# UP Board Class 12 Mathematics - 324(IY) - 2025 Question Paper with Solutions

| Time Allowed :3 Hours | Maximum Marks :100 | Total Questions :9 |
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**General Instructions**

**Read the following instructions very carefully and strictly follow them:**

1. There are in all nine questions in this question paper.
2. All questions are compulsory.
3. In the beginning of each question, the number of parts to be attempted are clearly mentioned.
4. Marks allotted to the questions are indicated against them.
5. Start solving from the first question and proceed to solve till the last one. Do not waste your time over a question you cannot solve.

**1. Do all parts.**
**Select the correct option of each part and write it on your answer-book.**

**a. If $f : \mathbb{R} \to \mathbb{R}$, is given by $f(x) = (3 - x^3)^{1/3}$, then $fof(x)$ is equal to :**

(A) $x^{1/3}$
(B) $x^3$
(C) $x$
(D) $(3 - x^3)$

**Correct Answer:** (C) x

**Solution:**

**Step 1: Understanding the Concept:**
The notation $fof(x)$ represents the composition of the function $f$ with itself, which is also written as $f(f(x))$. To find this, we substitute the entire expression for $f(x)$ into the variable $x$ within the function $f$ again.

**Step 2: Key Formula or Approach:**
Given $f(x) = (3 - x^3)^{1/3}$.
We need to calculate $f(f(x))$.

**Step 3: Detailed Explanation or Calculation:**
Start with the definition of $fof(x)$:

$$fof(x) = f(f(x))$$

Substitute the expression for $f(x)$:

$$fof(x) = f\left((3 - x^3)^{1/3}\right)$$

Now, apply the function $f$ to this new input. This means we replace $x$ in the expression $(3 - x^3)^{1/3}$ with the input $(3 - x^3)^{1/3}$.

$$fof(x) = \left(3 - \left[(3 - x^3)^{1/3}\right]^3\right)^{1/3}$$

The cube and the cube root cancel each other out:

$$fof(x) = \big(3 - (3 - x^3)\big)^{1/3}$$

Simplify the expression inside the parenthesis:

$$fof(x) = (3 - 3 + x^3)^{1/3}$$
$$fof(x) = (x^3)^{1/3}$$
$$fof(x) = x$$

**Step 4: Final Answer:**
The value of $fof(x)$ is $x$.

**Quick Tip**

When dealing with function composition, work from the inside out. First, evaluate the inner function, then use its result as the input for the outer function. For functions that seem to "undo" themselves, like this one, they are their own inverses. For any function $f$ that is its own inverse, $f(f(x)) = x$.

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**b. A relation R is defined in the set N as follows :**
**R = {(x, y) : x = y − 3, y ¿ 3}**
**Then which of the following is correct?**

(A) (2, 4) ∈ R
(B) (5, 8) ∈ R
(C) (3, 7) ∈ R
(D) (1, 5) ∈ R

**Correct Answer:** (B) (5, 8) ∈ R

**Solution:**

**Step 1: Understanding the Concept:**
An ordered pair (x, y) belongs to the relation R if and only if it satisfies both conditions given in the definition of R:

1. $y > 3$
2. $x = y - 3$
We need to test each of the given options against these two conditions.

**Step 2: Detailed Explanation or Calculation:**
**(A) (2, 4):** Here, $x = 2$ and $y = 4$.
- Check condition 1: $y > 3$. $4 > 3$ is true.
- Check condition 2: $x = y - 3$. $2 = 4 - 3$, which means $2 = 1$. This is false.
So, $(2, 4) \notin$ R.

**(B) (5, 8):** Here, $x = 5$ and $y = 8$.
- Check condition 1: $y > 3$. $8 > 3$ is true.
- Check condition 2: $x = y - 3$. $5 = 8 - 3$, which means $5 = 5$. This is true.
Since both conditions are satisfied, $(5, 8) \in$ R.

**(C) (3, 7):** Here, $x = 3$ and $y = 7$.
- Check condition 1: $y > 3$. $7 > 3$ is true.
- Check condition 2: $x = y - 3$. $3 = 7 - 3$, which means $3 = 4$. This is false.
So, $(3, 7) \notin$ R.

**(D) (1, 5):** Here, $x = 1$ and $y = 5$.
- Check condition 1: $y > 3$. $5 > 3$ is true.
- Check condition 2: $x = y - 3$. $1 = 5 - 3$, which means $1 = 2$. This is false.
So, $(1, 5) \notin$ R.

**Step 3: Final Answer:**
The only pair that satisfies both conditions is (5, 8).

**Quick Tip**

For questions involving relations, carefully check all conditions for each option. It's easy to make a mistake by only checking one of the conditions. A systematic check, as shown above, prevents errors.

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**c. The value of $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$ will be :**

(A) $\frac{\pi}{3}$
(B) $\frac{2\pi}{3}$
(C) $\frac{\pi}{6}$
(D) $\frac{\pi}{12}$

**Correct Answer:** (D) $\frac{\pi}{12}$

**Solution:**

**Step 1: Understanding the Concept:**
This is a definite integral problem. We need to find the antiderivative of the integrand and then evaluate it at the upper and lower limits of integration, according to the Fundamental Theorem of Calculus.

**Step 2: Key Formula or Approach:**
The standard integral formula required is:

$$\int \frac{1}{1+x^2} dx = \tan^{-1}(x) + C$$

The Fundamental Theorem of Calculus states:

$$\int_a^b f(x)dx = F(b) - F(a)$$

, where $F'(x) = f(x)$.

**Step 3: Detailed Explanation or Calculation:**
First, find the antiderivative of the integrand:

$$\int \frac{dx}{1+x^2} = \tan^{-1}(x)$$

Now, apply the limits of integration from 1 to $\sqrt{3}$:

$$\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = [\tan^{-1}(x)]_1^{\sqrt{3}}$$
$$= \tan^{-1}(\sqrt{3}) - \tan^{-1}(1)$$

We need to find the principal values for these inverse trigonometric functions.
- The angle whose tangent is $\sqrt{3}$ is $\frac{\pi}{3}$. So, $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$.
- The angle whose tangent is 1 is $\frac{\pi}{4}$. So, $\tan^{-1}(1) = \frac{\pi}{4}$.
Now, perform the subtraction:

$$\frac{\pi}{3} - \frac{\pi}{4} = \frac{4\pi - 3\pi}{12} = \frac{\pi}{12}$$

**Step 4: Final Answer:**
The value of the definite integral is $\frac{\pi}{12}$.

**Quick Tip**

Memorizing standard integral forms, especially those involving trigonometric and inverse trigonometric functions, is crucial for speed and accuracy. For definite integrals, be careful with the arithmetic when finding a common denominator.

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**d. The order of the differential equation $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + y\cos x = 0$ is :**

(A) 2
(B) 3
(C) 0
(D) Not defined

**Correct Answer:** (B) 3

**Solution:**

**Step 1: Understanding the Concept:**
The **order** of a differential equation is defined as the order of the highest derivative that appears in the equation.

**Step 2: Detailed Explanation or Calculation:**
The given differential equation is:

$$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + y\cos x = 0$$

Let's identify the derivatives present in the equation:
- $\frac{d^3y}{dx^3}$ is the third derivative (order 3).
- $\frac{d^2y}{dx^2}$ is the second derivative (order 2).
The highest order among all the derivatives in the equation is 3.

**Step 3: Final Answer:**
Therefore, the order of the differential equation is 3.

Quick Tip

Do not confuse the 'order' with the 'degree' of a differential equation. The 'degree' is the highest power of the highest-order derivative, after the equation has been cleared of radicals and fractions in its derivatives. The 'order' is simply the highest derivative itself.

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**e. The value of $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ is :**

(A) $\pi$
(B) $-\frac{\pi}{3}$
(C) $\frac{\pi}{3}$
(D) $\frac{2\pi}{3}$

**Correct Answer:** (B) $-\frac{\pi}{3}$

**Solution:**

**Step 1: Understanding the Concept:**
We need to evaluate the given expression by finding the principal values of the two inverse trigonometric functions. The principal value is the unique value of the angle within the defined range of the inverse trigonometric function.

**Step 2: Key Formula or Approach:**
The principal value ranges are:
- For $\tan^{-1}(x)$, the range is $(-\frac{\pi}{2}, \frac{\pi}{2})$.
- For $\sec^{-1}(x)$, the range is $[0, \pi] - \{\frac{\pi}{2}\}$.

**Step 3: Detailed Explanation or Calculation:**
**Part 1: Evaluate** $\tan^{-1}(\sqrt{3})$
Let $\theta = \tan^{-1}(\sqrt{3})$. This means $\tan(\theta) = \sqrt{3}$, where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.
The angle in this range for which the tangent is $\sqrt{3}$ is $\frac{\pi}{3}$.
So, $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$.

**Part 2: Evaluate** $\sec^{-1}(-2)$
Let $\phi = \sec^{-1}(-2)$. This means $\sec(\phi) = -2$, where $\phi \in [0, \pi]$ and $\phi \neq \frac{\pi}{2}$.
$\sec(\phi) = -2$ is equivalent to $\cos(\phi) = -\frac{1}{2}$.
We need to find the angle $\phi$ in the range $[0, \pi]$ where the cosine is $-\frac{1}{2}$. This occurs in the second quadrant.
The reference angle is $\cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}$. For the second quadrant, the angle is $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$.
So, $\sec^{-1}(-2) = \frac{2\pi}{3}$.

**Part 3: Combine the results**

$$\tan^{-1}\sqrt{3} - \sec^{-1}(-2) = \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$$

**Step 4: Final Answer:**
The value of the expression is $-\frac{\pi}{3}$.

**Quick Tip**

Always be mindful of the principal value ranges for inverse trigonometric functions. A common mistake is to choose an angle in the wrong quadrant. For $\sec^{-1}(x)$, it's often easier to convert the problem to $\cos^{-1}(1/x)$ and use its well-known range.

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**2. Do all parts.**

**a. Find the principal value of** $\sec^{-1}(-\sqrt{2})$**.**

**Solution:**

**Step 1: Understanding the Concept:**
We need to find the principal value of $\sec^{-1}(-\sqrt{2})$. The principal value is the unique angle $\theta$ within the defined range of the inverse secant function such that $\sec(\theta) = -\sqrt{2}$.

**Step 2: Key Formula or Approach:**
The principal value range for $y = \sec^{-1}(x)$ is $[0, \pi] - \{\frac{\pi}{2}\}$.
Let $\theta = \sec^{-1}(-\sqrt{2})$. This implies $\sec(\theta) = -\sqrt{2}$, which is equivalent to $\cos(\theta) = -\frac{1}{\sqrt{2}}$.

**Step 3: Detailed Explanation or Calculation:**
We need to find the angle $\theta$ in the interval $[0, \pi]$ for which $\cos(\theta) = -\frac{1}{\sqrt{2}}$.
The reference angle, for which $\cos(\alpha) = \frac{1}{\sqrt{2}}$, is $\alpha = \frac{\pi}{4}$.
Since the cosine value is negative, the angle $\theta$ must lie in the second quadrant.
The angle in the second quadrant with the reference angle $\frac{\pi}{4}$ is given by $\theta = \pi - \alpha$.

$$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

This value, $\frac{3\pi}{4}$, lies within the principal value range of $[0, \pi]$ and is not equal to $\frac{\pi}{2}$.

**Step 4: Final Answer:**
The principal value of $\sec^{-1}(-\sqrt{2})$ is $\frac{3\pi}{4}$.

**Quick Tip**

To find the principal value of $\sec^{-1}(x)$, it is often easier to convert the problem to its cosine equivalent, $\cos^{-1}(1/x)$, and find the angle within the range $[0, \pi]$.

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**b. Does the function** $f(x) = \begin{cases} x+5 & \textbf{if } x \leq 1 \\ x-5 & \textbf{if } x > 1 \end{cases}$ **continuous at** $x = 1$**?**

**Solution:**

**Step 1: Understanding the Concept:**
A function $f(x)$ is continuous at a point $x = c$ if the following three conditions are met:
1. $f(c)$ is defined.
2. The limit of $f(x)$ as $x$ approaches $c$ exists. This means the left-hand limit (LHL) must equal the right-hand limit (RHL).
3. The limit equals the function value: $\lim_{x\to c} f(x) = f(c)$.

**Step 2: Key Formula or Approach:**
We need to calculate the LHL, RHL, and $f(1)$ and check if they are equal.
LHL: $\lim_{x\to 1^-} f(x)$
RHL: $\lim_{x\to 1^+} f(x)$

**Step 3: Detailed Explanation or Calculation:**
**1. Calculate the Left-Hand Limit (LHL):**
For $x \to 1^-$, we use the definition $f(x) = x + 5$ because $x \leq 1$.

$$\text{LHL} = \lim_{x \to 1^-} (x + 5) = 1 + 5 = 6$$

**2. Calculate the Right-Hand Limit (RHL):**
For $x \to 1^+$, we use the definition $f(x) = x - 5$ because $x > 1$.

$$\text{RHL} = \lim_{x \to 1^+} (x - 5) = 1 - 5 = -4$$

**3. Compare the limits:**
Here, LHL = 6 and RHL = -4.
Since LHL $\neq$ RHL, the limit $\lim_{x \to 1} f(x)$ does not exist.

**Step 4: Final Answer:**
Because the limit of the function as $x$ approaches 1 does not exist, the function $f(x)$ is not continuous (it is discontinuous) at $x = 1$.

> **Quick Tip**
>
> For piecewise functions, the points where the definition changes are critical points to check for continuity. The first step should always be to check if the left-hand and right-hand limits are equal at these points.

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**c. Find the order and degree of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y \cdot \sin x = 0$.**

**Solution:**

**Step 1: Understanding the Concept:**
- **Order** of a differential equation is the order of the highest derivative present in the equation.
- **Degree** of a differential equation is the highest power of the highest-order derivative, provided the equation is a polynomial in its derivatives.

**Step 2: Key Formula or Approach:**
1. Identify all the derivatives in the equation.
2. Find the highest order among them. This is the 'order'.
3. Find the power to which this highest-order derivative is raised. This is the 'degree'.

**Step 3: Detailed Explanation or Calculation:**
The given differential equation is:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y \sin x = 0$$

**1. Identifying the order:**
The derivatives in the equation are $\frac{d^2y}{dx^2}$ (second derivative) and $\frac{dy}{dx}$ (first derivative).
The highest order of the derivatives is 2. Therefore, the **order** of the differential equation is 2.
**2. Identifying the degree:**
The equation is a polynomial in terms of its derivatives. The highest-order derivative is $\frac{d^2y}{dx^2}$, and its power is 1.
Therefore, the **degree** of the differential equation is 1.

**Step 4: Final Answer:**
The order of the differential equation is 2 and the degree is 1.

> **Quick Tip**
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> The order is determined by the "highest dash" (e.g., y", y"') while the degree is the exponent on that highest-order term. Ensure the equation is free from radicals or fractional powers of derivatives before determining the degree.

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**d. Find the direction cosines of a line joining two points $(-2, 4, -5)$ and $(1, 2, 3)$.**

**Solution:**

**Step 1: Understanding the Concept:**
The direction cosines of a line are the cosines of the angles the line makes with the positive x, y, and z axes. To find them, we first need the direction ratios of the line, which can be found from the coordinates of the two given points. Then, we normalize these ratios by dividing by the magnitude of the direction vector.

**Step 2: Key Formula or Approach:**
Let the two points be $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.
1. The direction ratios (a, b, c) are given by:
$a = x_2 - x_1, b = y_2 - y_1, c = z_2 - z_1$
2. The magnitude of the vector PQ is $r = \sqrt{a^2 + b^2 + c^2}$.
3. The direction cosines (l, m, n) are:
$l = \frac{a}{r}, m = \frac{b}{r}, n = \frac{c}{r}$

**Step 3: Detailed Explanation or Calculation:**
Let the given points be $P(-2, 4, -5)$ and $Q(1, 2, 3)$.
**1. Find the direction ratios (a, b, c):**

$$a = 1 - (-2) = 3$$

$$b = 2 - 4 = -2$$

$$c = 3 - (-5) = 8$$

So, the direction ratios are $\langle 3, -2, 8 \rangle$.

**2. Find the magnitude (r):**

$$r = \sqrt{3^2 + (-2)^2 + 8^2} = \sqrt{9 + 4 + 64} = \sqrt{77}$$

**3. Find the direction cosines (l, m, n):**

$$l = \frac{a}{r} = \frac{3}{\sqrt{77}}$$

$$m = \frac{b}{r} = \frac{-2}{\sqrt{77}}$$

$$n = \frac{c}{r} = \frac{8}{\sqrt{77}}$$

**Step 4: Final Answer:**

The direction cosines of the line are $\left(\frac{3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{8}{\sqrt{77}}\right)$.

**Quick Tip**

Remember that direction cosines are the components of the unit vector in the direction of the line. A quick check is that the sum of the squares of the direction cosines should always be equal to 1: $l^2 + m^2 + n^2 = 1$.

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**e. If P(A) = 0.3, P(B) = 0.4, then find P(A/B) if A and B are independent events.**

**Solution:**

**Step 1: Understanding the Concept:**

We need to find the conditional probability of event A given that event B has occurred, denoted as P(A/B). We are given that the events A and B are independent.

**Step 2: Key Formula or Approach:**

The definition of conditional probability is:

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

The definition of independent events is:

$$P(A \cap B) = P(A) \cdot P(B)$$

**Step 3: Detailed Explanation or Calculation:**

For independent events, the occurrence of one event does not affect the probability of the other. We can show this using the formulas.

Substitute the formula for independent events into the conditional probability formula:

$$P(A/B) = \frac{P(A) \cdot P(B)}{P(B)}$$

Since $P(B) = 0.4 \neq 0$, we can cancel $P(B)$ from the numerator and denominator:

$$P(A/B) = P(A)$$

We are given that $P(A) = 0.3$.
Therefore, $P(A/B) = 0.3$.

**Step 4: Final Answer:**
The value of P(A/B) is 0.3.

Quick Tip

A key property of independent events is that the conditional probability of one event given the other is simply the probability of the first event. That is, if A and B are independent, $P(A|B) = P(A)$ and $P(B|A) = P(B)$. Knowing this can provide a direct answer without calculation.

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**3. Do all parts.**

**a. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ and $A + A' = I$, then find the value of $\alpha$.**

**Solution:**

**Step 1: Understanding the Concept:**
We are given a matrix A and a condition involving the matrix, its transpose (A'), and the identity matrix (I). We need to solve the resulting matrix equation to find the value of the angle $\alpha$.

**Step 2: Key Formula or Approach:**
1. Find the transpose of A, denoted by A', by interchanging its rows and columns.
2. Add the matrices A and A'.
3. Set the resulting matrix equal to the 2x2 identity matrix, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
4. Equate the corresponding elements to form an equation and solve for $\alpha$.

**Step 3: Detailed Explanation or Calculation:**
The given matrix is:

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

Its transpose is:

$$A' = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

Now, add A and A':

$$A + A' = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} + \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$A + A' = \begin{bmatrix} \cos\alpha + \cos\alpha & -\sin\alpha + \sin\alpha \\ \sin\alpha - \sin\alpha & \cos\alpha + \cos\alpha \end{bmatrix} = \begin{bmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{bmatrix}$$

We are given that $A + A' = I$:

$$\begin{bmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating the corresponding elements (specifically, the diagonal elements):

$$2\cos\alpha = 1$$
$$\cos\alpha = \frac{1}{2}$$

The principal value of $\alpha$ for which $\cos\alpha = \frac{1}{2}$ is $\alpha = \frac{\pi}{3}$.

**Step 4: Final Answer:**
The value of $\alpha$ is $\frac{\pi}{3}$.

**Quick Tip**

Remember the basic properties of matrices: the transpose A' is found by flipping the matrix over its main diagonal, and matrix addition involves adding corresponding elements. For matrix equality, every corresponding element must be equal.

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**b. If $y = x^{x^{x^{\cdots\text{ad inf}}}}$, then prove that $x\frac{dy}{dx} = \frac{y^2}{1-y\log x}$.**

**Solution:**

**Step 1: Understanding the Concept:**
The given function is an infinite power tower. The key to solving such problems is to recognize the repeating pattern. Since the tower is infinite, the exponent of the first x is the same as the entire expression y. This allows us to write a simpler implicit equation, which can then be differentiated.

**Step 2: Key Formula or Approach:**
1. Rewrite the function using its self-similar property: $y = x^y$.

2. Use logarithmic differentiation because the function is of the form $[f(x)]^{g(x)}$.
3. Take the natural logarithm of both sides.
4. Differentiate implicitly with respect to x.
5. Rearrange the resulting equation to find $\frac{dy}{dx}$.

**Step 3: Detailed Explanation or Calculation:**
The given function is $y = x^{x^{x^{\cdots}}}$.
We can write this as:

$$y = x^y$$

Take the natural logarithm (log) of both sides:

$$\log y = \log(x^y)$$

Using the logarithm property $\log(a^b) = b \log a$:

$$\log y = y \log x$$

Now, differentiate both sides with respect to x, using implicit differentiation and the product rule on the right side:

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(y \log x)$$

$$\frac{1}{y}\frac{dy}{dx} = \left(\frac{dy}{dx} \cdot \log x\right) + \left(y \cdot \frac{1}{x}\right)$$

Rearrange the equation to isolate terms with $\frac{dy}{dx}$:

$$\frac{1}{y}\frac{dy}{dx} - \frac{dy}{dx}\log x = \frac{y}{x}$$

Factor out $\frac{dy}{dx}$:

$$\frac{dy}{dx}\left(\frac{1}{y} - \log x\right) = \frac{y}{x}$$

$$\frac{dy}{dx}\left(\frac{1 - y\log x}{y}\right) = \frac{y}{x}$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{y}{x} \cdot \frac{y}{1 - y\log x} = \frac{y^2}{x(1 - y\log x)}$$

Multiply both sides by x to get the required form:

$$x\frac{dy}{dx} = \frac{y^2}{1 - y\log x}$$

**Step 4: Final Answer:**
The relation $x\frac{dy}{dx} = \frac{y^2}{1-y\log x}$ is proved.

Quick Tip

Logarithmic differentiation is the standard technique for functions of the form $y = u(x)^{v(x)}$. The trick for infinite nested functions (like power towers or continued fractions) is to express the whole function in terms of itself to get a simpler, non-infinite equation.

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**c. Find the angle between the pair of lines $\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}$ and $\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$.**

**Solution:**

**Step 1: Understanding the Concept:**
The angle between two lines in 3D space is the angle between their direction vectors. We can find the direction ratios of each line from their Cartesian equations and then use the dot product formula to find the cosine of the angle between them.

**Step 2: Key Formula or Approach:**
If $\vec{b_1} = \langle a_1, b_1, c_1 \rangle$ and $\vec{b_2} = \langle a_2, b_2, c_2 \rangle$ are the direction vectors of two lines, the angle $\theta$ between them is given by:

$$\cos\theta = \frac{|\vec{b_1} \cdot \vec{b_2}|}{|\vec{b_1}||\vec{b_2}|} = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

**Step 3: Detailed Explanation or Calculation:**
From the equations of the lines, we can identify their direction ratios.
For the first line, $\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}$, the direction vector is:

$$\vec{b_1} = 3\hat{i} + 5\hat{j} + 4\hat{k}$$

For the second line, $\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$, the direction vector is:

$$\vec{b_2} = 1\hat{i} + 1\hat{j} + 2\hat{k}$$

Now, calculate the dot product $\vec{b_1} \cdot \vec{b_2}$:

$$\vec{b_1} \cdot \vec{b_2} = (3)(1) + (5)(1) + (4)(2) = 3 + 5 + 8 = 16$$

Next, calculate the magnitudes of the direction vectors:

$$|\vec{b_1}| = \sqrt{3^2 + 5^2 + 4^2} = \sqrt{9 + 25 + 16} = \sqrt{50} = 5\sqrt{2}$$
$$|\vec{b_2}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{1 + 1 + 4} = \sqrt{6}$$

Now, use the formula for $\cos\theta$:

$$\cos\theta = \frac{16}{(5\sqrt{2})(\sqrt{6})} = \frac{16}{5\sqrt{12}} = \frac{16}{5(2\sqrt{3})} = \frac{16}{10\sqrt{3}} = \frac{8}{5\sqrt{3}}$$

The angle $\theta$ is:

$$\theta = \cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$$

**Step 4: Final Answer:**
The angle between the pair of lines is $\cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$.

**Quick Tip**

The direction ratios of a line given in the form $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ are simply the denominators $\langle a, b, c\rangle$. Always ensure the coefficients of x, y, and z are 1 before reading the direction ratios.

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**d. For two invertible matrices A and B of order n, prove that $(AB)^{-1} = B^{-1}A^{-1}$.**

**Solution:**

**Step 1: Understanding the Concept:**
We need to prove the reversal law for the inverse of a product of matrices. The proof relies on the fundamental definition of a matrix inverse: a matrix M is the inverse of N if their product, in either order, is the identity matrix I (i.e., $MN = NM = I$).

**Step 2: Key Formula or Approach:**
To prove that $B^{-1}A^{-1}$ is the inverse of $(AB)$, we need to show that:
1. $(AB)(B^{-1}A^{-1}) = I$
2. $(B^{-1}A^{-1})(AB) = I$

**Step 3: Detailed Explanation or Calculation:**
Let's start with the product $(AB)$ and $(B^{-1}A^{-1})$.
**Proof of the first condition:**

$$(AB)(B^{-1}A^{-1})$$

Using the associative property of matrix multiplication, we can regroup the terms:

$$= A(BB^{-1})A^{-1}$$

Since B is an invertible matrix, by definition, $BB^{-1} = I$ (the identity matrix).

$$= A(I)A^{-1}$$

The product of any matrix with the identity matrix is the matrix itself ($AI = A$).

$$= AA^{-1}$$

Since A is an invertible matrix, by definition, $AA^{-1} = I$.

$$= I$$

So, we have shown that $(AB)(B^{-1}A^{-1}) = I$.

**Proof of the second condition:**
Now, let's check the product in the reverse order:

$$(B^{-1}A^{-1})(AB)$$

Using the associative property:

$$= B^{-1}(A^{-1}A)B$$

Since A is invertible, $A^{-1}A = I$.

$$= B^{-1}(I)B$$

Since $B^{-1}I = B^{-1}$:

$$= B^{-1}B$$

Since B is invertible, $B^{-1}B = I$.

$$= I$$

So, we have also shown that $(B^{-1}A^{-1})(AB) = I$.

**Step 4: Final Answer:**
Since $(AB)(B^{-1}A^{-1}) = I$ and $(B^{-1}A^{-1})(AB) = I$, by the definition of an inverse, the inverse of the matrix $(AB)$ is $B^{-1}A^{-1}$. Hence, $(AB)^{-1} = B^{-1}A^{-1}$ is proved.

Quick Tip

This property is often called the "socks and shoes" rule. To undo the process of putting on socks then shoes, you must first take off the shoes and then take off the socks (the reverse order). Similarly, the inverse of a product of matrices is the product of their inverses in the reverse order.

---

**4. Do all parts.**

**a. Show that the given function $f(x) = \cos x$ is increasing in $(\pi, 2\pi)$.**

**Solution:**

**Step 1: Understanding the Concept:**
A function is said to be increasing on an interval if its first derivative is positive throughout

that interval. To show that $f(x) = \cos x$ is increasing on $(\pi, 2\pi)$, we need to find its derivative, $f'(x)$, and show that $f'(x) > 0$ for all $x$ in the interval $(\pi, 2\pi)$.

**Step 2: Key Formula or Approach:**
The condition for a function $f(x)$ to be increasing is $f'(x) > 0$.
The derivative of $\cos x$ is $-\sin x$.

**Step 3: Detailed Explanation or Calculation:**
The given function is:

$$f(x) = \cos x$$

First, we find the derivative of the function:

$$f'(x) = \frac{d}{dx}(\cos x) = -\sin x$$

Now, we need to determine the sign of $f'(x)$ in the interval $(\pi, 2\pi)$.
The interval $(\pi, 2\pi)$ corresponds to the third and fourth quadrants of the unit circle.
- In the third quadrant, $x \in (\pi, 3\pi/2)$, the value of $\sin x$ is negative ($\sin x < 0$).
- In the fourth quadrant, $x \in (3\pi/2, 2\pi)$, the value of $\sin x$ is also negative ($\sin x < 0$).
Therefore, for the entire interval $x \in (\pi, 2\pi)$, we have $\sin x < 0$.
Now, let's look at the derivative $f'(x) = -\sin x$.
Since $\sin x$ is negative, $-\sin x$ will be positive.

$$f'(x) = -(\text{a negative value}) > 0$$

So, $f'(x) > 0$ for all $x \in (\pi, 2\pi)$.

**Step 4: Final Answer:**
Since the first derivative $f'(x) = -\sin x$ is positive on the interval $(\pi, 2\pi)$, the function $f(x) = \cos x$ is increasing on this interval. Hence proved.

**Quick Tip**

To determine if a function is increasing or decreasing, always analyze the sign of its first derivative. A positive derivative implies an increasing function, while a negative derivative implies a decreasing function. Visualizing the sine and cosine graphs or the unit circle can quickly help determine the sign of trigonometric functions in different intervals.

---

**b. If $[a_{ij}] = 2i - j$, then determine a matrix A of order $2 \times 3$.**

**Solution:**

**Step 1: Understanding the Concept:**
We are asked to construct a matrix A of order $2 \times 3$. This means the matrix will have 2 rows and 3 columns. The value of each element $a_{ij}$ in the matrix is given by the formula $2i - j$, where $i$ is the row number and $j$ is the column number.

**Step 2: Key Formula or Approach:**
A general $2 \times 3$ matrix A can be written as:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

We will calculate each element $a_{ij}$ using the given formula $a_{ij} = 2i - j$.

**Step 3: Detailed Explanation or Calculation:**
We calculate each of the six elements of the matrix:
**For the first row (i=1):**
- $a_{11} = 2(1) - 1 = 2 - 1 = 1$
- $a_{12} = 2(1) - 2 = 2 - 2 = 0$
- $a_{13} = 2(1) - 3 = 2 - 3 = -1$
**For the second row (i=2):**
- $a_{21} = 2(2) - 1 = 4 - 1 = 3$
- $a_{22} = 2(2) - 2 = 4 - 2 = 2$
- $a_{23} = 2(2) - 3 = 4 - 3 = 1$
Now, we assemble these elements into the matrix A.

**Step 4: Final Answer:**
The matrix A of order $2 \times 3$ is:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \end{bmatrix}$$

**Quick Tip**

When constructing a matrix from a formula, be systematic. Go row by row or column by column, carefully substituting the correct values of $i$ (row index) and $j$ (column index) into the given formula.

---

**c. Find the area of a parallelogram whose adjacent sides are given by vectors $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$.**

**Solution:**

**Step 1: Understanding the Concept:**
The area of a parallelogram whose adjacent sides are represented by two vectors is equal to the

magnitude of the cross product of those two vectors.

**Step 2: Key Formula or Approach:**
Area of parallelogram $= |\vec{a} \times \vec{b}|$.
The cross product is calculated using the determinant:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

**Step 3: Detailed Explanation or Calculation:**
The given vectors are $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$.
First, we compute the cross product $\vec{a} \times \vec{b}$:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 1 & 2 & -2 \end{vmatrix}$$

Expanding the determinant:

$$\begin{aligned} &= \hat{i}((1)(-2) - (2)(2)) - \hat{j}((3)(-2) - (2)(1)) + \hat{k}((3)(2) - (1)(1)) \\ &= \hat{i}(-2-4) - \hat{j}(-6-2) + \hat{k}(6-1) \\ &= \hat{i}(-6) - \hat{j}(-8) + \hat{k}(5) \\ &= -6\hat{i} + 8\hat{j} + 5\hat{k} \end{aligned}$$

Next, we find the magnitude of this resulting vector:

$$\begin{aligned} |\vec{a} \times \vec{b}| &= \sqrt{(-6)^2 + (8)^2 + (5)^2} \\ &= \sqrt{36 + 64 + 25} \\ &= \sqrt{125} = \sqrt{25 \times 5} = 5\sqrt{5} \end{aligned}$$

**Step 4: Final Answer:**
The area of the parallelogram is $5\sqrt{5}$ square units.

> **Quick Tip**
>
> Remember the geometric interpretation of the cross product: its magnitude gives the area of the parallelogram formed by the two vectors. Be careful with signs when calculating the determinant for the cross product.

---

**d. The volume of a cube is increasing at the rate of 9 $cm^3/s$. If the length of its edge is 10 cm, then its surface area is increasing with which rate?**

**Solution:**

**Step 1: Understanding the Concept:**
This is a related rates problem. We are given the rate of change of the volume of a cube ($\frac{dV}{dt}$) and need to find the rate of change of its surface area ($\frac{dS}{dt}$) at a specific instant when the edge length is known. The link between these rates is the rate of change of the edge length ($\frac{dx}{dt}$).

**Step 2: Key Formula or Approach:**
Let $x$ be the edge length of the cube.
Volume of the cube: $V = x^3$
Surface area of the cube: $S = 6x^2$
We will differentiate both formulas with respect to time (t) to relate the rates.

**Step 3: Detailed Explanation or Calculation:**
First, let's write down the given information:
- Rate of increase of volume: $\frac{dV}{dt} = 9 \text{ cm}^3/\text{s}$
- Edge length: $x = 10$ cm

We need to find $\frac{dS}{dt}$.
Differentiate the volume formula with respect to time $t$:

$$\frac{dV}{dt} = \frac{d}{dt}(x^3) = 3x^2\frac{dx}{dt}$$

We can use this equation to find $\frac{dx}{dt}$, the rate at which the edge length is increasing. Substitute the known values:

$$9 = 3(10)^2\frac{dx}{dt}$$
$$9 = 3(100)\frac{dx}{dt}$$
$$9 = 300\frac{dx}{dt}$$
$$\frac{dx}{dt} = \frac{9}{300} = \frac{3}{100} \text{ cm/s}$$

Now, differentiate the surface area formula with respect to time $t$:

$$\frac{dS}{dt} = \frac{d}{dt}(6x^2) = 12x\frac{dx}{dt}$$

Substitute the known values of $x$ and the calculated value of $\frac{dx}{dt}$:

$$\frac{dS}{dt} = 12(10)\left(\frac{3}{100}\right)$$
$$\frac{dS}{dt} = 120\left(\frac{3}{100}\right) = \frac{360}{100} = 3.6$$

**Step 4: Final Answer:**
The surface area of the cube is increasing at a rate of 3.6 $\text{cm}^2/\text{s}$.

Quick Tip

In related rates problems, the key is to find an intermediate rate that connects the given rate and the required rate. Here, $\frac{dx}{dt}$ was the crucial link between $\frac{dV}{dt}$ and $\frac{dS}{dt}$. Always start by writing the geometric formulas, then differentiate with respect to time.

---

**5. Do all parts.**

**a. Does the relation defined by R = {(x, y) : y is divisible by x} on the set A = {1, 2, 3, 4, 5, 6}, an equivalence relation?**

**Solution:**

**Step 1: Understanding the Concept:**
For a relation to be an equivalence relation, it must satisfy three properties: reflexivity, symmetry, and transitivity. We need to test the given relation R for each of these properties.

**Step 2: Key Definitions:**
- **Reflexive:** For all $a \in A$, $(a, a) \in R$. (Is every element related to itself?)
- **Symmetric:** If $(a, b) \in R$, then $(b, a) \in R$. (If a is related to b, is b related to a?)
- **Transitive:** If $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$. (If a is related to b and b is related to c, is a related to c?)

**Step 3: Detailed Explanation:**
**1. Test for Reflexivity:**
For any element $x \in A$, the pair $(x, x)$ is in R if 'x is divisible by x'. Every number is divisible by itself. Therefore, $(x, x) \in R$ for all $x \in A$. The relation is **reflexive**.

**2. Test for Symmetry:**
We need to check if $(x, y) \in R$ implies $(y, x) \in R$. This means, if 'y is divisible by x', does it imply that 'x is divisible by y'?
Let's take a counterexample from the set A.
Consider the pair (2, 4). Since 4 is divisible by 2, $(2, 4) \in R$.
Now, we must check if the reverse is true, i.e., if $(4, 2) \in R$. This would mean that 2 is divisible by 4. This is false.
Since we found a counterexample, the relation is **not symmetric**.

**3. Conclusion:**
An equivalence relation must be reflexive, symmetric, and transitive. Since the relation R is not symmetric, it cannot be an equivalence relation. We do not need to check for transitivity.

**Step 4: Final Answer:**
No, the given relation is not an equivalence relation because it is not symmetric.

**Quick Tip**

To disprove a property like symmetry or transitivity, you only need to find one counterexample. To prove it, you must show it holds for all possible cases. Forgetting a single property check can lead to an incorrect conclusion.

---

**b. Express the matrix $A = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.**

**Solution:**

**Step 1: Understanding the Concept:**
Any square matrix A can be uniquely expressed as the sum of a symmetric matrix P and a skew-symmetric matrix Q.

**Step 2: Key Formula or Approach:**
The symmetric part is given by $P = \frac{1}{2}(A + A^T)$.
The skew-symmetric part is given by $Q = \frac{1}{2}(A - A^T)$.
Then, $A = P + Q$.

**Step 3: Detailed Explanation or Calculation:**
The given matrix is $A = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$.
First, find the transpose of A:

$$A^T = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

**Calculate the symmetric matrix P:**

$$A + A^T = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix}$$

$$P = \frac{1}{2}(A + A^T) = \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{bmatrix}$$

**Calculate the skew-symmetric matrix Q:**

$$A - A^T = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$$

$$Q = \frac{1}{2}(A - A^T) = \frac{1}{2}\begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 5/2 & 3/2 \\ -5/2 & 0 & 3 \\ -3/2 & -3 & 0 \end{bmatrix}$$

**Step 4: Final Answer:**
The matrix A can be expressed as the sum $A = P + Q$, where:

$$A = \begin{bmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 5/2 & 3/2 \\ -5/2 & 0 & 3 \\ -3/2 & -3 & 0 \end{bmatrix}$$

**Quick Tip**

A quick check for your result: P should be symmetric ($P = P^T$) and Q should be skew-symmetric ($Q = -Q^T$). Also, the diagonal elements of any skew-symmetric matrix must be zero.

---

**c. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ for $-1 < x < 1$, then prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$.**

**Solution:**

**Step 1: Understanding the Concept:**
We are given an implicit relation between x and y and need to find its derivative, $\frac{dy}{dx}$. While implicit differentiation is possible, it is often simpler to first algebraically manipulate the equation to express y explicitly as a function of x.

**Step 2: Key Formula or Approach:**
1. Algebraically simplify the given equation to solve for y in terms of x.
2. Differentiate the resulting explicit function $y(x)$ using standard rules of differentiation (like the quotient rule).

**Step 3: Detailed Explanation or Calculation:**
The given equation is $x\sqrt{1+y} + y\sqrt{1+x} = 0$.
Rearrange the equation:

$$x\sqrt{1+y} = -y\sqrt{1+x}$$

Square both sides:

$$x^2(1+y) = (-y)^2(1+x)$$
$$x^2 + x^2y = y^2 + y^2x$$

Rearrange the terms to group x and y:

$$x^2 - y^2 = y^2x - x^2y$$

Factor both sides. The left side is a difference of squares, and we can factor -xy from the right side:

$$(x-y)(x+y) = -xy(x-y)$$

Assuming $x \neq y$, we can divide both sides by $(x-y)$:

$$x + y = -xy$$

Now, solve for y:

$$y + xy = -x$$
$$y(1+x) = -x$$
$$y = -\frac{x}{1+x}$$

Now we have an explicit function for y. We can differentiate it with respect to x using the quotient rule, $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v-uv'}{v^2}$:

$$\frac{dy}{dx} = -\left[\frac{(1)(1+x)-(x)(1)}{(1+x)^2}\right]$$
$$\frac{dy}{dx} = -\left[\frac{1+x-x}{(1+x)^2}\right]$$
$$\frac{dy}{dx} = -\frac{1}{(1+x)^2}$$

**Step 4: Final Answer:**
Hence, it is proved that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$.

**Quick Tip**

Before jumping into implicit differentiation, which can be complicated, always check if the equation can be algebraically simplified to get an explicit function. This often leads to a much easier differentiation process.

---

**d. Find the angle between the pair of lines:**
$\vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda(\hat{i} - \hat{j} - 2\hat{k})$ **and**
$\vec{r} = 2\hat{i} - \hat{j} - 56\hat{k} + \mu(3\hat{i} - 5\hat{j} - 4\hat{k})$**.**

**Solution:**

**Step 1: Understanding the Concept:**
The angle between two lines in vector form is the angle between their direction vectors. The direction vectors are the vectors multiplied by the scalar parameters ($\lambda$ and $\mu$). We use the dot

product formula to find the angle.

**Step 2: Key Formula or Approach:**
If $\vec{b_1}$ and $\vec{b_2}$ are the direction vectors of the two lines, the angle $\theta$ between them is given by:

$$\cos\theta = \frac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}||\vec{b_2}|}$$

**Step 3: Detailed Explanation or Calculation:**
From the given line equations, we identify the direction vectors:
For the first line, the direction vector is $\vec{b_1} = \hat{i} - \hat{j} - 2\hat{k}$.
For the second line, the direction vector is $\vec{b_2} = 3\hat{i} - 5\hat{j} - 4\hat{k}$.
**1. Calculate the dot product $\vec{b_1} \cdot \vec{b_2}$:**

$$\vec{b_1} \cdot \vec{b_2} = (1)(3) + (-1)(-5) + (-2)(-4) = 3 + 5 + 8 = 16$$

**2. Calculate the magnitudes of the vectors:**

$$|\vec{b_1}| = \sqrt{1^2 + (-1)^2 + (-2)^2} = \sqrt{1+1+4} = \sqrt{6}$$
$$|\vec{b_2}| = \sqrt{3^2 + (-5)^2 + (-4)^2} = \sqrt{9+25+16} = \sqrt{50} = 5\sqrt{2}$$

**3. Calculate $\cos\theta$:**

$$\cos\theta = \frac{16}{(\sqrt{6})(5\sqrt{2})} = \frac{16}{5\sqrt{12}} = \frac{16}{5(2\sqrt{3})} = \frac{16}{10\sqrt{3}} = \frac{8}{5\sqrt{3}}$$

Therefore, the angle is $\theta = \cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$.

**Step 4: Final Answer:**
The angle between the pair of lines is $\cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$.

Quick Tip

In the vector equation of a line $\vec{r} = \vec{a} + \lambda\vec{b}$, $\vec{a}$ is the position vector of a point on the line, and $\vec{b}$ is the direction vector. The angle between lines depends only on their direction vectors.

---

**e. Find the shortest distance between the lines:**
$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ **and**
$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$**.**

**Solution:**

**Step 1: Understanding the Concept:**
The given lines are skew (non-parallel and non-intersecting). The shortest distance between

two skew lines is found using a standard vector formula involving the position vectors of a point on each line and the direction vectors of the lines.

**Step 2: Key Formula or Approach:**
The shortest distance $d$ between two lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ is given by:

$$d = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

**Step 3: Detailed Explanation or Calculation:**
First, identify the vectors from the Cartesian equations:
**Line 1:** Passes through $P_1(-1, -1, -1)$ and has direction vector $\vec{b_1} = 7\hat{i} - 6\hat{j} + \hat{k}$. So, $\vec{a_1} = -\hat{i} - \hat{j} - \hat{k}$.
**Line 2:** Passes through $P_2(3, 5, 7)$ and has direction vector $\vec{b_2} = \hat{i} - 2\hat{j} + \hat{k}$. So, $\vec{a_2} = 3\hat{i} + 5\hat{j} + 7\hat{k}$.
**1. Calculate $\vec{a_2} - \vec{a_1}$:**

$$\vec{a_2} - \vec{a_1} = (3 - (-1))\hat{i} + (5 - (-1))\hat{j} + (7 - (-1))\hat{k} = 4\hat{i} + 6\hat{j} + 8\hat{k}$$

**2. Calculate the cross product $\vec{b_1} \times \vec{b_2}$:**

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix} = \hat{i}(-6 - (-2)) - \hat{j}(7 - 1) + \hat{k}(-14 - (-6))$$

$$= -4\hat{i} - 6\hat{j} - 8\hat{k}$$

**3. Calculate the dot product in the numerator:**

$$(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = (4)(-4) + (6)(-6) + (8)(-8) = -16 - 36 - 64 = -116$$

**4. Calculate the magnitude of the cross product:**

$$|\vec{b_1} \times \vec{b_2}| = \sqrt{(-4)^2 + (-6)^2 + (-8)^2} = \sqrt{16 + 36 + 64} = \sqrt{116}$$

**5. Calculate the shortest distance d:**

$$d = \left| \frac{-116}{\sqrt{116}} \right| = \sqrt{116} = \sqrt{4 \times 29} = 2\sqrt{29}$$

**Step 4: Final Answer:**
The shortest distance between the lines is $2\sqrt{29}$ units.

**Quick Tip**

To convert a line from Cartesian form $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ to vector form, identify the point $(x_1, y_1, z_1)$ to get $\vec{a}$ and the direction ratios $\langle a, b, c \rangle$ to get $\vec{b}$. Be careful with signs, for example, $x + 1$ means $x_1 = -1$.

---

**6. Do all parts.**

**a. Find the equations of the tangent and normal to the curve $x^{2/3} + y^{2/3} = 2$ at the point (1, 1).**

**Solution:**

**Step 1: Understanding the Concept:**
To find the equation of a tangent line to a curve at a given point, we first need to find the slope of the tangent, which is the value of the derivative $\frac{dy}{dx}$ at that point. The slope of the normal line is the negative reciprocal of the tangent's slope.

**Step 2: Key Formula or Approach:**
1. Differentiate the curve's equation implicitly with respect to x to find $\frac{dy}{dx}$.
2. Evaluate $\frac{dy}{dx}$ at the point (1, 1) to get the slope of the tangent, $m_t$.
3. Calculate the slope of the normal: $m_n = -1/m_t$.
4. Use the point-slope form of a line, $y - y_1 = m(x - x_1)$, to find the equations.

**Step 3: Detailed Explanation or Calculation:**
The equation of the curve is $x^{2/3} + y^{2/3} = 2$.
Differentiating both sides with respect to x:

$$\frac{d}{dx}(x^{2/3}) + \frac{d}{dx}(y^{2/3}) = \frac{d}{dx}(2)$$
$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

Divide by $\frac{2}{3}$:

$$x^{-1/3} + y^{-1/3}\frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\left(\frac{y}{x}\right)^{1/3}$$

Now, find the slope of the tangent at the point (1, 1):

$$m_t = \left.\frac{dy}{dx}\right|_{(1,1)} = -\left(\frac{1}{1}\right)^{1/3} = -1$$

The slope of the normal is:

$$m_n = -\frac{1}{m_t} = -\frac{1}{-1} = 1$$

**Equation of the Tangent:**
Using the point-slope form with point (1, 1) and slope $m_t = -1$:

$$y - 1 = -1(x - 1) \Rightarrow y - 1 = -x + 1 \Rightarrow x + y = 2$$

**Equation of the Normal:**
Using the point-slope form with point (1, 1) and slope $m_n = 1$:

$$y - 1 = 1(x - 1) \Rightarrow y - 1 = x - 1 \Rightarrow y = x \text{ or } x - y = 0$$

**Step 4: Final Answer:**
The equation of the tangent is $x + y = 2$.
The equation of the normal is $y = x$.

> **Quick Tip**
>
> Implicit differentiation is a powerful tool for finding derivatives when y is not explicitly defined as a function of x. Remember to apply the chain rule whenever you differentiate a term involving y.

---

**b. Find the area of the region enclosed by the parabola $y^2 = 4ax$ and its latus rectum.**

**Solution:**

**Step 1: Understanding the Concept:**
The parabola $y^2 = 4ax$ opens to the right with its vertex at the origin (0, 0). Its latus rectum is a line segment perpendicular to the axis of symmetry (the x-axis), passing through the focus at $(a, 0)$. The equation of the line containing the latus rectum is therefore $x = a$. The area is bounded by the parabola and this line.

**Step 2: Key Formula or Approach:**
The area of a region bounded by a curve $y = f(x)$, the x-axis, and the lines $x = c$ and $x = d$ is given by $\int_c^d f(x)dx$.
Since the parabola is symmetric about the x-axis, we can find the area in the first quadrant (from $x = 0$ to $x = a$) and multiply it by 2.

**Step 3: Detailed Explanation or Calculation:**
From $y^2 = 4ax$, we get $y = \pm\sqrt{4ax} = \pm 2\sqrt{a}\sqrt{x}$.
For the area in the first quadrant, we take $y = 2\sqrt{a}\sqrt{x}$.
The limits of integration are from the vertex $x = 0$ to the latus rectum $x = a$.
The area in the first quadrant is:

$$A_1 = \int_0^a 2\sqrt{a}\sqrt{x}\, dx = 2\sqrt{a}\int_0^a x^{1/2}\, dx$$

$$A_1 = 2\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^a = 2\sqrt{a}\left[\frac{2}{3}x^{3/2}\right]_0^a$$

$$A_1 = 2\sqrt{a}\left(\frac{2}{3}a^{3/2} - 0\right) = \frac{4\sqrt{a}}{3}a^{3/2} = \frac{4}{3}a^{1/2}a^{3/2} = \frac{4}{3}a^2$$

The total area is twice the area in the first quadrant:

$$A = 2 \times A_1 = 2 \times \frac{4}{3}a^2 = \frac{8}{3}a^2$$

**Step 4: Final Answer:**
The area of the region is $\frac{8}{3}a^2$ square units.

**Quick Tip**

Utilizing symmetry is a common strategy to simplify area calculations. Calculating the area in one quadrant and multiplying is often easier than dealing with positive and negative roots of y.

---

**c. Solve the differential equation $ydx - (x + 2y^2)dy = 0$.**

**Solution:**

**Step 1: Understanding the Concept:**
This differential equation is not separable or homogeneous. We can rearrange it to check if it's a linear differential equation. A linear differential equation can be written in the form $\frac{dy}{dx} + P(x)y = Q(x)$ or $\frac{dx}{dy} + P(y)x = Q(y)$.

**Step 2: Key Formula or Approach:**
1. Rearrange the equation into the standard linear form.
2. Identify the functions $P(y)$ and $Q(y)$.
3. Calculate the integrating factor (I.F.): $e^{\int P(y)dy}$.
4. The general solution is given by $x \cdot (\text{I.F.}) = \int Q(y) \cdot (\text{I.F.})dy + C$.

**Step 3: Detailed Explanation or Calculation:**
The given equation is $ydx - (x + 2y^2)dy = 0$.
Rearrange it as:

$$ydx = (x + 2y^2)dy$$
$$\frac{dx}{dy} = \frac{x + 2y^2}{y} = \frac{x}{y} + 2y$$
$$\frac{dx}{dy} - \frac{1}{y}x = 2y$$

This is a linear differential equation in x, with $P(y) = -\frac{1}{y}$ and $Q(y) = 2y$.
Now, find the integrating factor:

$$\text{I.F.} = e^{\int P(y)dy} = e^{\int -\frac{1}{y}dy} = e^{-\ln|y|} = e^{\ln|y|^{-1}} = \frac{1}{|y|}$$

Assuming $y > 0$, the I.F. is $\frac{1}{y}$.
The general solution is:

$$x \cdot (\text{I.F.}) = \int Q(y) \cdot (\text{I.F.}) dy + C$$
$$x \cdot \frac{1}{y} = \int (2y) \left(\frac{1}{y}\right) dy + C$$
$$\frac{x}{y} = \int 2\, dy + C$$
$$\frac{x}{y} = 2y + C$$
$$x = y(2y + C) = 2y^2 + Cy$$

**Step 4: Final Answer:**
The general solution of the differential equation is $x = 2y^2 + Cy$.

**Quick Tip**

If a differential equation is not linear in the form $\frac{dy}{dx}$, always check if it becomes linear by treating x as the dependent variable and y as the independent variable, i.e., by arranging it into the form $\frac{dx}{dy}$.

---

**d. Find the maximum value of Z = 8x + 5y under the constraints $5x + 3y \leq 15$, $2x + 5y \leq 10$ and $x \geq 0, y \geq 0$ by graphical method.**

**Solution:**

**Step 1: Understanding the Concept:**
This is a Linear Programming Problem (LPP). The solution involves finding the feasible region determined by the linear constraints and then evaluating the objective function Z at the vertices (corner points) of this region. The maximum value of Z will occur at one of these vertices.

**Step 2: Key Formula or Approach:**
1. Plot the boundary lines of the inequalities.
2. Identify the feasible region by finding the common area satisfying all constraints.
3. Determine the coordinates of the corner points of the feasible region.
4. Substitute the coordinates of each corner point into the objective function Z.
5. The largest value obtained is the maximum value of Z.

**Step 3: Detailed Explanation or Calculation:**
The constraints are: 1. $5x + 3y \leq 15$ 2. $2x + 5y \leq 10$ 3. $x \geq 0, y \geq 0$ The feasible region is in the first quadrant.

**Corner Points:**
- **O(0, 0):** The origin.
- **A:** The x-intercept of $5x + 3y = 15$. Let $y = 0$, then $5x = 15 \Rightarrow x = 3$. Point A is (3, 0).
- **C:** The y-intercept of $2x + 5y = 10$. Let $x = 0$, then $5y = 10 \Rightarrow y = 2$. Point C is (0, 2).
- **B:** The intersection of $5x + 3y = 15$ and $2x + 5y = 10$.
Multiply the first equation by 5 and the second by 3:
$25x + 15y = 75$
$6x + 15y = 30$
Subtract the second from the first: $19x = 45 \Rightarrow x = \frac{45}{19}$.
Substitute x back into $2x + 5y = 10$:
$2(\frac{45}{19}) + 5y = 10 \Rightarrow \frac{90}{19} + 5y = 10 \Rightarrow 5y = 10 - \frac{90}{19} = \frac{190-90}{19} = \frac{100}{19} \Rightarrow y = \frac{20}{19}$.
Point B is $(\frac{45}{19}, \frac{20}{19})$.
**Evaluate Z at corner points:**
- At O(0, 0): $Z = 8(0) + 5(0) = 0$
- At A(3, 0): $Z = 8(3) + 5(0) = 24$
- At C(0, 2): $Z = 8(0) + 5(2) = 10$
- At B$(\frac{45}{19}, \frac{20}{19})$: $Z = 8(\frac{45}{19}) + 5(\frac{20}{19}) = \frac{360}{19} + \frac{100}{19} = \frac{460}{19} \approx 24.21$

**Step 4: Final Answer:**
Comparing the values of Z at the corner points (0, 24, 10, and 460/19), the maximum value is $\frac{460}{19}$.

> **Quick Tip**
>
> When solving an LPP, carefully plot the lines and shade the feasible region. The corner points are the origin, the intercepts, and the intersections of the boundary lines that lie within the feasible region. Always check all corner points.

---

**e. There are three groups of children having 3 girls and one boy, 2 girls and 2 boys, one girl and 3 boys respectively. One child is selected at random from each group. Find the probability that the three selected children have one girl and 2 boys.**

**Solution:**

**Step 1: Understanding the Concept:**
We need to find the probability of a specific outcome (1 girl and 2 boys) when making independent selections from three different groups. This can happen in several mutually exclusive ways (e.g., girl from group 1, boys from groups 2 & 3; or boy from group 1, girl from group 2, boy from group 3, etc.). The total probability is the sum of the probabilities of these individual cases.

**Step 2: Key Formula or Approach:**
1. Determine the probability of selecting a girl (G) and a boy (B) from each group.

2. Identify all possible combinations of selections that result in one girl and two boys.
3. Calculate the probability for each combination by multiplying the probabilities of the independent selections.
4. Sum the probabilities of all combinations.

**Step 3: Detailed Explanation or Calculation:**
Let's denote the probabilities for each group:
- **Group 1:** 3 Girls, 1 Boy (Total 4). $P(G_1) = 3/4, P(B_1) = 1/4$.
- **Group 2:** 2 Girls, 2 Boys (Total 4). $P(G_2) = 2/4 = 1/2, P(B_2) = 2/4 = 1/2$.
- **Group 3:** 1 Girl, 3 Boys (Total 4). $P(G_3) = 1/4, P(B_3) = 3/4$.
The three possible ways to get exactly one girl and two boys are:
- **Case 1: Girl from Group 1, Boy from Group 2, Boy from Group 3 (G B B)**
$P(\text{Case 1}) = P(G_1) \times P(B_2) \times P(B_3) = \frac{3}{4} \times \frac{1}{2} \times \frac{3}{4} = \frac{9}{32}$
- **Case 2: Boy from Group 1, Girl from Group 2, Boy from Group 3 (B G B)**
$P(\text{Case 2}) = P(B_1) \times P(G_2) \times P(B_3) = \frac{1}{4} \times \frac{1}{2} \times \frac{3}{4} = \frac{3}{32}$
- **Case 3: Boy from Group 1, Boy from Group 2, Girl from Group 3 (B B G)**
$P(\text{Case 3}) = P(B_1) \times P(B_2) \times P(G_3) = \frac{1}{4} \times \frac{1}{2} \times \frac{1}{4} = \frac{1}{32}$
The total probability is the sum of the probabilities of these mutually exclusive cases:

$$P(\text{1 girl, 2 boys}) = P(\text{Case 1}) + P(\text{Case 2}) + P(\text{Case 3})$$
$$= \frac{9}{32} + \frac{3}{32} + \frac{1}{32} = \frac{13}{32}$$

**Step 4: Final Answer:**
The probability that the three selected children consist of one girl and two boys is $\frac{13}{32}$.

**Quick Tip**

For probability problems involving multiple independent events and specific combined outcomes, it's essential to list all the mutually exclusive ways the outcome can occur. Calculate the probability for each way and then add them up.

---

**7. Do any one part.**

**a. If** $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$**, then verify that** $A \cdot \text{adj}(A) = |A|I$ **and find** $A^{-1}$**.**

**Solution:**

**Step 1: Understanding the Concept:**
This problem involves two parts. First, we need to verify a fundamental property of matrices which states that the product of a matrix and its adjugate is equal to the determinant of the matrix times the identity matrix. Second, we need to find the inverse of the matrix A using

the formula involving the determinant and the adjugate.

**Step 2: Key Formula or Approach:**
1. Calculate the determinant of A, denoted as $|A|$.
2. Find the matrix of cofactors, then find the adjugate of A, $\text{adj}(A)$, which is the transpose of the cofactor matrix.
3. Calculate the product $A \cdot \text{adj}(A)$.
4. Calculate the product $|A|I$, where I is the 3x3 identity matrix.
5. Compare the results from steps 3 and 4 to verify the property.
6. Use the formula $A^{-1} = \frac{1}{|A|}\text{adj}(A)$ to find the inverse.

**Step 3: Detailed Explanation or Calculation:**
Given $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$.
**1. Calculate $|A|$:**

$$|A| = 1(16-9) - 3(4-3) + 3(3-4) = 1(7) - 3(1) + 3(-1) = 7 - 3 - 3 = 1$$

**2. Find adj$(A)$:**
The cofactors are:
$C_{11} = 7, C_{12} = -1, C_{13} = -1$
$C_{21} = -3, C_{22} = 1, C_{23} = 0$
$C_{31} = -3, C_{32} = 0, C_{33} = 1$
The cofactor matrix is $C = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$.
The adjugate matrix is $\text{adj}(A) = C^T = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.
**3. Calculate $A \cdot$ adj$(A)$:**

$$A \cdot \text{adj}(A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

**4. Calculate $|A|I$:**

$$|A|I = (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

**5. Verification:**
From steps 3 and 4, we see that $A \cdot \text{adj}(A) = I$ and $|A|I = I$. Thus, $A \cdot \text{adj}(A) = |A|I$ is verified.
**6. Find $A^{-1}$:**

$$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

**Step 4: Final Answer:**
The property is verified. The inverse of A is:

$$A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

**Quick Tip**

The relationship $A \cdot \text{adj}(A) = |A|I$ is a cornerstone of matrix algebra and directly leads to the formula for the inverse. Verifying it serves as a good check on your determinant and adjugate calculations before finding the inverse.

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**b. Solve the following system of linear equations by matrix method:**
$x + y + z = 6$
$y + 3z = 11$
$x + z = 2y$

**Solution:**

**Step 1: Understanding the Concept:**
We can solve a system of linear equations by representing it in the matrix form $AX = B$, where A is the coefficient matrix, X is the variable matrix, and B is the constant matrix. The solution is then found using the formula $X = A^{-1}B$.

**Step 2: Key Formula or Approach:**
1. Rewrite the equations in standard form and express the system as $AX = B$.
2. Find the inverse of the coefficient matrix A.
3. Multiply $A^{-1}$ by B to find the solution matrix X.

**Step 3: Detailed Explanation or Calculation:**
**1. Formulate the matrix equation:**
First, rewrite the third equation in standard form: $x + z = 2y \Rightarrow x - 2y + z = 0$.
The system is:
$x + y + z = 6$
$0x + y + 3z = 11$
$x - 2y + z = 0$
In matrix form $AX = B$:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

**2. Find the inverse of A:**
First, find the determinant of A:

$$|A| = 1(1-(-6)) - 1(0-3) + 1(0-1) = 7 + 3 - 1 = 9$$

Next, find the adjugate of A. The cofactor matrix is:

$$C = \begin{bmatrix} 7 & 3 & -1 \\ -3 & 0 & 3 \\ 2 & -3 & 1 \end{bmatrix}$$

$$\text{adj}(A) = C^T = \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

The inverse is:

$$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

**3. Solve for X:**

$$X = A^{-1}B = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

$$X = \frac{1}{9}\begin{bmatrix} (7)(6) + (-3)(11) + (2)(0) \\ (3)(6) + (0)(11) + (-3)(0) \\ (-1)(6) + (3)(11) + (1)(0) \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 42 - 33 \\ 18 \\ -6 + 33 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 \\ 18 \\ 27 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

**Step 4: Final Answer:**
The solution is $x = 1, y = 2, z = 3$.

**Quick Tip**

Before starting the matrix method, always rearrange the equations so that all variable terms are on the left and constants are on the right, with variables in the same order (e.g., x, then y, then z). This prevents errors when forming the coefficient matrix A.

**8. Do any one part.**

**a. Evaluate:** $\int_0^\pi \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x}$

**Solution:**

**Step 1: Understanding the Concept:**
This definite integral can be solved using the properties of definite integrals. The presence of 'x' in the numerator suggests using the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, which often helps to eliminate the 'x' term.

**Step 2: Key Formula or Approach:**
1. Use the property $I = \int_0^\pi f(x)dx = \int_0^\pi f(\pi - x)dx$.
2. Add the two forms of the integral to eliminate x from the numerator.
3. Use the property $\int_0^{2a} g(x)dx = 2\int_0^a g(x)dx$ if $g(2a-x) = g(x)$.
4. Evaluate the resulting integral by dividing the numerator and denominator by $\cos^2 x$ and using a substitution.

**Step 3: Detailed Explanation or Calculation:**
Let the given integral be $I$.

$$I = \int_0^\pi \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x} \quad \cdots (1)$$

Using the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$:

$$I = \int_0^\pi \frac{(\pi - x)\,dx}{a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)}$$

Since $\cos(\pi - x) = -\cos x$ and $\sin(\pi - x) = \sin x$, their squares are $\cos^2 x$ and $\sin^2 x$.

$$I = \int_0^\pi \frac{(\pi - x)\,dx}{a^2\cos^2 x + b^2\sin^2 x} \quad \cdots (2)$$

Adding equations (1) and (2):

$$2I = \int_0^\pi \frac{x + (\pi - x)}{a^2\cos^2 x + b^2\sin^2 x}\,dx = \int_0^\pi \frac{\pi}{a^2\cos^2 x + b^2\sin^2 x}\,dx$$
$$I = \frac{\pi}{2}\int_0^\pi \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

Let $g(x) = \frac{1}{a^2\cos^2 x + b^2\sin^2 x}$. Since $g(\pi - x) = g(x)$, we can use the property $\int_0^{2a} g(x)dx = 2\int_0^a g(x)dx$. Here $2a = \pi$.

$$I = \frac{\pi}{2}\cdot 2\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \pi\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

Divide numerator and denominator by $\cos^2 x$:

$$I = \pi\int_0^{\pi/2} \frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x}$$

Let $t = \tan x$, so $dt = \sec^2 x\, dx$. The limits change from $x = 0 \to t = 0$ and $x = \pi/2 \to t = \infty$.

$$I = \pi \int_0^\infty \frac{dt}{a^2 + (bt)^2}$$

This is a standard integral form:

$$I = \pi \left[ \frac{1}{b} \cdot \frac{1}{a} \tan^{-1}\left(\frac{bt}{a}\right) \right]_0^\infty = \frac{\pi}{ab} \left[ \tan^{-1}\left(\frac{bt}{a}\right) \right]_0^\infty$$

$$I = \frac{\pi}{ab} \left( \lim_{t\to\infty} \tan^{-1}\left(\frac{bt}{a}\right) - \tan^{-1}(0) \right) = \frac{\pi}{ab}\left(\frac{\pi}{2} - 0\right) = \frac{\pi^2}{2ab}$$

**Step 4: Final Answer:**
The value of the integral is $\frac{\pi^2}{2ab}$.

> **Quick Tip**
>
> Definite integrals of the form $\int_0^a x f(x)dx$ can often be simplified using the "King's property": $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. This is a very powerful tool for eliminating the 'x' term.

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**b. Show that $y = c_1 e^{ax}\cos(bx) + c_2 e^{ax}\sin(bx)$, where $c_1, c_2$ are constants, is a solution of the differential equation $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0$.**

**Solution:**

**Step 1: Understanding the Concept:**
To show that a given function is a solution to a differential equation, we must find the necessary derivatives of the function, substitute them into the equation, and verify that the equation holds true (i.e., it simplifies to $0 = 0$).

**Step 2: Key Formula or Approach:**
1. Factor the given function for easier differentiation: $y = e^{ax}(c_1\cos(bx) + c_2\sin(bx))$.
2. Find the first derivative $\frac{dy}{dx}$ using the product rule.
3. Find the second derivative $\frac{d^2y}{dx^2}$ by differentiating the first derivative.
4. Substitute $y$, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ into the given differential equation and simplify.

**Step 3: Detailed Explanation or Calculation:**
The given function is $y = e^{ax}(c_1\cos(bx) + c_2\sin(bx))$.
**First Derivative:**
Using the product rule:

$$\frac{dy}{dx} = (ae^{ax})(c_1\cos(bx) + c_2\sin(bx)) + (e^{ax})(-bc_1\sin(bx) + bc_2\cos(bx))$$

We can rewrite this by substituting $y$:

$$\frac{dy}{dx} = ay + e^{ax}(-bc_1 \sin(bx) + bc_2 \cos(bx)) \quad \cdots (1)$$

**Second Derivative:**
Differentiate equation (1) with respect to x:

$$\frac{d^2y}{dx^2} = a\frac{dy}{dx} + \frac{d}{dx}[e^{ax}(-bc_1 \sin(bx) + bc_2 \cos(bx))]$$

Apply the product rule to the second term:

$$\frac{d^2y}{dx^2} = a\frac{dy}{dx} + [(ae^{ax})(-bc_1 \sin(bx) + bc_2 \cos(bx)) + e^{ax}(-b^2c_1 \cos(bx) - b^2c_2 \sin(bx))]$$

From equation (1), we know that $e^{ax}(-bc_1 \sin(bx) + bc_2 \cos(bx)) = \frac{dy}{dx} - ay$. Substitute this into the first part of the bracket:

$$\frac{d^2y}{dx^2} = a\frac{dy}{dx} + \left[a(\frac{dy}{dx} - ay) + e^{ax}(-b^2(c_1 \cos(bx) + c_2 \sin(bx)))\right]$$

Recognize that $e^{ax}(c_1 \cos(bx) + c_2 \sin(bx))$ is $y$:

$$\frac{d^2y}{dx^2} = a\frac{dy}{dx} + a\frac{dy}{dx} - a^2y - b^2y$$

$$\frac{d^2y}{dx^2} = 2a\frac{dy}{dx} - (a^2 + b^2)y$$

**Verification:**
Rearrange the equation for the second derivative:

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0$$

This is identical to the given differential equation. Therefore, the function $y$ is a solution.

**Step 4: Final Answer:**
By finding the first and second derivatives and rearranging the terms, we have shown that the given function $y$ satisfies the differential equation $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0$. Hence proved.

**Quick Tip**

When differentiating complex products multiple times, look for opportunities to substitute back the original function (y) or its lower-order derivatives to simplify the expressions. This can significantly reduce the complexity of the algebra.

---

**9. Do any one part.**

**a. Prove that:** $\int_0^{\pi} \log(1 + \cos x)dx = -\pi \log_e 2$

**Solution:**

**Step 1: Understanding the Concept:**
This definite integral can be evaluated using the properties of definite integrals and trigonometric identities. The "King's property" $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ is a key tool for such problems.

**Step 2: Key Formula or Approach:**
1. Let $I = \int_0^\pi \log(1 + \cos x)dx$.
2. Apply the property $\int_0^a f(x)dx = \int_0^a f(a - x)dx$ with $a = \pi$.
3. Add the two expressions for I.
4. Use trigonometric and logarithm properties to simplify the resulting integral.
5. Use the property $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$ if $f(2a - x) = f(x)$.

**Step 3: Detailed Explanation or Calculation:**
Let $I = \int_0^\pi \log(1 + \cos x)dx \quad \cdots (1)$
Using the property $\int_0^\pi f(x)dx = \int_0^\pi f(\pi - x)dx$:

$$I = \int_0^\pi \log(1 + \cos(\pi - x))dx$$

Since $\cos(\pi - x) = -\cos x$:

$$I = \int_0^\pi \log(1 - \cos x)dx \quad \cdots (2)$$

Adding equations (1) and (2):

$$2I = \int_0^\pi [\log(1 + \cos x) + \log(1 - \cos x)]dx$$

Using the logarithm property $\log A + \log B = \log(AB)$:

$$2I = \int_0^\pi \log((1 + \cos x)(1 - \cos x))dx$$

$$2I = \int_0^\pi \log(1 - \cos^2 x)dx$$

Using the identity $\sin^2 x + \cos^2 x = 1$:

$$2I = \int_0^\pi \log(\sin^2 x)dx = \int_0^\pi 2\log(\sin x)dx$$

$$I = \int_0^\pi \log(\sin x)dx$$

Let's check if the integrand $f(x) = \log(\sin x)$ is symmetric about $\pi/2$. $f(\pi - x) = \log(\sin(\pi - x)) = \log(\sin x) = f(x)$.
So we can use the property $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$.

$$I = 2\int_0^{\pi/2} \log(\sin x)dx$$

There is a standard result for this integral: $\int_0^{\pi/2} \log(\sin x)dx = -\frac{\pi}{2}\log 2$.
Let's prove this result. Let $I_1 = \int_0^{\pi/2} \log(\sin x)dx$. Using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, $I_1 = \int_0^{\pi/2} \log(\sin(\pi/2 - x))dx = \int_0^{\pi/2} \log(\cos x)dx$. Adding these two forms of $I_1$:

$$2I_1 = \int_0^{\pi/2} (\log(\sin x) + \log(\cos x))dx = \int_0^{\pi/2} \log(\sin x \cos x)dx$$

$$2I_1 = \int_0^{\pi/2} \log\left(\frac{2\sin x\cos x}{2}\right)dx = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)dx$$

$$2I_1 = \int_0^{\pi/2} (\log(\sin 2x) - \log 2)dx = \int_0^{\pi/2} \log(\sin 2x)dx - \int_0^{\pi/2} \log 2dx$$

Let $t = 2x, dt = 2dx$. Limits: $x = 0 \to t = 0, x = \pi/2 \to t = \pi$.

$$\int_0^{\pi/2} \log(\sin 2x)dx = \frac{1}{2}\int_0^{\pi} \log(\sin t)dt = \frac{1}{2}(2\int_0^{\pi/2} \log(\sin t)dt) = \int_0^{\pi/2} \log(\sin t)dt = I_1$$

So, $2I_1 = I_1 - \frac{\pi}{2}\log 2$. This gives $I_1 = -\frac{\pi}{2}\log 2$.
Now substitute this result back into our main problem:

$$I = 2I_1 = 2\left(-\frac{\pi}{2}\log 2\right) = -\pi\log 2$$

**Step 4: Final Answer:**
Hence, it is proved that $\int_0^{\pi} \log(1+\cos x)dx = -\pi\log_e 2$.

**Quick Tip**

This problem relies on a famous result: $\int_0^{\pi/2} \log(\sin x)dx = -\frac{\pi}{2}\log 2$. Memorizing this and other standard definite integral results can save a lot of time and effort in exams.

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**b. Find the particular solution of the differential equation $\frac{dy}{dx} + y\cot x = 2x + x^2\cot x, (x \neq 0)$. It is given that $y = 0$ if $x = \frac{\pi}{2}$.**

**Solution:**

**Step 1: Understanding the Concept:**
The given equation is a first-order linear differential equation. We can solve it by finding an integrating factor. After finding the general solution, we use the given initial condition to determine the value of the integration constant and find the particular solution.

**Step 2: Key Formula or Approach:**
The equation is in the standard linear form $\frac{dy}{dx} + P(x)y = Q(x)$.
1. Identify $P(x)$ and $Q(x)$.

2. Calculate the Integrating Factor (I.F.): $e^{\int P(x)dx}$.
3. The general solution is given by $y \cdot (\text{I.F.}) = \int Q(x) \cdot (\text{I.F.})dx + C$.
4. Use the initial condition to find C.

**Step 3: Detailed Explanation or Calculation:**
The given equation is $\frac{dy}{dx} + y\cot x = 2x + x^2 \cot x$.
**1. Identify P(x) and Q(x):**
This is a linear differential equation with $P(x) = \cot x$ and $Q(x) = 2x + x^2 \cot x$.
**2. Calculate the Integrating Factor (I.F.):**

$$\text{I.F.} = e^{\int \cot x dx} = e^{\ln(\sin x)} = \sin x$$

**3. Find the General Solution:**

$$y \cdot (\sin x) = \int (2x + x^2 \cot x)(\sin x)dx + C$$

$$y \sin x = \int (2x \sin x + x^2 \cot x \sin x)dx + C$$

$$y \sin x = \int (2x \sin x + x^2 \cos x)dx + C$$

The integral $\int(2x\sin x + x^2 \cos x)dx$ is in the form $\int(f'(x)g(x) + f(x)g'(x))dx$, which is the derivative of a product. Let $f(x) = x^2$ and $g(x) = \sin x$. Then $f'(x) = 2x$ and $g'(x) = \cos x$. So, the integrand is the derivative of $x^2 \sin x$.

$$\int (2x \sin x + x^2 \cos x)dx = x^2 \sin x$$

Therefore, the general solution is:

$$y \sin x = x^2 \sin x + C$$

**4. Find the Particular Solution:**
We are given the initial condition $y = 0$ when $x = \frac{\pi}{2}$. Substitute these values:

$$(0) \sin\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}\right) + C$$

$$0 \cdot 1 = \frac{\pi^2}{4} \cdot 1 + C$$

$$0 = \frac{\pi^2}{4} + C \Rightarrow C = -\frac{\pi^2}{4}$$

Substitute the value of C back into the general solution:

$$y \sin x = x^2 \sin x - \frac{\pi^2}{4}$$

We can express y explicitly:

$$y = x^2 - \frac{\pi^2}{4 \sin x}$$

**Step 4: Final Answer:**
The particular solution is $y \sin x = x^2 \sin x - \frac{\pi^2}{4}$, or $y = x^2 - \frac{\pi^2}{4} \csc x$.

Quick Tip

When the integral on the right-hand side, $\int Q(x) \cdot (\text{I.F.}) dx$, looks complicated, check if the integrand is the result of a product rule differentiation. This "reverse product rule" is a common pattern in textbook problems involving linear differential equations.

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