UP Board Class 12 Mathematics - 324(IX) 2025 Question Paper with Solutions

Time Allowed :3 Hours	Maximum Marks :100	Total Questions :9

General Instructions

Read the following instructions very carefully and strictly follow them:

1. There are in all nine questions in this question paper.
2. All questions are compulsory.
3. In the beginning of each question, the number of parts to be attempted are clearly mentioned.
4. Marks allotted to the questions are indicated against them.
5. Start solving from the first question and proceed to solve till the last one. Do not waste your time over a question you cannot solve.

1. Do all parts.
Select the correct option of each part and write it on your answer-book.

(a) The modulus function f: R → R^+ given by f(x) = —x— is

(A) one-one and onto
(B) many-one and onto
(C) one-one but not onto
(D) neither one-one nor onto

Correct Answer: (B) many-one and onto

Solution:

Step 1: Understanding the Concept:
We need to determine if the function $f(x) = |x|$ with domain $\mathbb{R}$ (all real numbers) and codomain $\mathbb{R}^+$ (non-negative real numbers, i.e., $[0, \infty)$) is one-one (injective) and/or onto (surjective).
- A function is **one-one (injective)** if every distinct element in the domain maps to a distinct element in the codomain. That is, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.
- A function is **onto (surjective)** if its range is equal to its codomain. That is, for every element y in the codomain, there exists at least one element x in the domain such that $f(x) = y$.

Step 2: Detailed Explanation:
Checking for one-one property:
Let's take two distinct elements from the domain $\mathbb{R}$, for example, $x_1 = -2$ and $x_2 = 2$.

Now, let's find their images under f:

$$f(x_1) = f(-2) = |-2| = 2$$
$$f(x_2) = f(2) = |2| = 2$$

Here, we have $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.
Since two different inputs (-2 and 2) have the same output (2), the function is not one-one. It is a **many-one** function.

Checking for onto property:
The codomain is given as $\mathbb{R}^+$, which represents the set of non-negative real numbers $[0, \infty)$.
The range of the function $f(x) = |x|$ is the set of all possible output values. Since the absolute value of any real number is always non-negative, the range of $f(x)$ is also $[0, \infty)$.
Since Range = Codomain ($[0, \infty) = [0, \infty)$), the function is **onto**.

Step 3: Final Answer:
The function is many-one and onto.

Quick Tip

To quickly determine if a function is one-one, use the Horizontal Line Test on its graph. The graph of $y = |x|$ is a 'V' shape with its vertex at the origin. Any horizontal line $y = c$ (where $c > 0$) will intersect the graph at two points. This confirms the function is many-one.

(b) A relation R = {(a, b) : a = b - 1, b $\geq$ 3} is defined on set N, then

(A) (2, 4) $\in$ R
(B) (4, 5) $\in$ R
(C) (4, 6) $\in$ R
(D) (1, 3) $\in$ R

Correct Answer: (B) (4, 5) $\in$ R

Solution:

Step 1: Understanding the Concept:
The problem defines a relation R on the set of natural numbers $\mathbb{N}$. An ordered pair (a, b) belongs to this relation R if it satisfies two conditions simultaneously:

1. $a = b - 1$
2. $b \geq 3$

We need to check which of the given options satisfies both these conditions.

Step 2: Detailed Explanation:
Let's check each option against the two conditions.

(A) (2, 4) $\in$ **R**:
Here, $a = 2$ and $b = 4$.

- Condition 2: Is $b \geq 3$? Yes, $4 \geq 3$.
- Condition 1: Is $a = b - 1$? Is $2 = 4 - 1$? No, $2 \neq 3$.

Since one condition is not met, $(2, 4) \notin$ R.

(B) (4, 5) $\in$ **R**:
Here, $a = 4$ and $b = 5$.

- Condition 2: Is $b \geq 3$? Yes, $5 \geq 3$.
- Condition 1: Is $a = b - 1$? Is $4 = 5 - 1$? Yes, $4 = 4$.

Since both conditions are met, $(4, 5) \in$ R.

(C) (4, 6) $\in$ **R**:
Here, $a = 4$ and $b = 6$.

- Condition 2: Is $b \geq 3$? Yes, $6 \geq 3$.
- Condition 1: Is $a = b - 1$? Is $4 = 6 - 1$? No, $4 \neq 5$.

Since one condition is not met, $(4, 6) \notin$ R.

(D) (1, 3) $\in$ **R**:
Here, $a = 1$ and $b = 3$.

- Condition 2: Is $b \geq 3$? Yes, $3 \geq 3$.
- Condition 1: Is $a = b - 1$? Is $1 = 3 - 1$? No, $1 \neq 2$.

Since one condition is not met, $(1, 3) \notin$ R.

Step 3: Final Answer:
The only option that satisfies both conditions for the relation R is (4, 5).

Quick Tip

In questions involving relations defined by multiple conditions, methodically check each condition for every option. Do not stop after finding one condition is true; all conditions must hold for the element to be in the relation.

(c) The value of $\int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}}$ will be

(A) 0
(B) $\frac{\pi}{2}$
(C) $\frac{\pi}{4}$
(D) $\frac{\pi}{8}$

Correct Answer: (C) $\frac{\pi}{4}$

Solution:

Step 1: Understanding the Concept:
This is a standard problem in definite integrals that can be solved efficiently using a property of definite integrals, often known as the "King's Rule".

Step 2: Key Formula or Approach:
The property states that for a continuous function $f(x)$ on $[0, a]$:

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

Step 3: Detailed Calculation:
Let the given integral be I.

$$I = \int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}}$$

First, express $\tan x$ in terms of $\sin x$ and $\cos x$:

$$I = \int_0^{\pi/2} \frac{dx}{1+\frac{\sqrt{\sin x}}{\sqrt{\cos x}}} = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \quad \cdots (1)$$

Now, apply the property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ with $a = \pi/2$.

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos(\frac{\pi}{2}-x)}}{\sqrt{\cos(\frac{\pi}{2}-x)}+\sqrt{\sin(\frac{\pi}{2}-x)}}\,dx$$

Using the trigonometric identities $\cos(\frac{\pi}{2}-x) = \sin x$ and $\sin(\frac{\pi}{2}-x) = \cos x$, we get:

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx \quad \cdots (2)$$

Now, add equation (1) and equation (2):

$$I + I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$2I = \int_0^{\pi/2} \frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

$$2I = \int_0^{\pi/2} 1\,dx$$
$$2I = [x]_0^{\pi/2}$$
$$2I = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$
$$I = \frac{\pi}{4}$$

Step 4: Final Answer:
The value of the integral is $\frac{\pi}{4}$.

Quick Tip

Recognize this standard form. Any integral of the type $\int_0^{\pi/2} \frac{f(x)}{f(x)+f(\pi/2-x)}\,dx$ is equal to $\frac{\pi}{4}$. In this case, if we let $g(x) = \sqrt{\tan x}$, the denominator is $1+g(x)$. This might not seem to fit directly, but after converting to sine and cosine, the form $\int_0^{\pi/2} \frac{h(\cos x)}{h(\cos x)+h(\sin x)}\,dx$ appears, which also evaluates to $\frac{\pi}{4}$.

(d) The degree of differential equation $9\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}$ is

(A) 1
(B) 6
(C) 3
(D) 2

Correct Answer: (D) 2

Solution:

Step 1: Understanding the Concept:
The **order** of a differential equation is the order of the highest derivative appearing in it. The **degree** of a differential equation is the power of the highest order derivative, provided the equation is a polynomial in its derivatives (i.e., it is free from radicals and fractional powers of the derivatives).

Step 2: Key Approach:
To find the degree, we must first eliminate any fractional powers on the derivative terms. This is typically done by raising both sides of the equation to an appropriate integer power.

Step 3: Detailed Calculation:
The given differential equation is:

$$9\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}$$

1. Identify the highest order derivative:
The derivatives present are $\frac{dy}{dx}$ (order 1) and $\frac{d^2y}{dx^2}$ (order 2). The highest order is 2.
2. Eliminate the fractional power:
The equation has a fractional power of $\frac{3}{2}$. To eliminate it, we need to square both sides of the equation.

$$\left(9\frac{d^2y}{dx^2}\right)^2 = \left(\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}\right)^2$$

$$81\left(\frac{d^2y}{dx^2}\right)^2 = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^3$$

3. Determine the degree:
Now that the equation is a polynomial in its derivatives, we find the highest power of the highest order derivative. The highest order derivative is $\frac{d^2y}{dx^2}$, and its power is 2.

Step 4: Final Answer:
The degree of the given differential equation is 2.

Quick Tip

Always make sure to clear any radicals or fractional exponents involving derivatives before determining the degree. A common mistake is to state the degree without first converting the equation into a polynomial form of derivatives. The order can be found by simple inspection, but the degree requires this extra step.

(e) The value of expression $\hat{i} \cdot \hat{i} - \hat{j} \cdot \hat{j} + \hat{k} \times \hat{k}$ is

(A) 0
(B) 1
(C) 2
(D) 3

Correct Answer: (B) 1

Solution:

Step 1: Understanding the Concept:
This expression involves the dot product $(\cdot)$ and cross product $(\times)$ of the standard orthogonal unit vectors $\hat{i}, \hat{j}, \hat{k}$. A key issue is that the expression as written attempts to add scalars ($\hat{i} \cdot \hat{i}$ and $\hat{j} \cdot \hat{j}$) and a vector ($\hat{k} \times \hat{k}$), which is mathematically undefined. This strongly implies a typographical error in the original question paper.

Step 2: Key Formula or Approach:
The standard properties of unit vectors are:

- Dot product of a unit vector with itself: $\hat{i} \cdot \hat{i} = 1$, $\hat{j} \cdot \hat{j} = 1$, $\hat{k} \cdot \hat{k} = 1$.
- Cross product of any vector with itself is the zero vector: $\vec{A} \times \vec{A} = \vec{0}$. Therefore, $\hat{k} \times \hat{k} = \vec{0}$.

Given the invalidity of adding a scalar to a vector, we must assume the most likely typo. The pattern of the first two terms suggests the third term was also intended to be a dot product. Thus, we will solve the problem by assuming the expression is $\hat{i} \cdot \hat{i} - \hat{j} \cdot \hat{j} + \hat{k} \cdot \hat{k}$.

Step 3: Detailed Calculation:
Assuming the corrected expression is $\hat{i} \cdot \hat{i} - \hat{j} \cdot \hat{j} + \hat{k} \cdot \hat{k}$. Let's evaluate each term based on the properties above:

- $\hat{i} \cdot \hat{i} = |\hat{i}|^2 = 1^2 = 1$
- $\hat{j} \cdot \hat{j} = |\hat{j}|^2 = 1^2 = 1$
- $\hat{k} \cdot \hat{k} = |\hat{k}|^2 = 1^2 = 1$

Now, substitute these values back into the corrected expression:

$$\text{Value} = (\hat{i} \cdot \hat{i}) - (\hat{j} \cdot \hat{j}) + (\hat{k} \cdot \hat{k})$$

$$\text{Value} = 1 - 1 + 1$$

$$\text{Value} = 1$$

Step 4: Final Answer:
The calculated value is 1.

Quick Tip

In vector problems on multiple-choice tests, if an expression seems mathematically invalid (like adding scalars and vectors), check for a likely typo. Often, an operator is incorrect. Look for patterns in the expression to deduce the intended question. Here, the pattern of self-dot-products is the strongest clue.

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2. Do all parts.

(a) Write $\cot^{-1}\left\{\frac{1}{\sqrt{x^2-1}}\right\}; x > 1$ in the simplest form.

Correct Answer: $\sec^{-1} x$

Solution:

Step 1: Understanding the Concept:
The goal is to simplify the given inverse trigonometric expression. This is typically achieved by using a trigonometric substitution that simplifies the term inside the function, allowing us to use the property $\cot^{-1}(\cot\theta) = \theta$. The presence of the term $\sqrt{x^2-1}$ suggests the substitution $x = \sec\theta$.

Step 2: Key Formula or Approach:
We will use the substitution $x = \sec\theta$. This is chosen because of the trigonometric identity $\sec^2\theta - 1 = \tan^2\theta$. The given condition $x > 1$ implies that $\sec\theta > 1$, which means θ lies in the first quadrant, i.e., $0 < \theta < \frac{\pi}{2}$. This is important for determining the sign when taking the square root.

Step 3: Detailed Calculation:
Let the given expression be y.

$$y = \cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right)$$

Substitute $x = \sec\theta$. This implies $\theta = \sec^{-1} x$.

$$y = \cot^{-1}\left(\frac{1}{\sqrt{\sec^2\theta - 1}}\right)$$

Using the identity $\sec^2\theta - 1 = \tan^2\theta$:

$$y = \cot^{-1}\left(\frac{1}{\sqrt{\tan^2\theta}}\right)$$

Since $x > 1$, we have $\theta \in (0, \pi/2)$. In this interval, $\tan\theta$ is positive. Therefore, $\sqrt{\tan^2\theta} = \tan\theta$.

$$y = \cot^{-1}\left(\frac{1}{\tan\theta}\right)$$

We know that $\frac{1}{\tan\theta} = \cot\theta$.

$$y = \cot^{-1}(\cot\theta)$$

Since $\theta \in (0, \pi/2)$, which is within the principal value branch of $\cot^{-1}$, we can write:

$$y = \theta$$

Now, substitute back the value of θ in terms of x.

$$y = \sec^{-1} x$$

Step 4: Final Answer:
The simplest form of the expression is $\sec^{-1} x$.

Quick Tip

Memorize the standard substitutions for simplifying inverse trigonometric functions:

- For $\sqrt{a^2 - x^2}$, use $x = a\sin\theta$ or $x = a\cos\theta$.
- For $a^2 + x^2$ or $\sqrt{a^2 + x^2}$, use $x = a\tan\theta$ or $x = a\cot\theta$.
- For $\sqrt{x^2 - a^2}$, use $x = a\sec\theta$ or $x = a\csc\theta$.

(b) Prove that the function f(x) = —x— is continuous at x = 0.

Solution:

Step 1: Understanding the Concept:
A function $f(x)$ is said to be continuous at a point $x = a$ if three conditions are met:

1. $f(a)$ is defined.
2. The limit of the function as x approaches a exists. This means the Left-Hand Limit (LHL) equals the Right-Hand Limit (RHL).
$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$
3. The value of the limit is equal to the value of the function at that point.
$$\lim_{x\to a} f(x) = f(a)$$

Step 2: Detailed Explanation:
We need to check these three conditions for the function $f(x) = |x|$ at the point $x = 0$.

Condition 1: Value of the function at x = 0
The value of the function at $x = 0$ is:
$$f(0) = |0| = 0$$
The function is defined at $x = 0$.

Condition 2: Existence of the limit at x = 0
We need to calculate the Left-Hand Limit (LHL) and the Right-Hand Limit (RHL).
Left-Hand Limit (LHL):
$$\text{LHL} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} |x|$$
By definition, for $x < 0$, $|x| = -x$. So, we can write:
$$\text{LHL} = \lim_{x\to 0^-} (-x) = -(0) = 0$$
Alternatively, using $h \to 0^+$:
$$\text{LHL} = \lim_{h\to 0^+} f(0-h) = \lim_{h\to 0^+} |0-h| = \lim_{h\to 0^+} |-h| = \lim_{h\to 0^+} h = 0$$

Right-Hand Limit (RHL):

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} |x|$$

By definition, for $x > 0$, $|x| = x$. So, we can write:

$$\text{RHL} = \lim_{x \to 0^+} (x) = 0$$

Alternatively, using $h \to 0^+$:

$$\text{RHL} = \lim_{h \to 0^+} f(0+h) = \lim_{h \to 0^+} |0+h| = \lim_{h \to 0^+} |h| = \lim_{h \to 0^+} h = 0$$

Since LHL = RHL = 0, the limit exists and $\lim_{x \to 0} f(x) = 0$.

Condition 3: Comparing the limit and function value
We have found that:

$$\lim_{x \to 0} f(x) = 0 \quad \text{and} \quad f(0) = 0$$

Therefore, $\lim_{x \to 0} f(x) = f(0)$.

Step 3: Final Answer:
Since all three conditions for continuity are satisfied, the function $f(x) = |x|$ is continuous at $x = 0$.

Quick Tip

For proofs of continuity, always follow the three-step structure: calculate $f(a)$, calculate the LHL and RHL to find $\lim_{x \to a} f(x)$, and finally state that they are equal. This systematic approach ensures no steps are missed.

(c) Find the degree of the differential equation $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\left(\frac{dy}{dx}\right) = 2$

Correct Answer: 1

Solution:

Step 1: Understanding the Concept:
The **order** of a differential equation is the order of the highest derivative present in the equation. The **degree** of a differential equation is the highest power (positive integer exponent) of the highest order derivative, after the equation has been cleared of any radicals or fractional powers of the derivatives.

Step 2: Key Approach:
1. Identify the highest order derivative in the equation.

2. Check if the equation is a polynomial in its derivatives. If not, manipulate it to become one.
3. Find the power of the highest order derivative. This power is the degree.

Step 3: Detailed Explanation:
The given differential equation is:

$$xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\left(\frac{dy}{dx}\right) = 2$$

1. Identify the derivatives and their orders:

- $\frac{dy}{dx}$ is the first-order derivative.
- $\frac{d^2y}{dx^2}$ is the second-order derivative.

The highest order derivative is $\frac{d^2y}{dx^2}$, so the **order** of the equation is 2.

2. Check for polynomial form:
The equation is already expressed as a polynomial in terms of its derivatives $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$. There are no radicals or fractional exponents on the derivative terms.

3. Determine the degree:
We need to find the highest power of the highest order derivative, which is $\frac{d^2y}{dx^2}$. The term containing the highest order derivative is $xy\frac{d^2y}{dx^2}$. The power of $\frac{d^2y}{dx^2}$ in this term is 1.

$$xy\left(\frac{d^2y}{dx^2}\right)^1 + x\left(\frac{dy}{dx}\right)^2 - y\left(\frac{dy}{dx}\right)^1 = 2$$

The highest power of $\frac{d^2y}{dx^2}$ is 1.

Step 4: Final Answer:
The degree of the differential equation is 1.

Quick Tip

The degree is determined by the power of the highest order derivative only. Do not get confused by higher powers on lower order derivatives, such as the power of 2 on $\frac{dy}{dx}$ in this problem. The degree is 1, not 2.

(d) If $\mathbf{P(A) = 0.12}$, $\mathbf{P(B) = 0.15}$ and $\mathbf{P(B/A) = 0.18}$, then find the value of $\mathbf{P(A \cap B)}$.

Correct Answer: 0.0216

Solution:

Step 1: Understanding the Concept:
This problem involves conditional probability. The notation $P(B|A)$ represents the probability of event B occurring given that event A has already occurred. The relationship between conditional probability, joint probability, and individual probability is defined by the multiplication rule of probability.

Step 2: Key Formula or Approach:
The formula for the conditional probability of B given A is:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

To find the probability of the intersection of A and B, $P(A \cap B)$, we can rearrange this formula:

$$P(A \cap B) = P(B|A) \times P(A)$$

Step 3: Detailed Calculation:
We are given the following values:

- $P(A) = 0.12$
- $P(B) = 0.15$
- $P(B|A) = 0.18$

Using the rearranged formula:

$$P(A \cap B) = P(B|A) \times P(A)$$

Substitute the given values into the formula:

$$P(A \cap B) = 0.18 \times 0.12$$

Now, we perform the multiplication:

$$18 \times 12 = 216$$

Since there are a total of four decimal places in 0.18 and 0.12, the result will have four decimal places.

$$P(A \cap B) = 0.0216$$

Step 4: Final Answer:
The value of $P(A \cap B)$ is 0.0216.

Quick Tip

In conditional probability problems, identify what information is given and what is required. Notice that the value of $P(B)$ was extra information not needed to solve for $P(A \cap B)$ using $P(B|A)$. Always select the correct formula based on the given conditional probability.

(e) Find the angle between the vectors $-2\hat{i} + \hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$.

Correct Answer: $\theta = \cos^{-1}\left(-\frac{5}{14}\right)$

Solution:

Step 1: Understanding the Concept:
The angle θ between two non-zero vectors $\vec{a}$ and $\vec{b}$ can be found using the scalar (dot) product formula. The dot product relates the angle between the vectors to the product of their magnitudes.

Step 2: Key Formula or Approach:
The dot product of two vectors $\vec{a}$ and $\vec{b}$ is defined as:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

From this, we can find the angle θ using:

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

For vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$:

- $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$
- $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$
- $|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

Step 3: Detailed Calculation:
Let the given vectors be:

$$\vec{a} = -2\hat{i} + \hat{j} + 3\hat{k}$$
$$\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$$

1. Calculate the dot product $\vec{a} \cdot \vec{b}$:

$$\vec{a} \cdot \vec{b} = (-2)(3) + (1)(-2) + (3)(1)$$
$$\vec{a} \cdot \vec{b} = -6 - 2 + 3 = -5$$

2. Calculate the magnitude of $\vec{a}$:

$$|\vec{a}| = \sqrt{(-2)^2 + (1)^2 + (3)^2}$$
$$|\vec{a}| = \sqrt{4 + 1 + 9} = \sqrt{14}$$

3. Calculate the magnitude of $\vec{b}$:

$$|\vec{b}| = \sqrt{(3)^2 + (-2)^2 + (1)^2}$$
$$|\vec{b}| = \sqrt{9 + 4 + 1} = \sqrt{14}$$

4. Calculate $\cos\theta$:

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{-5}{\sqrt{14}\sqrt{14}}$$

$$\cos\theta = -\frac{5}{14}$$

5. Find the angle θ:

$$\theta = \cos^{-1}\left(-\frac{5}{14}\right)$$

Step 4: Final Answer:
The angle between the two vectors is $\cos^{-1}\left(-\frac{5}{14}\right)$.

Quick Tip

When the dot product $\vec{a}\cdot\vec{b}$ is negative, it implies that the angle θ between the vectors is obtuse ($90^\circ < \theta \leq 180^\circ$). If the dot product is positive, the angle is acute. If it is zero, the vectors are orthogonal (perpendicular). This can be a quick check on your answer.

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3. Do all parts.

(a) If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions defined by $f(x) = \cos x$ and $g(x) = 3x^2$ respectively, then prove that $gof \neq fog$.

Solution:

Step 1: Understanding the Concept:
This question requires an understanding of composite functions.
A composite function is created when one function is applied to the result of another function.
We need to find the expressions for gof (g-compose-f) and fog (f-compose-g) and then show that they are not identical.

Step 2: Key Formula or Approach:
The definitions of composite functions are:
1. $(gof)(x) = g(f(x))$
2. $(fog)(x) = f(g(x))$
To prove they are not equal, we need to show that $g(f(x)) \neq f(g(x))$ for at least one value of x, or that their general expressions are different.

Step 3: Detailed Explanation or Calculation:
First, let's find the expression for $(gof)(x)$.
By definition, $(gof)(x) = g(f(x))$.
We are given $f(x) = \cos x$.
So, we substitute $f(x)$ into $g(x)$:

$$(gof)(x) = g(\cos x)$$

The function $g(x)$ is defined as $g(x) = 3x^2$. To find $g(\cos x)$, we replace x with $\cos x$:

$$(gof)(x) = 3(\cos x)^2 = 3\cos^2 x$$

Next, let's find the expression for $(fog)(x)$.
By definition, $(fog)(x) = f(g(x))$.
We are given $g(x) = 3x^2$.
So, we substitute $g(x)$ into $f(x)$:

$$(fog)(x) = f(3x^2)$$

The function $f(x)$ is defined as $f(x) = \cos x$. To find $f(3x^2)$, we replace x with $3x^2$:

$$(fog)(x) = \cos(3x^2)$$

Finally, let's compare $(gof)(x)$ and $(fog)(x)$.
We have $(gof)(x) = 3\cos^2 x$ and $(fog)(x) = \cos(3x^2)$.
Clearly, the expressions are different. For example, let's test for $x = 0$:
$(gof)(0) = 3\cos^2(0) = 3(1)^2 = 3$.
$(fog)(0) = \cos(3 \cdot 0^2) = \cos(0) = 1$.
Since $3 \neq 1$, we have shown that $(gof)(x) \neq (fog)(x)$.

Step 4: Final Answer:
Since $3\cos^2 x \neq \cos(3x^2)$ for all $x \in \mathbb{R}$, it is proven that $gof \neq fog$.

Quick Tip

Composition of functions is generally not commutative, meaning the order in which you apply the functions matters. Always calculate $g(f(x))$ and $f(g(x))$ separately to check for equality. A single counterexample (testing a specific value of x) is sufficient to prove they are not equal.

(b) Find the general solution of differential equation $ydx + (x - y^2)dy = 0$.

Correct Answer: The general solution is $xy - \frac{y^3}{3} = C$ or $x = \frac{y^2}{3} + \frac{C}{y}$.

Solution:

Step 1: Understanding the Concept:
This is a first-order differential equation. We can check if it's an exact differential equation or rearrange it into a linear differential equation form.

Step 2: Key Formula or Approach:
The given equation can be rewritten in the form of a linear differential equation: $\frac{dx}{dy} + P(y)x =$

$Q(y)$.
The steps to solve this are:
1. Rearrange the equation into the standard linear form.
2. Identify $P(y)$ and $Q(y)$.
3. Calculate the Integrating Factor (I.F.) using the formula: $I.F. = e^{\int P(y)dy}$.
4. The general solution is given by: $x \cdot (I.F.) = \int Q(y) \cdot (I.F.)dy + C$.

Step 3: Detailed Explanation or Calculation:
The given differential equation is:

$$ydx + (x - y^2)dy = 0$$

Step 3.1: Rearrange the equation.
Divide the entire equation by dy:

$$y\frac{dx}{dy} + x - y^2 = 0$$

Move the $-y^2$ term to the right side:

$$y\frac{dx}{dy} + x = y^2$$

Divide by y to get the standard linear form $\frac{dx}{dy} + P(y)x = Q(y)$:

$$\frac{dx}{dy} + \frac{1}{y}x = y$$

Step 3.2: Identify $P(y)$ and $Q(y)$.
Comparing with the standard form, we have:
$P(y) = \frac{1}{y}$ and $Q(y) = y$.

Step 3.3: Calculate the Integrating Factor (I.F.).

$$I.F. = e^{\int P(y)dy} = e^{\int \frac{1}{y}dy}$$
$$I.F. = e^{\ln|y|} = |y|$$

Assuming $y > 0$, we can take $I.F. = y$.

Step 3.4: Find the general solution.
The solution is given by $x \cdot (I.F.) = \int Q(y) \cdot (I.F.)dy + C$.
Substitute the values of $I.F.$ and $Q(y)$:

$$x \cdot y = \int y \cdot y \, dy + C$$
$$xy = \int y^2 \, dy + C$$
$$xy = \frac{y^3}{3} + C$$

This is the general solution. We can also express x in terms of y:

$$x = \frac{y^2}{3} + \frac{C}{y}$$

Step 4: Final Answer:
The general solution of the given differential equation is $xy = \frac{y^3}{3} + C$.

Quick Tip

When a differential equation is in the form $M(x, y)dx + N(x, y)dy = 0$, first check if it's exact by testing if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. If it is, solving is straightforward. If not, try rearranging it into a linear form like $\frac{dy}{dx} + P(x)y = Q(x)$ or $\frac{dx}{dy} + P(y)x = Q(y)$. Recognizing the form is the key to solving first-order differential equations quickly.

(c) Prove that (4, 4, 2), (3, 5, 2) and (-1, -1, 2) are vertices of a right angle triangle.

Solution:

Step 1: Understanding the Concept:
A triangle is a right-angled triangle if two of its sides are perpendicular to each other. In vector terms, this means the dot product of the vectors representing those two sides is zero. Alternatively, we can use the distance formula to find the lengths of the three sides and check if they satisfy the Pythagorean theorem ($a^2 + b^2 = c^2$).

Step 2: Key Formula or Approach:
Let the given vertices be A(4, 4, 2), B(3, 5, 2), and C(-1, -1, 2).
1. Find the vectors representing the sides of the triangle: $\vec{AB}$, $\vec{BC}$, and $\vec{AC}$.
- $\vec{AB} = B - A = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$
2. Calculate the dot product of pairs of these vectors. If any dot product is zero, the corresponding vectors are perpendicular.
- $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$
3. If a dot product is zero, the triangle is right-angled.

Step 3: Detailed Explanation or Calculation:
Step 3.1: Define the vertices.
Let A = (4, 4, 2), B = (3, 5, 2), C = (-1, -1, 2).

Step 3.2: Find the side vectors.

$$\vec{AB} = B - A = (3 - 4, 5 - 4, 2 - 2) = (-1, 1, 0)$$

$$\vec{BC} = C - B = (-1-3, -1-5, 2-2) = (-4, -6, 0)$$
$$\vec{AC} = C - A = (-1-4, -1-4, 2-2) = (-5, -5, 0)$$

Step 3.3: Calculate the dot products.
Let's check if any pair of vectors is perpendicular.
Dot product of $\vec{AB}$ and $\vec{AC}$:

$$\vec{AB} \cdot \vec{AC} = (-1)(-5) + (1)(-5) + (0)(0)$$
$$= 5 - 5 + 0 = 0$$

Since the dot product of $\vec{AB}$ and $\vec{AC}$ is zero, the vectors are perpendicular. This means the angle between side AB and side AC is 90 degrees.

Alternative Method (Using Pythagorean Theorem):
Calculate the square of the lengths of the sides.

$$|\vec{AB}|^2 = (-1)^2 + 1^2 + 0^2 = 1 + 1 = 2$$
$$|\vec{BC}|^2 = (-4)^2 + (-6)^2 + 0^2 = 16 + 36 = 52$$
$$|\vec{AC}|^2 = (-5)^2 + (-5)^2 + 0^2 = 25 + 25 = 50$$

Check if the sum of the squares of the two smaller sides equals the square of the largest side:

$$|\vec{AB}|^2 + |\vec{AC}|^2 = 2 + 50 = 52$$

This is equal to $|\vec{BC}|^2$.
Since $|\vec{AB}|^2 + |\vec{AC}|^2 = |\vec{BC}|^2$, the Pythagorean theorem holds.

Step 4: Final Answer:
Both methods confirm that the triangle formed by the vertices A, B, and C is a right-angled triangle, with the right angle at vertex A.

Quick Tip

For problems involving proving geometric properties like right angles, using the dot product of vectors is often faster and less calculation-intensive than using the distance formula and the Pythagorean theorem. A dot product of zero immediately confirms perpendicularity.

(d) If $\begin{bmatrix} x+y & 2 \\ 5+z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$, then find the values of x, y, z.

Correct Answer: The solutions are $(x = 2, y = 4, z = 0)$ or $(x = 4, y = 2, z = 0)$.

Solution:

Step 1: Understanding the Concept:
This problem is based on the principle of equality of matrices. Two matrices are equal if and only if they have the same dimensions and their corresponding elements are equal.

Step 2: Key Formula or Approach:
Given the matrix equation $A = B$, where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, the equality implies:
- $a_{11} = b_{11}$
- $a_{12} = b_{12}$
- $a_{21} = b_{21}$
- $a_{22} = b_{22}$

We will set up a system of equations based on this principle and solve for the variables x, y, and z.

Step 3: Detailed Explanation or Calculation:
The given matrix equation is:
$$\begin{bmatrix} x + y & 2 \\ 5 + z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$$
By equating the corresponding elements, we get the following system of equations:
1. $x + y = 6$
2. $2 = 2$ (This is consistent and provides no new information)
3. $5 + z = 5$
4. $xy = 8$

Solve for z:
From equation (3):
$$5 + z = 5$$
$$z = 5 - 5$$
$$z = 0$$
Solve for x and y:
We have a system of two equations with two variables:
(1) $x + y = 6$
(4) $xy = 8$
From equation (1), express y in terms of x:
$$y = 6 - x$$
Substitute this expression for y into equation (4):
$$x(6 - x) = 8$$
$$6x - x^2 = 8$$
Rearrange into a standard quadratic equation form $ax^2 + bx + c = 0$:
$$x^2 - 6x + 8 = 0$$

Factor the quadratic equation:

$$(x-4)(x-2) = 0$$

This gives two possible values for x:
$x = 4$ or $x = 2$.

Now find the corresponding values for y using $y = 6 - x$:
Case 1: If $x = 4$

$$y = 6 - 4 = 2$$

Case 2: If $x = 2$

$$y = 6 - 2 = 4$$

So, we have two possible sets of solutions for (x, y).

Step 4: Final Answer:
The values of the variables are:
- $z = 0$
- Either $x = 4, y = 2$ or $x = 2, y = 4$.
The complete solutions are $(x = 4, y = 2, z = 0)$ and $(x = 2, y = 4, z = 0)$.

> Quick Tip
>
> When solving systems of equations derived from matrices, be systematic. Solve for the simplest variables first (like 'z' in this case). For a system like $x + y = a$ and $xy = b$, you are essentially looking for two numbers whose sum is 'a' and product is 'b'. This corresponds to finding the roots of the quadratic equation $t^2 - at + b = 0$.

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4. Do all parts.

a. Show that the function $f(x) = 7x^2 - 3$ is an increasing function when $x > 0$.

Solution:

Step 1: Understanding the Concept:
A function is considered increasing on an interval if its first derivative is positive on that interval. So, to show that $f(x)$ is increasing for $x > 0$, we need to find its derivative, $f'(x)$, and show that $f'(x) > 0$ for all $x > 0$.

Step 2: Key Formula or Approach:
The condition for an increasing function is:

$$f'(x) > 0$$

The power rule for differentiation states that $\frac{d}{dx}(x^n) = nx^{n-1}$.

Step 3: Detailed Explanation or Calculation:
First, we find the function given:

$$f(x) = 7x^2 - 3$$

Next, we differentiate $f(x)$ with respect to x to find $f'(x)$:

$$f'(x) = \frac{d}{dx}(7x^2 - 3)$$
$$f'(x) = 7 \cdot \frac{d}{dx}(x^2) - \frac{d}{dx}(3)$$
$$f'(x) = 7(2x) - 0$$
$$f'(x) = 14x$$

Now, we need to check the condition for the function to be increasing, which is $f'(x) > 0$.
We are given the condition that $x > 0$.
If $x > 0$, then multiplying by a positive constant (14) will not change the inequality sign.

$$14x > 14(0)$$
$$14x > 0$$

So, $f'(x) > 0$ for all $x > 0$.

Step 4: Final Answer:
Since the first derivative $f'(x) = 14x$ is positive for all $x > 0$, the function $f(x) = 7x^2 - 3$ is an increasing function when $x > 0$.

Quick Tip

To determine if a function is increasing or decreasing, always check the sign of its first derivative. A positive derivative means the function is increasing, a negative derivative means it's decreasing, and a zero derivative indicates a potential local maximum, minimum, or point of inflection.

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b. Find the unit vector perpendicular to each of the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ where $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.

Solution:

Step 1: Understanding the Concept:
A vector perpendicular to two given vectors can be found using their cross product. A unit vector is a vector with a magnitude of 1. The process involves first finding the two vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$, then finding their cross product, and finally normalizing the resulting vector to get the unit vector.

Step 2: Key Formula or Approach:
Let $\vec{u} = \vec{a} + \vec{b}$ and $\vec{v} = \vec{a} - \vec{b}$.
The vector perpendicular to both $\vec{u}$ and $\vec{v}$ is given by their cross product, $\vec{c} = \vec{u} \times \vec{v}$.
The unit vector in the direction of $\vec{c}$ is $\hat{c} = \frac{\vec{c}}{|\vec{c}|}$.

Step 3: Detailed Explanation or Calculation:
First, calculate $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$.
Given: $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.

$$\vec{a} + \vec{b} = (\hat{i} + \hat{j} + \hat{k}) + (\hat{i} + 2\hat{j} + 3\hat{k}) = (1+1)\hat{i} + (1+2)\hat{j} + (1+3)\hat{k} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$
$$\vec{a} - \vec{b} = (\hat{i} + \hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = (1-1)\hat{i} + (1-2)\hat{j} + (1-3)\hat{k} = 0\hat{i} - \hat{j} - 2\hat{k}$$

Let $\vec{u} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{v} = -\hat{j} - 2\hat{k}$.
Now, find the cross product $\vec{c} = \vec{u} \times \vec{v}$.

$$\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix}$$
$$\vec{c} = \hat{i}((3)(-2) - (4)(-1)) - \hat{j}((2)(-2) - (4)(0)) + \hat{k}((2)(-1) - (3)(0))$$
$$\vec{c} = \hat{i}(-6 - (-4)) - \hat{j}(-4 - 0) + \hat{k}(-2 - 0)$$
$$\vec{c} = \hat{i}(-2) - \hat{j}(-4) + \hat{k}(-2) = -2\hat{i} + 4\hat{j} - 2\hat{k}$$

Next, find the magnitude of $\vec{c}$.

$$|\vec{c}| = \sqrt{(-2)^2 + (4)^2 + (-2)^2} = \sqrt{4 + 16 + 4} = \sqrt{24} = 2\sqrt{6}$$

Finally, find the unit vector $\hat{c}$.

$$\hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{-2\hat{i} + 4\hat{j} - 2\hat{k}}{2\sqrt{6}} = \frac{-1}{\sqrt{6}}\hat{i} + \frac{2}{\sqrt{6}}\hat{j} - \frac{1}{\sqrt{6}}\hat{k}$$

Step 4: Final Answer:
The unit vector perpendicular to both $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ is $\frac{-1}{\sqrt{6}}\hat{i} + \frac{2}{\sqrt{6}}\hat{j} - \frac{1}{\sqrt{6}}\hat{k}$. Note that the opposite vector $\frac{1}{\sqrt{6}}\hat{i} - \frac{2}{\sqrt{6}}\hat{j} + \frac{1}{\sqrt{6}}\hat{k}$ is also a valid answer.

Quick Tip

Remember that the cross product $\vec{u} \times \vec{v}$ gives a vector perpendicular to the plane containing $\vec{u}$ and $\vec{v}$. To find a unit vector, always divide the resulting vector by its own magnitude. Be careful with the signs when calculating the determinant for the cross product.

c. If the Cartesian equation of a line is $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$, then find its equation in vector form.

Solution:

Step 1: Understanding the Concept:
The Cartesian equation and the vector equation are two ways to represent a line in three-dimensional space. The vector form requires a point on the line and a direction vector for the line. Both of these can be extracted directly from the given Cartesian form.

Step 2: Key Formula or Approach:
The standard Cartesian form of a line is:

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

This line passes through the point $P(x_0, y_0, z_0)$ and has direction ratios $\langle a, b, c\rangle$.
The corresponding vector form of the equation is:

$$\vec{r} = \vec{p} + \lambda\vec{d}$$

where $\vec{p}$ is the position vector of the point P, and $\vec{d}$ is the direction vector of the line.
So, $\vec{p} = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$ and $\vec{d} = a\hat{i} + b\hat{j} + c\hat{k}$.

Step 3: Detailed Explanation or Calculation:
The given Cartesian equation is:

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$$

We can rewrite the 'y' term to match the standard form $y - y_0$:

$$\frac{x-5}{3} = \frac{y-(-4)}{7} = \frac{z-6}{2}$$

Comparing this with the standard form $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$, we can identify:
The point on the line is $(x_0, y_0, z_0) = (5, -4, 6)$.
The direction ratios are $\langle a, b, c\rangle = \langle 3, 7, 2\rangle$.
Now, we construct the position vector of the point and the direction vector.
The position vector $\vec{p}$ is:

$$\vec{p} = 5\hat{i} - 4\hat{j} + 6\hat{k}$$

The direction vector $\vec{d}$ is:

$$\vec{d} = 3\hat{i} + 7\hat{j} + 2\hat{k}$$

Using the vector form $\vec{r} = \vec{p} + \lambda\vec{d}$, we get:

$$\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$$

where λ is a scalar parameter.

Step 4: Final Answer:
The equation of the line in vector form is $\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$.

> **Quick Tip**
>
> To quickly convert from Cartesian to vector form, identify the point by taking the values subtracted from x, y, and z (remember to flip the sign). The direction vector is formed by the denominators. This is a direct and fast conversion useful in timed exams.

---

d. There are 4 white and 2 black balls in a bag and in another bag 3 white and 5 black balls. Find the probability of getting both black balls if a ball is drawn from each bag.

Solution:

Step 1: Understanding the Concept:
This problem involves calculating the probability of two independent events occurring together. The outcome of drawing a ball from the first bag does not affect the outcome of drawing a ball from the second bag. For independent events, the probability that both occur is the product of their individual probabilities.

Step 2: Key Formula or Approach:
Let A be the event of drawing a black ball from the first bag, and B be the event of drawing a black ball from the second bag.
The probability of an event is given by:

$$P(\text{Event}) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$$

Since the events are independent, the probability of both events occurring is:

$$P(A \text{ and } B) = P(A) \times P(B)$$

Step 3: Detailed Explanation or Calculation:
First Bag:
Total number of balls = 4 white + 2 black = 6 balls.
Number of black balls = 2.
The probability of drawing a black ball from the first bag is $P(A)$.

$$P(A) = \frac{\text{Number of black balls}}{\text{Total number of balls}} = \frac{2}{6} = \frac{1}{3}$$

Second Bag:
Total number of balls = 3 white + 5 black = 8 balls.
Number of black balls = 5.

The probability of drawing a black ball from the second bag is $P(B)$.

$$P(B) = \frac{\text{Number of black balls}}{\text{Total number of balls}} = \frac{5}{8}$$

Combined Probability:
The probability of getting a black ball from both bags is the product of the individual probabilities.

$$P(\text{both black}) = P(A) \times P(B) = \frac{1}{3} \times \frac{5}{8} = \frac{5}{24}$$

Step 4: Final Answer:
The probability of getting both black balls is $\frac{5}{24}$.

> **Quick Tip**
>
> In probability, the word "and" for independent events usually means you should multiply the probabilities. The word "or" usually means you should add them (and subtract the intersection if they are not mutually exclusive). Recognizing these keywords can help you quickly set up the problem.

---

5. Do all parts.

a. If R_1 and R_2 be two equivalence relations on a set A, then prove that $R_1 \cap R_2$ be also an equivalence relation.

Solution:

Step 1: Understanding the Concept:
To prove that $R_1 \cap R_2$ is an equivalence relation, we must show that it satisfies the three properties of an equivalence relation: reflexivity, symmetry, and transitivity. We are given that R_1 and R_2 are already equivalence relations, so they satisfy these properties individually.

Step 2: Key Definitions:
Let R be a relation on a set A.
Reflexive: For all $a \in A$, $(a, a) \in R$.
Symmetric: If $(a, b) \in R$, then $(b, a) \in R$.
Transitive: If $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.
Intersection: $(a, b) \in R_1 \cap R_2$ if and only if $(a, b) \in R_1$ and $(a, b) \in R_2$.

Step 3: Detailed Explanation:
Let $R = R_1 \cap R_2$. We will prove that R is reflexive, symmetric, and transitive.
(i) Reflexivity:
Let a be an arbitrary element of A.
Since R_1 is an equivalence relation, it is reflexive. Therefore, $(a, a) \in R_1$.

Since R_2 is an equivalence relation, it is reflexive. Therefore, $(a, a) \in R_2$.
Since $(a, a) \in R_1$ and $(a, a) \in R_2$, by the definition of intersection, $(a, a) \in R_1 \cap R_2$.
Thus, $R_1 \cap R_2$ is reflexive.
(ii) Symmetry:
Let $(a, b) \in R_1 \cap R_2$.
This implies $(a, b) \in R_1$ and $(a, b) \in R_2$.
Since R_1 is symmetric, if $(a, b) \in R_1$, then $(b, a) \in R_1$.
Since R_2 is symmetric, if $(a, b) \in R_2$, then $(b, a) \in R_2$.
Since $(b, a) \in R_1$ and $(b, a) \in R_2$, by the definition of intersection, $(b, a) \in R_1 \cap R_2$.
Thus, $R_1 \cap R_2$ is symmetric.
(iii) Transitivity:
Let $(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2$.
From $(a, b) \in R_1 \cap R_2$, we have $(a, b) \in R_1$ and $(a, b) \in R_2$.
From $(b, c) \in R_1 \cap R_2$, we have $(b, c) \in R_1$ and $(b, c) \in R_2$.
Since R_1 is transitive, $(a, b) \in R_1$ and $(b, c) \in R_1$ implies $(a, c) \in R_1$.
Since R_2 is transitive, $(a, b) \in R_2$ and $(b, c) \in R_2$ implies $(a, c) \in R_2$.
Since $(a, c) \in R_1$ and $(a, c) \in R_2$, by the definition of intersection, $(a, c) \in R_1 \cap R_2$.
Thus, $R_1 \cap R_2$ is transitive.

Step 4: Final Answer:
Since $R_1 \cap R_2$ is reflexive, symmetric, and transitive, it is an equivalence relation. Hence proved.

Quick Tip

When proving properties for intersections (or unions) of sets, always break down the condition into its components for each set. Use the given properties of the individual sets and then combine the results using the definition of the intersection.

---

b. If $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$**, then prove that** $(I + A) = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$**.**

Solution:

Step 1: Understanding the Concept:
This problem requires proving a matrix equality. The strategy is to compute the left-hand side (LHS) and the right-hand side (RHS) of the equation separately and then show that they are equal. The proof will heavily rely on trigonometric half-angle identities.

Step 2: Key Formula or Approach:
We will use the half-angle identities for sine and cosine in terms of tangent:

$$\sin\alpha = \frac{2\tan(\alpha/2)}{1 + \tan^2(\alpha/2)}$$

$$\cos\alpha = \frac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)}$$

Let $t = \tan(\alpha/2)$. Then $\sin\alpha = \frac{2t}{1+t^2}$ and $\cos\alpha = \frac{1-t^2}{1+t^2}$.

Step 3: Detailed Explanation or Calculation:

Let $t = \tan(\alpha/2)$. The matrix A is $A = \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix}$.

Calculate the LHS:

$$I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix} = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}$$

Calculate the RHS:

First, find $I - A$:

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix} = \begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}$$

The rotation matrix is $R = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

Now, compute the product $(I - A)R$:

$$\text{RHS} = \begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\text{RHS} = \begin{bmatrix} \cos\alpha + t\sin\alpha & -\sin\alpha + t\cos\alpha \\ -t\cos\alpha + \sin\alpha & t\sin\alpha + \cos\alpha \end{bmatrix}$$

Now substitute the half-angle formulas for $\sin\alpha$ and $\cos\alpha$:

Entry (1,1): $\cos\alpha + t\sin\alpha = \frac{1-t^2}{1+t^2} + t\left(\frac{2t}{1+t^2}\right) = \frac{1-t^2+2t^2}{1+t^2} = \frac{1+t^2}{1+t^2} = 1$.

Entry (1,2): $-\sin\alpha + t\cos\alpha = -\frac{2t}{1+t^2} + t\left(\frac{1-t^2}{1+t^2}\right) = \frac{-2t+t-t^3}{1+t^2} = \frac{-t-t^3}{1+t^2} = \frac{-t(1+t^2)}{1+t^2} = -t$.

Entry (2,1): $-t\cos\alpha + \sin\alpha = -t\left(\frac{1-t^2}{1+t^2}\right) + \frac{2t}{1+t^2} = \frac{-t+t^3+2t}{1+t^2} = \frac{t+t^3}{1+t^2} = \frac{t(1+t^2)}{1+t^2} = t$.

Entry (2,2): $t\sin\alpha + \cos\alpha = t\left(\frac{2t}{1+t^2}\right) + \frac{1-t^2}{1+t^2} = \frac{2t^2+1-t^2}{1+t^2} = \frac{1+t^2}{1+t^2} = 1$.

So, the RHS matrix is:

$$\text{RHS} = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}$$

Step 4: Final Answer:

We have shown that $\text{LHS} = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}$ and $\text{RHS} = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}$.

Since LHS = RHS, the identity is proved.

Quick Tip

For matrix problems involving trigonometric functions, look for opportunities to use trigonometric identities. The half-angle tangent identities (Weierstrass substitution) are particularly powerful for simplifying expressions involving $\sin\alpha$ and $\cos\alpha$.

---

c. Differentiate $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$ with respect to x.

Solution:

Step 1: Understanding the Concept:
The problem asks for the derivative of a function involving an inverse trigonometric function. A direct application of the chain rule is possible but complicated. A more efficient method is to first simplify the expression inside the $\tan^{-1}$ using trigonometric identities.

Step 2: Key Formula or Approach:
We will use the following half-angle identities:
1. $\sin x = 2\sin(x/2)\cos(x/2)$
2. $1+\cos x = 2\cos^2(x/2)$
After simplification, we will use the derivative formula $\frac{d}{dx}(\tan^{-1}(\tan u)) = \frac{du}{dx}$, provided u is in the principal domain.

Step 3: Detailed Explanation or Calculation:
Let $y = \tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$.
First, simplify the argument of $\tan^{-1}$:

$$\frac{\sin x}{1+\cos x} = \frac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)}$$

Assuming $\cos(x/2) \neq 0$, we can cancel $2\cos(x/2)$ from the numerator and denominator:

$$\frac{\sin(x/2)}{\cos(x/2)} = \tan(x/2)$$

Now, substitute this simplified expression back into the function y:

$$y = \tan^{-1}(\tan(x/2))$$

For $x \in (-\pi, \pi)$, $x/2 \in (-\pi/2, \pi/2)$, which is the principal value range for the tangent function.
In this range, $\tan^{-1}(\tan u) = u$.
So, we can simplify the function to:

$$y = \frac{x}{2}$$

Now, differentiate y with respect to x:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}$$

Step 4: Final Answer:
The derivative of $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$ with respect to x is $\frac{1}{2}$.

Quick Tip

Before differentiating complex trigonometric or inverse trigonometric functions, always check if the expression can be simplified using identities. This can often reduce a complicated chain rule problem to a very simple derivative.

---

d. Find the shortest distance between the lines $\vec{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + \mu(3\hat{i} + \hat{j} + 2\hat{k})$.

Solution:

Step 1: Understanding the Concept:
The two given lines are skew (not parallel and not intersecting). The shortest distance between two skew lines is the length of the perpendicular line segment between them. We can find this using a standard vector formula.

Step 2: Key Formula or Approach:
The vector equations of the lines are of the form $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$.
The shortest distance (d) between them is given by the formula:

$$d = \left| \frac{(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

Step 3: Detailed Explanation or Calculation:
From the given equations, we identify the vectors:
$\vec{a}_1 = \hat{i} + \hat{j} + 0\hat{k}$
$\vec{b}_1 = 2\hat{i} - \hat{j} + \hat{k}$
$\vec{a}_2 = 2\hat{i} + \hat{j} - \hat{k}$
$\vec{b}_2 = 3\hat{i} + \hat{j} + 2\hat{k}$
1. Calculate $\vec{a}_2 - \vec{a}_1$:

$$\vec{a}_2 - \vec{a}_1 = (2\hat{i} + \hat{j} - \hat{k}) - (\hat{i} + \hat{j}) = (2-1)\hat{i} + (1-1)\hat{j} + (-1-0)\hat{k} = \hat{i} - \hat{k}$$

2. Calculate $\vec{b}_1 \times \vec{b}_2$:

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= \hat{i}((-1)(2) - (1)(1)) - \hat{j}((2)(2) - (1)(3)) + \hat{k}((2)(1) - (-1)(3))$$
$$= \hat{i}(-2-1) - \hat{j}(4-3) + \hat{k}(2+3) = -3\hat{i} - \hat{j} + 5\hat{k}$$

3. Calculate the dot product $(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)$:

$$(\hat{i} + 0\hat{j} - \hat{k}) \cdot (-3\hat{i} - \hat{j} + 5\hat{k}) = (1)(-3) + (0)(-1) + (-1)(5) = -3 - 5 = -8$$

4. Calculate the magnitude $|\vec{b}_1 \times \vec{b}_2|$:

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{(-3)^2 + (-1)^2 + 5^2} = \sqrt{9+1+25} = \sqrt{35}$$

5. Calculate the shortest distance d:

$$d = \left| \frac{-8}{\sqrt{35}} \right| = \frac{8}{\sqrt{35}}$$

Step 4: Final Answer:
The shortest distance between the two lines is $\frac{8}{\sqrt{35}}$ units.

Quick Tip

Memorize the formula for the shortest distance between skew lines. The numerator is the scalar triple product of $(\vec{a}_2 - \vec{a}_1)$, $\vec{b}_1$, and $\vec{b}_2$, which represents the volume of the parallelepiped formed by these vectors. The denominator is the magnitude of the cross product, representing the area of the base parallelogram.

---

e. If $y = e^{\tan^{-1} x}$, prove that $(1+x^2)\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} = 0$.

Solution:

Step 1: Understanding the Concept:
This problem requires showing that a given function satisfies a second-order linear differential equation. The process involves finding the first and second derivatives of the function and substituting them into the equation to verify the identity.

Step 2: Key Formula or Approach:
We will use the chain rule and product rule for differentiation.
- Chain Rule: $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$
- Product Rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
- Derivative of e^u: $\frac{d}{dx}e^u = e^u\frac{du}{dx}$
- Derivative of $\tan^{-1} x$: $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$

Step 3: Detailed Explanation or Calculation:
Given the function $y = e^{\tan^{-1} x}$.
First Derivative:
Differentiate y with respect to x using the chain rule:

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\tan^{-1} x}) = e^{\tan^{-1} x} \cdot \frac{d}{dx}(\tan^{-1} x)$$
$$\frac{dy}{dx} = e^{\tan^{-1} x} \cdot \frac{1}{1+x^2}$$

Since $y = e^{\tan^{-1} x}$, we can substitute y back into the equation:

$$\frac{dy}{dx} = \frac{y}{1+x^2}$$

To avoid fractions when differentiating again, rearrange this equation:

$$(1+x^2)\frac{dy}{dx} = y$$

Second Derivative:
Now, differentiate both sides of the rearranged equation with respect to x, using the product rule on the left side:

$$\frac{d}{dx}\left((1+x^2)\frac{dy}{dx}\right) = \frac{d}{dx}(y)$$

$$\left(\frac{d}{dx}(1+x^2)\right)\frac{dy}{dx} + (1+x^2)\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy}{dx}$$

$$(2x)\frac{dy}{dx} + (1+x^2)\frac{d^2y}{dx^2} = \frac{dy}{dx}$$

Now, rearrange the terms to match the required equation:

$$(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - \frac{dy}{dx} = 0$$

Factor out $\frac{dy}{dx}$ from the last two terms:

$$(1+x^2)\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} = 0$$

Step 4: Final Answer:
We have successfully derived the given differential equation starting from the function $y = e^{\tan^{-1} x}$. Hence, the relation is proved.

Quick Tip

When asked to prove a differential equation, after finding the first derivative, it's often a good strategy to rearrange the equation to eliminate fractions or complex terms (like substituting 'y' back in). Differentiating this simpler, rearranged form can make finding the second derivative much easier and lead directly to the required result.

---

6. Do all parts.

a. If $f(x) = \begin{cases} -2 & \textbf{if } x \le -1 \\ 2x & \textbf{if } -1 < x \le 1 \\ 2 & \textbf{if } x > 1 \end{cases}$**, then test the continuity of the function at** $x = -1$ **and at** $x = 1$**.**

Solution:

Step 1: Understanding the Concept:
For a function to be continuous at a point c, the left-hand limit (LHL), the right-hand limit (RHL), and the value of the function at that point, $f(c)$, must all be equal.

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$$

Step 2: Key Formula or Approach:
We will test the continuity at $x = -1$ and $x = 1$ by calculating the LHL, RHL, and the function's value at each point.

Step 3: Detailed Explanation or Calculation:
Testing continuity at $x = -1$:
1. **Left-Hand Limit (LHL):**
For $x < -1$, the function is $f(x) = -2$.

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (-2) = -2$$

2. **Right-Hand Limit (RHL):**
For $x > -1$, the function is $f(x) = 2x$.

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (2x) = 2(-1) = -2$$

3. **Value of the function at $x = -1$:**
For $x = -1$, the function is $f(x) = -2$.

$$f(-1) = -2$$

Since LHL = RHL = $f(-1) = -2$, the function is continuous at $x = -1$.

Testing continuity at $x = 1$:
1. **Left-Hand Limit (LHL):**
For $x < 1$, the function is $f(x) = 2x$.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x) = 2(1) = 2$$

2. **Right-Hand Limit (RHL):**
For $x > 1$, the function is $f(x) = 2$.

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2) = 2$$

3. **Value of the function at $x = 1$:**
For $x = 1$, the function is $f(x) = 2x$.

$$f(1) = 2(1) = 2$$

Since LHL = RHL = $f(1) = 2$, the function is continuous at $x = 1$.

Step 4: Final Answer:
The function $f(x)$ is continuous at both $x = -1$ and $x = 1$.

> Quick Tip
>
> When dealing with piecewise functions, be very careful to select the correct piece of the function definition corresponding to the limit you are evaluating (left-hand, right-hand) or the point you are evaluating.

---

b. If three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfying the condition $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. If $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{c}| = 2$, then find the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$.

Solution:

Step 1: Understanding the Concept:
This problem uses the properties of the dot product of vectors, specifically that the dot product of a vector with itself gives the square of its magnitude. We can use the given condition $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ to find the required value.

Step 2: Key Formula or Approach:
The key identity is $\vec{v} \cdot \vec{v} = |\vec{v}|^2$. We will take the dot product of the vector sum $(\vec{a} + \vec{b} + \vec{c})$ with itself.
We start with the given equation:

$$\vec{a} + \vec{b} + \vec{c} = \vec{0}$$

Taking the dot product of both sides with $(\vec{a} + \vec{b} + \vec{c})$:

$$(\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) = \vec{0} \cdot \vec{0}$$

Step 3: Detailed Explanation or Calculation:
Expanding the left side of the equation:

$$\vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} + \vec{c} \cdot \vec{c} = 0$$

Using $\vec{v} \cdot \vec{v} = |\vec{v}|^2$ and the commutative property of the dot product ($\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$):

$$|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$$

Now, substitute the given magnitudes: $|\vec{a}| = 3$, $|\vec{b}| = 4$, and $|\vec{c}| = 2$.

$$(3)^2 + (4)^2 + (2)^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$$
$$9 + 16 + 4 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$$
$$29 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$$

Now, solve for the required expression:

$$2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = -29$$
$$\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a} = -\frac{29}{2}$$

Step 4: Final Answer:
The value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$ is $-\frac{29}{2}$.

Quick Tip

Whenever you are given a sum of vectors equal to zero and their magnitudes, and asked to find a sum of dot products, the most direct method is to square the vector sum equation (i.e., take its dot product with itself).

---

c. The radius of an air bubble is increasing at the rate of $\frac{1}{2}$ cm/s. At what rate is the volume of the bubble increasing while the radius is 1 cm?

Solution:

Step 1: Understanding the Concept:
This is a related rates problem. We are given the rate of change of the radius of a sphere ($\frac{dr}{dt}$) and asked to find the rate of change of its volume ($\frac{dV}{dt}$) at a specific instant.

Step 2: Key Formula or Approach:
The volume (V) of a sphere with radius (r) is given by the formula:

$$V = \frac{4}{3}\pi r^3$$

To find the relationship between the rates, we differentiate this formula with respect to time (t) using the chain rule.

Step 3: Detailed Explanation or Calculation:
First, differentiate the volume formula with respect to time, t:

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

Applying the chain rule:

$$\frac{dV}{dt} = \frac{4}{3}\pi\cdot(3r^2)\cdot\frac{dr}{dt}$$
$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

We are given the following information:
- The rate of increase of the radius: $\frac{dr}{dt} = \frac{1}{2}$ cm/s

- The specific radius at which to find the rate of volume increase: $r = 1$ cm
Now, substitute these values into the differentiated equation:

$$\frac{dV}{dt} = 4\pi(1)^2\left(\frac{1}{2}\right)$$
$$\frac{dV}{dt} = 4\pi(1)\left(\frac{1}{2}\right)$$
$$\frac{dV}{dt} = 2\pi$$

Step 4: Final Answer:
The volume of the bubble is increasing at a rate of 2π cm^3/s when the radius is 1 cm.

Quick Tip

In related rates problems, first identify the formula that connects the quantities involved (here, volume and radius). Then, differentiate the entire formula with respect to time (t), making sure to use the chain rule for any variable that is a function of t. Finally, substitute the given values to find the unknown rate.

---

d. Minimize Z = 3x + 2y by graphical method under the following constraints:
x + y ≥ 8,
3x + 5y ≤ 15,
x ≥ 0, y ≥ 0

Solution:

Step 1: Understanding the Concept:
This is a Linear Programming Problem (LPP). The goal is to find the minimum value of an objective function Z, subject to a set of linear constraints. The graphical method involves plotting the constraints to find the feasible region (the set of all points satisfying all constraints) and then evaluating the objective function at the corner points of this region.

Step 2: Key Formula or Approach:
1. Treat each inequality as an equation to plot the boundary lines.
2. Determine the region represented by each inequality (by testing a point like (0,0)).
3. Identify the common region that satisfies all constraints simultaneously (the feasible region).
4. If a feasible region exists, find its vertices (corner points). The optimal solution (minimum or maximum) will occur at one of these vertices.

Step 3: Detailed Explanation or Calculation:
Let's analyze the constraints:
Constraint 1: $x + y \geq 8$. The boundary line is $x + y = 8$. This line passes through (8,0) and (0,8). Since $0 + 0 \geq 8$ is false, the region is the half-plane that does not contain the origin.

Constraint 2: $3x + 5y \leq 15$. The boundary line is $3x + 5y = 15$. This line passes through (5,0) and (0,3). Since $3(0) + 5(0) \leq 15$ is true, the region is the half-plane that contains the origin.
Constraint 3 & 4: $x \geq 0, y \geq 0$. This restricts the solution to the first quadrant.

Now, let's visualize the feasible region.
- The region for $3x + 5y \leq 15$ (along with $x \geq 0, y \geq 0$) is a triangle with vertices at (0,0), (5,0), and (0,3). All points in this region are close to the origin.
- The region for $x + y \geq 8$ (along with $x \geq 0, y \geq 0$) is the area above the line connecting (8,0) and (0,8). All points in this region are far from the origin.

There is no point (x, y) that can simultaneously be close to the origin (satisfying $3x + 5y \leq 15$) and far from the origin (satisfying $x + y \geq 8$). The two regions defined by the constraints are disjoint.
Therefore, there is no common region that satisfies all the given constraints. The feasible region is an empty set.

Step 4: Final Answer:
Since there is no feasible region (the intersection of the constraints is an empty set), there is no point that satisfies all the given conditions. Therefore, there is no solution to this minimization problem.

> **Quick Tip**
>
> When solving an LPP graphically, always shade the regions carefully. If the shaded areas for all constraints do not overlap, the feasible region is empty, and the problem has no solution. It's a good first step to quickly check if the regions could possibly overlap.

---

e. The probability of solving a question by the three students A, B, C are respectively $\frac{3}{10}, \frac{1}{5}$ and $\frac{1}{10}$. Find the probability of solving the question.

Solution:

Step 1: Understanding the Concept:
The phrase "solving the question" implies that at least one of the three students solves it. It is easier to calculate the probability of the complementary event, which is that none of the students solve the question, and then subtract this from 1.

Step 2: Key Formula or Approach:
Let A, B, and C be the events that students A, B, and C solve the question, respectively. We are given:
$P(A) = \frac{3}{10}$, $P(B) = \frac{1}{5}$, $P(C) = \frac{1}{10}$.
The probability that the question is solved is $P(A \cup B \cup C)$.

Using the complement rule:

$$P(\text{question is solved}) = 1 - P(\text{question is not solved by anyone})$$
$$P(A \cup B \cup C) = 1 - P(A' \cap B' \cap C')$$

Assuming the events are independent, $P(A' \cap B' \cap C') = P(A') \cdot P(B') \cdot P(C')$.

Step 3: Detailed Explanation or Calculation:
First, calculate the probabilities that each student fails to solve the question.
Probability that A fails: $P(A') = 1 - P(A) = 1 - \frac{3}{10} = \frac{7}{10}$.
Probability that B fails: $P(B') = 1 - P(B) = 1 - \frac{1}{5} = \frac{4}{5}$.
Probability that C fails: $P(C') = 1 - P(C) = 1 - \frac{1}{10} = \frac{9}{10}$.
Next, calculate the probability that all three of them fail. Since their attempts are independent events, we multiply their individual probabilities of failure.

$$P(A' \cap B' \cap C') = P(A') \times P(B') \times P(C')$$
$$P(A' \cap B' \cap C') = \frac{7}{10} \times \frac{4}{5} \times \frac{9}{10} = \frac{252}{500}$$

Simplify the fraction:

$$\frac{252}{500} = \frac{126}{250} = \frac{63}{125}$$

Finally, the probability that the question is solved (at least one person solves it) is 1 minus the probability that no one solves it.

$$P(\text{question is solved}) = 1 - \frac{63}{125} = \frac{125 - 63}{125} = \frac{62}{125}$$

Step 4: Final Answer:
The probability of solving the question is $\frac{62}{125}$.

Quick Tip

For problems asking for the probability of "at least one" event occurring, it's almost always simpler to calculate the probability of the complementary event ("none" of the events occurring) and subtract the result from 1.

---

7. Do any one part.

a. Find the inverse of the matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$.

Solution:

Step 1: Understanding the Concept:
To find the inverse of a matrix A, we use the formula $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$, where $\det(A)$ is the determinant of A and $\text{adj}(A)$ is the adjugate (or adjoint) of A. The inverse exists only if the determinant is non-zero.

Step 2: Key Formula or Approach:
1. Calculate the determinant of A, $\det(A)$.
2. Find the matrix of cofactors of A.
3. Find the adjugate of A by transposing the cofactor matrix, $\text{adj}(A) = C^T$.
4. Calculate the inverse using the formula $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$.

Step 3: Detailed Explanation or Calculation:
1. Calculate the Determinant:

$$\det(A) = \begin{vmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{vmatrix}$$

Expanding along the first row:

$$\det(A) = 2\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} - 0\begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} + (-1)\begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix}$$
$$\det(A) = 2((1)(3) - (0)(1)) - 0 + (-1)((5)(1) - (1)(0))$$
$$\det(A) = 2(3) - 1(5) = 6 - 5 = 1$$

Since $\det(A) = 1 \neq 0$, the inverse exists.

2. Find the Cofactor Matrix:
The cofactors $C_{ij} = (-1)^{i+j}M_{ij}$, where M_{ij} is the minor of the element a_{ij}.

$C_{11} = +\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} = 3$

$C_{12} = -\begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} = -15$

$C_{13} = +\begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix} = 5$

$C_{21} = -\begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} = -(0 - (-1)) = -1$

$C_{22} = +\begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} = 6$

$C_{23} = -\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = -2$

$C_{31} = +\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = (0 - (-1)) = 1$

$C_{32} = -\begin{vmatrix} 2 & -1 \\ 5 & 0 \end{vmatrix} = -(0 - (-5)) = -5$

$C_{33} = +\begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix} = 2$

The cofactor matrix is $C = \begin{bmatrix} 3 & -15 & 5 \\ -1 & 6 & -2 \\ 1 & -5 & 2 \end{bmatrix}$.

3. Find the Adjugate Matrix:

$$\text{adj}(A) = C^T = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

4. Calculate the Inverse:

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \frac{1}{1}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

Step 4: Final Answer:
The inverse of the matrix A is:

$$A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

Quick Tip

To verify your answer, you can multiply the original matrix A by its calculated inverse A^{-1}. The result should be the identity matrix I. $A \cdot A^{-1} = I$. This is a great way to catch calculation errors in an exam.

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b. Solve the system of equations by matrix method:
$3x - 2y + 3z = 8$
$2x + y - z = 1$
$4x - 3y + 2z = 4$

Solution:

Step 1: Understanding the Concept:
A system of linear equations can be represented in matrix form as $AX = B$, where A is the matrix of coefficients, X is the column matrix of variables, and B is the column matrix of constants. The solution can be found by pre-multiplying both sides by the inverse of A, which gives $X = A^{-1}B$.

Step 2: Key Formula or Approach:
1. Express the system of equations in the matrix form $AX = B$.
2. Find the inverse of the coefficient matrix A, using $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$.

3. Solve for the variable matrix X by computing the product $X = A^{-1}B$.

Step 3: Detailed Explanation or Calculation:
1. Formulate the matrix equation:
The given system is:
$3x - 2y + 3z = 8$
$2x + y - z = 1$
$4x - 3y + 2z = 4$
This can be written as $AX = B$, where:

$$A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

2. Find the inverse of A:
First, calculate the determinant of A:

$$\det(A) = 3((1)(2) - (-1)(-3)) - (-2)((2)(2) - (-1)(4)) + 3((2)(-3) - (1)(4))$$
$$\det(A) = 3(2 - 3) + 2(4 + 4) + 3(-6 - 4)$$
$$\det(A) = 3(-1) + 2(8) + 3(-10) = -3 + 16 - 30 = -17$$

Since $\det(A) \neq 0$, a unique solution exists.
Next, find the adjugate of A. The matrix of cofactors is:
$C_{11} = -1$, $C_{12} = -8$, $C_{13} = -10$
$C_{21} = -5$, $C_{22} = -6$, $C_{23} = 1$
$C_{31} = -1$, $C_{32} = 9$, $C_{33} = 7$

$$\text{Cofactor Matrix} = \begin{bmatrix} -1 & -8 & -10 \\ -5 & -6 & 1 \\ -1 & 9 & 7 \end{bmatrix}$$

$$\text{adj}(A) = C^T = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

3. Solve for X:

$$X = A^{-1}B = \frac{1}{-17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix} \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

Perform the matrix multiplication:

$$\begin{bmatrix} (-1)(8) + (-5)(1) + (-1)(4) \\ (-8)(8) + (-6)(1) + (9)(4) \\ (-10)(8) + (1)(1) + (7)(4) \end{bmatrix} = \begin{bmatrix} -8 - 5 - 4 \\ -64 - 6 + 36 \\ -80 + 1 + 28 \end{bmatrix} = \begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix}$$

Now, multiply by the scalar $\frac{1}{-17}$:

$$X = \frac{1}{-17}\begin{bmatrix}-17\\-34\\-51\end{bmatrix} = \begin{bmatrix}\frac{-17}{-17}\\\frac{-34}{-17}\\\frac{-51}{-17}\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

Step 4: Final Answer:
The solution to the system of equations is $x = 1, y = 2, z = 3$.

Quick Tip

The matrix method for solving systems of equations is systematic but prone to arithmetic errors, especially when finding the adjugate matrix. Always be careful with the signs of the cofactors. After finding the solution, quickly substitute the values of x, y, and z back into one of the original equations to check if it holds true.

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8. Do any one part.

a. Prove that the semi-vertical angle of a cone with given slant height and maximum volume is $\tan^{-1}(\sqrt{2})$.

Solution:

Step 1: Understanding the Concept:
This is an optimization problem where we need to maximize the volume of a cone given a constant slant height. We will express the volume as a function of the semi-vertical angle, then use calculus to find the angle that maximizes this volume.

Step 2: Key Formula or Approach:
Let l be the given slant height (a constant), h be the height, r be the radius, and θ be the semi-vertical angle of the cone.
The volume V of the cone is given by:

$$V = \frac{1}{3}\pi r^2 h$$

The relationships between the variables are:

$$r = l\sin\theta \quad \text{and} \quad h = l\cos\theta$$

We will express V as a function of θ, find its derivative $\frac{dV}{d\theta}$, set it to zero to find critical points, and use the second derivative test to confirm a maximum.

Step 3: Detailed Explanation or Calculation:
1. Express Volume in terms of θ:
Substitute the expressions for r and h into the volume formula:

$$V(\theta) = \frac{1}{3}\pi(l\sin\theta)^2(l\cos\theta)$$
$$V(\theta) = \frac{1}{3}\pi l^3 \sin^2\theta\cos\theta$$

2. Differentiate Volume with respect to θ:
To find the maximum volume, we differentiate V with respect to θ and set the result to zero. Let $K = \frac{1}{3}\pi l^3$ (a constant).

$$V(\theta) = K\sin^2\theta\cos\theta$$

Using the product rule $(uv)' = u'v + uv'$:

$$\frac{dV}{d\theta} = K\left[(2\sin\theta\cos\theta)\cos\theta + \sin^2\theta(-\sin\theta)\right]$$
$$\frac{dV}{d\theta} = K[2\sin\theta\cos^2\theta - \sin^3\theta]$$

3. Find Critical Points:
Set $\frac{dV}{d\theta} = 0$:

$$K[2\sin\theta\cos^2\theta - \sin^3\theta] = 0$$

Factor out $\sin\theta$:

$$K\sin\theta(2\cos^2\theta - \sin^2\theta) = 0$$

For a cone, $\theta \in (0, \pi/2)$, so $\sin\theta \neq 0$. Therefore, we must have:

$$2\cos^2\theta - \sin^2\theta = 0$$
$$2\cos^2\theta = \sin^2\theta$$

Divide by $\cos^2\theta$ (which is non-zero for the domain):

$$2 = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$$
$$\tan\theta = \sqrt{2} \quad (\text{since } \theta \text{ is acute, } \tan\theta > 0)$$
$$\theta = \tan^{-1}(\sqrt{2})$$

4. Second Derivative Test:
To confirm this is a maximum, we check the sign of the second derivative.

$$\frac{d^2V}{d\theta^2} = K\frac{d}{d\theta}[2\sin\theta\cos^2\theta - \sin^3\theta]$$
$$\frac{d^2V}{d\theta^2} = K[2\cos^3\theta - 4\sin^2\theta\cos\theta - 3\sin^2\theta\cos\theta]$$
$$\frac{d^2V}{d\theta^2} = K\cos\theta[2\cos^2\theta - 7\sin^2\theta]$$

At the critical point, we know $\sin^2\theta = 2\cos^2\theta$. Substituting this:

$$\frac{d^2V}{d\theta^2} = K\cos\theta[2\cos^2\theta - 7(2\cos^2\theta)] = K\cos\theta[-12\cos^2\theta] = -12K\cos^3\theta$$

Since $K > 0$ and $\cos\theta > 0$ for $\theta \in (0, \pi/2)$, we have $\frac{d^2V}{d\theta^2} < 0$.
This confirms that the volume is maximum at $\theta = \tan^{-1}(\sqrt{2})$.

Step 4: Final Answer:
The semi-vertical angle of a cone with a given slant height and maximum volume is $\tan^{-1}(\sqrt{2})$. Hence proved.

> **Quick Tip**
>
> In optimization problems involving geometric shapes, it's crucial to express the quantity to be optimized (like volume or area) as a function of a single variable. Using trigonometric relations is often the most efficient way to do this for problems involving angles.

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b. Find a particular solution of the differential equation $(x - y)(dx + dy) = dx - dy$ when $y = -1$ if $x = 0$.

Solution:

Step 1: Understanding the Concept:
The given differential equation is not immediately in a standard form. We need to rearrange it to identify its type. After rearranging, it becomes a homogeneous differential equation that can be solved using the substitution $v = x - y$. Once the general solution is found, we use the given initial conditions to find the particular solution.

Step 2: Key Formula or Approach:
1. Rearrange the equation to the form $\frac{dy}{dx} = f(x, y)$.
2. Use the substitution $v = x - y$, which implies $\frac{dv}{dx} = 1 - \frac{dy}{dx}$ or $\frac{dy}{dx} = 1 - \frac{dv}{dx}$.
3. Solve the resulting separable equation in v and x.
4. Substitute back $v = x - y$ to get the general solution.
5. Apply the initial condition $y(0) = -1$ to find the integration constant.

Step 3: Detailed Explanation or Calculation:
1. Rearrange the Equation:

$$(x - y)(dx + dy) = dx - dy$$
$$xdx + xdy - ydx - ydy = dx - dy$$

Group terms with dy and dx:

$$xdy - ydy + dy = dx - xdx + ydx$$

$$(x - y + 1)dy = (1 - x + y)dx$$
$$\frac{dy}{dx} = \frac{1 - (x - y)}{1 + (x - y)}$$

2. Substitute $v = x - y$:
Let $v = x - y$. Then $\frac{dy}{dx} = 1 - \frac{dv}{dx}$. The equation becomes:

$$1 - \frac{dv}{dx} = \frac{1 - v}{1 + v}$$
$$\frac{dv}{dx} = 1 - \frac{1 - v}{1 + v} = \frac{(1 + v) - (1 - v)}{1 + v} = \frac{1 + v - 1 + v}{1 + v} = \frac{2v}{1 + v}$$

3. Solve the Separable Equation:

$$\frac{1 + v}{2v} dv = dx$$

Integrate both sides:

$$\int \frac{1 + v}{2v} dv = \int dx$$
$$\int \left(\frac{1}{2v} + \frac{1}{2} \right) dv = \int dx$$
$$\frac{1}{2} \ln |v| + \frac{1}{2} v = x + C_0$$

Multiply by 2:

$$\ln |v| + v = 2x + 2C_0$$

Let $C = 2C_0$.

$$\ln |v| + v = 2x + C$$

4. Substitute Back:
Replace v with $x - y$:

$$\ln |x - y| + (x - y) = 2x + C$$
$$\ln |x - y| = x + y + C$$

This is the general solution.
5. Apply Initial Condition:
We are given $y = -1$ when $x = 0$. Substitute these values to find C:

$$\ln |0 - (-1)| = 0 + (-1) + C$$
$$\ln |1| = -1 + C$$
$$0 = -1 + C$$
$$C = 1$$

Step 4: Final Answer:
Substituting $C = 1$ back into the general solution gives the particular solution:

$$\ln|x - y| = x + y + 1$$

Quick Tip

When a differential equation contains terms like $(x - y)$ or $(x + y)$, it's a strong hint to try a substitution like $v = x - y$ or $v = x + y$. This often transforms the equation into a much simpler, separable form.

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9. Do any one part.

a. Integrate: $\int \left(\frac{2+\sin 2x}{1+\cos 2x}\right) e^x dx$

Solution:

Step 1: Understanding the Concept:
This integral is in the form $\int e^x g(x)dx$. This form often simplifies to a standard integral type $\int e^x(f(x)+f'(x))dx$, which has a direct solution. The strategy is to simplify the trigonometric function $g(x)$ and see if it can be expressed as a sum of a function and its derivative.

Step 2: Key Formula or Approach:
The key integration formula is:

$$\int e^x(f(x) + f'(x))dx = e^x f(x) + C$$

We will also use the trigonometric double-angle identities:
1. $\sin 2x = 2\sin x \cos x$
2. $1 + \cos 2x = 2\cos^2 x$

Step 3: Detailed Explanation or Calculation:
Let the integral be I. First, we simplify the trigonometric part of the integrand:

$$\frac{2 + \sin 2x}{1 + \cos 2x} = \frac{2 + 2\sin x \cos x}{2\cos^2 x}$$

Factor out 2 from the numerator and cancel with the denominator:

$$= \frac{2(1 + \sin x \cos x)}{2\cos^2 x} = \frac{1 + \sin x \cos x}{\cos^2 x}$$

Now, split the fraction into two parts:

$$= \frac{1}{\cos^2 x} + \frac{\sin x \cos x}{\cos^2 x}$$

Simplify each part:

$$= \sec^2 x + \tan x$$

Now, substitute this simplified expression back into the integral:

$$I = \int (\sec^2 x + \tan x)e^x dx$$

Let's check if this fits the form $\int e^x(f(x) + f'(x))dx$.
If we let $f(x) = \tan x$, then its derivative is $f'(x) = \frac{d}{dx}(\tan x) = \sec^2 x$.
So, the integrand is indeed in the form $e^x(f(x) + f'(x))$, where $f(x) = \tan x$.
Using the formula:

$$I = e^x f(x) + C = e^x \tan x + C$$

Step 4: Final Answer:
The result of the integration is $e^x \tan x + C$.

Quick Tip

When you see an integral involving the product of e^x and another function, always try to manipulate the other function to see if it can be written as $f(x) + f'(x)$. This special form is a common shortcut in integration problems.

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b. Solve: $\int \frac{(3x+5)dx}{x^3-x^2-x+1}$

Solution:

Step 1: Understanding the Concept:
This is an integral of a rational function. Since the degree of the numerator is less than the degree of the denominator, we can solve it using the method of partial fraction decomposition. This involves factoring the denominator and expressing the rational function as a sum of simpler fractions.

Step 2: Key Formula or Approach:
1. Factor the denominator polynomial.
2. Set up the partial fraction expansion based on the factors.
3. Solve for the unknown coefficients.
4. Integrate the resulting simpler fractions.

Step 3: Detailed Explanation or Calculation:
1. Factor the Denominator:
Let the denominator be $D(x) = x^3 - x^2 - x + 1$. We can factor it by grouping:

$$D(x) = x^2(x - 1) - 1(x - 1)$$

$$D(x) = (x^2 - 1)(x - 1)$$
$$D(x) = (x - 1)(x + 1)(x - 1) = (x - 1)^2(x + 1)$$
The denominator has a repeated linear factor $(x - 1)$ and a distinct linear factor $(x + 1)$.

2. Partial Fraction Decomposition:
We can write the integrand as:
$$\frac{3x + 5}{(x - 1)^2(x + 1)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x + 1}$$
To find the coefficients A, B, and C, multiply both sides by the denominator:
$$3x + 5 = A(x - 1)(x + 1) + B(x + 1) + C(x - 1)^2$$
Now, substitute strategic values for x:
- For $x = 1$: $3(1) + 5 = A(0) + B(1 + 1) + C(0) \Rightarrow 8 = 2B \Rightarrow B = 4$.
- For $x = -1$: $3(-1) + 5 = A(0) + B(0) + C(-1 - 1)^2 \Rightarrow 2 = 4C \Rightarrow C = \frac{1}{2}$.
- For $x = 0$: $3(0) + 5 = A(-1)(1) + B(1) + C(-1)^2 \Rightarrow 5 = -A + B + C$.
Substitute the values of B and C:
$$5 = -A + 4 + \frac{1}{2} \Rightarrow 5 = -A + \frac{9}{2} \Rightarrow A = \frac{9}{2} - 5 = \frac{9 - 10}{2} = -\frac{1}{2}$$
3. Integrate:
Now, we can rewrite the integral as:
$$\int \left(\frac{-1/2}{x - 1} + \frac{4}{(x - 1)^2} + \frac{1/2}{x + 1} \right) dx$$
$$= -\frac{1}{2} \int \frac{1}{x - 1} dx + 4 \int \frac{1}{(x - 1)^2} dx + \frac{1}{2} \int \frac{1}{x + 1} dx$$
$$= -\frac{1}{2} \ln|x - 1| + 4 \left(\frac{(x - 1)^{-1}}{-1} \right) + \frac{1}{2} \ln|x + 1| + C$$
$$= -\frac{1}{2} \ln|x - 1| - \frac{4}{x - 1} + \frac{1}{2} \ln|x + 1| + C$$
Step 4: Final Answer:
The solution is:
$$\frac{1}{2} \ln|x + 1| - \frac{1}{2} \ln|x - 1| - \frac{4}{x - 1} + C$$
This can also be written using logarithm properties as:
$$\frac{1}{2} \ln\left|\frac{x + 1}{x - 1}\right| - \frac{4}{x - 1} + C$$

Quick Tip

When performing partial fraction decomposition, using the "cover-up" method (substituting the roots of the denominator) is the fastest way to find coefficients for distinct linear factors and the coefficient of the highest power of a repeated linear factor. For the remaining coefficients, you can substitute any other convenient number (like x=0) or compare the coefficients of the highest power of x on both sides.
