean position to half the amplitude ($x = A/2$) is always $\frac{T}{12}$. Here, $t = \frac{6}{12} = 0.5 \text{ s}$.

91. If two bodies A and B of masses 5 kg and 10 kg are thrown vertically upwards from the surface of the earth with velocities $\sqrt{0.5gR}$ and $\sqrt{0.2gR}$ respectively, then the ratio of maximum heights reached by the bodies A and B is (R – Radius of the earth)

(A) 3 : 2

- (B) 5 : 2
(C) 3 : 1
(D) 2 : 1

Correct Answer: (C) 3 : 1

Solution:

Concept:

- When projection speeds are comparable to orbital/escape velocity, gravitational field variation with distance must be accounted for using the law of conservation of mechanical energy:

$$\frac{1}{2}mv^2 - \frac{GMm}{R} = -\frac{GMm}{R+h}$$

- Using $g = \frac{GM}{R^2} \implies GM = gR^2$, the maximum height h above the surface is related to projection speed v by:

$$h = \frac{v^2 R}{2gR - v^2}$$

- The maximum height is completely independent of the mass of the projectile.

Step 1: Derive the formula for the maximum height from conservation of energy

Let a particle of mass m be projected with initial speed v from the surface of the Earth. At the maximum height h , its instantaneous velocity is zero. By conservation of mechanical energy:

$$\frac{1}{2}mv^2 - \frac{GMm}{R} = 0 - \frac{GMm}{R+h}$$

Divide both sides by m :

$$\frac{1}{2}v^2 = GM \left(\frac{1}{R} - \frac{1}{R+h} \right) = \frac{GMh}{R(R+h)}$$

Substitute $GM = gR^2$:

$$\frac{1}{2}v^2 = \frac{gR^2h}{R(R+h)} = \frac{gRh}{R+h}$$

Solve for h :

$$v^2(R+h) = 2gRh \implies v^2R = (2gR - v^2)h \implies h = \frac{v^2R}{2gR - v^2}$$

Step 2: Calculate the maximum height reached by body A

For body A, the velocity is:

$$v_A = \sqrt{0.5gR} \implies v_A^2 = 0.5gR$$

Substitute into the height formula:

$$h_A = \frac{(0.5gR)R}{2gR - 0.5gR} = \frac{0.5gR^2}{1.5gR} = \frac{0.5}{1.5}R = \frac{1}{3}R$$

Step 3: Calculate the maximum height reached by body B

For body B, the velocity is:

$$v_B = \sqrt{0.2gR} \implies v_B^2 = 0.2gR$$

Substitute into the height formula:

$$h_B = \frac{(0.2gR)R}{2gR - 0.2gR} = \frac{0.2gR^2}{1.8gR} = \frac{0.2}{1.8}R = \frac{1}{9}R$$

Step 4: Compute the ratio of the maximum heights

Divide h_A by h_B :

$$\frac{h_A}{h_B} = \frac{\frac{1}{3}R}{\frac{1}{9}R} = \frac{9}{3} = 3$$

Thus, the ratio of maximum heights is 3 : 1.

Quick Tip: Remember the standard relation for large heights: $h = \frac{R}{\frac{2gR}{v^2} - 1}$. For $v_A^2 = \frac{1}{2}gR$, $\frac{2gR}{v_A^2} = 4 \implies h_A = \frac{R}{3}$. For $v_B^2 = \frac{1}{5}gR$, $\frac{2gR}{v_B^2} = 10 \implies h_B = \frac{R}{9}$. Their ratio is immediately $\frac{9}{3} = 3 : 1$.

92. A metal wire can withstand a maximum tension of 80 N. If the wire is cut into four parts of equal length, then each part can withstand a maximum tension of

- (A) 80 N
- (B) 20 N
- (C) 320 N
- (D) 40 N

Correct Answer: (A) 80 N

Solution:

Concept:

- Breaking stress (tensile strength) is an intrinsic material property that depends only on the material and is independent of dimensions:

$$\sigma_{\max} = \frac{T_{\max}}{A}$$

- The maximum tension (breaking load) a wire can withstand is directly proportional to its cross-sectional area A :

$$T_{\max} = \sigma_{\max} \cdot A$$

- Cutting a wire along its length does not alter its cross-sectional area.

Step 1: Relate maximum tension to the cross-sectional area

The maximum tension that a wire can support before breaking is given by:

$$T_{\max} = \sigma_b \cdot A$$

where σ_b is the breaking stress of the material and A is the area of cross-section of the wire.

Step 2: Analyze the effect of cutting the wire into parts

The original wire is cut into four parts of equal length:

- The length of each part becomes $L' = \frac{L}{4}$.
- The cross-sectional area A remains completely unchanged.
- The material of the wire remains the same, so the breaking stress σ_b remains unchanged.

Step 3: Determine the maximum tension for each piece

Since neither σ_b nor A changes for each piece:

$$T'_{\max} = \sigma_b \cdot A = T_{\max} = 80 \text{ N}$$

Each part can still withstand a maximum tension of 80 N.

Quick Tip: Breaking force depends exclusively on the cross-sectional area, not on the length of the wire. Cutting the length into any number of pieces leaves the maximum breaking tension unchanged.

93. A uniform cylindrical vessel is filled with two immiscible liquids A and B of densities

750 kg m⁻³ and 1000 kg m⁻³ respectively. The thickness of the liquid layer A is 10 m and the thickness of the liquid layer B which lies at the bottom is 15 m. If the atmospheric pressure is 10⁵ N m⁻², then the ratio of the absolute pressures at the bottom of the vessel and at the interface of the two liquids is (Acceleration due to gravity = 10 ms⁻²)

- (A) 10 : 7
- (B) 2 : 1
- (C) 13 : 7
- (D) 3 : 1

Correct Answer: (C) 13 : 7

Solution:

Concept:

- The lighter liquid (lower density) floats on top of the heavier liquid (higher density).
- Absolute pressure at any depth is the sum of atmospheric pressure and the gauge pressure of all liquid columns lying above that depth:

$$P = P_0 + \sum \rho_i g h_i$$

- Absolute pressure at the interface between liquid A and liquid B is:

$$P_{\text{interface}} = P_0 + \rho_A g h_A$$

- Absolute pressure at the bottom of the vessel is:

$$P_{\text{bottom}} = P_{\text{interface}} + \rho_B g h_B$$

Step 1: State the given parameters

The parameters of the liquids and environment are:

$$P_0 = 10^5 \text{ N m}^{-2}$$

$$\rho_A = 750 \text{ kg m}^{-3}, \quad h_A = 10 \text{ m}$$

$$\rho_B = 1000 \text{ kg m}^{-3}, \quad h_B = 15 \text{ m}$$

$$g = 10 \text{ ms}^{-2}$$

Step 2: Calculate the absolute pressure at the interface

The interface is located at depth $h_A = 10 \text{ m}$ below the free surface:

$$P_{\text{interface}} = P_0 + \rho_A g h_A$$

Compute the hydrostatic pressure of layer A:

$$\rho_A g h_A = 750 \times 10 \times 10 = 75000 \text{ N m}^{-2} = 0.75 \times 10^5 \text{ N m}^{-2}$$

Thus:

$$P_{\text{interface}} = 10^5 + 0.75 \times 10^5 = 1.75 \times 10^5 \text{ N m}^{-2}$$

Step 3: Calculate the absolute pressure at the bottom of the vessel

The bottom of the vessel lies below both liquid layers:

$$P_{\text{bottom}} = P_{\text{interface}} + \rho_B g h_B$$

Compute the hydrostatic pressure of layer B:

$$\rho_B g h_B = 1000 \times 10 \times 15 = 150000 \text{ N m}^{-2} = 1.50 \times 10^5 \text{ N m}^{-2}$$

Thus:

$$P_{\text{bottom}} = 1.75 \times 10^5 + 1.50 \times 10^5 = 3.25 \times 10^5 \text{ N m}^{-2}$$

Step 4: Compute the ratio of pressures

Find the ratio of P_{bottom} to $P_{\text{interface}}$:

$$\frac{P_{\text{bottom}}}{P_{\text{interface}}} = \frac{3.25 \times 10^5}{1.75 \times 10^5} = \frac{3.25}{1.75}$$

Multiply numerator and denominator by 4:

$$\frac{3.25 \times 4}{1.75 \times 4} = \frac{13}{7}$$

Thus, the ratio is 13 : 7.

Quick Tip: Work in units of 10^5 Pa: $P_0 = 1$, $\Delta P_A = 0.75 \Rightarrow P_{\text{interface}} = 1.75$. $\Delta P_B = 1.50 \Rightarrow P_{\text{bottom}} = 3.25$. The ratio is immediately $\frac{3.25}{1.75} = \frac{13}{7}$.

94. If W is the energy required to form a soap bubble of radius R , then the energy required to increase the radius of that bubble from R to $3R$ is

- (A) $9W$
- (B) $3W$
- (C) $8W$
- (D) $4W$

Correct Answer: (C) $8W$

Solution:

Concept:

- A soap bubble in air has two free liquid-air interfaces (inner and outer surfaces).
- The total surface area of a spherical soap bubble of radius r is:

$$A = 2 \times (4\pi r^2) = 8\pi r^2$$

- The surface energy of the bubble is directly proportional to its total surface area:

$$U = T \cdot A = 8\pi T r^2$$

where T is the surface tension of the soap solution.

- The work done to change the radius from R_1 to R_2 is the difference in surface energies:

$$\Delta W = U_2 - U_1 = 8\pi T (R_2^2 - R_1^2)$$

Step 1: Express the initial energy W in terms of radius R

The work done to form a soap bubble of radius R from zero radius is:

$$W = T \cdot \Delta A = T \cdot (2 \times 4\pi R^2) = 8\pi T R^2$$

Step 2: Calculate the energy required to increase the radius from R to $3R$

The initial radius is $R_1 = R$, and the final radius is $R_2 = 3R$. The increase in the total surface area is:

$$\Delta A = 2 \times (4\pi(3R)^2 - 4\pi R^2) = 8\pi(9R^2 - R^2) = 8\pi(8R^2)$$

Step 3: Relate the required energy to W

The work done to expand the bubble is:

$$W' = T \cdot \Delta A = T \cdot [8 \times (8\pi R^2)] = 8(8\pi TR^2)$$

Substitute $W = 8\pi TR^2$:

$$W' = 8W$$

Quick Tip: Surface energy is proportional to the square of the radius: $U \propto R^2$. Expanding from R to $3R$ changes the energy by $(3)^2 - (1)^2 = 9 - 1 = 8$, so the required energy is simply $8W$.

95. If Q_1 is the heat required to convert 2 g of ice at a temperature of 0°C to water at a temperature of 40°C and Q_2 is the heat required to convert 4 g of ice at a temperature of 0°C to water at a temperature of 80°C , then $Q_1 : Q_2 =$ (Latent heat of fusion of ice = 80 cal g^{-1} and specific heat capacity of water = $1 \text{ cal g}^{-1}\text{C}^{-1}$)

- (A) 3 : 8
- (B) 1 : 2
- (C) 1 : 4
- (D) 3 : 4

Correct Answer: (A) 3 : 8

Solution:

Concept:

- The total heat required to convert ice at 0°C into water at temperature T consists of two stages:

1. Phase change (melting) from ice at 0°C to water at 0°C :

$$Q_{\text{latent}} = mL_f$$

2. Temperature increase from 0°C to T :

$$Q_{\text{sensible}} = mc_w\Delta T$$

- Total heat required:

$$Q = m(L_f + c_w\Delta T)$$

Step 1: Calculate the heat Q_1

For the first case:

$$m_1 = 2 \text{ g}, \quad \Delta T_1 = 40 - 0 = 40^{\circ}\text{C}$$

The total heat required is:

$$Q_1 = m_1(L_f + c_w\Delta T_1)$$

Substitute $L_f = 80 \text{ cal g}^{-1}$ and $c_w = 1 \text{ cal g}^{-1}\text{C}^{-1}$:

$$Q_1 = 2 \times (80 + 1 \times 40) = 2 \times (80 + 40) = 2 \times 120 = 240 \text{ cal}$$

Step 2: Calculate the heat Q_2

For the second case:

$$m_2 = 4 \text{ g}, \quad \Delta T_2 = 80 - 0 = 80^{\circ}\text{C}$$

The total heat required is:

$$Q_2 = m_2(L_f + c_w\Delta T_2)$$

Substitute the numerical values:

$$Q_2 = 4 \times (80 + 1 \times 80) = 4 \times (80 + 80) = 4 \times 160 = 640 \text{ cal}$$

Step 3: Compute the ratio $Q_1 : Q_2$

Divide Q_1 by Q_2 :

$$\frac{Q_1}{Q_2} = \frac{240}{640} = \frac{24}{64}$$

Divide numerator and denominator by 8:

$$\frac{Q_1}{Q_2} = \frac{3}{8}$$

Thus, the ratio is 3 : 8.

Quick Tip: Calculate the per-gram heat requirements first: $q_1 = 80 + 40 = 120 \text{ cal/g}$, $q_2 = 80 + 80 = 160 \text{ cal/g}$. Then: $\frac{Q_1}{Q_2} = \frac{2 \times 120}{4 \times 160} = \frac{240}{640} = \frac{3}{8}$.

96. Two rectangular metal boxes A and B having same dimensions are made of different materials of thermal conductivities $180 \text{ W m}^{-1}\text{K}^{-1}$ and $270 \text{ W m}^{-1}\text{K}^{-1}$ respectively. The two boxes are completely filled with ice and are placed in identical surroundings. If the time taken by the ice in box A to completely melt is 21 minutes, then the time taken in minutes for half of the mass of ice in box B to melt is

- (A) 14
- (B) 7
- (C) 10.5
- (D) 42

Correct Answer: (B) 7

Solution:

Concept:

- The rate of heat conduction through the walls of a container is governed by Fourier's Law of heat conduction:

$$\frac{dQ}{dt} = \frac{KA\Delta T}{x}$$

where K is thermal conductivity, A is surface area, ΔT is temperature difference, and x is wall thickness.

- The heat required to melt a mass m of ice is:

$$Q = mL_f$$

- Since the containers have identical dimensions, surface areas, wall thicknesses, and surroundings:

$$Q = \left(\frac{dQ}{dt} \right) t \Rightarrow mL_f \propto K \cdot t \Rightarrow t \propto \frac{m}{K}$$

Step 1: Relate melting time to mass and thermal conductivity

Since the dimensions, wall thickness, and temperature difference between surroundings and ice (0°C) are identical:

$$\frac{dQ}{dt} \propto K$$

The heat required to melt a mass m of ice is:

$$Q = mL_f$$

Therefore, the time t required to melt a mass m is:

$$t = \frac{mL_f}{\frac{dQ}{dt}} \propto \frac{m}{K}$$

Step 2: Set up the proportionality equation for boxes A and B

Let M be the total mass of ice completely filling each box. For box A:

$$m_A = M, \quad K_A = 180 \text{ W m}^{-1}\text{K}^{-1}, \quad t_A = 21 \text{ min}$$

For box B:

$$m_B = \frac{M}{2}, \quad K_B = 270 \text{ W m}^{-1}\text{K}^{-1}, \quad t_B = ?$$

Step 3: Calculate the time t_B

Using the proportionality relation:

$$\frac{t_B}{t_A} = \left(\frac{m_B}{m_A} \right) \times \left(\frac{K_A}{K_B} \right)$$

Substitute the values:

$$\frac{t_B}{21} = \left(\frac{M/2}{M} \right) \times \left(\frac{180}{270} \right)$$

Simplify the fractions:

$$\frac{t_B}{21} = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$$

Solve for t_B :

$$t_B = \frac{21}{3} = 7 \text{ minutes}$$

Quick Tip: Direct ratio method: $t \propto \frac{m}{K}$. Since mass is halved ($\times \frac{1}{2}$) and conductivity is $\frac{270}{180} = \frac{3}{2}$ times greater ($\times \frac{2}{3}$), the time scales by $\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$. Hence, $t_B = \frac{21}{3} = 7$ minutes.

97. The temperatures of source and sink of a Carnot heat engine are 127°C and 27°C respectively. If the working substance of the engine is 2 moles of a rigid diatomic gas, then the decrease in the internal energy of the substance during adiabatic expansion process is (Universal gas constant = $8.31 \text{ J mol}^{-1}\text{K}^{-1}$)

- (A) 2493 J
- (B) 4986 J
- (C) 3324 J
- (D) 4155 J

Correct Answer: (D) 4155 J

Solution:

Concept:

- In a Carnot cycle, adiabatic expansion occurs between the source temperature T_1 and the sink temperature T_2 .
- The change in internal energy of an ideal gas during any process depends solely on the initial and final temperatures:

$$\Delta U = nC_v\Delta T = nC_v(T_2 - T_1)$$

- For a rigid diatomic gas (degrees of freedom $f = 5$):

$$C_v = \frac{5}{2}R$$

- The decrease in internal energy is:

$$-\Delta U = nC_v(T_1 - T_2)$$

Step 1: Convert temperatures to the absolute Kelvin scale

Source temperature:

$$T_1 = 127 + 273 = 400 \text{ K}$$

Sink temperature:

$$T_2 = 27 + 273 = 300 \text{ K}$$

Temperature difference during adiabatic expansion:

$$\Delta T = T_1 - T_2 = 400 - 300 = 100 \text{ K}$$

Step 2: Determine the molar heat capacity at constant volume

For a rigid diatomic gas, vibrational modes are not active, so degrees of freedom $f = 5$:

$$C_v = \frac{5}{2}R$$

Step 3: Calculate the decrease in internal energy

The decrease in internal energy during adiabatic expansion is:

$$-\Delta U = nC_v(T_1 - T_2) = n\left(\frac{5}{2}R\right)(T_1 - T_2)$$

Substitute the given values $n = 2$, $R = 8.31 \text{ J mol}^{-1}\text{K}^{-1}$, and $\Delta T = 100 \text{ K}$:

$$-\Delta U = 2 \times \left(\frac{5}{2} \times 8.31\right) \times 100$$

Cancel the factor of 2:

$$-\Delta U = 5 \times 8.31 \times 100 = 5 \times 831 = 4155 \text{ J}$$

Quick Tip: In any adiabatic expansion, work is done entirely at the expense of internal energy: $W_{\text{adiab}} = -\Delta U = nC_v\Delta T$. For 2 moles of rigid diatomic gas, $-\Delta U = 2 \times \frac{5}{2}R \times 100 = 500R = 500 \times 8.31 = 4155 \text{ J}$.

98. If the average kinetic energy of the molecules of a gas at a temperature of 30°C is U , then the temperature at which the average kinetic energy of the molecules of the gas becomes $2U$ is

- (A) 939°C
- (B) 303°C
- (C) 333°C
- (D) 60°C

Correct Answer: (C) 333°C

Solution:

Concept:

- According to the kinetic theory of gases, the average translational kinetic energy of a molecule is directly proportional to absolute temperature T in Kelvin:

$$E_k = \frac{3}{2}k_B T \implies E_k \propto T$$

- Temperature values must always be converted to the Kelvin scale before applying proportionality:

$$T(\text{K}) = T(^{\circ}\text{C}) + 273$$

Step 1: Convert the initial temperature to Kelvin

Given initial temperature:

$$T_1 = 30^{\circ}\text{C} = 30 + 273 = 303 \text{ K}$$

The corresponding average kinetic energy is:

$$U_1 = U$$

Step 2: Apply the proportionality between kinetic energy and absolute temperature

Since average kinetic energy is directly proportional to absolute temperature:

$$\frac{U_2}{U_1} = \frac{T_2}{T_1}$$

We require the new kinetic energy to be:

$$U_2 = 2U$$

Substitute into the relation:

$$\frac{2U}{U} = \frac{T_2}{303} \implies \frac{T_2}{303} = 2$$

Step 3: Find the final absolute temperature

Solve for T_2 :

$$T_2 = 2 \times 303 = 606 \text{ K}$$

Step 4: Convert the final temperature back to Celsius

Convert T_2 to degrees Celsius:

$$t_2 = 606 - 273 = 333^\circ\text{C}$$

Quick Tip: Never simply double the Celsius temperature! Doubling kinetic energy doubles the Kelvin temperature: $2 \times (30 + 273) \text{ K} = 606 \text{ K} \implies 606 - 273 = 333^\circ\text{C}$.

99. A wire of length 100 cm is clamped between two rigid supports and is made to vibrate in its fundamental mode. If the amplitude at the midpoint of the wire is A , then the distance between two points having amplitude of $\frac{A}{\sqrt{2}}$ is

- (A) 50 cm
- (B) 60 cm
- (C) 40 cm
- (D) 25 cm

Correct Answer: (A) 50 cm

Solution:

Concept:

- A string clamped at both ends forms nodes at $x = 0$ and $x = L$.

- The standing wave displacement for the fundamental mode ($n = 1$) is:

$$y(x, t) = [A_0 \sin(kx)] \cos(\omega t)$$

where the position-dependent amplitude is $A(x) = A_0 \sin(kx)$.

- For the fundamental mode, length $L = \frac{\lambda}{2} \implies k = \frac{2\pi}{\lambda} = \frac{\pi}{L}$.
- The maximum amplitude occurs at the antinode (midpoint $x = \frac{L}{2}$), where $A_{\text{mid}} = A_0 \sin\left(\frac{\pi}{2}\right) = A_0 = A$.

Step 1: Formulate the amplitude profile for the fundamental mode

With nodes at $x = 0$ and $x = L = 100$ cm:

$$k = \frac{\pi}{L} = \frac{\pi}{100} \text{ cm}^{-1}$$

The amplitude as a function of position x from one fixed end is:

$$A(x) = A \sin\left(\frac{\pi x}{100}\right)$$

Step 2: Set the amplitude equal to $\frac{A}{\sqrt{2}}$

We need to find points where:

$$A(x) = \frac{A}{\sqrt{2}}$$

Substitute into the amplitude formula:

$$A \sin\left(\frac{\pi x}{100}\right) = \frac{A}{\sqrt{2}} \implies \sin\left(\frac{\pi x}{100}\right) = \frac{1}{\sqrt{2}}$$

Step 3: Find the positions of the two points on the wire

For $x \in [0, 100]$, the sine function takes the value $\frac{1}{\sqrt{2}}$ at two angles in the first and second quadrants:

$$\frac{\pi x_1}{100} = \frac{\pi}{4} \implies x_1 = \frac{100}{4} = 25 \text{ cm}$$

$$\frac{\pi x_2}{100} = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \implies x_2 = \frac{3 \times 100}{4} = 75 \text{ cm}$$

Step 4: Compute the distance between the two points

The distance between these two symmetric positions is:

$$d = x_2 - x_1 = 75 - 25 = 50 \text{ cm}$$

Quick Tip: In the fundamental mode of a clamped string of length L , the amplitude profile is $A\sin(\frac{\pi x}{L})$. The two points where amplitude is $\frac{A}{\sqrt{2}}$ lie at $\frac{L}{4}$ and $\frac{3L}{4}$. Their separation is always $\frac{3L}{4} - \frac{L}{4} = \frac{L}{2} = 50$ cm.

100. The lengths of two open pipes are in the ratio 5:6. If 8 beats are heard per second when both the pipes are vibrated in their fundamental modes, then the frequency of third harmonic of the longer pipe is

- (A) 144 Hz
- (B) 240 Hz
- (C) 120 Hz
- (D) 60 Hz

Correct Answer: (C) 120 Hz

Solution:

Concept:

- The fundamental frequency of an open organ pipe of length L (with sound speed v) is:

$$f = \frac{v}{2L}$$

- Fundamental frequency is inversely proportional to pipe length:

$$f \propto \frac{1}{L}$$

- The beat frequency between two simultaneously sounding fundamental notes is:

$$f_b = |f_1 - f_2|$$

- For an open organ pipe, all harmonics are present, and the frequency of the n -th harmonic is:

$$f_n = nf_1$$

Step 1: Relate the fundamental frequencies of the two pipes

Let the lengths of the two open pipes be L_1 and L_2 . We are given:

$$\frac{L_1}{L_2} = \frac{5}{6}$$

Since fundamental frequency $f \propto \frac{1}{L}$:

$$\frac{f_1}{f_2} = \frac{L_2}{L_1} = \frac{6}{5}$$

Here, L_2 is the longer pipe, and its fundamental frequency is f_2 .

Step 2: Use the beat frequency to determine the fundamental frequencies

Let:

$$f_1 = 6x, \quad f_2 = 5x$$

The beat frequency is:

$$f_b = f_1 - f_2 = 6x - 5x = x$$

We are given that 8 beats per second are heard:

$$x = 8 \text{ Hz}$$

Thus, the fundamental frequency of the longer pipe is:

$$f_2 = 5x = 5 \times 8 = 40 \text{ Hz}$$

Step 3: Calculate the frequency of the third harmonic of the longer pipe

For an open pipe, the frequency of the n -th harmonic is:

$$f_n = n \cdot f_{\text{fundamental}}$$

For the third harmonic ($n = 3$) of the longer pipe:

$$f_3 = 3 \times f_2 = 3 \times 40 = 120 \text{ Hz}$$

Quick Tip: For open pipes with length ratio 5 : 6, their fundamental frequencies are in the ratio 6 : 5. The difference is 1 unit = 8 Hz, so the longer pipe's fundamental frequency is $5 \times 8 = 40$ Hz. Its 3rd harmonic is $3 \times 40 = 120$ Hz.

101. A point object is moving towards a concave mirror of focal length 20 cm along the principal axis with a uniform speed of 4 cm s^{-1} . The speed of the image when the object is at a distance of 60 cm from the mirror is

- (A) 3 cm s^{-1}
- (B) 1 cm s^{-1}
- (C) 2 cm s^{-1}
- (D) 4 cm s^{-1}

Correct Answer: (B) 1 cm s^{-1}

Solution:

Concept:

- The position of the image formed by a spherical mirror is given by the mirror equation:

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

- Differentiating the mirror equation with respect to time gives the longitudinal velocity of the image:

$$-\frac{1}{v^2} \frac{dv}{dt} - \frac{1}{u^2} \frac{du}{dt} = 0 \implies v_I = -\left(\frac{v}{u}\right)^2 v_O = -m^2 v_O$$

where $m = -\frac{v}{u}$ is the transverse magnification.

Step 1: Apply the sign convention to given values

For a concave mirror:

$$f = -20 \text{ cm}$$

The object is located in front of the mirror:

$$u = -60 \text{ cm}$$

The speed of the object is:

$$v_O = \frac{du}{dt} = 4 \text{ cm s}^{-1}$$

Step 2: Find the position of the image v

Using the mirror formula:

$$\frac{1}{v} = \frac{1}{f} - \frac{1}{u}$$

Substitute the values:

$$\frac{1}{v} = \frac{1}{-20} - \frac{1}{-60} = -\frac{1}{20} + \frac{1}{60}$$

Find a common denominator:

$$\frac{1}{v} = \frac{-3+1}{60} = -\frac{2}{60} = -\frac{1}{30} \text{ cm}^{-1}$$

Thus:

$$v = -30 \text{ cm}$$

Step 3: Calculate the longitudinal magnification

Compute the ratio $\frac{v}{u}$:

$$\frac{v}{u} = \frac{-30}{-60} = \frac{1}{2}$$

The square of this ratio relates the speeds:

$$m^2 = \left(\frac{v}{u}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

Step 4: Determine the speed of the image

The magnitude of the velocity of the image is:

$$|v_I| = \left(\frac{v}{u}\right)^2 |v_O|$$

Substitute the values:

$$|v_I| = \frac{1}{4} \times 4 \text{ cm s}^{-1} = 1 \text{ cm s}^{-1}$$

Quick Tip: The speed of an image moving along the principal axis of a mirror is given directly by $v_I = \left(\frac{f}{u-f}\right)^2 v_O$. Here, $v_I = \left(\frac{-20}{-60-(-20)}\right)^2 \times 4 = \left(\frac{-20}{-40}\right)^2 \times 4 = \frac{1}{4} \times 4 = 1 \text{ cm s}^{-1}$.

102. If the focal lengths of the objective and eyepiece of a giant refracting telescope are 20 m and 1 cm respectively, then the angular magnification of the telescope is

- (A) 5
- (B) 500
- (C) 20
- (D) 2000

Correct Answer: (D) 2000

Solution:

Concept:

- An astronomical refracting telescope consists of an objective lens of large focal length f_o and an eyepiece of short focal length f_e .
- In normal adjustment (image formed at infinity), the angular magnification m is given by:

$$m = \frac{f_o}{f_e}$$

- Both focal lengths must be expressed in identical units before computing the ratio.

Step 1: Convert all focal lengths to the same unit

Focal length of the objective:

$$f_o = 20 \text{ m} = 20 \times 100 \text{ cm} = 2000 \text{ cm}$$

Focal length of the eyepiece:

$$f_e = 1 \text{ cm}$$

Step 2: Compute the angular magnification

Substitute the focal lengths into the telescope magnification formula:

$$m = \frac{f_o}{f_e}$$

$$m = \frac{2000 \text{ cm}}{1 \text{ cm}} = 2000$$

Quick Tip: Always check units carefully in optical instrument formulas. Converting meters to centimeters gives $\frac{2000 \text{ cm}}{1 \text{ cm}} = 2000$ instantly.

103. When monochromatic light of wavelength λ is used in Young's double slit experiment, if I is the intensity of light at a point on the screen where path difference is $\frac{\lambda}{3}$, then the intensity of light at a point on the screen where path difference becomes λ is

- (A) $2I$
- (B) $3I$
- (C) $4I$
- (D) I

Correct Answer: (C) $4I$

Solution:

Concept:

- The phase difference ϕ between two interfering coherent beams is related to their path difference Δx by:

$$\phi = \frac{2\pi}{\lambda} \Delta x$$

- The resultant intensity due to two identical coherent sources of equal individual intensity I_0 is:

$$I(\phi) = 4I_0 \cos^2\left(\frac{\phi}{2}\right) = I_{\max} \cos^2\left(\frac{\phi}{2}\right)$$

- When path difference is an integral multiple of λ ($\Delta x = n\lambda$), fully constructive interference occurs, yielding maximum intensity $I_{\max}$.

Step 1: Find the phase difference corresponding to a path difference of $\lambda/3$

For the first point on the screen:

$$\Delta x_1 = \frac{\lambda}{3}$$

The corresponding phase difference is:

$$\phi_1 = \frac{2\pi}{\lambda} \Delta x_1 = \frac{2\pi}{\lambda} \left(\frac{\lambda}{3}\right) = \frac{2\pi}{3}$$

Step 2: Relate the given intensity I to the maximum intensity $I_{\max}$

The intensity at this point is given by:

$$I = I_{\max} \cos^2\left(\frac{\phi_1}{2}\right) = I_{\max} \cos^2\left(\frac{\pi}{3}\right)$$

Since $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$:

$$I = I_{\max} \left(\frac{1}{2}\right)^2 = \frac{I_{\max}}{4}$$

Therefore, the maximum intensity is:

$$I_{\max} = 4I$$

Step 3: Find the intensity when the path difference is λ

For the second point:

$$\Delta x_2 = \lambda$$

The corresponding phase difference is:

$$\phi_2 = \frac{2\pi}{\lambda}(\lambda) = 2\pi$$

Calculate the intensity at this point:

$$I_2 = I_{\max} \cos^2\left(\frac{\phi_2}{2}\right) = I_{\max} \cos^2(\pi) = I_{\max}(-1)^2 = I_{\max}$$

Step 4: Express the final intensity in terms of I

From Step 2:

$$I_2 = I_{\max} = 4I$$

Quick Tip: A path difference of λ corresponds to a central or principal maximum where $I = I_{\max}$. Since $\Delta x = \lambda/3$ gives $\cos^2(60^\circ) = 1/4$, we have $I = I_{\max}/4 \implies I_{\max} = 4I$.

104. If ϕ_A and ϕ_B are the electric fluxes leaving and entering a Gaussian surface respectively, then the charge enclosed in the surface is

(A) $(\phi_A - \phi_B)\epsilon_0$

(B) $\frac{(\phi_A + \phi_B)}{\epsilon_0}$

(C) $(\phi_A + \phi_B)\epsilon_0$

(D) $\frac{(\phi_B - \phi_A)}{\epsilon_0}$

Correct Answer: (A) $(\phi_A - \phi_B)\epsilon_0$

Solution:

Concept:

- By standard sign convention in electrostatics, electric flux directed outward (leaving a closed Gaussian surface) is taken as positive.
- Electric flux directed inward (entering a closed Gaussian surface) is taken as negative.
- According to Gauss's Law:

$$\Phi_{\text{net}} = \frac{q_{\text{enclosed}}}{\epsilon_0}$$

where q_{enclosed} is the total net charge enclosed by the Gaussian surface.

Step 1: Determine the net electric flux passing through the surface

The flux leaving the surface is:

$$\Phi_{\text{out}} = +\phi_A$$

The flux entering the surface is:

$$\Phi_{\text{in}} = -\phi_B$$

The total net flux through the Gaussian surface is:

$$\Phi_{\text{net}} = \Phi_{\text{out}} + \Phi_{\text{in}} = \phi_A - \phi_B$$

Step 2: Apply Gauss's Law to find the enclosed charge

According to Gauss's Law:

$$\Phi_{\text{net}} = \frac{q_{\text{enclosed}}}{\epsilon_0}$$

Substitute $\Phi_{\text{net}} = \phi_A - \phi_B$:

$$\phi_A - \phi_B = \frac{q_{\text{enclosed}}}{\epsilon_0}$$

Step 3: Solve for the enclosed charge

Multiply both sides by the permittivity of free space ϵ_0 :

$$q_{\text{enclosed}} = (\phi_A - \phi_B)\epsilon_0$$

Quick Tip: Remember: Leaving flux is positive ($+\phi_A$) and entering flux is negative ($-\phi_B$). Net flux is $\phi_A - \phi_B$, which gives $q = (\phi_A - \phi_B)\epsilon_0$ by Gauss's Law.

105. The spheres A, B and C have radii R, R and 2R respectively. Initially A has a charge -Q, B is neutral and C is positively charged with a surface charge density equal to that of A. If the three spheres are placed in contact with each other and later separated, then the charges on the three spheres A, B and C respectively are

- (A) $\frac{5Q}{4}, \frac{5Q}{4}, \frac{5Q}{2}$
(B) Q, Q, Q
(C) $\frac{6Q}{5}, \frac{6Q}{5}, \frac{3Q}{5}$
(D) $\frac{3Q}{4}, \frac{3Q}{4}, \frac{3Q}{2}$

Correct Answer: (D) $\frac{3Q}{4}, \frac{3Q}{4}, \frac{3Q}{2}$

Solution:

Concept:

- Surface charge density is defined as charge divided by surface area:

$$\sigma = \frac{q}{4\pi r^2}$$

- When conducting spheres are placed in contact, charge redistributes until all spheres reach the same electric potential V .
- The capacitance of an isolated spherical conductor is proportional to its radius:

$$C = 4\pi\epsilon_0 r \implies C \propto r$$

- Consequently, after contact, the total charge divides in direct proportion to the radii of the spheres:

$$q'_i = \frac{r_i}{\sum r_k} Q_{\text{total}}$$

Step 1: Find the initial charge on sphere C

Sphere A has radius R and charge $q_A = -Q$. The magnitude of its surface charge density is:

$$\sigma_A = \frac{Q}{4\pi R^2}$$

Sphere C has radius $2R$ and positive surface charge density equal to σ_A :

$$\sigma_C = +\sigma_A = \frac{Q}{4\pi R^2}$$

The total charge on sphere C is:

$$q_C = \sigma_C \times (4\pi(2R)^2) = \left(\frac{Q}{4\pi R^2}\right) \times (16\pi R^2) = +4Q$$

Sphere B is initially neutral:

$$q_B = 0$$

Step 2: Calculate the total charge of the three-sphere system

By the principle of conservation of electric charge:

$$Q_{\text{total}} = q_A + q_B + q_C = -Q + 0 + 4Q = +3Q$$

Step 3: Determine the ratio of charge distribution after contact

When placed in contact, all three spheres share a common potential V . Since $q = CV$ and $C \propto r$:

$$q'_A : q'_B : q'_C = R_A : R_B : R_C = R : R : 2R = 1 : 1 : 2$$

Sum of the ratio parts:

$$1 + 1 + 2 = 4$$

Step 4: Calculate the individual charges after separation

For sphere A:

$$q'_A = \frac{1}{4}Q_{\text{total}} = \frac{1}{4}(3Q) = \frac{3Q}{4}$$

For sphere B:

$$q'_B = \frac{1}{4}Q_{\text{total}} = \frac{1}{4}(3Q) = \frac{3Q}{4}$$

For sphere C:

$$q'_C = \frac{2}{4}Q_{\text{total}} = \frac{2}{4}(3Q) = \frac{3Q}{2}$$

Thus, the charges are $\frac{3Q}{4}, \frac{3Q}{4}, \frac{3Q}{2}$.

Quick Tip: Charge on sphere C is $\sigma \times 4\pi(2R)^2 = 4Q$. Total charge is $-Q + 0 + 4Q = 3Q$. Since radii are in the ratio 1 : 1 : 2, the charges are $\frac{1}{4}(3Q), \frac{1}{4}(3Q), \frac{2}{4}(3Q)$.

106. If e , m and τ are charge, mass and average collision time for an electron respectively, then the magnitude of drift velocity per unit electric field is

- (A) $\frac{em}{\tau}$
- (B) $\frac{e\tau}{m}$
- (C) $\frac{m\tau}{e}$
- (D) $\frac{\tau}{me}$

Correct Answer: (B) $\frac{e\tau}{m}$

Solution:

Concept:

- Under an external electric field E , each electron experiences an electrostatic force:

$$F = eE$$

- The acceleration experienced by an electron of mass m is:

$$a = \frac{eE}{m}$$

- The average drift velocity acquired over relaxation time τ is:

$$v_d = a\tau = \frac{eE\tau}{m}$$

- The magnitude of drift velocity per unit electric field is defined as the electron mobility μ :

$$\mu = \frac{v_d}{E}$$

Step 1: Derive the expression for drift velocity

From Newton's second law, the acceleration of the conduction electron is:

$$a = \frac{F}{m} = \frac{eE}{m}$$

Since the initial thermal velocities average to zero, the average drift velocity gained during relaxation time τ is:

$$v_d = a\tau = \left(\frac{eE}{m}\right)\tau = \frac{e\tau}{m}E$$

Step 2: Calculate the drift velocity per unit electric field

The magnitude of drift velocity per unit electric field is:

$$\frac{v_d}{E} = \frac{\frac{e\tau}{m}E}{E} = \frac{e\tau}{m}$$

Quick Tip: Drift velocity per unit electric field is simply the definition of mobility: $\mu = \frac{v_d}{E} = \frac{e\tau}{m}$.

107. In a meter bridge with resistors R_1 and R_2 in the left and right gaps respectively, the balancing point is obtained at 25 cm from the left end of the bridge wire. If shunt resistances of 2Ω each are connected to both R_1 and R_2 , the balancing point shifts by 15 cm. The initial values of R_1 and R_2 respectively are

- (A) 2Ω , 6Ω
- (B) 1Ω , 3Ω
- (C) 4Ω , 12Ω
- (D) 5Ω , 15Ω

Correct Answer: (A) 2Ω , 6Ω

Solution:

Concept:

- In a balanced meter bridge of wire length 100 cm:

$$\frac{R_{\text{left}}}{R_{\text{right}}} = \frac{l}{100-l}$$

- When a shunt resistor S is connected in parallel with a resistor R , the equivalent resistance becomes:

$$R' = \frac{R \cdot S}{R + S}$$

Step 1: Relate R_1 and R_2 using the initial balancing point

The initial balancing length from the left end is:

$$l_1 = 25 \text{ cm}$$

The remaining length on the right is:

$$100 - l_1 = 100 - 25 = 75 \text{ cm}$$

Apply the meter bridge balance condition:

$$\frac{R_1}{R_2} = \frac{l_1}{100 - l_1} = \frac{25}{75} = \frac{1}{3}$$

This gives:

$$R_2 = 3R_1 \quad \text{--- (1)}$$

Step 2: Find the effective resistances after shunting

A shunt resistance of 2Ω is placed in parallel across both resistors:

$$R'_1 = \frac{2R_1}{R_1 + 2}$$

$$R'_2 = \frac{2R_2}{R_2 + 2} = \frac{2(3R_1)}{3R_1 + 2} = \frac{6R_1}{3R_1 + 2}$$

Step 3: Determine the new balancing point

Since $R_1 < R_2$, connecting identical small shunts decreases the ratio R'_2/R'_1 , bringing it closer to unity. Thus, the balancing point shifts to the right by 15 cm:

$$l_2 = 25 + 15 = 40 \text{ cm}$$

The remaining length is:

$$100 - l_2 = 100 - 40 = 60 \text{ cm}$$

The new bridge balance condition is:

$$\frac{R'_1}{R'_2} = \frac{l_2}{100 - l_2} = \frac{40}{60} = \frac{2}{3} \quad \text{--- (2)}$$

Step 4: Solve for R_1 and R_2

Substitute the expressions for R'_1 and R'_2 into equation (2):

$$\frac{\left(\frac{2R_1}{R_1+2}\right)}{\left(\frac{6R_1}{3R_1+2}\right)} = \frac{2}{3}$$

Simplify the fraction:

$$\frac{2R_1}{R_1+2} \times \frac{3R_1+2}{6R_1} = \frac{2}{3}$$

Cancel $2R_1$:

$$\frac{3R_1+2}{3(R_1+2)} = \frac{2}{3}$$

Multiply both sides by 3:

$$\frac{3R_1+2}{R_1+2} = 2$$

Cross-multiply and solve:

$$3R_1 + 2 = 2(R_1 + 2) = 2R_1 + 4$$

$$3R_1 - 2R_1 = 4 - 2 \implies R_1 = 2 \Omega$$

From equation (1):

$$R_2 = 3R_1 = 3(2) = 6 \Omega$$

Quick Tip: Initially $R_2 = 3R_1$. Test Option (A): $R_1 = 2 \Omega \implies R'_1 = \frac{2 \times 2}{2+2} = 1 \Omega$. $R_2 = 6 \Omega \implies R'_2 = \frac{6 \times 2}{6+2} = 1.5 \Omega$. The ratio is $\frac{R'_1}{R'_2} = \frac{1}{1.5} = \frac{2}{3}$, giving $l = 40$ cm, exactly a 15 cm shift.

108. A long solenoid having 15 cm circumference and 70 turns per metre is carrying a current of 2A. The magnetic field inside the solenoid at a distance 1 cm from the surface and the magnetic field outside the solenoid respectively are

(A) 0, 1.76×10^{-4} T

(B) 8.8×10^{-3} T, 0

(C) 1.76×10^{-4} T, 0

(D) 0, 8.8×10^{-3} T

Correct Answer: (C) 1.76×10^{-4} T, 0

Solution:

Concept:

- For an ideal, tightly wound long solenoid:
 - The magnetic field outside the solenoid is practically zero:

$$B_{\text{outside}} = 0$$

- The magnetic field inside the solenoid is uniform everywhere and directed parallel to the axis:

$$B_{\text{inside}} = \mu_0 n I$$

- Here, n is the number of turns per unit length, I is the current, and $\mu_0 = 4\pi \times 10^{-7} \text{ T m A}^{-1}$.

Step 1: Verify that the point 1 cm from the surface lies inside the solenoid

The circumference of the solenoid is:

$$C = 2\pi r = 15 \text{ cm} \implies r = \frac{15}{2\pi} \approx \frac{15}{6.28} \approx 2.39 \text{ cm}$$

Since the distance from the outer surface is 1 cm, which is less than the radius $r \approx 2.39$ cm, the point is strictly inside the solenoid.

Step 2: Calculate the magnetic field inside the solenoid

Given:

$$n = 70 \text{ turns/m}, \quad I = 2 \text{ A}, \quad \mu_0 = 4\pi \times 10^{-7} \text{ T m A}^{-1}$$

Use Ampere's law formula for an ideal long solenoid:

$$B_{\text{inside}} = \mu_0 n I$$

Substitute the values:

$$B_{\text{inside}} = (4\pi \times 10^{-7}) \times 70 \times 2 = 560\pi \times 10^{-7} \text{ T}$$

Substitute $\pi \approx \frac{22}{7}$:

$$B_{\text{inside}} = 560 \times \left(\frac{22}{7}\right) \times 10^{-7} = 80 \times 22 \times 10^{-7} = 1760 \times 10^{-7} \text{ T}$$

Express in scientific notation:

$$B_{\text{inside}} = 1.76 \times 10^{-4} \text{ T}$$

Step 3: State the magnetic field outside the solenoid

For an infinitely long solenoid, the exterior magnetic field is zero:

$$B_{\text{outside}} = 0$$

Thus, the respective values are $1.76 \times 10^{-4} \text{ T}$ and 0.

Quick Tip: For an ideal solenoid, the magnetic field outside is always zero, which immediately eliminates options (A) and (D). Calculating $B = \mu_0 nI = 4\pi \times 10^{-7} \times 70 \times 2 = 1.76 \times 10^{-4} \text{ T}$ selects option (C).

109. A proton and an alpha particle enter a uniform magnetic field with the same kinetic energy making the same angle of 30° with the direction of the field. The ratio of the pitches of the paths of proton and alpha particle is

- (A) 2 : 1
- (B) 1 : 1
- (C) 1 : 2
- (D) 8 : 1

Correct Answer: (B) 1 : 1

Solution:

Concept:

- When a charged particle enters a uniform magnetic field at an angle θ , it follows a helical path.
- The pitch p of the helix is the distance advanced parallel to the magnetic field in one full

revolution:

$$p = v_{\parallel} \cdot T = (v \cos \theta) \cdot \left(\frac{2\pi m}{qB} \right)$$

- Expressing the velocity in terms of linear momentum $P = mv = \sqrt{2mK}$:

$$p = \frac{2\pi \sqrt{2mK} \cos \theta}{qB}$$

- For constant kinetic energy K , magnetic field B , and angle θ :

$$p \propto \frac{\sqrt{m}}{q}$$

Step 1: State the mass and charge ratios for proton and alpha particle

For a proton:

$$m_p = m, \quad q_p = e$$

For an alpha particle (helium-4 nucleus):

$$m_{\alpha} = 4m, \quad q_{\alpha} = 2e$$

Step 2: Express pitch in terms of mass and charge

Using the formula for the pitch of a helix:

$$p = \frac{2\pi \sqrt{2mK} \cos \theta}{qB}$$

Since K , θ , and B are identical for both particles:

$$p \propto \frac{\sqrt{m}}{q}$$

Step 3: Evaluate $\frac{\sqrt{m}}{q}$ for both particles

For the proton:

$$\left(\frac{\sqrt{m}}{q} \right)_p = \frac{\sqrt{m}}{e}$$

For the alpha particle:

$$\left(\frac{\sqrt{m}}{q} \right)_{\alpha} = \frac{\sqrt{4m}}{2e} = \frac{2\sqrt{m}}{2e} = \frac{\sqrt{m}}{e}$$

Step 4: Compute the ratio of pitches

Divide the two pitch values:

$$\frac{p_p}{p_\alpha} = \frac{\left(\frac{\sqrt{m}}{q}\right)_p}{\left(\frac{\sqrt{m}}{q}\right)_\alpha} = \frac{\frac{\sqrt{m}}{e}}{\frac{\sqrt{m}}{e}} = 1$$

Thus, the ratio is 1 : 1.

Quick Tip: Remember that pitch depends on $\frac{\sqrt{m}}{q}$. Since an alpha particle has 4 times the mass and 2 times the charge of a proton, $\frac{\sqrt{4}}{2} = 1$, making the pitch ratio identical: 1 : 1.

110. If the magnetic field at the equator of the earth is 0.4 G, then the earth's magnetic dipole moment (in A m^2) is of the order of (Radius of the Earth = 6400 km)

- (A) 10^{18}
- (B) 10^{17}
- (C) 10^{26}
- (D) 10^{23}

Correct Answer: (D) 10^{23}

Solution:

Concept:

- The equator of the Earth corresponds to the broadside-on (equatorial) position of the Earth's magnetic dipole.
- The magnetic field on the magnetic equator of a magnetic dipole of dipole moment M is:

$$B_E = \frac{\mu_0}{4\pi} \frac{M}{R^3}$$

where R is the radius of the Earth.

- Convert magnetic field from Gauss to Tesla:

$$1 \text{ G} = 10^{-4} \text{ T}$$

- Use $\frac{\mu_0}{4\pi} = 10^{-7} \text{ T m A}^{-1}$.

Step 1: Convert all quantities to SI units

Magnetic field at the equator:

$$B_E = 0.4 \text{ G} = 0.4 \times 10^{-4} \text{ T} = 4 \times 10^{-5} \text{ T}$$

Radius of the Earth:

$$R = 6400 \text{ km} = 6.4 \times 10^6 \text{ m}$$

Constant:

$$\frac{\mu_0}{4\pi} = 10^{-7} \text{ T m A}^{-1}$$

Step 2: Rearrange the formula for the dipole moment M

From the equatorial magnetic field formula:

$$B_E = \frac{\mu_0}{4\pi} \frac{M}{R^3}$$

Solve for M:

$$M = \frac{B_E \cdot R^3}{\left(\frac{\mu_0}{4\pi}\right)} = \frac{B_E \cdot R^3}{10^{-7}}$$

Step 3: Calculate R^3

Compute the cube of the radius:

$$R^3 = (6.4 \times 10^6)^3 = (6.4)^3 \times 10^{18} \approx 262.14 \times 10^{18} = 2.62 \times 10^{20} \text{ m}^3$$

Step 4: Calculate M and find its order of magnitude

Substitute B_E and R^3 :

$$M = \frac{(4 \times 10^{-5}) \times (2.62 \times 10^{20})}{10^{-7}}$$

Simplify exponents:

$$M = 4 \times 2.62 \times 10^{-5+20+7} = 10.48 \times 10^{22} = 1.05 \times 10^{23} \text{ A m}^2$$

The order of magnitude is:

$$10^{23}$$

Quick Tip: Order-of-magnitude estimation: $M = \frac{BR^3}{10^{-7}} \approx \frac{(4 \times 10^{-5}) \times (2.6 \times 10^{20})}{10^{-7}} \approx 10^{23} \text{ A m}^2$.

111. The magnetic flux (ϕ in weber) linked with a coil varies with time (t in second) as per the equation $\phi = 2.5t^2 + 5t + 7$. If the induced electric power in the coil at a time 3 s is 2 W, then the induced electric power in the coil at a time 5 s is

- (A) 5.5 W
- (B) 3.5 W
- (C) 2.5 W
- (D) 4.5 W

Correct Answer: (D) 4.5 W

Solution:

Concept:

- By Faraday's Law of Electromagnetic Induction, the magnitude of induced electromotive force (EMF) is:

$$e = \left| \frac{d\phi}{dt} \right|$$

- The electric power dissipated due to the induced EMF in a coil of resistance R is:

$$P = \frac{e^2}{R}$$

- For a fixed coil resistance R , the induced power is proportional to the square of the induced EMF:

$$P \propto e^2$$

Step 1: Find the expression for the induced EMF as a function of time

The magnetic flux linked with the coil is:

$$\phi(t) = 2.5t^2 + 5t + 7 \text{ Wb}$$

Differentiate $\phi(t)$ with respect to time t :

$$e(t) = \frac{d\phi}{dt} = \frac{d}{dt}(2.5t^2 + 5t + 7) = 5t + 5 \text{ V}$$

Step 2: Evaluate the induced EMF at $t = 3 \text{ s}$ and $t = 5 \text{ s}$

At $t = 3$ s:

$$e(3) = 5(3) + 5 = 15 + 5 = 20 \text{ V}$$

At $t = 5$ s:

$$e(5) = 5(5) + 5 = 25 + 5 = 30 \text{ V}$$

Step 3: Determine the resistance of the coil

We are given that at $t = 3$ s, the induced electric power is $P_1 = 2$ W. Using the power formula:

$$P_1 = \frac{[e(3)]^2}{R} \implies 2 = \frac{20^2}{R} = \frac{400}{R}$$

Solve for R :

$$R = \frac{400}{2} = 200 \Omega$$

Step 4: Calculate the induced power at $t = 5$ s

Substitute $e(5) = 30$ V and $R = 200 \Omega$ into the power equation:

$$P_2 = \frac{[e(5)]^2}{R} = \frac{30^2}{200} = \frac{900}{200} = 4.5 \text{ W}$$

Quick Tip: Since power satisfies $P \propto e^2$, directly use the ratio: $\frac{P_2}{P_1} = \left(\frac{e_2}{e_1}\right)^2 = \left(\frac{30}{20}\right)^2 = \frac{9}{4}$. Then $P_2 = 2 \times \frac{9}{4} = 4.5$ W.

112. If a resistor of resistance $\frac{125}{\sqrt{3}} \Omega$ and a capacitor of capacitance $\frac{40}{\pi} \mu\text{F}$ are connected in series with an ac supply of frequency 100 Hz, then the phase difference between current and voltage in the circuit is

- (A) 30°
- (B) 60°
- (C) 45°
- (D) 90°

Correct Answer: (B) 60°

Solution:**Concept:**

- In a series RC circuit connected to an alternating supply, capacitive reactance is:

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

- The phase difference ϕ by which the current leads the voltage is given by:

$$\tan \phi = \frac{X_C}{R}$$

Step 1: Calculate the angular frequency ω

Given linear frequency:

$$f = 100 \text{ Hz}$$

The angular frequency is:

$$\omega = 2\pi f = 2\pi(100) = 200\pi \text{ rad s}^{-1}$$

Step 2: Calculate the capacitive reactance X_C

Given capacitance:

$$C = \frac{40}{\pi} \mu\text{F} = \frac{40}{\pi} \times 10^{-6} \text{ F}$$

Using the reactance formula:

$$X_C = \frac{1}{\omega C} = \frac{1}{(200\pi) \times \left(\frac{40}{\pi} \times 10^{-6}\right)}$$

Cancel π and simplify:

$$X_C = \frac{1}{200 \times 40 \times 10^{-6}} = \frac{1}{8000 \times 10^{-6}} = \frac{1}{8 \times 10^{-3}} = \frac{1000}{8} = 125 \Omega$$

Step 3: Compute the tangent of the phase difference

Given resistance:

$$R = \frac{125}{\sqrt{3}} \Omega$$

Substitute X_C and R into the phase angle formula:

$$\tan \phi = \frac{X_C}{R} = \frac{125}{\frac{125}{\sqrt{3}}} = 125 \times \frac{\sqrt{3}}{125} = \sqrt{3}$$

Step 4: Determine the phase difference ϕ

Since $\tan \phi = \sqrt{3}$:

$$\phi = \tan^{-1}(\sqrt{3}) = 60^\circ$$

Quick Tip: Recognize that π cancels in $\omega C = 2\pi(100) \times \frac{40}{\pi} \times 10^{-6} = 8 \times 10^{-3}$, immediately giving $X_C = 125 \Omega$. Then $\frac{X_C}{R} = \sqrt{3} \implies \phi = 60^\circ$.

113. If the rms value of the electric field of an electromagnetic wave propagating in free space is $30\sqrt{\pi} \text{ V m}^{-1}$, then the intensity of the wave is

- (A) 30 W m^{-2}
- (B) 3.75 W m^{-2}
- (C) 15 W m^{-2}
- (D) 7.5 W m^{-2}

Correct Answer: (D) 7.5 W m^{-2}

Solution:

Concept:

- The intensity I of an electromagnetic plane wave in free space is the average energy transmitted per unit area per second:

$$I = \epsilon_0 c E_{\text{rms}}^2$$

where $c = 3 \times 10^8 \text{ m s}^{-1}$ is the speed of light in vacuum and ϵ_0 is the permittivity of free space.

- Using the electrostatic constant relation:

$$\frac{1}{4\pi\epsilon_0} = 9 \times 10^9 \text{ N m}^2 \text{C}^{-2} \implies \epsilon_0 = \frac{1}{4\pi \times 9 \times 10^9} = \frac{1}{36\pi \times 10^9} \text{ F m}^{-1}$$

Step 1: State the given values

RMS value of the electric field:

$$E_{\text{rms}} = 30\sqrt{\pi} \text{ V m}^{-1}$$

Square of the RMS electric field:

$$E_{\text{rms}}^2 = (30\sqrt{\pi})^2 = 900\pi \text{ V}^2\text{m}^{-2}$$

Step 2: Substitute physical constants into the intensity formula

The formula for the intensity is:

$$I = \epsilon_0 c E_{\text{rms}}^2$$

Substitute $\epsilon_0 = \frac{1}{36\pi \times 10^9}$, $c = 3 \times 10^8$, and $E_{\text{rms}}^2 = 900\pi$:

$$I = \left(\frac{1}{36\pi \times 10^9} \right) \times (3 \times 10^8) \times (900\pi)$$

Step 3: Simplify and calculate the numerical value

Cancel π from numerator and denominator:

$$I = \frac{3 \times 10^8 \times 900}{36 \times 10^9}$$

Simplify the constants:

$$I = \frac{2700 \times 10^8}{36 \times 10^9} = \frac{270 \times 10^9}{36 \times 10^9} = \frac{270}{36}$$

Divide both numerator and denominator by 18:

$$I = \frac{15}{2} = 7.5 \text{ W m}^{-2}$$

Quick Tip: Remember the product $\epsilon_0 c = \frac{1}{\mu_0 c} \approx \frac{1}{377} \Omega^{-1}$. Using $\epsilon_0 = \frac{1}{36\pi \times 10^9}$ allows the factor of π to cancel directly with $(\sqrt{\pi})^2$.

114. When monochromatic photons incident on a photosensitive material of work function 1.8 eV, photoelectrons are emitted with a maximum kinetic energy of 2.2 eV. If the frequency of the incident photons is doubled, then the maximum kinetic energy of the emitted photoelectrons is

- (A) 7.2 eV
- (B) 6.2 eV
- (C) 4.4 eV
- (D) 3.6 eV

Correct Answer: (B) 6.2 eV

Solution:

Concept:

- According to Einstein's photoelectric equation:

$$K_{\max} = h\nu - \Phi$$

where $h\nu$ is the energy of the incident photon, Φ is the work function of the metal, and $K_{\max}$ is the maximum kinetic energy of the emitted photoelectrons.

- When the frequency of incident radiation is doubled ($\nu' = 2\nu$), the energy of each incident photon also doubles:

$$E' = h\nu' = 2(h\nu)$$

Step 1: Find the initial photon energy

Given:

$$\Phi = 1.8 \text{ eV}$$

$$K_{\max,1} = 2.2 \text{ eV}$$

Using Einstein's photoelectric equation:

$$K_{\max,1} = E_1 - \Phi$$

Substitute the values:

$$2.2 = E_1 - 1.8 \implies E_1 = 2.2 + 1.8 = 4.0 \text{ eV}$$

Thus, the energy of each initial photon is $h\nu = 4.0 \text{ eV}$.

Step 2: Determine the new photon energy after doubling frequency

When the frequency of incident photons is doubled:

$$\nu' = 2\nu$$

The new incident energy per photon is:

$$E_2 = h\nu' = 2(h\nu) = 2(4.0 \text{ eV}) = 8.0 \text{ eV}$$

Step 3: Calculate the new maximum kinetic energy

The work function Φ remains unchanged for the photosensitive material:

$$K_{\max,2} = E_2 - \Phi$$

Substitute the values:

$$K_{\max,2} = 8.0 - 1.8 = 6.2 \text{ eV}$$

Quick Tip: Doubling the frequency doubles the photon energy, NOT the kinetic energy: $K_2 = 2(K_1 + \Phi) - \Phi = 2K_1 + \Phi = 2(2.2) + 1.8 = 6.2 \text{ eV}$.

115. The ratio of the wavelengths of first and third spectral lines of Lyman series of hydrogen atom is

- (A) 5 : 4
- (B) 9 : 4
- (C) 7 : 3
- (D) 16 : 3

Correct Answer: (A) 5 : 4

Solution:

Concept:

- The Rydberg formula for the spectral lines of the hydrogen atom is:

$$\frac{1}{\lambda} = R \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right)$$

where R is the Rydberg constant.

- For the Lyman series, transitions terminate at the ground level: $n_1 = 1$.
- The first line corresponds to a transition from $n_2 = 2$ to $n_1 = 1$.
- The third line corresponds to a transition from $n_2 = 4$ to $n_1 = 1$.

Step 1: Calculate the wavelength of the first line of the Lyman series

For the first line of the Lyman series:

$$n_1 = 1, \quad n_2 = 2$$

Apply the Rydberg formula:

$$\frac{1}{\lambda_1} = R \left(\frac{1}{1^2} - \frac{1}{2^2} \right) = R \left(1 - \frac{1}{4} \right) = \frac{3}{4}R$$

Thus:

$$\lambda_1 = \frac{4}{3R}$$

Step 2: Calculate the wavelength of the third line of the Lyman series

For the third line of the Lyman series:

$$n_1 = 1, \quad n_2 = 4$$

Apply the Rydberg formula:

$$\frac{1}{\lambda_3} = R \left(\frac{1}{1^2} - \frac{1}{4^2} \right) = R \left(1 - \frac{1}{16} \right) = \frac{15}{16}R$$

Thus:

$$\lambda_3 = \frac{16}{15R}$$

Step 3: Compute the ratio $\lambda_1 : \lambda_3$

Divide λ_1 by λ_3 :

$$\frac{\lambda_1}{\lambda_3} = \frac{\frac{4}{3R}}{\frac{16}{15R}} = \frac{4}{3} \times \frac{15}{16}$$

Simplify the fraction:

$$\frac{\lambda_1}{\lambda_3} = \frac{4}{16} \times \frac{15}{3} = \frac{1}{4} \times 5 = \frac{5}{4}$$

Thus, the ratio is 5 : 4.

Quick Tip: Since wavelength is inversely proportional to transition energy: $\frac{\lambda_1}{\lambda_3} = \frac{\Delta E_3}{\Delta E_1} = \frac{1-1/16}{1-1/4} = \frac{15/16}{3/4} = \frac{5}{4}$.

116. When an electron and a positron at rest annihilate each other, the momentum of each of the photons emitted is (Rest mass of electron = 9×10^{-31} kg)

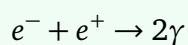
- (A) $36 \times 10^{-23} \text{ kg ms}^{-1}$
- (B) $9 \times 10^{-23} \text{ kg ms}^{-1}$
- (C) $27 \times 10^{-23} \text{ kg ms}^{-1}$
- (D) $18 \times 10^{-23} \text{ kg ms}^{-1}$

Correct Answer: (C) $27 \times 10^{-23} \text{ kg ms}^{-1}$

Solution:

Concept:

- When an electron (e^-) and a positron (e^+) at rest annihilate, their total initial linear momentum is zero.
- To conserve momentum and energy simultaneously, two identical photons must be produced traveling in opposite directions:



- The total rest mass energy converted into radiation is:

$$E_{\text{total}} = 2m_e c^2$$

- The energy carried away by each photon is:

$$E_\gamma = m_e c^2$$

- The linear momentum of each photon is given by de Broglie/Einstein relation:

$$p = \frac{E_\gamma}{c} = m_e c$$

Step 1: State the mass of the electron and the speed of light

Given:

$$m_e = 9 \times 10^{-31} \text{ kg}$$

The speed of light in vacuum is:

$$c = 3 \times 10^8 \text{ ms}^{-1}$$

Step 2: Relate photon momentum to electron rest mass

Each photon carries an energy equal to the rest mass energy of one electron:

$$E = m_e c^2$$

The momentum of a photon is related to its energy by:

$$p = \frac{E}{c} = \frac{m_e c^2}{c} = m_e c$$

Step 3: Calculate the numerical value of momentum

Substitute the numerical values:

$$p = (9 \times 10^{-31} \text{ kg}) \times (3 \times 10^8 \text{ ms}^{-1})$$

$$p = 27 \times 10^{-31+8} = 27 \times 10^{-23} \text{ kg ms}^{-1}$$

Quick Tip: For pair annihilation at rest, the momentum of each emitted gamma photon is simply $p = m_e c = (9 \times 10^{-31})(3 \times 10^8) = 27 \times 10^{-23} \text{ kg ms}^{-1}$.

117. If a thorium nucleus of mass number 232 and atomic number 90 emits six alpha particles and four β^- particles, then the ratio of the number of protons and the number of neutrons in the resulting final nucleus is

(A) 39 : 65

- (B) 41 : 63
(C) 41 : 104
(D) 63 : 104

Correct Answer: (B) 41 : 63

Solution:

Concept:

- An alpha particle (α) is a helium nucleus: ${}^4_2\text{He}$. Emission of each α -particle decreases mass number A by 4 and atomic number Z by 2.
- A beta-minus particle (β^-) is an electron: ${}^0_{-1}e$. Emission of each β^- -particle leaves mass number A unchanged and increases atomic number Z by 1.
- The number of protons in a nucleus is Z , and the number of neutrons is $N = A - Z$.

Step 1: State initial values of mass number and atomic number

For the parent thorium nucleus:

$$A_0 = 232, \quad Z_0 = 90$$

Step 2: Determine the changes resulting from decay

The nucleus emits 6 α -particles and 4 β^- -particles. Change in mass number A :

$$\Delta A = -6(4) - 4(0) = -24$$

Change in atomic number Z :

$$\Delta Z = -6(2) + 4(+1) = -12 + 4 = -8$$

Step 3: Calculate final proton and neutron numbers

The final atomic number (number of protons) is:

$$Z = Z_0 + \Delta Z = 90 - 8 = 82$$

The final mass number is:

$$A = A_0 + \Delta A = 232 - 24 = 208$$

The final number of neutrons is:

$$N = A - Z = 208 - 82 = 126$$

Step 4: Compute the ratio of protons to neutrons

Form the ratio of Z to N :

$$\frac{Z}{N} = \frac{82}{126}$$

Divide numerator and denominator by 2:

$$\frac{Z}{N} = \frac{41}{63}$$

Thus, the ratio is 41 : 63.

Quick Tip: Quick check: Final nucleus has $Z = 90 - 2(6) + 4 = 82$ (Lead, Pb) and $A = 232 - 4(6) = 208$. Neutrons $N = 208 - 82 = 126$. Ratio $P/N = 82/126 = 41/63$.

118. An electron in n-region of a p-n junction diode moves towards the junction with a speed of $5 \times 10^5 \text{ ms}^{-1}$ and after penetration through the barrier, the electron enters the p-region with a speed of $3 \times 10^5 \text{ ms}^{-1}$. The barrier potential of the p-n junction is (Mass of electron $= 9 \times 10^{-31} \text{ kg}$ and charge of electron $= 1.6 \times 10^{-19} \text{ C}$)

- (A) 0.5 V
- (B) 0.45 V
- (C) 0.7 V
- (D) 0.65 V

Correct Answer: (B) 0.45 V

Solution:

Concept:

- As an electron crosses the junction depletion region from the n-side to the p-side, it experiences an opposing electric field produced by the barrier potential.
- By the work-energy theorem, the decrease in the electron's kinetic energy equals the

work done against the potential barrier:

$$eV_0 = \Delta K = \frac{1}{2}m(v_1^2 - v_2^2)$$

where V_0 is the barrier potential.

Step 1: Calculate the difference between the squares of velocities

Initial speed:

$$v_1 = 5 \times 10^5 \text{ ms}^{-1} \implies v_1^2 = 25 \times 10^{10} \text{ m}^2\text{s}^{-2}$$

Final speed:

$$v_2 = 3 \times 10^5 \text{ ms}^{-1} \implies v_2^2 = 9 \times 10^{10} \text{ m}^2\text{s}^{-2}$$

Subtract the squares:

$$v_1^2 - v_2^2 = (25 - 9) \times 10^{10} = 16 \times 10^{10} \text{ m}^2\text{s}^{-2}$$

Step 2: Compute the loss in kinetic energy

Substitute the mass $m = 9 \times 10^{-31} \text{ kg}$:

$$\Delta K = \frac{1}{2}(9 \times 10^{-31}) \times (16 \times 10^{10})$$

$$\Delta K = 9 \times 8 \times 10^{-21} = 72 \times 10^{-21} \text{ J}$$

Step 3: Calculate the barrier potential V_0

Using the relation $eV_0 = \Delta K$:

$$V_0 = \frac{\Delta K}{e}$$

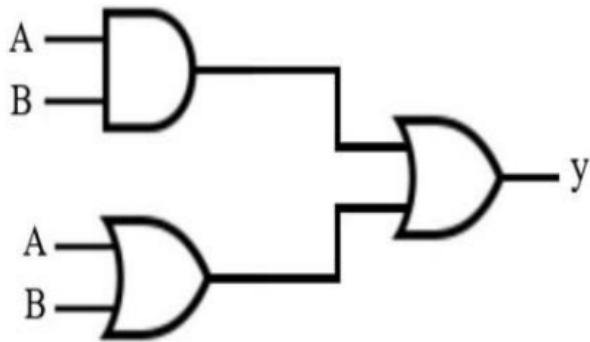
Substitute $e = 1.6 \times 10^{-19} \text{ C}$:

$$V_0 = \frac{72 \times 10^{-21} \text{ J}}{1.6 \times 10^{-19} \text{ C}} = \frac{72}{160} = \frac{9}{20} = 0.45 \text{ V}$$

Quick Tip: Barrier potential in volts is numerically equal to the kinetic energy loss expressed in electron-volts (eV): $\Delta K = \frac{1}{2} \frac{m}{e} (v_1^2 - v_2^2) = 0.45 \text{ eV} \implies V_0 = 0.45 \text{ V}$.

119. Two OR gates and one AND gate are connected as shown in the figure. The correct truth

table of the combination of the logic gates is



Options :

1. ✖

A	B	y
0	0	0
0	1	0
1	0	0
1	1	1

2. ✖

A	B	y
0	0	1
0	1	0
1	0	0
1	1	1

3. ✔

A	B	y
0	0	0
0	1	1
1	0	1
1	1	1

4. ✖

A	B	y
0	0	0
0	1	1
1	0	1
1	1	0

- (A) Option 1
- (B) Option 2
- (C) Option 3
- (D) Option 4

A	B	y
0	0	0
0	1	1
1	0	1
1	1	1

Correct Answer: (C) 

Solution:

Concept:

- The output of a two-input AND gate is high (1) only when both inputs are high:

$$y_{\text{AND}} = A \cdot B$$

- The output of a two-input OR gate is high (1) if at least one input is high:

$$y_{\text{OR}} = A + B$$

- By Boolean algebra absorption laws:

$$(A \cdot B) + (A + B) = A + B$$

Step 1: Identify the logic functions of each stage

From the circuit diagram:

- The inputs A and B are connected to an AND gate:

$$y_1 = A \cdot B$$

- The inputs A and B are simultaneously connected to an OR gate:

$$y_2 = A + B$$

- The outputs y_1 and y_2 serve as the inputs to a final OR gate:

$$y = y_1 + y_2 = (A \cdot B) + (A + B)$$

Step 2: Simplify using Boolean algebra

Using the absorption law of Boolean algebra:

$$A + (A \cdot B) = A$$

Thus:

$$y = (A \cdot B) + A + B = A + B$$

The overall logic function of the entire circuit simplifies directly to an OR operation between A and B .

Step 3: Construct the complete truth table

Evaluate the simplified expression $y = A + B$ for all binary combinations:

- For $A = 0, B = 0$:

$$y = 0 + 0 = 0$$

- For $A = 0, B = 1$:

$$y = 0 + 1 = 1$$

- For $A = 1, B = 0$:

$$y = 1 + 0 = 1$$

- For $A = 1, B = 1$:

$$y = 1 + 1 = 1$$

Quick Tip: By Boolean absorption, $(A \cdot B) + (A + B) = A + B$. The whole network behaves identically to a simple OR gate, immediately giving output 0 only for $A = 0, B = 0$ and 1 for all other input combinations.

120. The approximate bandwidth of frequency required to transmit music is

- (A) 20 kHz
- (B) 2800 Hz
- (C) 4.2 MHz
- (D) 100 MHz

Correct Answer: (A) 20 kHz

Solution:

Concept:

- Bandwidth is the range of frequencies over which an information signal spreads or the channel capacity required to transmit it without significant distortion.
- Different types of communication signals require specific transmission bandwidths:
 - Speech signals: Frequency range from 300 Hz to 3100 Hz, requiring a bandwidth of ≈ 2800 Hz.
 - Music transmission: High-fidelity audio encompasses high-frequency harmonics produced by musical instruments extending up to 20 kHz, requiring a bandwidth of ≈ 20 kHz.
 - Video signals: Transmission of visual pictures requires a bandwidth of ≈ 4.2 MHz.
 - TV signals: Combined video and audio broadcast channels utilize a bandwidth of ≈ 6 MHz.

Step 1: Compare frequency bandwidth requirements for communication signals

Review the standard frequency bandwidth requirements specified for communication systems:

- Speech signal bandwidth: $3100 \text{ Hz} - 300 \text{ Hz} = 2800 \text{ Hz}$
- Music signal bandwidth: up to 20 kHz due to the rich presence of acoustic harmonics
- Television video signal bandwidth: 4.2 MHz

Step 2: Identify the bandwidth required for music

Because musical instruments generate acoustic overtones that span the entire audible range of the human ear (20 Hz to 20 kHz), the transmission bandwidth required for faithful reproduction of music is approximately 20 kHz.

Quick Tip: Standard communication bandwidth reference: Speech ≈ 2.8 kHz, Music ≈ 20 kHz, and Video ≈ 4.2 MHz.

Chemistry

121. The energy of orbit X of Li^{2+} ($Z = 3$) is -2.18×10^{-18} J. What is the radius of the same orbit (in Å)?

- (A) 2.116
- (B) 2.105
- (C) 1.587
- (D) 2.645

Correct Answer: (C) 1.587

Solution:

Concept:

- For a hydrogen-like species with atomic number Z , the energy of the n -th Bohr orbit is:

$$E_n = -R_H \frac{Z^2}{n^2} = -2.18 \times 10^{-18} \times \frac{Z^2}{n^2} \text{ J}$$

- The radius of the n -th Bohr orbit is given by:

$$r_n = 0.529 \times \frac{n^2}{Z} \text{ Å}$$

Step 1: Determine the principal quantum number n of orbit X

For lithium ion Li^{2+} , the atomic number is $Z = 3$. The given energy of orbit X is:

$$E_n = -2.18 \times 10^{-18} \text{ J}$$

Substitute into Bohr's energy formula:

$$-2.18 \times 10^{-18} \times \frac{3^2}{n^2} = -2.18 \times 10^{-18}$$

Divide both sides by -2.18×10^{-18} :

$$\frac{9}{n^2} = 1 \implies n^2 = 9 \implies n = 3$$

Step 2: Calculate the radius of the orbit

Using the formula for the radius of the n -th orbit of a hydrogen-like atom:

$$r_n = 0.529 \times \frac{n^2}{Z} \text{ \AA}$$

Substitute $n = 3$ and $Z = 3$:

$$r_3 = 0.529 \times \frac{3^2}{3} = 0.529 \times 3 \text{ \AA}$$

Perform the multiplication:

$$r_3 = 1.587 \text{ \AA}$$

Quick Tip: Since $E_n \propto \frac{Z^2}{n^2}$, having $E_n = -2.18 \times 10^{-18} \text{ J}$ immediately implies $\frac{Z}{n} = 1$. For $Z = 3$, this gives $n = 3$. Then $r_n = 0.529 \times \frac{3^2}{3} = 0.529 \times 3 = 1.587 \text{ \AA}$.

122. If ν_0 and ν represent the threshold frequency and frequency of incident light respectively, then the correct equation for the velocity of photoelectrons emitted from the metal surface will be

- (A) $v = \sqrt{\frac{2h}{m}(\nu - \nu_0)}$
(B) $v = \sqrt{\frac{2h}{m}(\nu_0 - \nu)}$
(C) $v = \sqrt{\frac{m}{2h}(\nu - \nu_0)}$
(D) $v = \sqrt{\frac{m}{2hc}(\nu_0 - \nu)}$

Correct Answer: (A) $v = \sqrt{\frac{2h}{m}(\nu - \nu_0)}$

Solution:

Concept:

- According to Einstein's photoelectric equation:

$$h\nu = h\nu_0 + K_{\max}$$

where $h\nu$ is the energy of the incident photon, $h\nu_0$ is the work function, and $K_{\max}$ is the maximum kinetic energy of the emitted photoelectrons.

- The kinetic energy of an electron of mass m moving with velocity v is:

$$K_{\max} = \frac{1}{2}mv^2$$

Step 1: Isolate the kinetic energy term in Einstein's equation

Einstein's photoelectric equation is:

$$K_{\max} = h\nu - h\nu_0 = h(\nu - \nu_0)$$

Step 2: Relate kinetic energy to electron velocity

Substitute $K_{\max} = \frac{1}{2}mv^2$:

$$\frac{1}{2}mv^2 = h(\nu - \nu_0)$$

Step 3: Solve for the velocity v

Multiply both sides by 2 and divide by m :

$$v^2 = \frac{2h}{m}(\nu - \nu_0)$$

Taking the square root on both sides:

$$v = \sqrt{\frac{2h}{m}(\nu - \nu_0)}$$

Quick Tip: Since photoelectric emission occurs only when the incident frequency exceeds the threshold frequency ($\nu > \nu_0$), the term under the square root must be positive, which eliminates options with $(\nu_0 - \nu)$.

123. Which of the following statements are not correct?

- I. The process of adding an electron to a gaseous atom is always exothermic
- II. The process of removing an electron from a gaseous atom is always endothermic
- III. The first ionization enthalpy of oxygen is greater than nitrogen
- IV. In group I elements lithium is the least electropositive

(A) I, III, IV only

(B) I & II only

(C) I & III only

(D) II & IV only

Correct Answer: (C) I & III only

Solution:

Concept:

- Electron gain enthalpy can be positive (endothermic) for noble gases, alkaline earth metals, or second electron additions.
- Ionization enthalpy is always endothermic because energy must be supplied to overcome the electrostatic attraction between the electron and the nucleus.
- Nitrogen has a stable half-filled $2p^3$ electronic configuration, making its first ionization enthalpy higher than that of oxygen ($2p^4$).
- Down Group 1, atomic size increases and ionization enthalpy decreases, making lithium the least electropositive (lowest tendency to lose an electron) among the alkali metals.

Step 1: Analyze Statement I

The process of adding an electron (electron gain enthalpy) is not always exothermic:

- For noble gases and alkaline earth metals (such as Be, Mg), the electron gain enthalpy is positive (endothermic).
- The second electron gain enthalpy for any atom is always endothermic due to electrostatic repulsion from the existing negative ion.

Hence, Statement I is incorrect.

Step 2: Analyze Statement II

Ionization enthalpy involves overcoming the attractive force of the nucleus on an electron: Energy is always required to remove an electron from an isolated gaseous atom. Hence, the process is always endothermic. Statement II is correct.

Step 3: Analyze Statement III

Electronic configuration of Nitrogen ($Z = 7$): $1s^2 2s^2 2p^3$ (half-filled, extra stable). Electronic configuration of Oxygen ($Z = 8$): $1s^2 2s^2 2p^4$. Due to the extra stability of the half-filled $2p$ subshell in nitrogen, nitrogen has a higher first ionization enthalpy than oxygen:

$$IE_1(N) > IE_1(O)$$

Hence, Statement III is incorrect.

Step 4: Analyze Statement IV

In Group 1 (alkali metals), electropositive character increases down the group from Li to Cs as ionization energy decreases: Lithium has the highest ionization enthalpy in the group and is therefore the least electropositive alkali metal. Statement IV is correct.

Step 5: Identify the incorrect statements

Statements I and III are not correct.

Quick Tip: Remember the anomaly across Period 2: $IE_1(N) > IE_1(O)$ because nitrogen possesses an extra-stable half-filled $2p^3$ subshell.

124. Identify the correct orders with respect to the given property

I. $B < Al < Mg < K$ – metallic character

II. $Si < P < C < N$ – electronegativity

III. $Si < C < N < F$ – non-metallic character

IV. $Al < Mg < S < P$ – first ionization enthalpy

The correct answer is

- (A) I, II, III only
- (B) II, III, IV only
- (C) I, III only
- (D) I, II, III, IV

Correct Answer: (D) I, II, III, IV

Solution:

Concept:

- Metallic character increases down a group and decreases across a period from left to right.
- Electronegativity increases across a period from left to right and decreases down a group.
- Non-metallic character increases across a period and decreases down a group.
- First ionization enthalpy generally increases across a period, but group 15 elements

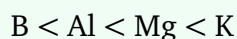
(half-filled p^3) have higher ionization enthalpy than group 16 elements (p^4), and group 2 elements (s^2) have higher ionization enthalpy than group 13 elements (p^1).

Step 1: Evaluate Order I (Metallic character)

Metallic character increases down a group and towards the left of the periodic table:

- Boron is a metalloid/non-metal.
- Aluminium is a post-transition metal.
- Magnesium is an alkaline earth metal.
- Potassium is an alkali metal located in Period 4.

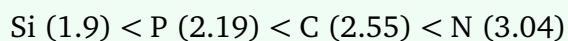
Thus, the metallic character order is:



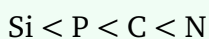
Hence, Order I is correct.

Step 2: Evaluate Order II (Electronegativity)

Paulings electronegativity values:



Thus, the order of electronegativity is:



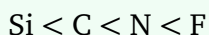
Hence, Order II is correct.

Step 3: Evaluate Order III (Non-metallic character)

Non-metallic character increases towards the top-right of the periodic table:

- Silicon is a metalloid in period 3.
- Carbon is a non-metal in group 14.
- Nitrogen is in group 15.
- Fluorine is the most non-metallic element in group 17.

Thus, the order of non-metallic character is:



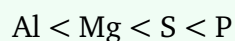
Hence, Order III is correct.

Step 4: Evaluate Order IV (First ionization enthalpy)

Consider the elements of Period 3: Na, Mg, Al, Si, P, S, Cl, Ar.

- $\text{Mg}([\text{Ne}]3s^2)$ has a fully filled subshell, so $\text{IE}_1(\text{Al}) < \text{IE}_1(\text{Mg})$.
- $\text{P}([\text{Ne}]3s^23p^3)$ has a half-filled subshell, so $\text{IE}_1(\text{S}) < \text{IE}_1(\text{P})$.

Comparing these four elements gives:



Hence, Order IV is correct.

Quick Tip: For period 3 first ionization enthalpies, remember the two inversions: $\text{Mg} > \text{Al}$ (due to stable $3s^2$) and $\text{P} > \text{S}$ (due to stable half-filled $3p^3$).

125. Identify the correct orders of covalent character of given molecules

I. $\text{KI} > \text{KBr} > \text{KCl} > \text{KF}$

II. $\text{LiCl} > \text{NaCl} > \text{KCl} > \text{RbCl}$

III. $\text{CsCl} > \text{CaCl}_2 > \text{MgCl}_2 > \text{AlCl}_3$

The correct answer is

- (A) I, II, III
(B) I, II only
(C) II, III only
(D) I, III only

Correct Answer: (B) I, II only

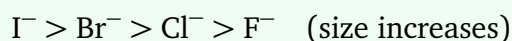
Solution:

Concept:

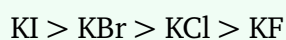
- According to Fajan's Rules, covalent character in an ionic compound increases with:
 1. Increasing polarizability of the anion (larger size of anion).
 2. Increasing polarizing power of the cation (smaller size and higher charge of cation).

Step 1: Analyze Order I

The cation K^+ is common to all molecules. The anions belong to Group 17:



A larger anion has a more easily polarizable electron cloud, leading to greater covalent character. Therefore, the correct order of covalent character is:



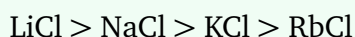
Hence, Order I is correct.

Step 2: Analyze Order II

The anion Cl^- is common to all molecules. The cations belong to Group 1 with equal charge (+1):



A smaller cation has higher charge density and stronger polarizing power, producing greater covalent character:



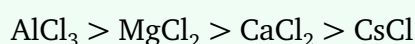
Hence, Order II is correct.

Step 3: Analyze Order III

The cations and their charges are:

- Cs^+ (charge +1, large size)
- Ca^{2+} (charge +2)
- Mg^{2+} (charge +2, smaller than Ca^{2+})
- Al^{3+} (charge +3, smallest size and highest charge)

Higher positive charge and smaller cation size lead to much greater polarizing power and covalent character:



The given order lists the exact reverse of the correct order. Hence, Order III is incorrect.

Quick Tip: By Fajan's rules: Covalent character $\propto \frac{\text{charge on cation}}{\text{size of cation}} \times \text{size of anion}$.

126. Match the following

List-1 (Molecule) فہرست-1 (سار)		List-2 (Geometry) فہرست-2 (جیومیٹری)	
A	BrF ₅	I	T-Shape T-شکل
B	XeF ₄	II	See-Saw سی سا (آرائی)
C	ClF ₃	III	Square planar مربع مستوی
D	SF ₄	IV	Square pyramid مربع ابراہی

The correct answer is

- (A) A – IV, B – III, C – II, D – I
 (B) A – III, B – IV, C – II, D – I
 (C) A – IV, B – I, C – III, D – II
 (D) A – IV, B – III, C – I, D – II

Correct Answer: (D) A – IV, B – III, C – I, D – II

Solution:

Concept:

- According to VSEPR theory, molecular geometry is determined by the number of bond pairs (bp) and lone pairs (lp) on the central atom:

$$\text{Steric Number} = \text{bp} + \text{lp}$$

- Steric number = 5 (sp^3d):

- 4 bp + 1 lp → See-Saw
- 3 bp + 2 lp → T-Shape
- Steric number = 6 (sp^3d^2):
 - 5 bp + 1 lp → Square pyramidal
 - 4 bp + 2 lp → Square planar

Step 1: Determine the geometry of BrF_5 (A)

Bromine has 7 valence electrons:

- Forms 5 single bonds with fluorine: 5 bond pairs.
- Leaves $7 - 5 = 2$ non-bonding electrons: 1 lone pair.
- Steric number = $5 + 1 = 6$ (sp^3d^2 hybridization).

The geometry for an AB_5E type molecule is Square pyramid. Hence, A corresponds to IV.

Step 2: Determine the geometry of XeF_4 (B)

Xenon has 8 valence electrons:

- Forms 4 single bonds with fluorine: 4 bond pairs.
- Leaves $8 - 4 = 4$ non-bonding electrons: 2 lone pairs.
- Steric number = $4 + 2 = 6$ (sp^3d^2 hybridization).

The geometry for an AB_4E_2 type molecule is Square planar. Hence, B corresponds to III.

Step 3: Determine the geometry of ClF_3 (C)

Chlorine has 7 valence electrons:

- Forms 3 single bonds with fluorine: 3 bond pairs.
- Leaves $7 - 3 = 4$ non-bonding electrons: 2 lone pairs.
- Steric number = $3 + 2 = 5$ (sp^3d hybridization).

The geometry for an AB_3E_2 type molecule is T-Shape. Hence, C corresponds to I.

Step 4: Determine the geometry of SF_4 (D)

Sulfur has 6 valence electrons:

- Forms 4 single bonds with fluorine: 4 bond pairs.

- Leaves $6 - 4 = 2$ non-bonding electrons: 1 lone pair.
- Steric number $= 4 + 1 = 5$ (sp^3d hybridization).

The geometry for an AB_4E type molecule is See-Saw. Hence, D corresponds to II.

Step 5: Match lists

The matching is: A – IV, B – III, C – I, D – II.

Quick Tip: Quick VSEPR recall: ClF_3 is a standard T-shaped interhalogen, and XeF_4 is the classic square planar noble gas fluoride.

127. An organic compound on analysis is found to have 10.06% carbon, 0.84% hydrogen and 89.10% chlorine by weight. The simplest whole number ratio of C, H and Cl is

- (A) 1 : 2 : 3
 (B) 1 : 1 : 3
 (C) 1 : 2 : 2
 (D) 1 : 3 : 1

Correct Answer: (B) 1 : 1 : 3

Solution:

Concept:

- To determine the simplest whole number ratio (empirical formula):
 1. Divide the percentage of each element by its atomic mass to find the relative number of moles.
 2. Divide each mole value by the smallest mole value obtained.
 3. Simplify to the nearest integer whole number ratio.
- Atomic masses: C = 12 g/mol, H = 1 g/mol, Cl = 35.5 g/mol.

Step 1: Calculate the relative number of moles of each element

For Carbon (C):

$$n_C = \frac{10.06}{12} \approx 0.8383$$

For Hydrogen (H):

$$n_{\text{H}} = \frac{0.84}{1} = 0.8400$$

For Chlorine (Cl):

$$n_{\text{Cl}} = \frac{89.10}{35.5} \approx 2.5098$$

Step 2: Divide each mole value by the smallest value

The smallest number of moles is $n_{\text{C}} \approx 0.8383$:

$$\text{Ratio for C} = \frac{0.8383}{0.8383} = 1$$

$$\text{Ratio for H} = \frac{0.8400}{0.8383} \approx 1.002 \approx 1$$

$$\text{Ratio for Cl} = \frac{2.5098}{0.8383} \approx 2.994 \approx 3$$

Step 3: State the simplest whole number ratio

The simplest whole number ratio of C : H : Cl is:

$$1 : 1 : 3$$

This corresponds to the empirical formula CHCl_3 (chloroform).

Quick Tip: Notice that the percentages correspond to chloroform (CHCl_3): Molar mass of $\text{CHCl}_3 = 12 + 1 + 3(35.5) = 119.5 \text{ g/mol}$. Percentage of Cl = $\frac{106.5}{119.5} \times 100\% \approx 89.1\%$, confirming 1 : 1 : 3.

128. At 300 K, a 10 L vessel contains 0.4 g of He, 1.6 g of O_2 and 1.4 g of N_2 . The partial pressure of He gas (in atm) is (Assume ideal behaviour for all gases) ($R = 0.0821 \text{ L atm K}^{-1} \text{ mol}^{-1}$)

- (A) 0.123
- (B) 0.246
- (C) 0.323
- (D) 0.133

Correct Answer: (B) 0.246

Solution:**Concept:**

- By Dalton's Law of Partial Pressures and the ideal gas equation:

$$p_i = \frac{n_i RT}{V}$$

where n_i is the number of moles of gas i , R is the universal gas constant, T is absolute temperature, and V is the container volume.

- The partial pressure of helium depends only on its own moles, volume, and temperature, completely independent of the presence of other non-reacting gases.
- Molar mass of Helium: $M_{\text{He}} = 4 \text{ g/mol}$.

Step 1: Calculate the number of moles of Helium

Given mass of helium:

$$w_{\text{He}} = 0.4 \text{ g}$$

Molar mass of helium:

$$M_{\text{He}} = 4 \text{ g/mol}$$

Number of moles:

$$n_{\text{He}} = \frac{0.4}{4} = 0.1 \text{ mol}$$

Step 2: Apply the ideal gas equation to helium

Given:

$$V = 10 \text{ L}, \quad T = 300 \text{ K}, \quad R = 0.0821 \text{ L atm K}^{-1} \text{ mol}^{-1}$$

The partial pressure of helium is:

$$p_{\text{He}} = \frac{n_{\text{He}} RT}{V}$$

Substitute the values:

$$p_{\text{He}} = \frac{0.1 \times 0.0821 \times 300}{10}$$

Step 3: Compute the numerical value

Calculate the numerator:

$$0.1 \times 300 = 30$$

$$30 \times 0.0821 = 2.463$$

Divide by the volume $V = 10$ L:

$$p_{\text{He}} = \frac{2.463}{10} = 0.2463 \text{ atm} \approx 0.246 \text{ atm}$$

Quick Tip: To find the partial pressure of a specific component in a mixture, apply $p = \frac{nRT}{V}$ directly to that component alone, ignoring the other gases.

129. Identify the sets which contain either only extensive properties or only intensive properties

I. Density, specific heat, surface tension

II. Mole fraction, Internal energy, viscosity

III. Molality, pressure, Entropy

IV. Number of moles, Enthalpy, Heat capacity

The correct answer is

- (A) I, IV only
- (B) I, II only
- (C) II, III only
- (D) II, IV only

Correct Answer: (A) I, IV only

Solution:

Concept:

- **Intensive properties** are independent of the amount or size of matter in the system (e.g., temperature, pressure, density, viscosity, specific heat, surface tension, molarity, molality, mole fraction).
- **Extensive properties** depend directly on the mass or quantity of matter present in the system (e.g., mass, volume, internal energy, enthalpy, entropy, heat capacity, number of moles).

Step 1: Classify the properties in Set I

- Density: Intensive

- Specific heat: Intensive
- Surface tension: Intensive

Set I contains **only intensive properties**.

Step 2: Classify the properties in Set II

- Mole fraction: Intensive
- Internal energy: Extensive
- Viscosity: Intensive

Set II contains a mixture of intensive and extensive properties.

Step 3: Classify the properties in Set III

- Molality: Intensive
- Pressure: Intensive
- Entropy: Extensive

Set III contains a mixture of intensive and extensive properties.

Step 4: Classify the properties in Set IV

- Number of moles: Extensive
- Enthalpy: Extensive
- Heat capacity: Extensive

Set IV contains **only extensive properties**.

Step 5: Select the correct sets

Sets I (only intensive) and IV (only extensive) satisfy the condition.

Quick Tip: Remember: "Specific" or "molar" properties are always intensive (e.g., specific heat), while total thermal capacities (heat capacity, enthalpy, entropy, internal energy) are extensive.

130. A certain buffer solution contains $[X^-]$ and $[HX]$ in the ratio of 1 : 10. If K_b of X^- is 10^{-10} , then the pH value of the buffer is

- (A) 9
- (B) 5
- (C) 3
- (D) 11

Correct Answer: (C) 3

Solution:

Concept:

- For a conjugate acid-base pair in aqueous solution:

$$K_a \cdot K_b = K_w = 10^{-14} \implies pK_a + pK_b = 14$$

- According to the Henderson-Hasselbalch equation for an acidic buffer:

$$pH = pK_a + \log\left(\frac{[\text{conjugate base}]}{[\text{acid}]}\right) = pK_a + \log\left(\frac{[X^-]}{[HX]}\right)$$

Step 1: Find pK_b and pK_a

The dissociation constant of the conjugate base is:

$$K_b = 10^{-10}$$

Calculate pK_b :

$$pK_b = -\log(10^{-10}) = 10$$

Using the conjugate relationship:

$$pK_a = 14 - pK_b = 14 - 10 = 4$$

Step 2: Calculate the pH using the Henderson-Hasselbalch equation

Given ratio of conjugate base to acid:

$$\frac{[X^-]}{[HX]} = \frac{1}{10}$$

Substitute into the buffer equation:

$$\text{pH} = \text{p}K_a + \log\left(\frac{[X^-]}{[HX]}\right)$$

$$\text{pH} = 4 + \log\left(\frac{1}{10}\right)$$

Since $\log\left(\frac{1}{10}\right) = \log(10^{-1}) = -1$:

$$\text{pH} = 4 + (-1) = 3$$

Quick Tip: Direct calculation: $K_a = \frac{10^{-14}}{10^{-10}} = 10^{-4} \Rightarrow \text{p}K_a = 4$. With a 1 : 10 base-to-acid ratio, the solution is more acidic than its $\text{p}K_a$, giving $\text{pH} = 4 - 1 = 3$.

131. Identify the correct statements from the following

I. Hydrogen bonding is present in liquid water and solid water (ice)

II. Hydrogenation of vegetable oils give a fat called vanaspati

III. Water present in $\text{BaCl}_2 \cdot 2\text{H}_2\text{O}$ belongs to the type “interstitial water”

The correct answer is

- (A) I, III only
- (B) I, II, III
- (C) I, II only
- (D) II, III only

Correct Answer: (B) I, II, III

Solution:

Concept:

- Intermolecular hydrogen bonding exists extensively in both liquid water and solid water (ice). In ice, hydrogen bonds form a three-dimensional open cage-like lattice.
- Catalytic hydrogenation of polyunsaturated vegetable oils using finely divided nickel

at elevated temperatures produces saturated solid fats, commercially termed vanaspati ghee.

- Hydrated salts can contain water in three distinct forms:

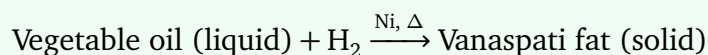
- Coordinated water, e.g., $[\text{Cr}(\text{H}_2\text{O})_6]^{3+} 3\text{Cl}^-$
- Interstitial water, e.g., $\text{BaCl}_2 \cdot 2\text{H}_2\text{O}$
- Hydrogen-bonded water, e.g., $[\text{Cu}(\text{H}_2\text{O})_4]\text{SO}_4 \cdot \text{H}_2\text{O}$

Step 1: Evaluate Statement I

Water molecules contain highly electronegative oxygen atoms covalently bonded to hydrogen atoms. In both liquid water and solid ice, intermolecular hydrogen bonding is present. In ice, each oxygen atom is tetrahedrally coordinated to four hydrogen atoms (two by covalent bonds and two by hydrogen bonds). Hence, Statement I is correct.

Step 2: Evaluate Statement II

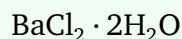
Unsaturated vegetable oils contain carbon-carbon double bonds ($-\text{CH} = \text{CH}-$). Passing hydrogen gas through vegetable oil in the presence of a nickel catalyst at 473 K reduces double bonds to single bonds:



Hence, Statement II is correct.

Step 3: Evaluate Statement III

In the crystalline lattice of barium chloride dihydrate, water molecules occupy interstitial voids between the ions without direct coordination to the barium ion:



Therefore, water molecules in $\text{BaCl}_2 \cdot 2\text{H}_2\text{O}$ are classified as interstitial water. Hence, Statement III is correct.

Step 4: Conclusion

All three statements I, II, and III are correct.

Quick Tip: Remember the three types of water of crystallization: $\text{BaCl}_2 \cdot 2\text{H}_2\text{O}$ represents interstitial water, $[\text{Cr}(\text{H}_2\text{O})_6]\text{Cl}_3$ represents coordinated water, and $\text{CuSO}_4 \cdot 5\text{H}_2\text{O}$ contains both coordinated and hydrogen-bonded water.

132. The pair of elements which do not give super oxides with excess of oxygen is

- (A) K, Rb
- (B) Na, K
- (C) Li, Mg
- (D) K, Ba

Correct Answer: (C) Li, Mg

Solution:

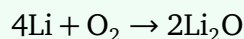
Concept:

- The stability of oxides, peroxides, and superoxides of s-block metals depends on the lattice energy and the matching of cation and anion sizes.
- Smaller cations have high polarizing power and stabilize smaller oxide anions (O^{2-}).
- Larger cations have lower polarizing power and stabilize larger polyatomic anions like peroxide (O_2^{2-}) and superoxide (O_2^-).
- Superoxides are formed primarily by heavy alkali metals (K, Rb, Cs).

Step 1: Analyze the reaction of Group 1 elements with oxygen

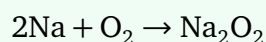
Alkali metals react with excess air/oxygen as follows:

- Lithium (Li): Due to its small ionic size and high charge density, it forms predominantly normal monoxide:

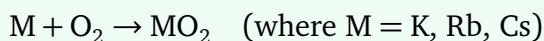


It does not form superoxide.

- Sodium (Na): Forms mainly peroxide:



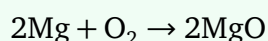
- Potassium (K), Rubidium (Rb), and Cesium (Cs): Form stable superoxides:



Step 2: Analyze the reaction of Group 2 elements with oxygen

Alkaline earth metals react with oxygen as follows:

- Magnesium (Mg): Due to its small size and high positive charge density (+2), it stabilizes only the normal oxide (O^{2-}):



It does not form peroxides or superoxides under normal conditions.

- Barium (Ba): Being larger, it can form peroxide (BaO_2), but not superoxides.

Step 3: Identify the pair that does not form superoxides

Both Lithium (Li) and Magnesium (Mg) have high charge densities and small ionic radii (diagonal relationship), preventing the stabilization of the large O_2^- superoxide anion. Therefore, the pair Li and Mg does not form superoxides.

Quick Tip: Superoxide formation requires a large, weakly polarizing cation. Only large alkali metals (K, Rb, Cs) form superoxides (MO_2). Lithium and magnesium form exclusively normal monoxides (Li_2O and MgO).

133. Which of the following is not correctly matched with its common name?

- (A) Dead burnt plaster – $CaSO_4$
- (B) Quick lime – CaO
- (C) Slaked lime – $Ca(OH)_2$
- (D) Gypsum – $CaSO_4 \cdot \frac{1}{2}H_2O$

Correct Answer: (D) Gypsum – $CaSO_4 \cdot \frac{1}{2}H_2O$

Solution:

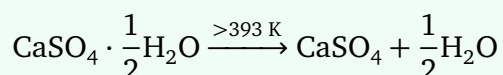
Concept:

- Calcium compounds have well-established industrial and common commercial names:

- Quick lime: Calcium oxide, CaO
- Slaked lime: Calcium hydroxide, Ca(OH)_2
- Gypsum: Calcium sulphate dihydrate, $\text{CaSO}_4 \cdot 2\text{H}_2\text{O}$
- Plaster of Paris: Calcium sulphate hemihydrate, $\text{CaSO}_4 \cdot \frac{1}{2}\text{H}_2\text{O}$
- Dead burnt plaster: Anhydrous calcium sulphate, CaSO_4

Step 1: Verify Option (A)

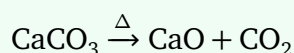
Heating Plaster of Paris above 393 K drives off all water of crystallization, leaving anhydrous calcium sulphate:



Anhydrous CaSO_4 is known as dead burnt plaster because it loses the property of setting with water. This match is correct.

Step 2: Verify Option (B)

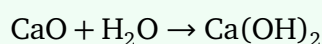
Thermal decomposition of limestone yields calcium oxide:



Calcium oxide (CaO) is commonly called quick lime. This match is correct.

Step 3: Verify Option (C)

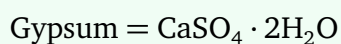
Reaction of quick lime with water produces calcium hydroxide:



Calcium hydroxide (Ca(OH)_2) is commonly called slaked lime. This match is correct.

Step 4: Verify Option (D)

Gypsum is the naturally occurring mineral calcium sulphate dihydrate:



The formula $\text{CaSO}_4 \cdot \frac{1}{2}\text{H}_2\text{O}$ corresponds to Plaster of Paris (hemihydrate), not gypsum. Therefore, Option (D) is not correctly matched.

Quick Tip: Remember the hydration states of calcium sulfate: Gypsum has 2 molecules of water ($\text{CaSO}_4 \cdot 2\text{H}_2\text{O}$), Plaster of Paris has half a molecule ($\text{CaSO}_4 \cdot \frac{1}{2}\text{H}_2\text{O}$), and dead burnt plaster has zero (CaSO_4).

134. The composition of Borax is $\text{Na}_2\text{B}_4\text{O}_7 \cdot x\text{H}_2\text{O}$ and that of Kernite is $\text{Na}_2\text{B}_4\text{O}_7 \cdot y\text{H}_2\text{O}$. (x-y) is equal to

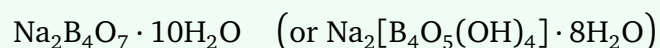
- (A) 4
- (B) 3
- (C) 2
- (D) 6

Correct Answer: (D) 6

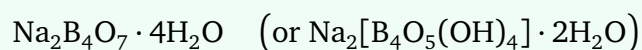
Solution:

Concept:

- Borax and kernite are naturally occurring hydrated sodium tetraborate minerals.
- Borax (tincal) crystallizes as a decahydrate:

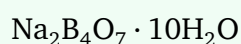


- Kernite (also called rasorite) crystallizes as a tetrahydrate:



Step 1: Identify the number of water molecules x in Borax

The chemical formula of borax is:

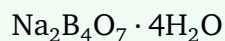


Comparing with $\text{Na}_2\text{B}_4\text{O}_7 \cdot x\text{H}_2\text{O}$:

$$x = 10$$

Step 2: Identify the number of water molecules y in Kernite

The chemical formula of kernite is:



Comparing with $\text{Na}_2\text{B}_4\text{O}_7 \cdot y\text{H}_2\text{O}$:

$$y = 4$$

Step 3: Calculate $(x - y)$

Subtract y from x :

$$x - y = 10 - 4 = 6$$

Quick Tip: Borax is a decahydrate ($10\text{H}_2\text{O}$) while Kernite is a tetrahydrate ($4\text{H}_2\text{O}$). The difference in hydration number is simply $10 - 4 = 6$.

135. Given below are two statements

Statement-I: CO reduces Al_2O_3 to Al

Statement-II: CO is the neutral ligand with one σ and 2π bonds between carbon and oxygen

The correct answer is

- (A) Both statements I and II are correct
- (B) Statement I is correct, but statement II is not correct
- (C) Statement I is not correct, but statement II is correct
- (D) Both statements I and II are not correct

Correct Answer: (C) Statement I is not correct, but statement II is correct

Solution:

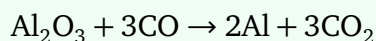
Concept:

- According to the Ellingham diagram, aluminium has an extremely high affinity for oxygen, and the Gibbs free energy line for the formation of Al_2O_3 lies far below that of CO or CO_2 at all practically achievable metallurgical temperatures.
- Carbon monoxide (CO) cannot reduce Al_2O_3 to aluminium metal. Aluminium must be extracted electrolytically (Hall-Héroult process).

- In carbon monoxide ($\text{:C} \equiv \text{O:}$), the carbon-oxygen triple bond consists of one σ bond and two π bonds. It acts as a neutral monodentate ligand that bonds through carbon.

Step 1: Evaluate Statement I

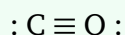
The standard free energy of formation of Al_2O_3 is extremely negative ($\Delta G^\circ \ll 0$). For the reduction reaction:



The overall change in free energy ΔG° is positive at all feasible temperatures because the $\text{Al}-\text{O}$ bond is substantially stronger than the $\text{C}-\text{O}$ bond in CO_2 . Hence, CO cannot reduce Al_2O_3 to Al. Statement I is not correct.

Step 2: Evaluate Statement II

The Lewis structure of carbon monoxide is:



Between the carbon and oxygen atoms:

- There is one σ bond (formed by sp hybrid orbital overlap).
- There are two π bonds (formed by sideways overlap of unhybridized p orbitals, one being a coordinate dative bond from oxygen to carbon).

CO carries no net formal charge, making it a neutral ligand (carbonyl ligand) capable of forming synergic bonds with transition metals. Hence, Statement II is correct.

Quick Tip: Active metals with high electropositivity (like Al, Mg, Na) cannot be reduced by carbon or CO because their oxides lie at the bottom of the Ellingham diagram. They require electrolytic reduction.

136. The oxides of which element are responsible for photochemical smog?

- (A) Sulphur
- (B) Nitrogen
- (C) Carbon
- (D) Chlorine

Correct Answer: (B) Nitrogen

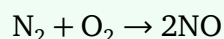
Solution:

Concept:

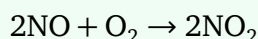
- **Photochemical smog** (oxidizing smog or Los Angeles smog) is formed in warm, dry, and sunny climates by the action of solar ultraviolet light on unsaturated hydrocarbons and nitrogen oxides (NO_x).
- **Classical smog** (reducing smog or London smog) occurs in cool, humid climates and is primarily composed of smoke, fog, and sulphur dioxide (SO_2).

Step 1: Identify the primary precursors of photochemical smog

High temperatures in automobile engines and thermal power plants cause nitrogen and oxygen from air to combine, producing nitric oxide:

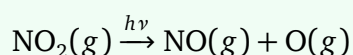


Nitric oxide rapidly oxidizes in air to nitrogen dioxide:

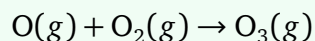


Step 2: Analyze the photochemical reaction mechanism

Nitrogen dioxide absorbs solar radiation in the ultraviolet region and dissociates into nitric oxide and reactive nascent oxygen:



The nascent oxygen combines with atmospheric oxygen to form ozone:



Ozone then reacts with unburnt hydrocarbons to generate acrolein, formaldehyde, and peroxyacetyl nitrate (PAN), which are the defining components of photochemical smog.

Step 3: Conclusion

The oxides of nitrogen (NO_x) are the essential primary pollutants responsible for the initiation and propagation of photochemical smog.

Quick Tip: Distinguish the two types of smog: Photochemical smog $\rightarrow$ Oxides of Nitrogen (NO_2) + Hydrocarbons + Sunlight. Classical smog $\rightarrow$ Oxides of Sulphur (SO_2) + Smoke + Fog.

137. 1 mole of a hydrocarbon $\text{A}(\text{C}_5\text{H}_{10})$ on ozonolysis gives two compounds X and Y. Both X and Y respond to iodoform test. X gives test with ammoniacal AgNO_3 solution but not with Y. What are X and Y respectively?

- (A) CH_2O ; $\text{CH}_3\text{COC}_2\text{H}_5$
(B) CH_3CHO ; $(\text{CH}_3)_2\text{CO}$
(C) $(\text{CH}_3)_2\text{C}=\text{O}$; $\text{CH}_3\text{CH}=\text{O}$
(D) CH_3COOH ; $(\text{CH}_3)_2\text{C}=\text{O}$

Correct Answer: (B) CH_3CHO ; $(\text{CH}_3)_2\text{CO}$

Solution:

Concept:

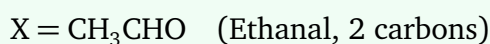
- Reductive ozonolysis of an alkene cleaves the double bond to produce two carbonyl compounds.
- Ammoniacal silver nitrate solution (Tollens' reagent) is reduced to a silver mirror by aldehydes, but ketones do not give this test.
- The iodoform test gives a yellow precipitate of CHI_3 with compounds containing either a methyl ketone group ($\text{CH}_3-\text{C}=\text{O}$) or a $\text{CH}_3-\text{CH}(\text{OH})-$ group.
- Acetaldehyde (CH_3CHO) is the only aldehyde that gives a positive iodoform test.

Step 1: Deduce the functional nature of compound X

We are given:

- X gives a positive Tollens' test (reacts with ammoniacal AgNO_3), which proves that X is an aldehyde.
- X also gives a positive iodoform test.

Since acetaldehyde (CH_3CHO) is the unique aldehyde containing the CH_3-CHO unit, compound X must be:

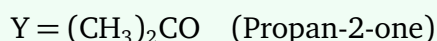


Step 2: Deduce the functional nature of compound Y

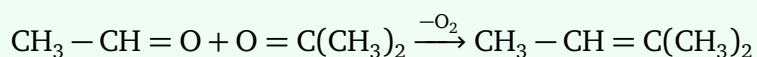
We are given:

- Y does not react with ammoniacal AgNO_3 , which indicates that Y is a ketone.
- Y gives a positive iodoform test, meaning it contains a $\text{CH}_3 - \text{C} = \text{O}$ group.
- The total number of carbon atoms in hydrocarbon A is 5.

Since X contains 2 carbons, Y must contain $5 - 2 = 3$ carbons. A 3-carbon methyl ketone is acetone:

**Step 3: Reconstruct the structure of hydrocarbon A**

Combine the fragments by removing oxygen atoms and joining with a double bond:



Hydrocarbon A is 2-methylbut-2-ene (C_5H_{10}). Respectively, X is CH_3CHO and Y is $(\text{CH}_3)_2\text{CO}$.

Quick Tip: Key deduction: Acetaldehyde (CH_3CHO) is the ONLY aldehyde that gives a positive iodoform test. Since X gives both Tollens' and iodoform tests, X must be CH_3CHO .

138. The compound which is not isomeric with ethoxyethane?

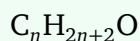
- (A) Propylmethylether
- (B) Butan-2-ol
- (C) Butanone
- (D) 2-Methylpropan-1-ol

Correct Answer: (C) Butanone

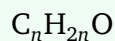
Solution:**Concept:**

- Isomers are compounds that share the exact same molecular formula but have different structural arrangements or functional groups.

- Saturated open-chain monoethers and monohydric alcohols have the general molecular formula:



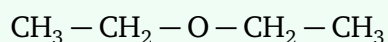
- Saturated aliphatic ketones and aldehydes have the general molecular formula:



possessing a degree of unsaturation (double bond equivalent) of 1.

Step 1: Determine the molecular formula of ethoxyethane

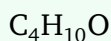
Ethoxyethane (diethyl ether) has the structure:



Count the atoms:

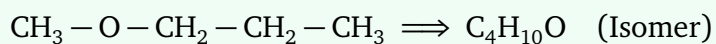
- Carbon: 4
- Hydrogen: $3 + 2 + 2 + 3 = 10$
- Oxygen: 1

Molecular formula:

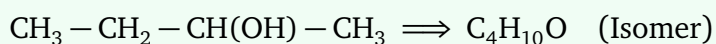


Step 2: Examine the molecular formula of each option

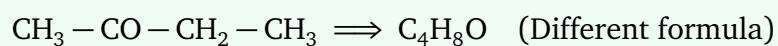
- Option (A) Propylmethylether:



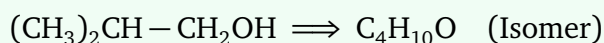
- Option (B) Butan-2-ol:



- Option (C) Butanone:



- Option (D) 2-Methylpropan-1-ol:



Step 3: Identify the non-isomeric compound

Butanone has molecular formula $\text{C}_4\text{H}_8\text{O}$, which contains two fewer hydrogen atoms than ethoxyethane ($\text{C}_4\text{H}_{10}\text{O}$). Therefore, butanone is not isomeric with ethoxyethane.

Quick Tip: Ethers and alcohols are saturated ($\text{C}_n\text{H}_{2n+2}\text{O}$), while ketones contain a carbonyl double bond ($\text{C}_n\text{H}_{2n}\text{O}$). A ketone cannot be an isomer of a saturated ether.

139. Identify the pair of alkanes which undergo aromatization from the following

I. n-Hexane

II. Isopentane

III. n-Heptane

IV. Neohexane

The correct answer is

- (A) I & II only
- (B) I & III only
- (C) II & III only
- (D) III & IV only

Correct Answer: (B) I & III only

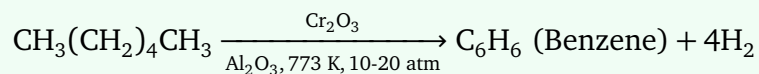
Solution:

Concept:

- **Aromatization** (catalytic reforming) is the conversion of unbranched open-chain alkanes containing 6 to 8 carbon atoms into benzene and its homologous alkyl derivatives.
- The reaction occurs under high temperature ($\approx 773\text{ K}$) and pressure (10–20 atm) in the presence of supported transition metal oxide catalysts such as Cr_2O_3 , V_2O_5 , or Mo_2O_3 over alumina (Al_2O_3).
- Alkanes with fewer than 6 carbon atoms cannot form an aromatic six-membered ring.

Step 1: Analyze n-Hexane (I)

n-Hexane has an unbranched continuous 6-carbon chain:



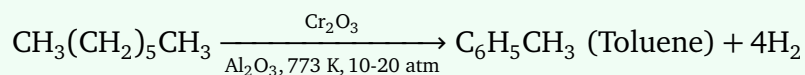
It undergoes cyclization and dehydrogenation to yield benzene. Hence, n-hexane undergoes aromatization.

Step 2: Analyze Isopentane (II)

Isopentane (2-methylbutane) has only 5 carbon atoms. Since at least 6 carbon atoms in a suitable chain are strictly required to form a stable aromatic benzene ring, isopentane cannot undergo aromatization.

Step 3: Analyze n-Heptane (III)

n-Heptane has an unbranched continuous 7-carbon chain:



It undergoes cyclization to form methylcyclohexane, followed by aromatization to give toluene. Hence, n-heptane undergoes aromatization.

Step 4: Analyze Neohexane (IV)

Neohexane (2,2-dimethylbutane) contains a quaternary carbon atom with a 4-carbon parent chain. Due to steric hindrance and the absence of a contiguous 6-carbon backbone, it does not undergo direct aromatization.

Step 5: Select the valid pair

The unbranched alkanes that undergo aromatization are n-hexane (I) and n-heptane (III).

Quick Tip: Aromatization requires unbranched alkanes with ≥ 6 carbon atoms: n-hexane yields benzene, n-heptane yields toluene, and n-octane yields o-xylene.

140. An organic compound contains 69.4% C, 5.8% H, $x\%$ N and $y\%$ O. A sample of 0.30 g of this compound was analyzed for nitrogen by Kjeldahl's method. The ammonia evolved was absorbed in 50 mL of 0.05 M H_2SO_4 . The excess acid required 25 mL of 0.1 M NaOH for neutralization. The empirical formula of the compound is

(A) $\text{C}_3\text{H}_3\text{NO}$

- (B) $C_6H_6N_2O$
(C) C_7H_7NO
(D) $C_8H_7N_2O$

Correct Answer: (C) C_7H_7NO

Solution:

Concept:

- In Kjeldahl's method for estimation of nitrogen:

$$\%N = \frac{1.4 \times N_{\text{acid}} \times V_{\text{acid consumed}}}{w}$$

where $V_{\text{acid consumed}}$ is the volume in mL of normal acid neutralized by evolved ammonia, and w is the mass of the organic compound in grams.

- Normality of H_2SO_4 is $N = M \times \text{basicity} = M \times 2$.
- Percentage of oxygen is calculated by difference:

$$\%O = 100 - (\%C + \%H + \%N)$$

- Empirical formula is obtained by finding the simplest molar ratio of the elements.

Step 1: Calculate the milliequivalents of acid consumed by ammonia

Initial volume and normality of H_2SO_4 :

$$N_{\text{acid}} = 2 \times 0.05 \text{ M} = 0.1 \text{ N}$$

$$\text{Initial meq of } H_2SO_4 = 50 \text{ mL} \times 0.1 \text{ N} = 5.0 \text{ meq}$$

Milliequivalents of NaOH used to neutralize excess acid:

$$\text{meq of NaOH} = 25 \text{ mL} \times 0.1 \text{ N} = 2.5 \text{ meq}$$

Milliequivalents of H_2SO_4 neutralized by evolved NH_3 :

$$\text{meq consumed} = 5.0 - 2.5 = 2.5 \text{ meq}$$

Step 2: Calculate the percentage of nitrogen in the compound

Using the Kjeldahl formula with sample mass $w = 0.30$ g:

$$\%N = \frac{1.4 \times (\text{meq of acid consumed})}{w}$$

$$\%N = \frac{1.4 \times 2.5}{0.30} = \frac{3.5}{0.30} \approx 11.67\%$$

Step 3: Calculate the percentage of oxygen

Subtract the sum of the known percentages from 100%:

$$\%O = 100 - (69.4\% + 5.8\% + 11.67\%) = 100 - 86.87\% = 13.13\%$$

Step 4: Determine the atomic molar ratios

Divide the percentage of each element by its atomic mass:

$$\text{C: } \frac{69.4}{12} = 5.783$$

$$\text{H: } \frac{5.8}{1} = 5.800$$

$$\text{N: } \frac{11.67}{14} = 0.834$$

$$\text{O: } \frac{13.13}{16} = 0.821$$

Step 5: Find the simplest whole number ratio

Divide each ratio by the smallest value (0.821):

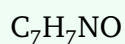
$$\text{C: } \frac{5.783}{0.821} \approx 7.04 \approx 7$$

$$\text{H: } \frac{5.800}{0.821} \approx 7.06 \approx 7$$

$$\text{N: } \frac{0.834}{0.821} \approx 1.01 \approx 1$$

$$\text{O: } \frac{0.821}{0.821} = 1$$

Therefore, the empirical formula of the compound is:



Quick Tip: Remember to convert molarity of sulfuric acid to normality ($N = 2M$). Milliequivalents of acid reacted with $\text{NH}_3 = (50 \times 0.1) - (25 \times 0.1) = 2.5$ meq. This immediately leads to $\%N \approx 11.7\%$, giving the formula $\text{C}_7\text{H}_7\text{NO}$.

141. Given below are two statements

Statement-I: The percent compositions of Ni^{2+} and Ni^{3+} in $\text{Ni}_{0.98}\text{O}$ is 96% and 4% respectively

Statement-II: The fraction of Fe^{3+} and Fe^{2+} ions in 1 mole of $\text{Fe}_{0.93}\text{O}$ is 0.14 and 0.79 respectively

The correct answer is

- (A) Both statements I and II are correct
(B) Statement I is correct, but statement II is not correct
(C) Statement I is not correct, but statement II is correct
(D) Both statements I and II are not correct

Correct Answer: (A) Both statements I and II are correct

Solution:

Concept:

- Non-stoichiometric metal deficiency defects arise when metal cations exist in multiple oxidation states within the crystal lattice.
- Electrical neutrality of the crystal must be conserved: the total positive charge of all metal cations must balance the total negative charge of the oxide anions (O^{2-}).
- For a formula M_xO , 1 mole of the oxide contains x moles of metal ions and 1 mole of O^{2-} ions (total negative charge = -2).

Step 1: Evaluate Statement-I for $\text{Ni}_{0.98}\text{O}$

In 1 mole of $\text{Ni}_{0.98}\text{O}$, there are 0.98 moles of Ni cations and 1 mole of O^{2-} ions. Let the moles of Ni^{2+} be a . Then the moles of Ni^{3+} are $0.98 - a$. Apply the condition of electrical neutrality:

$$2(a) + 3(0.98 - a) = 2$$

Expand and solve for a :

$$2a + 2.94 - 3a = 2 \implies -a = 2 - 2.94 = -0.94 \implies a = 0.94$$

Thus:

$$n(\text{Ni}^{2+}) = 0.94, \quad n(\text{Ni}^{3+}) = 0.98 - 0.94 = 0.04$$

Calculate the percentage of each ion out of the total nickel ions:

$$\% \text{Ni}^{2+} = \frac{0.94}{0.98} \times 100\% \approx 95.92\% \approx 96\%$$

$$\% \text{Ni}^{3+} = \frac{0.04}{0.98} \times 100\% \approx 4.08\% \approx 4\%$$

Hence, Statement-I is correct.

Step 2: Evaluate Statement-II for $\text{Fe}_{0.93}\text{O}$

In 1 mole of $\text{Fe}_{0.93}\text{O}$, there are 0.93 moles of iron cations and 1 mole of O^{2-} ions. Let the moles of Fe^{3+} be y . Then the moles of Fe^{2+} are $0.93 - y$. Apply electrical neutrality:

$$3(y) + 2(0.93 - y) = 2$$

Expand and solve for y :

$$3y + 1.86 - 2y = 2 \implies y = 2 - 1.86 = 0.14$$

Thus:

$$n(\text{Fe}^{3+}) = 0.14 \text{ mol}$$

$$n(\text{Fe}^{2+}) = 0.93 - 0.14 = 0.79 \text{ mol}$$

The fractions of Fe^{3+} and Fe^{2+} in 1 mole of $\text{Fe}_{0.93}\text{O}$ are 0.14 and 0.79 respectively. Hence, Statement-II is correct.

Step 3: Conclusion

Both Statement-I and Statement-II are correct.

Quick Tip: For M_{1-x}O , the fraction of trivalent metal ion M^{3+} is always $2x$. For $\text{Ni}_{0.98}\text{O}$, $x = 0.02 \implies \text{Ni}^{3+} = 0.04$ (out of 0.98, giving $\approx 4\%$). For $\text{Fe}_{0.93}\text{O}$, $x = 0.07 \implies \text{Fe}^{3+} = 0.14$ and $\text{Fe}^{2+} = 0.93 - 0.14 = 0.79$.

142. Henry's law constant for argon at 298 K is 40 k bar. The mass of argon (in g) dissolved in 2.0 L water, when the pressure applied is 3.0 bar at the same temperature is (Molar mass of

argon = 40 g mol⁻¹)

- (A) 0.66
- (B) 3.33
- (C) 4.33
- (D) 0.33

Correct Answer: (D) 0.33

Solution:

Concept:

- Henry's Law states that the solubility of a gas in a liquid is directly proportional to the partial pressure of the gas above the liquid:

$$p = K_H \cdot x$$

where p is the partial pressure, K_H is the Henry's law constant, and x is the mole fraction of the dissolved gas.

- For a dilute solution, the mole fraction can be approximated as:

$$x_{\text{Ar}} = \frac{n_{\text{Ar}}}{n_{\text{Ar}} + n_{\text{H}_2\text{O}}} \approx \frac{n_{\text{Ar}}}{n_{\text{H}_2\text{O}}}$$

- Mass of dissolved gas is given by:

$$m_{\text{Ar}} = n_{\text{Ar}} \times M_{\text{Ar}}$$

Step 1: Calculate the mole fraction of dissolved argon

Given:

$$p = 3.0 \text{ bar}$$

$$K_H = 40 \text{ k bar} = 40 \times 10^3 \text{ bar} = 4 \times 10^4 \text{ bar}$$

By Henry's Law:

$$x_{\text{Ar}} = \frac{p}{K_H} = \frac{3.0}{4 \times 10^4} = 0.75 \times 10^{-4} = 7.5 \times 10^{-5}$$

Step 2: Calculate the moles of water in 2.0 L

The density of water is 1 kg/L, so 2.0 L of water has mass:

$$m_{\text{H}_2\text{O}} = 2000 \text{ g}$$

Molar mass of water is 18 g/mol:

$$n_{\text{H}_2\text{O}} = \frac{2000}{18} = \frac{1000}{9} \text{ mol} \approx 111.11 \text{ mol}$$

Step 3: Determine the moles of dissolved argon

Since $x_{\text{Ar}} \ll 1$:

$$x_{\text{Ar}} \approx \frac{n_{\text{Ar}}}{n_{\text{H}_2\text{O}}} \implies n_{\text{Ar}} = x_{\text{Ar}} \times n_{\text{H}_2\text{O}}$$

Substitute the values:

$$n_{\text{Ar}} = (7.5 \times 10^{-5}) \times \left(\frac{1000}{9} \right) = \frac{7.5 \times 10^{-2}}{9} = \frac{0.075}{9} = \frac{1}{120} \text{ mol}$$

Step 4: Compute the mass of dissolved argon

Using the molar mass of argon ($M_{\text{Ar}} = 40 \text{ g/mol}$):

$$m_{\text{Ar}} = n_{\text{Ar}} \times M_{\text{Ar}} = \left(\frac{1}{120} \text{ mol} \right) \times 40 \text{ g/mol}$$

$$m_{\text{Ar}} = \frac{40}{120} = \frac{1}{3} \text{ g} \approx 0.33 \text{ g}$$

Quick Tip: Direct formula: $m = \frac{p}{K_H} \times \frac{V_{\text{water}} (\text{in g})}{18} \times M_{\text{gas}}$. Here, $m = \frac{3}{40000} \times \frac{2000}{18} \times 40 = \frac{3 \times 2 \times 40}{40 \times 18} = \frac{6}{18} = \frac{1}{3} \approx 0.33 \text{ g}$.

143. At infinite dilution, the molar conductivity of aluminum sulphate is $858 \text{ S cm}^2 \text{ mol}^{-1}$. If $\lambda_{\text{SO}_4^{2-}}^\circ$ is $160 \text{ S cm}^2 \text{ mol}^{-1}$, what is the molar conductivity of Al^{3+} ion (in $\text{S cm}^2 \text{ mol}^{-1}$)?

- (A) 198
- (B) 918
- (C) 189
- (D) 378

Solution:

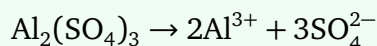
Concept:

- According to Kohlrausch's Law of Independent Migration of Ions, the limiting molar conductivity of an electrolyte is the sum of the individual contributions of the cations and anions:

$$\Lambda_m^\circ = \nu_+ \lambda_+^\circ + \nu_- \lambda_-^\circ$$

where ν_+ and ν_- are the stoichiometric numbers of cations and anions per formula unit of electrolyte.

- For aluminium sulphate, $\text{Al}_2(\text{SO}_4)_3$, complete dissociation yields 2 Al^{3+} ions and 3 SO_4^{2-} ions:



Step 1: Write the Kohlrausch's law equation for aluminium sulphate

For $\text{Al}_2(\text{SO}_4)_3$:

$$\Lambda_m^\circ(\text{Al}_2(\text{SO}_4)_3) = 2\lambda^\circ(\text{Al}^{3+}) + 3\lambda^\circ(\text{SO}_4^{2-})$$

Step 2: Substitute the given conductivity values

We are given:

$$\Lambda_m^\circ(\text{Al}_2(\text{SO}_4)_3) = 858 \text{ S cm}^2\text{mol}^{-1}$$

$$\lambda^\circ(\text{SO}_4^{2-}) = 160 \text{ S cm}^2\text{mol}^{-1}$$

Substitute these into the equation:

$$858 = 2\lambda^\circ(\text{Al}^{3+}) + 3(160)$$

Step 3: Solve for the limiting molar conductivity of Al^{3+}

Compute the contribution of the sulphate anions:

$$3 \times 160 = 480 \text{ S cm}^2\text{mol}^{-1}$$

Subtract from the total limiting molar conductivity:

$$2\lambda^\circ(\text{Al}^{3+}) = 858 - 480 = 378$$

Divide by 2:

$$\lambda^\circ(\text{Al}^{3+}) = \frac{378}{2} = 189 \text{ S cm}^2\text{mol}^{-1}$$

Quick Tip: Remember the stoichiometric coefficients from the chemical formula $\text{Al}_2(\text{SO}_4)_3$: $\lambda^\circ(\text{Al}^{3+}) = \frac{\Lambda_m^\circ - 3\lambda^\circ(\text{SO}_4^{2-})}{2} = \frac{858 - 480}{2} = 189 \text{ S cm}^2\text{mol}^{-1}$.

144. The difference between energy of an activated complex and the average energy of reactants is called?

- (A) Threshold energy
- (B) Lattice energy
- (C) Activation energy
- (D) Kinetic energy

Correct Answer: (C) Activation energy

Solution:

Concept:

- **Threshold energy** is the minimum total energy that reacting molecules must possess in order to undergo an effective collision and form products.
- **Activation energy (E_a)** is the extra minimum amount of energy that reactant molecules must absorb to form the high-energy intermediate (activated complex):

$$E_a = E_{\text{threshold}} - E_{\text{reactants}}$$

- **Activated complex** is an unstable arrangement of atoms with highest potential energy along the reaction coordinate.

Step 1: Define the energy terms in chemical kinetics

- **Average energy of reactants (E_R):** The intrinsic thermal energy possessed by reactant molecules before reaction.

- Energy of activated complex / transition state (E_{complex}): The peak energy of the potential energy barrier, also called threshold energy.
- The difference $E_{\text{complex}} - E_R$ represents the barrier height.

Step 2: Identify the correct terminology

By definition, the difference between the energy of the activated complex and the average kinetic/potential energy of the reacting species is the **activation energy** (E_a).

Quick Tip: Energy difference breakdown: Total energy of activated complex = Threshold Energy.
Energy of activated complex – Energy of reactants = Activation Energy (E_a).

145. The graph drawn between $\log \frac{x}{m}$ and $\log P$ for an adsorption process is a straight line at an angle of 45° , with intercept equal to 0.3010. The extent of adsorption $\left(\frac{x}{m}\right)$ at a pressure of 0.2 atm is ($\log 2 = 0.3010$; $\tan 45^\circ = 1$)

- (A) 0.2
- (B) 0.3
- (C) 0.4
- (D) 0.8

Correct Answer: (C) 0.4

Solution:

Concept:

- According to Freundlich's adsorption isotherm:

$$\frac{x}{m} = kP^{1/n}$$

- Taking the logarithm on both sides yields the linear equation:

$$\log \left(\frac{x}{m} \right) = \log k + \frac{1}{n} \log P$$

- Comparing with the straight-line form $y = c + mx$:

$$\text{Slope } m = \tan \theta = \frac{1}{n}$$

$$\text{Intercept } c = \log k$$

Step 1: Determine the value of $\frac{1}{n}$ from the slope

The angle of inclination of the straight line is 45° . The slope of the graph is:

$$\text{Slope} = \tan 45^\circ = 1$$

Since slope $= \frac{1}{n}$:

$$\frac{1}{n} = 1$$

Step 2: Determine the constant k from the intercept

The intercept on the vertical axis is:

$$\text{Intercept} = \log k = 0.3010$$

Since $\log 2 = 0.3010$:

$$k = 10^{0.3010} = 2$$

Step 3: Calculate the extent of adsorption at $P = 0.2$ atm

Substitute $k = 2$ and $\frac{1}{n} = 1$ into Freundlich's isotherm:

$$\frac{x}{m} = kP^{1/n} = 2 \times P^1 = 2P$$

At pressure $P = 0.2$ atm:

$$\frac{x}{m} = 2 \times 0.2 = 0.4$$

Quick Tip: Since slope is 1, $\frac{x}{m}$ is directly proportional to P : $\frac{x}{m} = kP$. With intercept $\log k = \log 2 \Rightarrow k = 2$, we immediately get $\frac{x}{m} = 2(0.2) = 0.4$.

146. In the metallurgy of silver, silver is leached with a dilute solution of KCN in the presence of air to form a complex ion X. This in the presence of zinc converts into another complex ion

Y. The ratio of CN^- ligands in X and Y is

- (A) 1 : 1
- (B) 1 : 2
- (C) 2 : 1
- (D) 2 : 3

Correct Answer: (B) 1 : 2

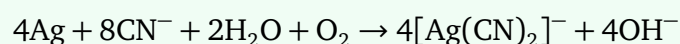
Solution:

Concept:

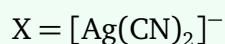
- Extraction of silver by the MacArthur-Forrest cyanide process involves hydrometallurgical leaching followed by displacement with a more electropositive metal (zinc).
- Leaching of silver with cyanide in the presence of atmospheric oxygen produces a soluble dicyanoargentate(I) complex ion.
- Subsequent reduction with zinc powder forms a tetracyanozincate(II) complex ion while precipitating metallic silver.

Step 1: Identify the leaching reaction and complex ion X

Silver is leached with aqueous KCN in the presence of air (oxygen acting as an oxidizing agent):



The soluble complex ion formed is:

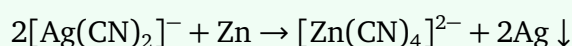


The number of CN^- ligands attached to silver in complex X is:

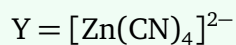
$$n_{\text{X}} = 2$$

Step 2: Identify the displacement reaction and complex ion Y

Zinc dust is added to the solution to displace silver:



The resulting complex ion formed is:



The number of CN^- ligands coordinated to zinc in complex Y is:

$$n_Y = 4$$

Step 3: Compute the ratio of CN^- ligands in X and Y

Form the ratio of ligands in X and Y:

$$\frac{n_X}{n_Y} = \frac{2}{4} = \frac{1}{2}$$

Thus, the ratio is 1 : 2.

Quick Tip: Silver(I) has coordination number 2 in $[\text{Ag}(\text{CN})_2]^-$, while Zinc(II) has coordination number 4 in $[\text{Zn}(\text{CN})_4]^{2-}$. The ratio of cyanide ligands is directly $2 : 4 = 1 : 2$.

147. Four sets of reactants required for the preparation of four different oxoacids of phosphorous are given below

I. PCl_3 , H_3PO_3

II. P_2O_3 , H_2O

III. Red P_4 , alkali

IV. P_4O_{10} , H_2O

The oxoacids formed from which sets of reactants have phosphorus in +3 oxidation state?

(A) I, II only

(B) III, IV only

(C) II, III only

(D) I, IV only

Correct Answer: (A) I, II only

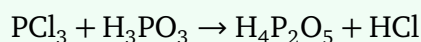
Solution:

Concept:

- Oxidation state of phosphorus in its oxoacids depends on the specific chemical reactions and hydration states of its halides and oxides.
- Oxoacids of phosphorus with +3 oxidation state include:
 - Orthophosphorous acid (H_3PO_3): Oxidation state = +3
 - Pyrophosphorous acid ($\text{H}_4\text{P}_2\text{O}_5$): Oxidation state = +3
- Hypophosphorous acid (H_3PO_2) has phosphorus in +1 oxidation state.
- Hypophosphoric acid ($\text{H}_4\text{P}_2\text{O}_6$) has phosphorus in +4 oxidation state.
- Orthophosphoric acid (H_3PO_4) has phosphorus in +5 oxidation state.

Step 1: Analyze Set I ($\text{PCl}_3 + \text{H}_3\text{PO}_3$)

Reaction between phosphorus trichloride and orthophosphorous acid produces pyrophosphorous acid:



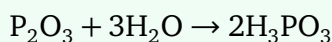
Calculate the oxidation state of P in $\text{H}_4\text{P}_2\text{O}_5$:

$$4(+1) + 2x + 5(-2) = 0 \implies 2x - 6 = 0 \implies x = +3$$

Phosphorus is in the +3 oxidation state. Set I is valid.

Step 2: Analyze Set II ($\text{P}_2\text{O}_3 + \text{H}_2\text{O}$)

Phosphorus trioxide (P_4O_6 or empirical formula P_2O_3) on dissolution in cold water yields orthophosphorous acid:



Calculate the oxidation state of P in H_3PO_3 :

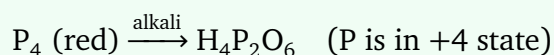
$$3(+1) + x + 3(-2) = 0 \implies x - 3 = 0 \implies x = +3$$

Phosphorus is in the +3 oxidation state. Set II is valid.

Step 3: Analyze Set III (Red P_4 + alkali)

Reaction of red phosphorus with aqueous alkali yields hypophosphoric acid salts (oxidation

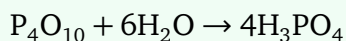
state +4) or related sub-valent species:



This does not yield a +3 oxoacid. Set III is invalid.

Step 4: Analyze Set IV ($\text{P}_4\text{O}_{10} + \text{H}_2\text{O}$)

Phosphorus pentoxide reacts with water to form orthophosphoric acid:



Calculate the oxidation state of P in H_3PO_4 :

$$3(+1) + x + 4(-2) = 0 \implies x = +5$$

Phosphorus is in the +5 oxidation state. Set IV is invalid.

Step 5: Conclusion

Only sets I and II yield oxoacids with phosphorus in the +3 oxidation state.

Quick Tip: P_2O_3 is the anhydride of H_3PO_3 (both have +3 state). $\text{PCl}_3 + \text{H}_3\text{PO}_3$ gives $\text{H}_4\text{P}_2\text{O}_5$ (also +3 state). P_4O_{10} is the anhydride of H_3PO_4 (+5 state).

148. Ethyl alcohol on reaction with PCl_3 gives ethyl chloride and an oxoacid X. This on reaction with PCl_3 again forms another oxoacid Y. The number of $-\text{OH}$ groups in X and Y are respectively

- (A) 2, 3
- (B) 2, 2
- (C) 2, 4
- (D) 3, 4

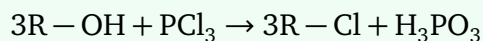
Correct Answer: (B) 2, 2

Solution:

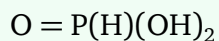
Concept:

- Reaction of alcohols with phosphorus trichloride gives alkyl halides along with orthophos-

phorous acid:



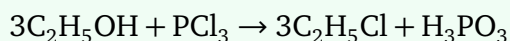
- Orthophosphorous acid (H_3PO_3) is diprotic because it possesses two ionizable P – OH groups and one non-ionizable P – H bond:



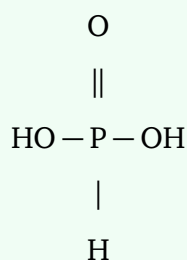
- Pyrophosphorous acid ($H_4P_2O_5$) is formed by condensation of H_3PO_3 with PCl_3 , and has two P – OH groups.

Step 1: Identify oxoacid X and its structure

The reaction of ethanol with PCl_3 is:



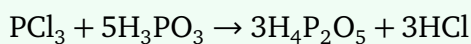
Thus, oxoacid X is orthophosphorous acid (H_3PO_3). Its structural formula is:



Number of –OH groups in X = 2.

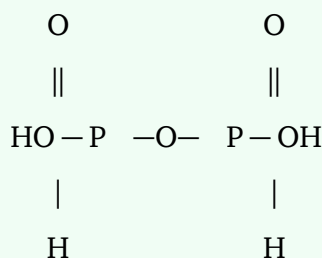
Step 2: Identify oxoacid Y and its structure

When orthophosphorous acid reacts with PCl_3 , condensation occurs to form pyrophosphorous acid:



Thus, oxoacid Y is pyrophosphorous acid ($H_4P_2O_5$). Its structural formula contains a P – O – P

linkage:



Each phosphorus atom carries one $-\text{OH}$ group, one $-\text{H}$ group, and one $=\text{O}$. Total number of $-\text{OH}$ groups in Y = $1 + 1 = 2$.

Step 3: Conclusion

The number of $-\text{OH}$ groups in X and Y are 2 and 2 respectively.

Quick Tip: Remember that H_3PO_3 is a dibasic acid containing 2 $-\text{OH}$ groups, and its dimer anhydride $\text{H}_4\text{P}_2\text{O}_5$ is also dibasic with 2 $-\text{OH}$ groups.

149. In which of the following the order is not correctly matched with the property mentioned?

- (A) $\text{HF} < \text{HCl} < \text{HBr} < \text{HI}$ – Acidic strength
- (B) $\text{NH}_3 > \text{PH}_3 > \text{AsH}_3 > \text{SbH}_3$ – Basic strength
- (C) $\text{H}_2\text{O} < \text{H}_2\text{S} < \text{H}_2\text{Se} < \text{H}_2\text{Te}$ – Acidic strength
- (D) $\text{H}_2\text{O} < \text{H}_2\text{S} < \text{H}_2\text{Se} < \text{H}_2\text{Te}$ – Bond angle

Correct Answer: (D) $\text{H}_2\text{O} < \text{H}_2\text{S} < \text{H}_2\text{Se} < \text{H}_2\text{Te}$ – Bond angle

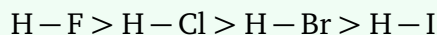
Solution:

Concept:

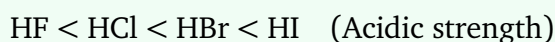
- Down a group for hydrides of non-metals, bond length increases, and the $\text{E}-\text{H}$ bond dissociation enthalpy decreases, which increases acidic strength.
- For group 15 hydrides, electron density on the central atom decreases with increasing size, causing basic strength to decrease down the group.
- Bond angles of hydrides decrease sharply down a group as central atom electronegativity decreases and the central atom bonding transitions to predominantly unhybridized pure p-orbitals (Drago's rule).

Step 1: Evaluate Option (A)

For hydrogen halides, bond dissociation enthalpy follows:



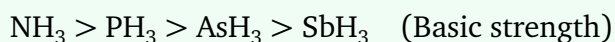
Since the H—I bond is weakest, it releases H^+ most readily in water:



This match is correct.

Step 2: Evaluate Option (B)

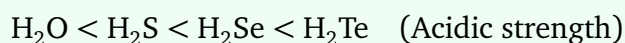
For group 15 hydrides, the lone pair on nitrogen is localized in a compact sp^3 orbital. Down the group from N to Sb, atomic radius increases and lone pair density decreases, diminishing proton-accepting ability:



This match is correct.

Step 3: Evaluate Option (C)

For group 16 hydrides, the E—H bond dissociation enthalpy decreases from oxygen to tellurium:



This match is correct.

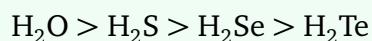
Step 4: Evaluate Option (D)

The experimental bond angles of group 16 hydrides are:

- $\text{H}_2\text{O} \approx 104.5^\circ$
- $\text{H}_2\text{S} \approx 92.1^\circ$
- $\text{H}_2\text{Se} \approx 91^\circ$
- $\text{H}_2\text{Te} \approx 90^\circ$

Because the electronegativity of the central atom decreases down the group, bonding electron pairs are drawn further from the nucleus, reducing bond pair-bond pair repulsion. Hence,

bond angle decreases down the group:



Option (D) incorrectly provides the reverse order.

Quick Tip: Bond angles of hydrides decrease down any group: $\text{H}_2\text{O}(104.5^\circ) > \text{H}_2\text{S}(92^\circ) > \text{H}_2\text{Se}(91^\circ) > \text{H}_2\text{Te}(90^\circ)$. Any statement claiming bond angles increase down the group is incorrect.

150. The colour and magnetic nature of the compound formed, when MnO_2 is fused with a mixture of KOH and KNO_3 are respectively

- (A) Green, paramagnetic
- (B) Blue, paramagnetic
- (C) Green, diamagnetic
- (D) Violet, diamagnetic

Correct Answer: (A) Green, paramagnetic

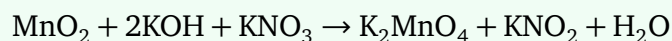
Solution:

Concept:

- Fusion of manganese dioxide (MnO_2) with an alkali metal hydroxide (KOH) in the presence of an oxidizing agent (KNO_3 or atmospheric O_2) produces potassium manganate (K_2MnO_4).
- In the manganate ion ($[\text{MnO}_4]^{2-}$), manganese is in the +6 oxidation state.
- The electronic configuration of Mn^{6+} determines the presence of unpaired electrons, and hence its magnetic nature.

Step 1: Write the chemical reaction for the oxidative fusion

When MnO_2 is fused with molten KOH in the presence of the oxidizing agent KNO_3 :



The compound formed is potassium manganate (K_2MnO_4).

Step 2: Identify the characteristic color

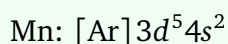
Potassium manganate consists of K^+ and the manganate ion $[MnO_4]^{2-}$. The manganate ion exhibits an intense, characteristic **dark green** color.

Step 3: Determine the oxidation state and electronic configuration of Mn

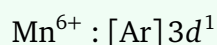
In $[MnO_4]^{2-}$:

$$x + 4(-2) = -2 \implies x = +6$$

Manganese has atomic number $Z = 25$:



For Mn^{6+} , six electrons are removed:

**Step 4: Determine the magnetic behavior**

Since Mn^{6+} possesses one unpaired electron in the $3d$ subshell ($n = 1$): The compound is **paramagnetic** with a spin-only magnetic moment:

$$\mu = \sqrt{1(1+2)} = \sqrt{3} \approx 1.73 \text{ BM}$$

Therefore, the compound formed is green and paramagnetic.

Quick Tip: Remember the two key manganese oxoions: $[MnO_4]^{2-}$ (manganate, +6) is green and paramagnetic ($3d^1$). $[MnO_4]^-$ (permanganate, +7) is purple/violet and diamagnetic ($3d^0$).

151. Which one of the following is an outer orbital complex and exhibits paramagnetic behaviour?

- (A) $[Co(C_2O_4)_3]^{3-}$
- (B) $[MnCl_6]^{3-}$
- (C) $[Mn(CN)_6]^{3-}$
- (D) $[Fe(CN)_6]^{3-}$

Correct Answer: (B) $[MnCl_6]^{3-}$

Solution:

Concept:

- According to Valence Bond Theory (VBT), octahedral complexes are classified based on the d-orbitals used in hybridization:
 - **Inner orbital complex:** Involves inner $(n - 1)d$ orbitals giving d^2sp^3 hybridization (typically favored by strong field ligands).
 - **Outer orbital complex:** Involves outer nd orbitals giving sp^3d^2 hybridization (typically favored by weak field ligands).
- A complex is paramagnetic if it possesses one or more unpaired electrons, and diamagnetic if all electrons are paired.

Step 1: Analyze Option (A): $[\text{Co}(\text{C}_2\text{O}_4)_3]^{3-}$

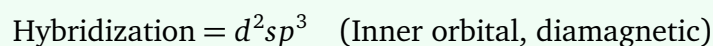
Cobalt is in the +3 oxidation state:



Although oxalate (ox^{2-}) is normally a weak-to-intermediate field ligand, with Co^{3+} it causes electron pairing due to high effective nuclear charge and chelation:

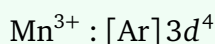


The complex uses inner 3d orbitals:



Step 2: Analyze Option (B): $[\text{MnCl}_6]^{3-}$

Manganese is in the +3 oxidation state:



Chloride (Cl^-) is a weak field ligand ($\Delta_o < P$), so no electron pairing occurs:



Hybridization utilizes outer $4d$ orbitals:

$$\text{Hybridization} = sp^3d^2 \quad (\text{Outer orbital complex})$$

Number of unpaired electrons $n = 4$, hence it is strongly **paramagnetic**.

Step 3: Analyze Option (C): $[\text{Mn}(\text{CN})_6]^{3-}$

Manganese is in the +3 oxidation state ($3d^4$). Cyanide (CN^-) is a strong field ligand ($\Delta_o > P$), forcing pairing:

$$t_{2g}^4 e_g^0$$

Two inner $3d$ orbitals are vacant, resulting in:

$$\text{Hybridization} = d^2sp^3 \quad (\text{Inner orbital complex})$$

Step 4: Analyze Option (D): $[\text{Fe}(\text{CN})_6]^{3-}$

Iron is in the +3 oxidation state:

$$\text{Fe}^{3+} : [\text{Ar}] 3d^5$$

Cyanide (CN^-) is a strong field ligand, forcing pairing to leave one unpaired electron:

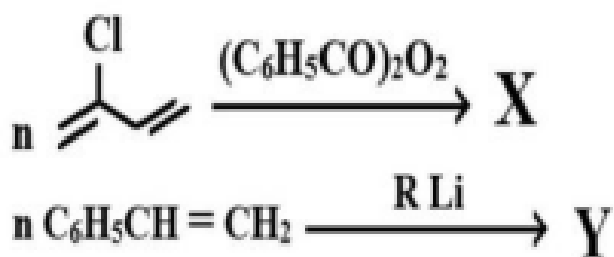
$$t_{2g}^5 e_g^0$$

Hybridization utilizes inner $3d$ orbitals:

$$\text{Hybridization} = d^2sp^3 \quad (\text{Inner orbital complex})$$

Quick Tip: Weak field ligands like Cl^- do not cause pairing and always form outer orbital complexes (sp^3d^2). With Mn^{3+} ($3d^4$), $[\text{MnCl}_6]^{3-}$ has 4 unpaired electrons and is paramagnetic.

152. Observe the following polymerization reactions and statements given about them



I. Chain termination step is present in both reactions

II. X is an elastomer and Y is a thermoplastic polymer

III. X is used in preparation of conveyer belts and Y is used in preparation of television cabinets

Correct statements are

- (A) I, II, III
- (B) I, III only
- (C) I, II only
- (D) II, III only

Correct Answer: (D) II, III only

Solution:

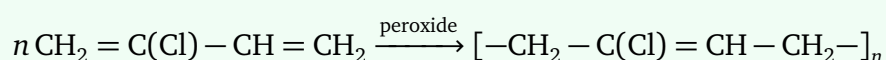
Concept:

- Free-radical addition polymerization initiated by peroxides involves three standard steps: chain initiation, chain propagation, and chain termination.
- Anionic polymerization initiated by organolithium reagents (RLi) produces living polymers in which chain termination steps are completely absent under inert conditions.
- Neoprene (polychloroprene) is a synthetic rubber (elastomer) with high resistance to oils and heat, widely used for making conveyer belts and gaskets.
- Polystyrene is a linear thermoplastic polymer used in packaging, manufacturing toys, and television cabinets.

Step 1: Analyze reaction 1 and identify polymer X

Monomer: Chloroprene (2-chloro-1,3-butadiene). Initiator: Benzoyl peroxide, $(\text{C}_6\text{H}_5\text{CO})_2\text{O}_2$.

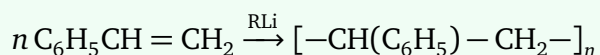
Mechanism: Free radical addition polymerization. Product X: Neoprene (polychloroprene).



Neoprene is an elastomer possessing high tensile strength and elasticity. It is commercially used in making conveyer belts, gaskets, and hoses.

Step 2: Analyze reaction 2 and identify polymer Y

Monomer: Styrene ($\text{C}_6\text{H}_5\text{CH}=\text{CH}_2$). Initiator: Organolithium (RLi). Mechanism: Anionic living polymerization. Product Y: Polystyrene.



Polystyrene is a classic thermoplastic polymer. It is widely utilized in the manufacture of television cabinets, radio cases, and insulating materials.

Step 3: Evaluate Statement I

In the anionic polymerization of styrene using RLi , the growing carbanionic chain ends cannot undergo mutual combination or disproportionation: There is no inherent chain-termination step (living polymer). Hence, Statement I is incorrect.

Step 4: Evaluate Statements II and III

- Statement II: X (Neoprene) is an elastomer and Y (Polystyrene) is a thermoplastic polymer. (Correct)
- Statement III: Neoprene is used in conveyer belts, and polystyrene is used in television cabinets. (Correct)

Therefore, only statements II and III are correct.

Quick Tip: Anionic polymerization using alkyllithium is a living polymerization where chain termination is absent. Neoprene is an elastomer (conveyer belts) and polystyrene is a thermoplastic (TV cabinets).

153. The number of amino acids present in insulin is

- (A) 21
- (B) 30
- (C) 51
- (D) 41

Correct Answer: (C) 51

Solution:

Concept:

- Insulin is a peptide hormone produced by the beta cells of the pancreatic islets that regulates blood glucose levels.
- It was the first protein whose primary structure (amino acid sequence) was fully elucidated (by Frederick Sanger in 1953).
- Insulin is composed of two polypeptide chains: Chain A and Chain B, cross-linked by interchain disulfide bonds.

Step 1: Analyze the chain structure of human insulin

The structure of insulin consists of two polypeptide chains:

- Chain A: Contains 21 amino acid residues.
- Chain B: Contains 30 amino acid residues.

Step 2: Calculate the total number of amino acids

Add the number of amino acid residues in both chains:

$$\text{Total amino acids} = 21 + 30 = 51$$

These two chains are held together by two interchain disulfide bridges (between Cys-A7 and Cys-B7, and between Cys-A20 and Cys-B19), with an additional intrachain disulfide ring within chain A.

Quick Tip: Standard fact in biochemistry: Insulin contains 51 amino acids (Chain A has 21, Chain B has 30).

154. Match the following

List-1 فہرست-1		List-2 فہرست-2	
A	Antioxidant ضد تکسیدی عامل	I	Antiseptic اینٹی سپٹک (جراثیم کش)
B	Food preservative غذائی تحفظاتی عامل	II	Artificial sweetener مصنوعی میٹھا س پیدا کرنے والے
C	Sucralose سکرالوز	III	Butylated hydroxy toluene بیوٹیلٹیڈ ہائیڈروکسی ٹالوین
D	Bithionol بیتھیونول	IV	Sodium benzoate سودیئم بینزویٹ

The correct answer is

- (A) A – III, B – I, C – IV, D – II
 (B) A – IV, B – III, C – II, D – I
 (C) A – III, B – IV, C – I, D – II
 (D) A – III, B – IV, C – II, D – I

Correct Answer: (D) A – III, B – IV, C – II, D – I

Solution:

Concept:

- **Antioxidants:** Chemical substances that retard the action of oxygen on food, preventing rancidity (e.g., BHT, BHA).
- **Food preservatives:** Chemicals that prevent spoilage of food due to microbial growth (e.g., sodium benzoate, salts of sorbic acid).
- **Artificial sweeteners:** Non-nutritive chemical compounds that impart a sweet taste without adding calories (e.g., aspartame, saccharin, sucralose).
- **Antiseptics:** Chemical agents applied to living tissues to inhibit microbial growth (e.g., bithionol, chloroxlenol).

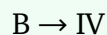
Step 1: Match Item A (Antioxidant)

Butylated hydroxy toluene (BHT) is an effective synthetic antioxidant commonly added to fats and packaged foods to inhibit radical oxidation:

A → III

Step 2: Match Item B (Food preservative)

Sodium benzoate ($\text{C}_6\text{H}_5\text{COONa}$) is the most widely utilized food preservative, metabolizing in the body as hippuric acid:



Step 3: Match Item C (Sucralose)

Sucralose is a trichloro derivative of sucrose that acts as an artificial sweetener stable at cooking temperatures:



Step 4: Match Item D (Bithionol)

Bithionol is added to soaps to impart antiseptic properties and reduce body odor produced by bacterial decomposition:



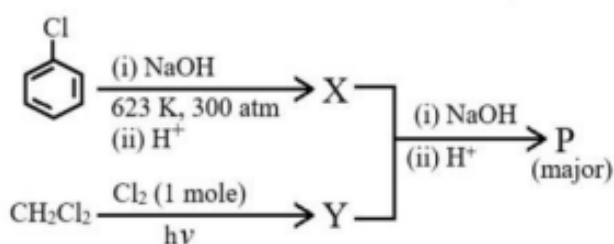
Step 5: Identify the correct combination

The complete match is:



Quick Tip: Remember: BHT = Antioxidant, Sodium benzoate = Food preservative, Sucralose = Artificial sweetener, Bithionol = Antiseptic in soap.

155. What is the end product P in the given sequence of reactions?



- (A) p-Hydroxybenzaldehyde
- (B) m-Hydroxybenzaldehyde
- (C) o-Hydroxybenzaldehyde
- (D) o-Hydroxybenzoic acid

Correct Answer: (C) o-Hydroxybenzaldehyde

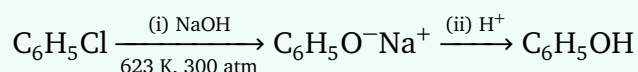
Solution:

Concept:

- Dow's Process: Chlorobenzene undergoes nucleophilic aromatic substitution under drastic conditions (NaOH, 623 K, 300 atm) to yield sodium phenoxide, which on acidification gives phenol.
- Free-radical photochemical chlorination of dichloromethane (CH_2Cl_2) with 1 mole of Cl_2 gives chloroform (CHCl_3).
- The Reimer-Tiemann reaction of phenol with chloroform in the presence of aqueous alkali introduces a formyl group ($-\text{CHO}$) predominantly at the ortho position due to intramolecular hydrogen bonding.

Step 1: Identify intermediate X

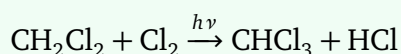
Chlorobenzene reacts with molten NaOH at 623 K and 300 atm:



Thus, intermediate X is **phenol** ($\text{C}_6\text{H}_5\text{OH}$).

Step 2: Identify intermediate Y

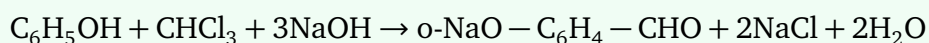
Monochlorination of dichloromethane (CH_2Cl_2) under ultraviolet light:



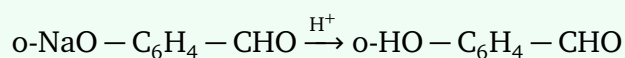
Thus, intermediate Y is **chloroform** (CHCl_3).

Step 3: Determine the product P of the reaction between X and Y

Phenol (X) reacts with chloroform (Y) in the presence of aqueous NaOH (Reimer-Tiemann reaction):



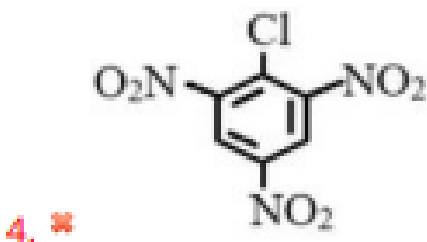
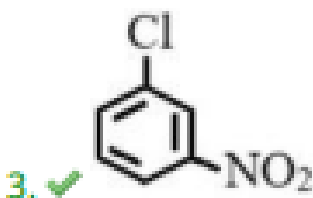
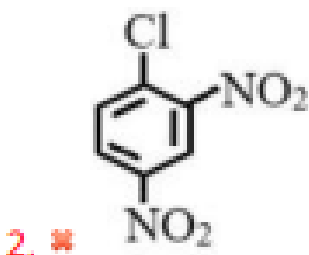
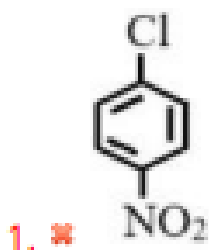
Subsequent acidification yields:



The major product P is salicylaldehyde (**o-hydroxybenzaldehyde**), stabilized by strong intramolecular hydrogen bonding between the phenolic $-\text{OH}$ and the formyl oxygen.

Quick Tip: Phenol + Chloroform + NaOH is the standard Reimer-Tiemann reaction yielding salicylaldehyde (o-hydroxybenzaldehyde) as the major product.

156. The halogen compound which is least reactive towards nucleophilic substitution reactions is



- (A) 4-chloronitrobenzene
(B) 2,4-dinitrochlorobenzene
(C) 3-chloronitrobenzene
(D) 2,4,6-trinitrochlorobenzene

Correct Answer: (C) 3-chloronitrobenzene

Solution:

Concept:

- Nucleophilic aromatic substitution (S_NAr) on aryl halides proceeds through an addition-elimination mechanism involving a resonance-stabilized carbanionic intermediate (Meisenheimer complex).
- Strong electron-withdrawing groups (like $-\text{NO}_2$) facilitate nucleophilic substitution by withdrawing electron density from the aromatic ring.
- The negative charge in the Meisenheimer intermediate is delocalized exclusively to the ortho and para positions relative to the leaving group:
 - Nitro groups at **ortho** and **para** positions stabilize the carbanion directly via powerful resonance ($-M$) and inductive ($-I$) effects.
 - A nitro group at the **meta** position stabilizes the intermediate only through its weaker inductive ($-I$) effect, with no resonance stabilization.

Step 1: Compare the activation by nitro groups at different positions

The rate of S_NAr increases with the number of nitro groups located at ortho and para positions:

- 2,4,6-trinitrochlorobenzene (Option D): Three $-\text{NO}_2$ groups at ortho and para positions. Extremely reactive (hydrolyzes in warm water alone).
- 2,4-dinitrochlorobenzene (Option B): Two $-\text{NO}_2$ groups at ortho and para positions. Highly reactive (hydrolyzes with warm aqueous Na_2CO_3).
- 4-chloronitrobenzene (Option A): One $-\text{NO}_2$ group at the para position. Substantially activated towards substitution compared to chlorobenzene.

Step 2: Analyze 3-chloronitrobenzene (Option C)

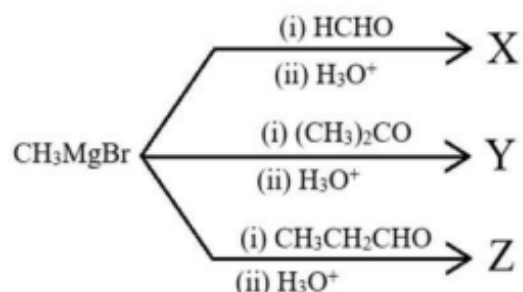
In 3-chloronitrobenzene (m-chloronitrobenzene), the $-\text{NO}_2$ group is at the meta position relative to the chlorine atom. During nucleophilic attack, none of the resonating structures of the Meisenheimer intermediate place the negative charge on the ring carbon bearing the meta-nitro group: The strong $-M$ resonance effect of the nitro group is completely ineffective at the meta position. Consequently, activation towards nucleophilic attack is minimal.

Step 3: Conclusion

Among the given nitrochlorobenzenes, 3-chloronitrobenzene is the least reactive towards nucleophilic substitution.

Quick Tip: The $-\text{NO}_2$ group activates aryl halides towards $\text{S}_{\text{N}}\text{Ar}$ only from ortho and para positions. At the meta position, resonance stabilization of the carbanion is absent, making it the least reactive.

157. The order of reactivity of X, Y and Z towards the Lucas reagent is



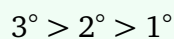
- (A) $\text{Y} > \text{X} > \text{Z}$
(B) $\text{Y} > \text{Z} > \text{X}$
(C) $\text{X} > \text{Y} > \text{Z}$
(D) $\text{Z} > \text{X} > \text{Y}$

Correct Answer: (B) $\text{Y} > \text{Z} > \text{X}$

Solution:

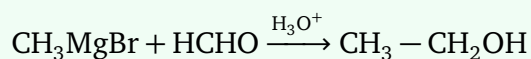
Concept:

- Addition of a Grignard reagent (RMgX) to carbonyl compounds yields alcohols upon hydrolysis:
 - Formaldehyde (HCHO) yields a primary (1°) alcohol.
 - Higher aldehydes ($\text{R}'\text{CHO}$) yield secondary (2°) alcohols.
 - Ketones ($\text{R}'_2\text{CO}$) yield tertiary (3°) alcohols.
- Lucas reagent consists of concentrated HCl and anhydrous ZnCl_2 .
- The reaction proceeds through a carbocation intermediate via an $\text{S}_{\text{N}}1$ mechanism.
- Reactivity of alcohols towards Lucas reagent follows carbocation stability:



Step 1: Identify alcohol X

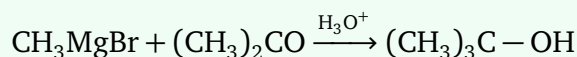
Reaction of methylmagnesium bromide with formaldehyde:



Product X is ethanol, which is a **primary (1°) alcohol**.

Step 2: Identify alcohol Y

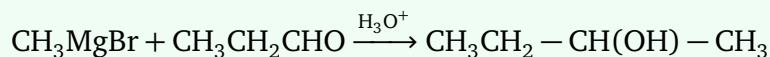
Reaction of methylmagnesium bromide with acetone:



Product Y is 2-methylpropan-2-ol (tert-butanol), which is a **tertiary (3°) alcohol**.

Step 3: Identify alcohol Z

Reaction of methylmagnesium bromide with propanal:



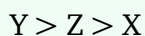
Product Z is butan-2-ol, which is a **secondary (2°) alcohol**.

Step 4: Determine the reactivity towards Lucas reagent

The reaction with Lucas reagent involves carbocation formation:

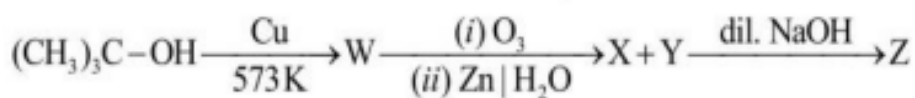
- 3° alcohol (Y) reacts immediately at room temperature (turbidity appears instantly).
- 2° alcohol (Z) reacts in about 5 minutes.
- 1° alcohol (X) does not react at room temperature (turbidity appears only on heating).

Therefore, the order of reactivity is:



Quick Tip: Grignard addition: Formaldehyde gives 1° (X), higher aldehyde gives 2° (Z), ketone gives 3° (Y). Reactivity with Lucas reagent is always 3° > 2° > 1°, so Y > Z > X.

158. The IUPAC name of the end product Z is



- (A) 4-Hydroxybutan-2-one
(B) 3-Hydroxybutan-2-one
(C) 4-Hydroxybutanal
(D) But-3-en-2-one

Correct Answer: (A) 4-Hydroxybutan-2-one

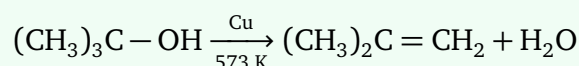
Solution:

Concept:

- Tertiary alcohols heated with copper at 573 K do not undergo dehydrogenation; instead, they undergo dehydration to form alkenes.
- Reductive ozonolysis (O_3 , $\text{Zn}/\text{H}_2\text{O}$) cleaves the carbon-carbon double bond to form carbonyl compounds.
- A crossed aldol condensation between formaldehyde (having no α -hydrogen) and acetone (possessing α -hydrogens) yields a β -hydroxy ketone.

Step 1: Identify intermediate W

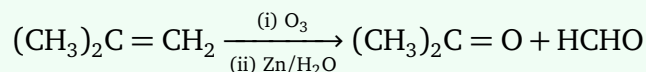
When tert-butyl alcohol ($(\text{CH}_3)_3\text{C}-\text{OH}$) is passed over heated copper at 573 K, dehydration occurs:



Intermediate W is 2-methylpropene (isobutylene).

Step 2: Identify compounds X and Y from ozonolysis

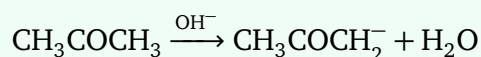
Ozonolysis of 2-methylpropene followed by reduction with zinc dust and water:



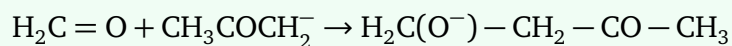
The products are acetone (propan-2-one) and formaldehyde (methanal).

Step 3: Perform crossed aldol reaction to obtain Z

In the presence of dilute NaOH, formaldehyde acts exclusively as an electrophilic acceptor since it lacks α -hydrogens. Acetone forms the nucleophilic enolate ion:



The enolate attacks formaldehyde:



Protonation gives the aldol product:



Product Z is $\text{HO} - \text{CH}_2 - \text{CH}_2 - \text{CO} - \text{CH}_3$.

Step 4: Assign the IUPAC name to product Z

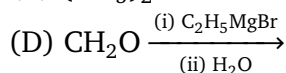
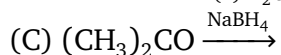
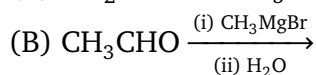
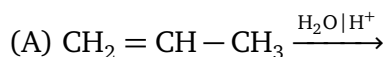
Number the 4-carbon chain giving priority to the ketone functional group:

- C1: $-\text{CH}_3$
- C2: $-\text{CO}-$ (carbonyl group at position 2)
- C3: $-\text{CH}_2-$
- C4: $-\text{CH}_2\text{OH}$ (hydroxyl group at position 4)

The IUPAC name is **4-hydroxybutan-2-one**.

Quick Tip: 3° alcohol with $\text{Cu}/573\text{ K}$ dehydrates to isobutylene $(\text{CH}_3)_2\text{C}=\text{CH}_2$. Ozonolysis gives $\text{HCHO} + \text{CH}_3\text{COCH}_3$. Crossed aldol between them gives $\text{HO} - \text{CH}_2\text{CH}_2\text{COCH}_3$ (4-hydroxybutan-2-one).

159. Identify the reaction, which does not give isopropyl alcohol

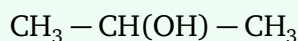


Correct Answer: (D) $\text{CH}_2\text{O} \xrightarrow[\text{(ii) H}_2\text{O}]{\text{(i) C}_2\text{H}_5\text{MgBr}}$

Solution:

Concept:

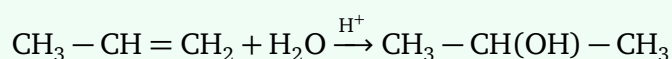
- Isopropyl alcohol (propan-2-ol) is a secondary alcohol with structure:



- Acid-catalyzed hydration of propene follows Markovnikov's rule to yield propan-2-ol.
- Reaction of Grignard reagents with acetaldehyde yields secondary alcohols.
- Reduction of ketones using metal hydrides (such as NaBH_4) yields secondary alcohols.
- Reaction of Grignard reagents with formaldehyde (HCHO) always yields primary alcohols.

Step 1: Analyze Option (A)

Acid-catalyzed hydration of propene:



The proton adds to the terminal carbon to form a stable 2° carbocation, followed by water addition to yield isopropyl alcohol.

Step 2: Analyze Option (B)

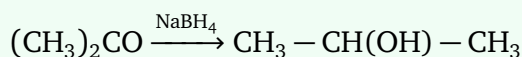
Nucleophilic addition of methylmagnesium bromide to acetaldehyde:



This yields isopropyl alcohol.

Step 3: Analyze Option (C)

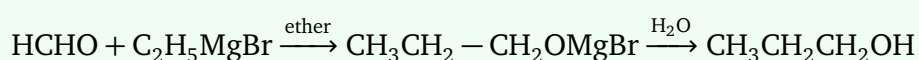
Reduction of acetone (propan-2-one) with sodium borohydride:



Hydride attack on the carbonyl carbon followed by protonation gives isopropyl alcohol.

Step 4: Analyze Option (D)

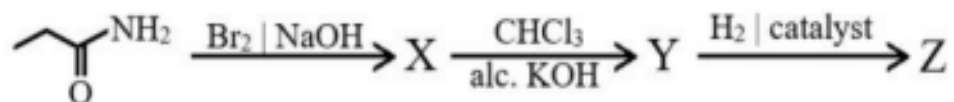
Addition of ethylmagnesium bromide to formaldehyde:



The product formed is propan-1-ol (n-propyl alcohol), which is a primary alcohol, not isopropyl alcohol.

Quick Tip: Formaldehyde + Grignard reagent always yields a primary (1°) alcohol. $\text{HCHO} + \text{C}_2\text{H}_5\text{MgBr}$ gives 1-propanol, never isopropyl alcohol.

160. What is the end product Z of the following reaction sequence?



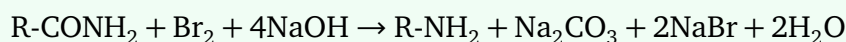
- (A) $\text{CH}_3 - \text{NH}_2$
(B) $\text{CH}_3\text{CH}_2\text{NHCH}_3$
(C) $(\text{CH}_3)_3\text{N}$
(D) $\text{CH}_3 - \text{NH} - \text{OH}$

Correct Answer: (B) $\text{CH}_3\text{CH}_2\text{NHCH}_3$

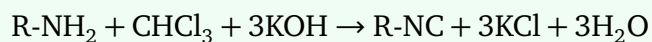
Solution:

Concept:

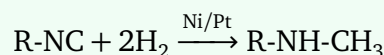
- **Hoffmann Bromamide Degradation:** Primary acid amides react with bromine and alkali to yield primary amines with one less carbon atom:



- **Carbylamine Reaction:** Primary aliphatic and aromatic amines react with chloroform and alcoholic KOH to produce foul-smelling isocyanides (carbylamines):

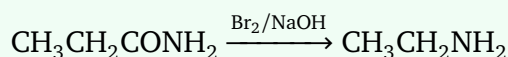


- **Reduction of Isocyanides:** Catalytic hydrogenation of isocyanides yields secondary methylamines:



Step 1: Identify intermediate X

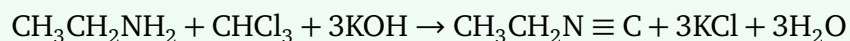
The starting material is propanamide ($\text{CH}_3\text{CH}_2\text{CONH}_2$). Reaction with Br_2 and aqueous NaOH causes degradation by loss of the carbonyl group:



Product X is ethanamine (ethylamine), a primary amine with 2 carbon atoms.

Step 2: Identify intermediate Y

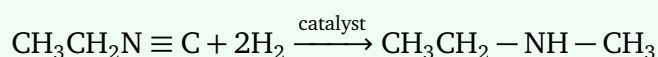
Ethylamine (X) undergoes the carbylamine test when heated with chloroform and alcoholic KOH:



Product Y is ethyl isocyanide (ethyl carbylamine).

Step 3: Identify final product Z

Catalytic reduction of ethyl isocyanide with hydrogen gas in the presence of a metal catalyst:



Product Z is N-methylethanamine (ethylmethanamine), a secondary amine.

Quick Tip: Sequential reaction path: Amide $\xrightarrow{\text{Hoffmann}}$ Primary amine ($-\text{NH}_2$) $\xrightarrow{\text{Carbylamine}}$ Isocyanide ($-\text{NC}$) $\xrightarrow{\text{Reduction}}$ Secondary methylamine ($-\text{NH} - \text{CH}_3$).