UP Board Class 12 Mathematics - 324(JA) - 2025 Question Paper with Solutions

Time Allowed : 3 Hours	Maximum Marks : 100	Total Questions : 9

General Instructions

Read the following instructions very carefully and strictly follow them:

1. There are in all nine questions in this question paper.
2. All questions are compulsory.
3. In the beginning of each question, the number of parts to be attempted are clearly mentioned.
4. Marks allotted to the questions are indicated against them.
5. Start solving from the first question and proceed to solve till the last one. Do not waste your time over a question you cannot solve.

1. Do all parts.
Select the correct option of each part and write it on your answer-book.

a. The relation R = { (a,b): b = a + 2 } defined in the set A = { 1, 2, 3, 4, 5 } is

(A) not reflexive and symmetric, but transitive
(B) not reflexive and transitive, but symmetric
(C) not symmetric and transitive, but reflexive
(D) not reflexive, not symmetric and not also transitive

Correct Answer: (D) not reflexive, not symmetric and not also transitive

Solution:

Step 1: Understanding the Concept:
To determine the properties of the relation R on the set A, we need to check for three conditions: reflexivity, symmetry, and transitivity.
First, let's list the ordered pairs in the relation R based on the rule $b = a + 2$.
For $a = 1, b = 1 + 2 = 3$, so $(1, 3) \in R$.
For $a = 2, b = 2 + 2 = 4$, so $(2, 4) \in R$.
For $a = 3, b = 3 + 2 = 5$, so $(3, 5) \in R$.
For $a = 4, b = 4 + 2 = 6$, which is not in set A.
For $a = 5, b = 5 + 2 = 7$, which is not in set A.
So, the relation is $R = \{(1, 3), (2, 4), (3, 5)\}$.

Step 2: Key Formula or Approach:
Reflexivity: A relation is reflexive if $(a, a) \in R$ for every $a \in A$.
Symmetry: A relation is symmetric if $(a, b) \in R$ implies $(b, a) \in R$ for all $a, b \in A$.
Transitivity: A relation is transitive if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

Step 3: Detailed Explanation:
Checking for Reflexivity:
For R to be reflexive, $(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)$ must be in R.
However, none of these pairs are in R. For example, for $a = 1$, $b = 1 \neq 1 + 2$, so $(1, 1) \notin R$.
Thus, R is **not reflexive**.

Checking for Symmetry:
For R to be symmetric, if $(a, b) \in R$, then (b, a) must also be in R.
We have $(1, 3) \in R$. For symmetry, $(3, 1)$ should be in R.
But for $(3, 1)$, we have $a = 3, b = 1$. The condition $b = a + 2$ becomes $1 = 3 + 2$, which is false.
So $(3, 1) \notin R$.
Thus, R is **not symmetric**.

Checking for Transitivity:
For R to be transitive, if $(a, b) \in R$ and $(b, c) \in R$, then (a, c) must be in R.
Let's check for pairs that satisfy the condition. We have $(1, 3) \in R$. We also have a pair starting with 3, which is $(3, 5) \in R$.
So we have $(a, b) = (1, 3)$ and $(b, c) = (3, 5)$. For transitivity, $(a, c) = (1, 5)$ must be in R.
Let's check if $(1, 5)$ satisfies the condition $b = a + 2$. Here $a = 1, b = 5$. The condition is $5 = 1 + 2$, which is false. So $(1, 5) \notin R$.
Thus, R is **not transitive**.

Step 4: Final Answer:
The relation R is not reflexive, not symmetric, and not transitive.

> **Quick Tip**
>
> When testing properties of a relation on a finite set, it is often easiest to first list all the elements of the relation. Then, check for reflexivity (all (a,a) pairs), symmetry (if (a,b) exists, does (b,a) exist?), and transitivity (if (a,b) and (b,c) exist, does (a,c) exist?) by examining the list.

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b. If the orders of the matrices A and B are $m \times n$ and $n \times p$ respectively, then the order of AB will be

(A) $m \times p$
(B) $p \times m$

(C) $m \times n$
(D) $n \times p$

Correct Answer: (A) $m \times p$

Solution:

Step 1: Understanding the Concept:
This question tests the fundamental rule for the multiplication of two matrices. The product of two matrices, AB, is defined only if the number of columns in the first matrix (A) is equal to the number of rows in the second matrix (B). The resulting matrix has an order determined by the outer dimensions of the original matrices.

Step 2: Key Formula or Approach:
Let matrix A have order $m \times n$.
Let matrix B have order $n \times p$.
The condition for multiplication is that the inner dimensions must match: $n = n$. This is satisfied.
The order of the product matrix AB is given by the outer dimensions: $m \times p$.

Step 3: Detailed Explanation:
We are given:
Order of matrix A $= m \times n$ (m rows, n columns).
Order of matrix B $= n \times p$ (n rows, p columns).
To find the order of the product AB, we look at the dimensions:

$$(\text{A})_{m\times n} \times (\text{B})_{n\times p}$$

The number of columns of A is n, and the number of rows of B is n. Since they are equal, the product AB is defined.
The order of the resulting matrix AB is the number of rows of A by the number of columns of B.
Therefore, the order of AB is $m \times p$.

Step 4: Final Answer:
The order of the product matrix AB is $m \times p$.

Quick Tip

A simple way to remember the rule for matrix multiplication order is to write the dimensions side-by-side: $(m \times n) \times (n \times p)$. If the two "inner" numbers are the same (n), multiplication is possible. The "outer" numbers (m and p) give the dimensions of the resulting matrix: $m \times p$.

c. The degree of the differential equation $7\left(\frac{d^3y}{dx^3}\right)^2 + 5\left(\frac{d^2y}{dx^2}\right)^3 + x\frac{dy}{dx} + y = 0$ **will be**

(A) 3
(B) 2
(C) 6
(D) 5

Correct Answer: (B) 2

Solution:

Step 1: Understanding the Concept:
The question asks for the degree of a given differential equation. The degree is defined as the highest power (or exponent) of the highest-order derivative in the equation, after the equation has been cleared of any radicals and fractions in its derivatives.

Step 2: Key Formula or Approach:
1. Identify the highest-order derivative in the differential equation. This gives the **order** of the equation.
2. Identify the power (exponent) of this highest-order derivative. This power is the **degree** of the equation.

Step 3: Detailed Explanation:
The given differential equation is:

$$7\left(\frac{d^3y}{dx^3}\right)^2 + 5\left(\frac{d^2y}{dx^2}\right)^3 + x\frac{dy}{dx} + y = 0$$

1. Find the highest-order derivative:
The derivatives present in the equation are $\frac{d^3y}{dx^3}$ (third order), $\frac{d^2y}{dx^2}$ (second order), and $\frac{dy}{dx}$ (first order).
The highest order among these is 3. So, the order of the differential equation is 3.

2. Find the degree:
The degree is the power of the highest-order derivative, which is $\frac{d^3y}{dx^3}$.
The term containing the highest-order derivative is $7\left(\frac{d^3y}{dx^3}\right)^2$.
The power of $\frac{d^3y}{dx^3}$ in this term is 2.
Therefore, the degree of the differential equation is 2.

Step 4: Final Answer:
The degree of the given differential equation is 2.

Quick Tip

Do not get confused between the order and the degree. The order is determined by the highest derivative (e.g., d^3y/dx^3), while the degree is the power of that highest derivative. A common mistake is to pick the highest power in the entire equation (which is 3 in this case, on the second derivative), but the degree is only related to the highest *order* derivative.

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d. The direction cosines of the vector $\hat{i} + \hat{j} - 2\hat{k}$ are

(A) (1, 1, -2)
(B) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$
(C) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$
(D) $\left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$

Correct Answer: (B) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$

Solution:

Step 1: Understanding the Concept:
Direction cosines of a vector are the cosines of the angles that the vector makes with the positive x, y, and z axes. They are also the components of the unit vector in the direction of the given vector.

Step 2: Key Formula or Approach:
For a vector $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$: 1. Find the magnitude of the vector: $|\vec{v}| = \sqrt{a^2 + b^2 + c^2}$.
2. The direction cosines (l, m, n) are given by:

$$l = \frac{a}{|\vec{v}|}, \quad m = \frac{b}{|\vec{v}|}, \quad n = \frac{c}{|\vec{v}|}$$

Step 3: Detailed Explanation:
The given vector is $\vec{v} = \hat{i} + \hat{j} - 2\hat{k}$.
The components of the vector are $a = 1$, $b = 1$, and $c = -2$. These are the direction ratios.
1. Calculate the magnitude of $\vec{v}$:

$$|\vec{v}| = \sqrt{1^2 + 1^2 + (-2)^2}$$

$$|\vec{v}| = \sqrt{1 + 1 + 4} = \sqrt{6}$$

2. Calculate the direction cosines:

$$l = \frac{a}{|\vec{v}|} = \frac{1}{\sqrt{6}}$$

$$m = \frac{b}{|\vec{v}|} = \frac{1}{\sqrt{6}}$$
$$n = \frac{c}{|\vec{v}|} = \frac{-2}{\sqrt{6}}$$

So, the direction cosines are $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$.

Step 4: Final Answer:
The direction cosines of the vector $\hat{i} + \hat{j} - 2\hat{k}$ are $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$.

Quick Tip

The direction ratios of a vector are simply its scalar components (a, b, c). To find the direction cosines, you just need to divide each direction ratio by the magnitude of the vector. Remember that the sum of the squares of the direction cosines is always 1: $l^2 + m^2 + n^2 = 1$. This can be used for a quick verification.

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e. If R* be the set of all non-zero real numbers, then the mapping $f : R^* \to R^*$ defined by $f(x) = \frac{1}{x}$ is

(A) one-one and onto
(B) many-one and onto
(C) one-one, but not onto
(D) neither one-one nor onto

Correct Answer: (A) one-one and onto

Solution:

Step 1: Understanding the Concept:
The question asks to classify the function $f(x) = 1/x$ based on whether it is one-one (injective) and onto (surjective). The domain and codomain are both R*, the set of all non-zero real numbers.
One-one (Injective): A function is one-one if every distinct element in the domain maps to a distinct element in the codomain. Formally, $f(x_1) = f(x_2) \implies x_1 = x_2$.
Onto (Surjective): A function is onto if every element in the codomain is the image of at least one element from the domain. Formally, for every $y \in$ Codomain, there exists an $x \in$ Domain such that $f(x) = y$.

Step 2: Key Formula or Approach:
1. **Test for one-one:** Assume $f(x_1) = f(x_2)$ for $x_1, x_2 \in R^*$. If this implies $x_1 = x_2$, the function is one-one.

2. **Test for onto:** Let y be an arbitrary element in the codomain R^*. Try to find an x in the domain R^* such that $f(x) = y$. If such an x always exists, the function is onto.

Step 3: Detailed Explanation:
The function is $f(x) = 1/x$, with $f : R^* \to R^*$.

1. Checking for One-one (Injectivity):
Let $x_1, x_2 \in R^*$. Assume $f(x_1) = f(x_2)$.

$$\frac{1}{x_1} = \frac{1}{x_2}$$

By taking the reciprocal of both sides (which is valid since $x_1, x_2 \neq 0$), we get:

$$x_1 = x_2$$

Since $f(x_1) = f(x_2)$ implies $x_1 = x_2$, the function is **one-one**.

2. Checking for Onto (Surjectivity):
Let y be an arbitrary element in the codomain R^*. We need to find if there is an x in the domain R^* such that $f(x) = y$.

$$f(x) = y$$
$$\frac{1}{x} = y$$

Solving for x, we get:

$$x = \frac{1}{y}$$

Since $y \in R^*$, it means y is a non-zero real number. Therefore, $x = 1/y$ will also be a non-zero real number. This means x is in the domain R^*.
So, for any y in the codomain, we can find a pre-image $x = 1/y$ in the domain.
Thus, the function is **onto**.

Step 4: Final Answer:
Since the function is both one-one and onto, it is a bijective function. The correct option is one-one and onto.

Quick Tip

The properties of a function heavily depend on its specified domain and codomain. For example, if the function was $f : R \to R$, it would not even be a function because $f(0)$ is undefined. If it were $f : R^* \to R$, it would be one-one but not onto, because you could never get $y = 0$. Always pay close attention to the domain and codomain.

2. Do all parts.

a. Find the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$.

Solution:

Step 1: Understanding the Concept:
The principal value of an inverse trigonometric function, $\cos^{-1}(x)$, is the unique angle θ in the range $[0, \pi]$ such that $\cos(\theta) = x$. For negative inputs, we use the identity $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$.

Step 2: Key Formula or Approach:
1. Use the identity: $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$.
2. Find the value of $\cos^{-1}(x)$.
3. Substitute this value back into the identity to find the principal value.

Step 3: Detailed Explanation or Calculation:
We need to find the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$.
Using the identity, we have:

$$\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \pi - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

We know that the angle θ in $[0, \pi]$ for which $\cos(\theta) = \frac{1}{\sqrt{2}}$ is $\frac{\pi}{4}$.
So, $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$.
Substituting this back:

$$\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \pi - \frac{\pi}{4}$$
$$= \frac{4\pi - \pi}{4} = \frac{3\pi}{4}$$

The value $\frac{3\pi}{4}$ lies in the principal value range of $\cos^{-1}(x)$, which is $[0, \pi]$.

Step 4: Final Answer:
The principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ is $\frac{3\pi}{4}$.

Quick Tip

Remember the principal value ranges for inverse trigonometric functions. For $\cos^{-1}(x)$, it's $[0, \pi]$ (Quadrants I and II). For $\sin^{-1}(x)$ and $\tan^{-1}(x)$, it's $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (Quadrants I and IV). This is crucial for finding the correct principal value.

b. Test whether the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} x+5, & \textbf{if } x \leq 1 \\ x-5, & \textbf{if } x > 1 \end{cases}$ is continuous at $x = 1$.

Solution:

Step 1: Understanding the Concept:
A function $f(x)$ is continuous at a point $x = c$ if three conditions are met: 1. $f(c)$ is defined.
2. The limit of $f(x)$ as x approaches c exists. This means the left-hand limit (LHL) must equal the right-hand limit (RHL).
3. The limit equals the function's value: $\lim_{x\to c} f(x) = f(c)$.

Step 2: Key Formula or Approach:
To test for continuity at $x = 1$, we must calculate: 1. Left-Hand Limit (LHL): $\lim_{x\to 1^-} f(x)$ 2. Right-Hand Limit (RHL): $\lim_{x\to 1^+} f(x)$ 3. Value of the function: $f(1)$ If LHL = RHL = $f(1)$, the function is continuous. Otherwise, it is discontinuous.

Step 3: Detailed Explanation or Calculation:
The function is defined as $f(x) = \begin{cases} x+5, & \text{if } x \leq 1 \\ x-5, & \text{if } x > 1 \end{cases}$.

1. Calculate the Left-Hand Limit (LHL):
As x approaches 1 from the left ($x \leq 1$), we use the function $f(x) = x + 5$.

$$\text{LHL} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x+5) = 1+5 = 6$$

2. Calculate the Right-Hand Limit (RHL):
As x approaches 1 from the right ($x > 1$), we use the function $f(x) = x - 5$.

$$\text{RHL} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-5) = 1-5 = -4$$

3. Calculate the value of the function at $x = 1$:
For $x = 1$, the definition is $f(x) = x + 5$.

$$f(1) = 1+5 = 6$$

Since the Left-Hand Limit (6) is not equal to the Right-Hand Limit (-4), the limit $\lim_{x\to 1} f(x)$ does not exist.

Step 4: Final Answer:
The function $f(x)$ is not continuous (discontinuous) at $x = 1$ because the left-hand limit and the right-hand limit are not equal.

Quick Tip

For piecewise functions, the points where the definition changes are the most likely points of discontinuity. Always check the continuity at these points by comparing the left-hand limit, right-hand limit, and the function's value.

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c. Find the value of $\int x^3 e^{x^4} dx$.

Solution:

Step 1: Understanding the Concept:
This is an indefinite integral that can be solved using the method of substitution (u-substitution). We look for a part of the integrand whose derivative is also present (up to a constant factor).

Step 2: Key Formula or Approach:
1. Identify a suitable substitution, u. Let $u = x^4$.
2. Differentiate u with respect to x to find du.
3. Rewrite the integral entirely in terms of u and du.
4. Integrate with respect to u.
5. Substitute the original expression for u back into the result.

Step 3: Detailed Explanation or Calculation:
Let the integral be $I = \int x^3 e^{x^4} dx$.
1. Substitution:
Notice that the derivative of the exponent x^4 is $4x^3$, which is a constant multiple of the other factor x^3 in the integrand. So, we choose: Let $u = x^4$.
2. Differentiate:
Then, $du = 4x^3 dx$.
This implies $x^3 dx = \frac{1}{4} du$.
3. Rewrite the integral:
Substitute u and $\frac{1}{4} du$ into the integral:

$$I = \int e^{x^4} (x^3 dx) = \int e^u \left(\frac{1}{4} du\right)$$

$$I = \frac{1}{4} \int e^u du$$

4. Integrate:
The integral of e^u is e^u.

$$I = \frac{1}{4} e^u + C$$

5. Substitute back:
Replace u with x^4:

$$I = \frac{1}{4} e^{x^4} + C$$

Step 4: Final Answer:
The value of the integral is $\frac{1}{4}e^{x^4} + C$, where C is the constant of integration.

> **Quick Tip**
>
> When using substitution, always look for a function-derivative pair. In integrals involving e, a good first guess is to substitute for the exponent. Check if its derivative (or something close to it) is present as a factor in the integrand.

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d. If $P(B) = \frac{9}{13}$ and $P(A \cap B) = \frac{4}{13}$, find $P(A|B)$.

Solution:

Step 1: Understanding the Concept:
This question asks for the conditional probability of event A given that event B has occurred, denoted as $P(A|B)$. This is a direct application of the definition of conditional probability.

Step 2: Key Formula or Approach:
The formula for the conditional probability of A given B is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

where $P(A \cap B)$ is the probability of both A and B occurring, and $P(B)$ is the probability of B occurring, with the condition that $P(B) \neq 0$.

Step 3: Detailed Explanation or Calculation:
We are given the following probabilities:

$$P(B) = \frac{9}{13}$$

$$P(A \cap B) = \frac{4}{13}$$

Using the formula for conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{4/13}{9/13}$$

The denominators (13) cancel out:

$$P(A|B) = \frac{4}{9}$$

Step 4: Final Answer:
The value of $P(A|B)$ is $\frac{4}{9}$.

Quick Tip

The notation $P(A|B)$ can be read as "the probability of A, given B". The event that is "given" always goes in the denominator of the conditional probability formula. It represents the reduced sample space.

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e. Let $\vec{a} = \hat{i} + 2\hat{j}$ and $\vec{b} = 2\hat{i} + \hat{j}$. Is $|\vec{a}| = |\vec{b}|$? Are the vectors $\vec{a}$ and $\vec{b}$ equal?

Solution:

Step 1: Understanding the Concept:
This question checks the understanding of two different properties of vectors: magnitude and equality.
Magnitude of a vector: The length of the vector. For $\vec{v} = x\hat{i} + y\hat{j}$, the magnitude is $|\vec{v}| = \sqrt{x^2 + y^2}$.
Equality of vectors: Two vectors are equal if and only if all their corresponding components are identical.

Step 2: Key Formula or Approach:
1. Calculate $|\vec{a}|$ and $|\vec{b}|$ using the magnitude formula and compare them.
2. Compare the corresponding components of $\vec{a}$ and $\vec{b}$ to check for equality.

Step 3: Detailed Explanation or Calculation:
Given vectors:

$$\vec{a} = 1\hat{i} + 2\hat{j}$$
$$\vec{b} = 2\hat{i} + 1\hat{j}$$

1. Check if Magnitudes are Equal:
Calculate the magnitude of $\vec{a}$:

$$|\vec{a}| = \sqrt{(1)^2 + (2)^2} = \sqrt{1+4} = \sqrt{5}$$

Calculate the magnitude of $\vec{b}$:

$$|\vec{b}| = \sqrt{(2)^2 + (1)^2} = \sqrt{4+1} = \sqrt{5}$$

Since both magnitudes are equal to $\sqrt{5}$, we can conclude that $|\vec{a}| = |\vec{b}|$.

2. Check if Vectors are Equal:
For two vectors to be equal, their corresponding components must be equal.
Vector $\vec{a}$ can be written as $(1, 2)$.
Vector $\vec{b}$ can be written as $(2, 1)$.
Comparing the $\hat{i}$ components: $1 \neq 2$.
Comparing the $\hat{j}$ components: $2 \neq 1$.

Since the components are not identical, the vectors $\vec{a}$ **and** $\vec{b}$ **are not equal**.

Step 4: Final Answer:
Yes, the magnitudes are equal: $|\vec{a}| = |\vec{b}| = \sqrt{5}$.
No, the vectors are not equal because their corresponding components are different.

> **Quick Tip**
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> Having the same magnitude does not imply that two vectors are equal. Equal vectors must have both the same magnitude and the same direction. Different vectors can have the same length but point in different directions.

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3. Do all parts.

a. Find the value of $\frac{dy}{dx}$ if $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$, where $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$.

Solution:

Step 1: Understanding the Concept:
The problem asks for the derivative of an inverse trigonometric function. The expression inside the $\tan^{-1}$ function is a hint to use a trigonometric substitution to simplify the function before differentiating. The expression is identical in form to the triple angle formula for tangent.

Step 2: Key Formula or Approach:
1. Recognize the trigonometric identity: $\tan(3\theta) = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$.
2. Use the substitution $x = \tan\theta$, which implies $\theta = \tan^{-1} x$.
3. Use the given domain of x to find the range of θ.
4. Simplify the expression for y using the identity and the property $\tan^{-1}(\tan z) = z$ (within the principal value range).
5. Differentiate the simplified expression for y with respect to x.

Step 3: Detailed Explanation or Calculation:
Let $x = \tan\theta$. Then $\theta = \tan^{-1} x$.
The given domain for x is $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$.
Substituting $x = \tan\theta$, we get $-\frac{1}{\sqrt{3}} < \tan\theta < \frac{1}{\sqrt{3}}$.
This implies $-\frac{\pi}{6} < \theta < \frac{\pi}{6}$.
Now, substitute $x = \tan\theta$ into the expression for y:

$$y = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right)$$

Using the triple angle identity, this simplifies to:

$$y = \tan^{-1}(\tan(3\theta))$$

To simplify this further, we must check if 3θ lies within the principal value range of $\tan^{-1}$, which is $(-\frac{\pi}{2}, \frac{\pi}{2})$.
From our range for θ, $-\frac{\pi}{6} < \theta < \frac{\pi}{6}$, we can find the range for 3θ:

$$3 \times (-\frac{\pi}{6}) < 3\theta < 3 \times (\frac{\pi}{6}) \implies -\frac{\pi}{2} < 3\theta < \frac{\pi}{2}$$

Since 3θ is within the principal value range, we can write:

$$y = 3\theta$$

Now, substitute back $\theta = \tan^{-1} x$:

$$y = 3\tan^{-1} x$$

Finally, differentiate y with respect to x:

$$\frac{dy}{dx} = \frac{d}{dx}(3\tan^{-1} x) = 3 \cdot \frac{1}{1+x^2} = \frac{3}{1+x^2}$$

Step 4: Final Answer:
The value of $\frac{dy}{dx}$ is $\frac{3}{1+x^2}$.

Quick Tip

Whenever you see expressions like $\frac{2x}{1-x^2}$, $\frac{1-x^2}{1+x^2}$, $\frac{3x-x^3}{1-3x^2}$ inside inverse trigonometric functions, always consider substituting $x = \tan\theta$. Similarly, for expressions involving $\sqrt{1-x^2}$, try $x = \sin\theta$ or $x = \cos\theta$. This often simplifies the problem dramatically.

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b. Find the value of $\int \frac{1}{x^2-a^2} dx$.

Solution:

Step 1: Understanding the Concept:
This is a standard indefinite integral of a rational function. The standard method to solve it from first principles is using partial fraction decomposition. Alternatively, one can state the well-known formula directly. We will show the derivation using partial fractions.

Step 2: Key Formula or Approach:
1. Factor the denominator: $x^2 - a^2 = (x-a)(x+a)$.
2. Decompose the fraction $\frac{1}{(x-a)(x+a)}$ into partial fractions: $\frac{A}{x-a} + \frac{B}{x+a}$.
3. Solve for the constants A and B.

4. Integrate the resulting simpler fractions.

Step 3: Detailed Explanation or Calculation:
1. Partial Fraction Decomposition:

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$$

To find A and B, we multiply by the common denominator:

$$1 = A(x+a) + B(x-a)$$

We can find the coefficients by substituting convenient values for x.
Let $x = a$:

$$1 = A(a+a) + B(a-a) \implies 1 = 2aA \implies A = \frac{1}{2a}$$

Let $x = -a$:

$$1 = A(-a+a) + B(-a-a) \implies 1 = -2aB \implies B = -\frac{1}{2a}$$

So, the decomposition is:

$$\frac{1}{x^2-a^2} = \frac{1}{2a(x-a)} - \frac{1}{2a(x+a)}$$

2. Integration: Now we integrate the expression:

$$\int \frac{1}{x^2-a^2}dx = \int \left(\frac{1}{2a(x-a)} - \frac{1}{2a(x+a)}\right) dx$$

$$= \frac{1}{2a} \int \left(\frac{1}{x-a} - \frac{1}{x+a}\right) dx$$

Using the standard integral $\int \frac{1}{u}du = \ln|u|$:

$$= \frac{1}{2a}(\ln|x-a| - \ln|x+a|) + C$$

Using the logarithm property $\ln M - \ln N = \ln(M/N)$:

$$= \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$$

Step 4: Final Answer:
The value of the integral is $\frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$.

Quick Tip

This is one of the fundamental integration formulas and is worth memorizing to save time.
$\int \frac{1}{x^2-a^2}dx = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$
Also remember the related formula:
$\int \frac{1}{a^2-x^2}dx = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + C$

c. If R_1 and R_2 are equivalence relations in the set A, prove that $R_1 \cap R_2$ is also an equivalence relation in A.

Solution:

Step 1: Understanding the Concept:
An equivalence relation is a binary relation that is reflexive, symmetric, and transitive. To prove that the intersection of two equivalence relations is also an equivalence relation, we must show that the intersection itself satisfies all three of these properties, assuming the original relations do.

Step 2: Key Formula or Approach:
Let R_1 and R_2 be two equivalence relations on a set A. Let $R = R_1 \cap R_2$. We need to prove the following: 1. **Reflexivity:** For all $a \in A$, $(a, a) \in R$. 2. **Symmetry:** If $(a, b) \in R$, then $(b, a) \in R$. 3. **Transitivity:** If $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

Step 3: Detailed Explanation or Calculation:
1. Proof of Reflexivity:
Let a be an arbitrary element of A.
Since R_1 is an equivalence relation, it is reflexive. Therefore, $(a, a) \in R_1$.
Since R_2 is an equivalence relation, it is reflexive. Therefore, $(a, a) \in R_2$.
By the definition of intersection, since (a, a) is in both R_1 and R_2, we have $(a, a) \in R_1 \cap R_2$.
Thus, $R_1 \cap R_2$ is reflexive.

2. Proof of Symmetry:
Let (a, b) be an arbitrary element of $R_1 \cap R_2$.
This means $(a, b) \in R_1$ and $(a, b) \in R_2$.
Since R_1 is symmetric, if $(a, b) \in R_1$, then $(b, a) \in R_1$.
Since R_2 is symmetric, if $(a, b) \in R_2$, then $(b, a) \in R_2$.
Because (b, a) is in both R_1 and R_2, it must be in their intersection: $(b, a) \in R_1 \cap R_2$.
Thus, $R_1 \cap R_2$ is symmetric.

3. Proof of Transitivity:
Let $(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2$.
From $(a, b) \in R_1 \cap R_2$, we know $(a, b) \in R_1$ and $(a, b) \in R_2$.
From $(b, c) \in R_1 \cap R_2$, we know $(b, c) \in R_1$ and $(b, c) \in R_2$.
Now consider R_1: We have $(a, b) \in R_1$ and $(b, c) \in R_1$. Since R_1 is transitive, it follows that $(a, c) \in R_1$.
Now consider R_2: We have $(a, b) \in R_2$ and $(b, c) \in R_2$. Since R_2 is transitive, it follows that $(a, c) \in R_2$.
Since (a, c) is in both R_1 and R_2, it must be in their intersection: $(a, c) \in R_1 \cap R_2$.
Thus, $R_1 \cap R_2$ is transitive.

Step 4: Final Answer:
Since $R_1 \cap R_2$ is reflexive, symmetric, and transitive, it is an equivalence relation in the set A.

> Quick Tip
>
> This property holds for the intersection of any number of equivalence relations. However, the union of two equivalence relations is not necessarily an equivalence relation (it is reflexive and symmetric, but may not be transitive).

---

d. If two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\vec{a} \cdot \vec{b} = 4$, find $|\vec{a} - \vec{b}|$.

Solution:

Step 1: Understanding the Concept:
The problem asks for the magnitude of the difference between two vectors, given their individual magnitudes and their dot product. The key is to relate the magnitude of a vector to the dot product of the vector with itself.

Step 2: Key Formula or Approach:
We will use the property that for any vector $\vec{v}$, its magnitude squared is given by $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$.
1. Apply this property to the vector $(\vec{a} - \vec{b})$: $|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})$.
2. Expand the dot product using its distributive property.
3. Substitute the given values of $|\vec{a}|$, $|\vec{b}|$, and $\vec{a} \cdot \vec{b}$.
4. Take the square root to find $|\vec{a} - \vec{b}|$.

Step 3: Detailed Explanation or Calculation:
We want to find $|\vec{a} - \vec{b}|$. Let's start by calculating its square:

$$|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})$$

Expand the dot product:

$$= \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b}$$

Using the properties $\vec{v} \cdot \vec{v} = |\vec{v}|^2$ and the commutative property of the dot product ($\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$):

$$= |\vec{a}|^2 - 2(\vec{a} \cdot \vec{b}) + |\vec{b}|^2$$

Now, substitute the given values: $|\vec{a}| = 2$, $|\vec{b}| = 3$, and $\vec{a} \cdot \vec{b} = 4$.

$$|\vec{a} - \vec{b}|^2 = (2)^2 - 2(4) + (3)^2$$

$$= 4 - 8 + 9$$

$$= 5$$

Now, take the square root of both sides. Since magnitude must be non-negative:

$$|\vec{a} - \vec{b}| = \sqrt{5}$$

Step 4: Final Answer:
The value of $|\vec{a} - \vec{b}|$ is $\sqrt{5}$.

> **Quick Tip**
>
> This is a very common type of vector problem. Remember the two key expansion formulas:
> $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + 2(\vec{a} \cdot \vec{b}) + |\vec{b}|^2$
> $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 - 2(\vec{a} \cdot \vec{b}) + |\vec{b}|^2$
> These are analogous to the algebraic identities $(x + y)^2$ and $(x - y)^2$.

---

4. Do all parts.

a. The Cartesian equation of a line is $\frac{x+3}{2} = \frac{y-5}{4} = \frac{z+6}{2}$. Find its vector equation.

Solution:

Step 1: Understanding the Concept:
The Cartesian equation of a line in 3D space, given by $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$, describes a line that passes through the point (x_1, y_1, z_1) and has direction ratios $\langle a, b, c \rangle$. The vector equation of a line is given by $\vec{r} = \vec{a} + \lambda\vec{b}$, where $\vec{a}$ is the position vector of a point on the line and $\vec{b}$ is a vector parallel to the line. We need to convert from one form to the other.

Step 2: Key Formula or Approach:
1. Identify the point (x_1, y_1, z_1) through which the line passes from the Cartesian equation. This gives the position vector $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$.
2. Identify the direction ratios $\langle a, b, c \rangle$ from the denominators of the Cartesian equation. This gives the parallel vector $\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$.
3. Write the vector equation in the form $\vec{r} = \vec{a} + \lambda\vec{b}$, where λ is a scalar parameter.

Step 3: Detailed Explanation:
The given Cartesian equation is:

$$\frac{x+3}{2} = \frac{y-5}{4} = \frac{z+6}{2}$$

We can rewrite this in the standard form $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$:

$$\frac{x-(-3)}{2} = \frac{y-5}{4} = \frac{z-(-6)}{2}$$

1. Identify the point on the line:
Comparing with the standard form, the line passes through the point $(x_1, y_1, z_1) = (-3, 5, -6)$.
The position vector of this point is $\vec{a} = -3\hat{i} + 5\hat{j} - 6\hat{k}$.

2. Identify the direction vector:
The direction ratios are given by the denominators, so $\langle a, b, c \rangle = \langle 2, 4, 2 \rangle$.
The vector parallel to the line is $\vec{b} = 2\hat{i} + 4\hat{j} + 2\hat{k}$.

3. Write the vector equation:
The vector equation of the line is $\vec{r} = \vec{a} + \lambda\vec{b}$.

$$\vec{r} = (-3\hat{i} + 5\hat{j} - 6\hat{k}) + \lambda(2\hat{i} + 4\hat{j} + 2\hat{k})$$

We can also use a simpler direction vector by factoring out a common scalar. Here, we can factor out 2 from $\vec{b}$: $2(\hat{i} + 2\hat{j} + \hat{k})$. Since we are using a parameter λ, any scalar multiple of the direction vector works. Let's call the new parameter $\mu = 2\lambda$. An equally valid vector equation is:

$$\vec{r} = (-3\hat{i} + 5\hat{j} - 6\hat{k}) + \mu(\hat{i} + 2\hat{j} + \hat{k})$$

Step 4: Final Answer:
The vector equation of the line is $\vec{r} = (-3\hat{i} + 5\hat{j} - 6\hat{k}) + \lambda(2\hat{i} + 4\hat{j} + 2\hat{k})$.

Quick Tip

To convert from Cartesian to vector form, simply pick out the coordinates of the point (remembering to flip the signs, e.g., $x + 3$ means $x_1 = -3$) to form vector $\vec{a}$, and the denominators to form the direction vector $\vec{b}$.

---

b. A die is thrown once. If the event 'the number obtained on the die is a multiple of 3' is represented by E and 'the number obtained on the die is even' is represented by F, tell whether the events E and F are independent.

Solution:

Step 1: Understanding the Concept:
Two events, E and F, are said to be independent if the occurrence of one event does not affect the probability of the other event occurring. Mathematically, two events are independent if and only if $P(E \cap F) = P(E) \times P(F)$. We need to calculate these three probabilities and check if the condition holds.

Step 2: Key Formula or Approach:
1. Define the sample space S for throwing a single die.
2. Identify the outcomes for event E (a multiple of 3).

3. Identify the outcomes for event F (an even number).
4. Identify the outcomes for the intersection event $E \cap F$ (an even multiple of 3).
5. Calculate the probabilities $P(E)$, $P(F)$, and $P(E \cap F)$.
6. Check if $P(E \cap F) = P(E) \times P(F)$.

Step 3: Detailed Explanation:

1. Sample Space:

When a die is thrown once, the sample space is S = {1, 2, 3, 4, 5, 6}. The total number of outcomes is $n(S) = 6$.

2. Event E:

E is the event that the number is a multiple of 3. The outcomes are E = {3, 6}.
The number of favorable outcomes is $n(E) = 2$.
The probability is $P(E) = \frac{n(E)}{n(S)} = \frac{2}{6} = \frac{1}{3}$.

3. Event F:

F is the event that the number is even. The outcomes are F = {2, 4, 6}.
The number of favorable outcomes is $n(F) = 3$.
The probability is $P(F) = \frac{n(F)}{n(S)} = \frac{3}{6} = \frac{1}{2}$.

4. Intersection Event $E \cap F$:

$E \cap F$ is the event that the number is both a multiple of 3 and even. The only outcome is $E \cap F = \{6\}$.
The number of favorable outcomes is $n(E \cap F) = 1$.
The probability is $P(E \cap F) = \frac{n(E \cap F)}{n(S)} = \frac{1}{6}$.

5. Check for Independence:

We need to check if $P(E \cap F) = P(E) \times P(F)$.
Let's calculate the product of $P(E)$ and $P(F)$:

$$P(E) \times P(F) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

We see that $P(E \cap F) = \frac{1}{6}$, which is equal to $P(E) \times P(F)$.

Step 4: Final Answer:

Since $P(E \cap F) = P(E) \times P(F)$, the events E and F are independent.

Quick Tip

The concept of independence is crucial in probability. Always remember the test: $P(E \cap F) = P(E) \times P(F)$. If they are equal, the events are independent. If they are not equal, the events are dependent.

c. If $A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$, find $(A + 2B)'$.

Solution:

Step 1: Understanding the Concept:
This question involves matrix operations, specifically addition, scalar multiplication, and transposition. The notation A' denotes the transpose of matrix A. We need to use the properties of matrix transpose, one of which is $(X + Y)' = X' + Y'$ and another is $(kX)' = kX'$ where k is a scalar.

Step 2: Key Formula or Approach:
We can use the property of transposes: $(A + 2B)' = A' + (2B)' = A' + 2B'$. 1. We are already given A'.
2. Find the transpose of B, which is B'.
3. Calculate 2B'.
4. Add A' and 2B' to find the result.

Step 3: Detailed Explanation:
We are given:

$$A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$$

We need to find $(A + 2B)'$. Using the properties of transpose:

$$(A + 2B)' = A' + (2B)' = A' + 2B'$$

First, let's find B', the transpose of B. To find the transpose, we interchange the rows and columns.

$$B' = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$$

Next, calculate 2B':

$$2B' = 2\begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2(-1) & 2(1) \\ 2(0) & 2(2) \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 0 & 4 \end{bmatrix}$$

Finally, add A' and 2B':

$$(A + 2B)' = A' + 2B' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 2 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2 + (-2) & 3 + 2 \\ 1 + 0 & 2 + 4 \end{bmatrix} = \begin{bmatrix} -4 & 5 \\ 1 & 6 \end{bmatrix}$$

Step 4: Final Answer:

The value of $(A + 2B)'$ is $\begin{bmatrix} -4 & 5 \\ 1 & 6 \end{bmatrix}$.

Quick Tip

Using the properties of transpose $(X + Y)' = X' + Y'$ is often faster than first calculating A, then 2B, then A+2B, and finally taking the transpose. Using properties can reduce the number of calculations and potential errors.

---

d. Show that the function $f(x) = \log \sin x$ is increasing in the interval $(0, \frac{\pi}{2})$ and decreasing in $(\frac{\pi}{2}, \pi)$.

Solution:

Step 1: Understanding the Concept:

To determine the intervals where a function is increasing or decreasing, we use the first derivative test. A differentiable function $f(x)$ is: - **Increasing** on an interval if its first derivative, $f'(x)$, is positive $(f'(x) > 0)$ for all x in that interval. - **Decreasing** on an interval if its first derivative, $f'(x)$, is negative $(f'(x) < 0)$ for all x in that interval.

Step 2: Key Formula or Approach:

1. Find the first derivative of the function, $f'(x)$.
2. Determine the sign of $f'(x)$ in the interval $(0, \frac{\pi}{2})$.
3. Determine the sign of $f'(x)$ in the interval $(\frac{\pi}{2}, \pi)$.
4. Conclude whether the function is increasing or decreasing based on the sign of the derivative in each interval.

Step 3: Detailed Explanation:

The given function is $f(x) = \log(\sin x)$. The domain of this function requires $\sin x > 0$, which is true for $x \in (0, \pi)$.

1. Find the first derivative, $f'(x)$:

Using the chain rule, $\frac{d}{dx}(\log u) = \frac{1}{u} \cdot \frac{du}{dx}$.

$$f'(x) = \frac{d}{dx}(\log(\sin x)) = \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x)$$

$$f'(x) = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x} = \cot x$$

2. Analyze the sign of $f'(x)$ in $(0, \frac{\pi}{2})$:

The interval $(0, \frac{\pi}{2})$ corresponds to the first quadrant.

In the first quadrant, both $\sin x$ and $\cos x$ are positive.

Therefore, $f'(x) = \cot x = \frac{\cos x}{\sin x}$ is positive for all $x \in (0, \frac{\pi}{2})$.
Since $f'(x) > 0$, the function $f(x)$ is **increasing** in the interval $(0, \frac{\pi}{2})$.

3. Analyze the sign of $f'(x)$ in $(\frac{\pi}{2}, \pi)$:
The interval $(\frac{\pi}{2}, \pi)$ corresponds to the second quadrant.
In the second quadrant, $\sin x$ is positive, but $\cos x$ is negative.
Therefore, $f'(x) = \cot x = \frac{\cos x}{\sin x} = \frac{\text{negative}}{\text{positive}}$ is negative for all $x \in (\frac{\pi}{2}, \pi)$.
Since $f'(x) < 0$, the function $f(x)$ is **decreasing** in the interval $(\frac{\pi}{2}, \pi)$.

Step 4: Final Answer:
We have shown that $f'(x) > 0$ in $(0, \frac{\pi}{2})$ and $f'(x) < 0$ in $(\frac{\pi}{2}, \pi)$. Therefore, the function $f(x) = \log \sin x$ is increasing in the first interval and decreasing in the second.

> Quick Tip
>
> The sign of the first derivative determines whether a function is increasing or decreasing. A helpful way to remember the signs of trigonometric functions is the "All Students Take Calculus" (ASTC) rule for quadrants I, II, III, and IV, respectively.

---

5. Do all parts.

a. Find the value of the determinant $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$.

Solution:

Step 1: Understanding the Concept:
The problem is to evaluate a 3x3 determinant. The key to solving such problems efficiently is to look for symmetries and apply properties of determinants to simplify the expression before expanding. A common technique is to perform row or column operations to create zeros or common factors.

Step 2: Key Formula or Approach:
1. Use the property that adding a multiple of one column (or row) to another does not change the value of the determinant. Specifically, apply the column operation $C_1 \to C_1 + C_2 + C_3$.
2. Factor out the common term from the first column.
3. Apply row operations to create zeros in the first column, making expansion easier.
4. Expand the determinant along the first column.

Step 3: Detailed Explanation:

Let Δ be the given determinant:

$$\Delta = \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$$

Apply the column operation $C_1 \to C_1 + C_2 + C_3$:

$$\Delta = \begin{vmatrix} (x+y+2z)+x+y & x & y \\ z+(y+z+2x)+y & y+z+2x & y \\ z+x+(z+x+2y) & x & z+x+2y \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 2x+2y+2z & x & y \\ 2x+2y+2z & y+z+2x & y \\ 2x+2y+2z & x & z+x+2y \end{vmatrix}$$

Factor out the common term $2(x+y+z)$ from the first column:

$$\Delta = 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$

Now, apply row operations to create zeros. Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$:

$$\Delta = 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 1-1 & (y+z+2x)-x & y-y \\ 1-1 & x-x & (z+x+2y)-y \end{vmatrix}$$

$$\Delta = 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 0 & x+y+z & 0 \\ 0 & 0 & x+y+z \end{vmatrix}$$

Now, expand the determinant along the first column:

$$\Delta = 2(x+y+z) \left[1 \begin{vmatrix} x+y+z & 0 \\ 0 & x+y+z \end{vmatrix} - 0 + 0 \right]$$

$$\Delta = 2(x+y+z)[(x+y+z)(x+y+z) - (0)(0)]$$

$$\Delta = 2(x+y+z)(x+y+z)^2$$

$$\Delta = 2(x+y+z)^3$$

Step 4: Final Answer:

The value of the determinant is $2(x+y+z)^3$.

Quick Tip

When evaluating determinants, always look for an opportunity to apply the operation $R_1 \to R_1 + R_2 + R_3$ or $C_1 \to C_1 + C_2 + C_3$. If this creates a common factor in a row or column, it will simplify the determinant significantly.

b. If $y = x^{x^{x^{\dots\text{to infinity}}}}$, prove that $x\frac{dy}{dx} = \frac{y^2}{1-y\log x}$.

Solution:

Step 1: Understanding the Concept:
This problem involves an infinite power tower. The key to solving such problems is to recognize the self-similar nature of the expression. The expression in the exponent is the same as the original expression for y. This allows us to write a simpler, implicit equation for y, which can then be differentiated using logarithmic differentiation.

Step 2: Key Formula or Approach:
1. Rewrite the infinite tower in a finite form: $y = x^y$.
2. Take the natural logarithm of both sides to bring the exponent down: $\log y = y \log x$.
3. Differentiate both sides of the equation implicitly with respect to x.
4. Solve the resulting equation for $\frac{dy}{dx}$.
5. Rearrange the expression to match the required form.

Step 3: Detailed Explanation:
Given the function: $y = x^{x^{x^{\cdots}}}$.
Due to the infinite nature of the tower, we can write:

$$y = x^y$$

Take the natural logarithm on both sides:

$$\log y = \log(x^y)$$

Using the logarithm property $\log(a^b) = b\log a$, we get:

$$\log y = y\log x$$

Now, differentiate both sides with respect to x, using the product rule on the right side:

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(y\log x)$$

$$\frac{1}{y}\frac{dy}{dx} = \left(\frac{dy}{dx}\cdot\log x\right) + \left(y\cdot\frac{1}{x}\right)$$

Now, we need to solve for $\frac{dy}{dx}$. Group the terms containing $\frac{dy}{dx}$ on one side:

$$\frac{1}{y}\frac{dy}{dx} - (\log x)\frac{dy}{dx} = \frac{y}{x}$$

Factor out $\frac{dy}{dx}$:

$$\frac{dy}{dx}\left(\frac{1}{y} - \log x\right) = \frac{y}{x}$$

$$\frac{dy}{dx}\left(\frac{1-y\log x}{y}\right)=\frac{y}{x}$$

Isolate $\frac{dy}{dx}$:

$$\frac{dy}{dx}=\frac{y}{x}\cdot\frac{y}{1-y\log x}=\frac{y^2}{x(1-y\log x)}$$

Finally, multiply both sides by x to get the desired form:

$$x\frac{dy}{dx}=\frac{y^2}{1-y\log x}$$

Step 4: Final Answer:

The relation $x\frac{dy}{dx}=\frac{y^2}{1-y\log x}$ has been proven.

Quick Tip

Logarithmic differentiation is the go-to method for functions that have variables in both the base and the exponent (like $f(x)^{g(x)}$) or for complex products/quotients. The trick for infinite towers or nested radicals is always to express the function in terms of itself, e.g., $y=f(x,y)$.

---

c. Evaluate: $\int\sqrt{3-2x-x^2}dx$.

Solution:

Step 1: Understanding the Concept:

This is an integral of a square root of a quadratic expression. The standard technique is to complete the square for the quadratic expression inside the root to transform it into the form $\sqrt{a^2-u^2}$, $\sqrt{u^2-a^2}$, or $\sqrt{u^2+a^2}$. This allows the use of a standard integration formula.

Step 2: Key Formula or Approach:

1. Complete the square for the quadratic $3-2x-x^2$.
2. Rewrite the integral in the standard form $\int\sqrt{a^2-u^2}du$.
3. Apply the standard integration formula:

$$\int\sqrt{a^2-u^2}du=\frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right)+C$$

Step 3: Detailed Explanation:

First, complete the square for the expression inside the square root:

$$\begin{aligned}3-2x-x^2&=-(x^2+2x-3)\\&=-((x^2+2x+1)-1-3)\end{aligned}$$

$$= -((x+1)^2 - 4)$$
$$= 4 - (x+1)^2$$

So, the integral becomes:

$$\int \sqrt{4-(x+1)^2}dx$$

This is now in the form $\int \sqrt{a^2-u^2}du$, where: $a^2 = 4 \implies a = 2$
$u = x+1 \implies du = dx$
Now, apply the standard formula:

$$\int \sqrt{a^2-u^2}du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right) + C$$

Substitute back $a = 2$ and $u = x+1$:

$$\int \sqrt{4-(x+1)^2}dx = \frac{x+1}{2}\sqrt{4-(x+1)^2} + \frac{4}{2}\sin^{-1}\left(\frac{x+1}{2}\right) + C$$
$$= \frac{x+1}{2}\sqrt{3-2x-x^2} + 2\sin^{-1}\left(\frac{x+1}{2}\right) + C$$

Step 4: Final Answer:
The value of the integral is $\frac{x+1}{2}\sqrt{3-2x-x^2} + 2\sin^{-1}\left(\frac{x+1}{2}\right) + C$.

Quick Tip

When completing the square for a quadratic ax^2+bx+c, it's often easiest to factor out 'a' first, especially if it's negative. Here, we factored out -1 from $-x^2-2x$ to get $-(x^2+2x)$, making it easier to see that we need to add and subtract 1 inside the parenthesis.

---

d. Prove that $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}$**,** $x \in (0, \pi/4)$**.**

Solution:

Step 1: Understanding the Concept:
The problem requires simplifying a complex trigonometric expression involving an inverse function. The key is to simplify the argument of the $\cot^{-1}$ function using trigonometric identities. The identities for $1 \pm \sin x$ are particularly useful.

Step 2: Key Formula or Approach:
1. Use the identities:

$$1 + \sin x = \cos^2(x/2) + \sin^2(x/2) + 2\sin(x/2)\cos(x/2) = (\cos(x/2) + \sin(x/2))^2$$
$$1 - \sin x = \cos^2(x/2) + \sin^2(x/2) - 2\sin(x/2)\cos(x/2) = (\cos(x/2) - \sin(x/2))^2$$

2. Substitute these into the expression and simplify the square roots, paying attention to the signs based on the given interval for x.
3. Simplify the resulting fraction inside the $\cot^{-1}$ function.
4. Use the property $\cot^{-1}(\cot\theta) = \theta$ for θ in the appropriate range.

Step 3: Detailed Explanation:
Let's simplify the terms under the square roots.

$$\sqrt{1+\sin x} = \sqrt{(\cos(x/2)+\sin(x/2))^2} = |\cos(x/2)+\sin(x/2)|$$

$$\sqrt{1-\sin x} = \sqrt{(\cos(x/2)-\sin(x/2))^2} = |\cos(x/2)-\sin(x/2)|$$

The problem states that $x \in (0, \pi/4)$. This means $x/2 \in (0, \pi/8)$.
In this interval $(0, \pi/8)$, both $\cos(x/2)$ and $\sin(x/2)$ are positive. Also, $\cos(x/2) > \sin(x/2)$.
Therefore, $(\cos(x/2)+\sin(x/2))$ is positive, and $(\cos(x/2)-\sin(x/2))$ is also positive.
So we can remove the absolute value signs:

$$\sqrt{1+\sin x} = \cos(x/2)+\sin(x/2)$$

$$\sqrt{1-\sin x} = \cos(x/2)-\sin(x/2)$$

Now, substitute these into the main expression's argument:

$$\frac{(\cos(x/2)+\sin(x/2))+(\cos(x/2)-\sin(x/2))}{(\cos(x/2)+\sin(x/2))-(\cos(x/2)-\sin(x/2))}$$

Simplify the numerator and the denominator:

$$\frac{2\cos(x/2)}{2\sin(x/2)} = \cot(x/2)$$

So the original expression becomes:

$$\cot^{-1}(\cot(x/2))$$

Since $x/2 \in (0, \pi/8)$, which is within the principal value range of $\cot^{-1}$ (which is $(0, \pi)$), we can simplify this to:

$$\cot^{-1}(\cot(x/2)) = \frac{x}{2}$$

Step 4: Final Answer:
We have proven that $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}$ for $x \in (0, \pi/4)$.

Quick Tip

The identities $1+\sin x = (\cos(x/2)+\sin(x/2))^2$ and $1-\sin x = (\cos(x/2)-\sin(x/2))^2$ are extremely useful for simplifying expressions involving $\sqrt{1 \pm \sin x}$. Always check the given interval for x to correctly determine the signs when taking the square root.

e. For the differential equation $xy\frac{dy}{dx} = (x+2)(y+2)$, find the curve passing through the point $(1, -1)$.

Solution:

Step 1: Understanding the Concept:
This is a first-order differential equation. It can be solved using the method of separation of variables. After finding the general solution (containing an arbitrary constant C), we use the given point $(1, -1)$ to find the specific value of C, which gives the particular solution or the equation of the specific curve.

Step 2: Key Formula or Approach:
1. Separate the variables by moving all terms involving y to one side with dy and all terms involving x to the other side with dx.
2. Integrate both sides of the separated equation.
3. Use the initial condition (the point $(1, -1)$) to solve for the constant of integration, C.
4. Substitute C back into the general solution to get the equation of the curve.

Step 3: Detailed Explanation:
The given differential equation is:

$$xy\frac{dy}{dx} = (x+2)(y+2)$$

1. Separate the variables:

$$\frac{y}{y+2}dy = \frac{x+2}{x}dx$$

The left side can be simplified for easier integration: $\frac{y}{y+2} = \frac{y+2-2}{y+2} = 1 - \frac{2}{y+2}$.
The right side can also be simplified: $\frac{x+2}{x} = 1 + \frac{2}{x}$.
So the separated equation becomes:

$$\left(1 - \frac{2}{y+2}\right) dy = \left(1 + \frac{2}{x}\right) dx$$

2. Integrate both sides:

$$\int \left(1 - \frac{2}{y+2}\right) dy = \int \left(1 + \frac{2}{x}\right) dx$$

$$y - 2\ln|y+2| = x + 2\ln|x| + C$$

This is the general solution.

3. Use the initial condition $(1, -1)$:
Substitute $x = 1$ and $y = -1$ into the general solution to find C.

$$-1 - 2\ln|-1+2| = 1 + 2\ln|1| + C$$

$$-1 - 2\ln(1) = 1 + 2(0) + C$$

Since $\ln(1) = 0$:

$$-1 - 2(0) = 1 + 0 + C$$
$$-1 = 1 + C \implies C = -2$$

4. Write the particular solution:
Substitute $C = -2$ back into the general solution:

$$y - 2\ln|y + 2| = x + 2\ln|x| - 2$$

This can be rearranged as:

$$y - x + 2 = 2\ln|x| + 2\ln|y + 2|$$
$$y - x + 2 = 2\ln|x(y + 2)|$$

Step 4: Final Answer:
The equation of the curve passing through the point $(1, -1)$ is $y - 2\ln|y+2| = x + 2\ln|x| - 2$.

> **Quick Tip**
>
> When separating variables, if you have a fraction where the degree of the numerator is equal to or greater than the degree of the denominator (like $\frac{y}{y+2}$), perform algebraic division (or a simple manipulation as shown) to simplify it before integrating.

---

6. Do all parts.

a. A person has taken the contract of a construction work. The probability of strike is 0.65. The probabilities of the construction work being completed on time in the circumstances of no strike and strike are respectively 0.80 and 0.32. Find the probability of the construction work being completed on time.

Solution:

Step 1: Understanding the Concept:
This problem can be solved using the Law of Total Probability. The event of the work being completed on time depends on whether there is a strike or not. We need to consider both scenarios (strike and no strike), calculate the probability of completion in each case, and then sum them up to get the overall probability of completion.

Step 2: Key Formula or Approach:
Let S be the event that a strike occurs.
Let S' be the event that there is no strike.
Let C be the event that the construction work is completed on time.
The Law of Total Probability states:

$$P(C) = P(S) \cdot P(C|S) + P(S') \cdot P(C|S')$$

where $P(C|S)$ is the probability of completion given a strike, and $P(C|S')$ is the probability of completion given no strike.

Step 3: Detailed Explanation:
From the problem statement, we have the following probabilities:
Probability of a strike, $P(S) = 0.65$.
Probability of no strike, $P(S') = 1 - P(S) = 1 - 0.65 = 0.35$.
Probability of completion given no strike, $P(C|S') = 0.80$.
Probability of completion given a strike, $P(C|S) = 0.32$.

Now, we apply the Law of Total Probability:

$$P(C) = P(S) \cdot P(C|S) + P(S') \cdot P(C|S')$$

Substitute the given values into the formula:

$$P(C) = (0.65 \times 0.32) + (0.35 \times 0.80)$$

$$P(C) = 0.208 + 0.280$$

$$P(C) = 0.488$$

Step 4: Final Answer:
The probability of the construction work being completed on time is 0.488.

> **Quick Tip**
>
> Problems involving conditional events and partitions of the sample space (like 'strike' and 'no strike') are often solved using the Law of Total Probability or Bayes' Theorem. Identify the events and conditional probabilities clearly before applying the formula.

---

b. Minimize Z = 200x + 500y by graphical method subject to the following constraints:
$x + 2y \geq 10$, $3x + 4y \leq 24$, $x \geq 0$, $y \geq 0$.

Solution:

Step 1: Understanding the Concept:
This is a Linear Programming Problem (LPP). The goal is to find the minimum value of a linear objective function Z, subject to a set of linear inequalities called constraints. The graphical method involves plotting these constraints to find the feasible region and then evaluating the objective function at the corner points (vertices) of this region. The optimal solution (minimum or maximum) will occur at one of these vertices.

Step 2: Key Formula or Approach:
1. Convert the inequalities into equations to plot the lines.
2. Graph the lines on a 2D plane.
3. Identify the feasible region, which is the area that satisfies all the given constraints simultaneously.
4. Determine the coordinates of the corner points of the feasible region.
5. Evaluate the objective function Z = 200x + 500y at each corner point.
6. The smallest value of Z will be the minimum value.

Step 3: Detailed Explanation:
The constraints are:

$$x + 2y \geq 10$$

$$3x + 4y \leq 24$$

$$x \geq 0, y \geq 0$$

1. Graph the lines:
Line 1: $x + 2y = 10$. It passes through (10, 0) and (0, 5). The region $x + 2y \geq 10$ is the area on and above this line.
Line 2: $3x + 4y = 24$. It passes through (8, 0) and (0, 6). The region $3x + 4y \leq 24$ is the area on and below this line.
The constraints $x \geq 0, y \geq 0$ restrict the feasible region to the first quadrant.

2. Find the feasible region and corner points:
The feasible region is the area in the first quadrant that is above the line $x+2y = 10$ and below the line $3x + 4y = 24$. The vertices (corner points) of this region are:
Point A: Intersection of $x = 0$ and $3x + 4y = 24$. $3(0) + 4y = 24 \implies y = 6$. So, A = (0, 6).
Point B: Intersection of $x = 0$ and $x + 2y = 10$. $0 + 2y = 10 \implies y = 5$. So, B = (0, 5).
Point C: Intersection of $x + 2y = 10$ and $3x + 4y = 24$.
From $x + 2y = 10$, we have $x = 10 - 2y$. Substitute this into the second equation:

$$3(10 - 2y) + 4y = 24$$

$$30 - 6y + 4y = 24$$

$$30 - 2y = 24 \implies 2y = 6 \implies y = 3$$

Substitute $y = 3$ back into $x = 10 - 2y$:

$$x = 10 - 2(3) = 4$$

So, C = (4, 3).

3. Evaluate Z at corner points:
The objective function is Z = 200x + 500y.
At A(0, 6): $Z = 200(0) + 500(6) = 3000$.
At B(0, 5): $Z = 200(0) + 500(5) = 2500$.
At C(4, 3): $Z = 200(4) + 500(3) = 800 + 1500 = 2300$.

4. Find the minimum value:
Comparing the values of Z, the minimum value is 2300.

Step 4: Final Answer:
The minimum value of Z is 2300, which occurs at the point (x, y) = (4, 3).

> **Quick Tip**
>
> Always shade the feasible region clearly on the graph to avoid confusion. To find which side of a line satisfies an inequality, pick a test point (like the origin (0,0), if the line doesn't pass through it) and check if it satisfies the inequality.

---

c. Find the shortest distance between the lines whose vector equations are the following:
$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$ **and** $\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$**.**

Solution:

Step 1: Understanding the Concept:
The problem is to find the shortest distance between two skew lines given in vector form. The standard method involves using a formula that uses the position vectors of a point on each line and the direction vectors of the lines.

Step 2: Key Formula or Approach:
First, rewrite the line equations in the standard form $\vec{r} = \vec{a} + \lambda\vec{b}$.
Line 1: $\vec{r} = \vec{a_1} + t\vec{b_1}$
Line 2: $\vec{r} = \vec{a_2} + s\vec{b_2}$
The formula for the shortest distance (SD) between two skew lines is:

$$\text{SD} = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

Step 3: Detailed Explanation:
1. Identify vectors from the line equations:
Line 1: $\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k} = (\hat{i} - 2\hat{j} + 3\hat{k}) + t(-\hat{i} + \hat{j} - 2\hat{k})$.
So, $\vec{a_1} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b_1} = -\hat{i} + \hat{j} - 2\hat{k}$.
Line 2: $\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k} = (\hat{i} - \hat{j} - \hat{k}) + s(\hat{i} + 2\hat{j} - 2\hat{k})$.
So, $\vec{a_2} = \hat{i} - \hat{j} - \hat{k}$ and $\vec{b_2} = \hat{i} + 2\hat{j} - 2\hat{k}$.

2. Calculate $\vec{a_2} - \vec{a_1}$:

$$\vec{a_2} - \vec{a_1} = (\hat{i} - \hat{j} - \hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k}) = 0\hat{i} + \hat{j} - 4\hat{k}$$

3. Calculate $\vec{b_1} \times \vec{b_2}$:

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix} = \hat{i}(-2 - (-4)) - \hat{j}(2 - (-2)) + \hat{k}(-2 - 1) = 2\hat{i} - 4\hat{j} - 3\hat{k}$$

4. Calculate $|\vec{b_1} \times \vec{b_2}|$:

$$|\vec{b_1} \times \vec{b_2}| = \sqrt{2^2 + (-4)^2 + (-3)^2} = \sqrt{4 + 16 + 9} = \sqrt{29}$$

5. Calculate $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})$:

$$(0\hat{i} + \hat{j} - 4\hat{k}) \cdot (2\hat{i} - 4\hat{j} - 3\hat{k}) = (0)(2) + (1)(-4) + (-4)(-3) = 0 - 4 + 12 = 8$$

6. Calculate the Shortest Distance:

$$\text{SD} = \left| \frac{8}{\sqrt{29}} \right| = \frac{8}{\sqrt{29}}$$

Step 4: Final Answer:
The shortest distance between the two lines is $\frac{8}{\sqrt{29}}$ units.

Quick Tip

Before using the skew lines formula, quickly check if the lines are parallel by seeing if their direction vectors ($\vec{b_1}$ and $\vec{b_2}$) are scalar multiples of each other. If they are, a different formula for parallel lines must be used. In this case, $\vec{b_1}$ and $\vec{b_2}$ are clearly not parallel.

---

d. For two vectors $\vec{a}$ and $\vec{b}$ prove that $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$.

Solution:

Step 1: Understanding the Concept:
This inequality is known as the triangle inequality for vectors. It states that the magnitude of the sum of two vectors is less than or equal to the sum of their individual magnitudes. Geometrically, it means that the length of any side of a triangle is less than or equal to the sum of the lengths of the other two sides. The proof relies on the properties of the dot product and the Cauchy-Schwarz inequality.

Step 2: Key Formula or Approach:
1. Start with the square of the magnitude, $|\vec{a} + \vec{b}|^2$.
2. Use the property $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$.
3. Expand the dot product $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})$.
4. Apply the Cauchy-Schwarz inequality, which states $\vec{a} \cdot \vec{b} \leq |\vec{a}||\vec{b}|$.

5. Show that $|\vec{a}+\vec{b}|^2 \le (|\vec{a}|+|\vec{b}|)^2$.
6. Take the square root of both sides to obtain the final inequality.

Step 3: Detailed Explanation:
We consider the square of the magnitude of $\vec{a}+\vec{b}$:

$$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$$

Expanding the dot product:

$$|\vec{a}+\vec{b}|^2 = \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$$

Using the properties $\vec{v}\cdot\vec{v} = |\vec{v}|^2$ and $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$:

$$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + 2(\vec{a}\cdot\vec{b}) + |\vec{b}|^2$$

From the definition of the dot product, we know $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between the vectors. The Cauchy-Schwarz inequality states that $\vec{a}\cdot\vec{b} \le |\vec{a}||\vec{b}|$ (since $\cos\theta \le 1$).
Applying this to our expression:

$$|\vec{a}|^2 + 2(\vec{a}\cdot\vec{b}) + |\vec{b}|^2 \le |\vec{a}|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2$$

The right side of the inequality is the expansion of $(|\vec{a}|+|\vec{b}|)^2$.
So, we have:

$$|\vec{a}+\vec{b}|^2 \le (|\vec{a}|+|\vec{b}|)^2$$

Since magnitudes are non-negative, we can take the square root of both sides without changing the inequality:

$$|\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$$

Step 4: Final Answer:
The triangle inequality for vectors, $|\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$, has been proven. The equality holds when the vectors $\vec{a}$ and $\vec{b}$ are in the same direction ($\theta = 0$).

Quick Tip

Proving vector identities and inequalities often involves starting with the square of the magnitude and using the identity $|\vec{v}|^2 = \vec{v}\cdot\vec{v}$. This converts the problem from one of magnitudes (which involve square roots) to one of dot products, which are often easier to manipulate algebraically.

---

e. Solve: $ydx - (x+2y^2)dy = 0$.

Solution:

Step 1: Understanding the Concept:
This is a first-order differential equation. It is not immediately separable, but we can rearrange it to see if it fits the form of a linear differential equation. By rewriting it with x as the dependent variable and y as the independent variable ($\frac{dx}{dy}$), we can identify it as a linear equation of the form $\frac{dx}{dy} + P(y)x = Q(y)$.

Step 2: Key Formula or Approach:
1. Rearrange the equation into the standard linear form $\frac{dx}{dy} + P(y)x = Q(y)$.
2. Identify $P(y)$ and $Q(y)$.
3. Calculate the integrating factor (I.F.) using the formula: I.F. $= e^{\int P(y)dy}$.
4. The general solution is given by: $x \cdot (\text{I.F.}) = \int Q(y) \cdot (\text{I.F.})dy + C$.
5. Solve the integral and find the final solution.

Step 3: Detailed Explanation:
The given equation is:

$$ydx - (x + 2y^2)dy = 0$$

Rearrange to solve for $\frac{dx}{dy}$:

$$ydx = (x + 2y^2)dy$$

$$\frac{dx}{dy} = \frac{x + 2y^2}{y} = \frac{x}{y} + 2y$$

Arrange this into the standard linear form:

$$\frac{dx}{dy} - \frac{1}{y}x = 2y$$

This is a linear differential equation in x.
1. Identify P(y) and Q(y):
$P(y) = -\frac{1}{y}$ and $Q(y) = 2y$.
2. Calculate the Integrating Factor (I.F.):

$$\text{I.F.} = e^{\int P(y)dy} = e^{\int -\frac{1}{y}dy} = e^{-\ln|y|} = e^{\ln|y^{-1}|} = e^{\ln|1/y|} = \frac{1}{y}$$

(Assuming $y > 0$, we can drop the absolute value).
3. Apply the solution formula:

$$x \cdot (\text{I.F.}) = \int Q(y) \cdot (\text{I.F.})dy + C$$

$$x \cdot \frac{1}{y} = \int (2y) \cdot \left(\frac{1}{y}\right) dy + C$$

$$\frac{x}{y} = \int 2dy + C$$

4. Integrate and find the solution:

$$\frac{x}{y} = 2y + C$$

$$x = y(2y + C) = 2y^2 + Cy$$

Step 4: Final Answer:
The general solution to the differential equation is $x = 2y^2 + Cy$, where C is an arbitrary constant.

Quick Tip

If a differential equation is not easily separable or linear in the form $\frac{dy}{dx}$, try rearranging it into the form $\frac{dx}{dy}$. Sometimes, the equation is linear with respect to the other variable.

---

7. Do any one part.

a.) If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$**, find** A^{-1}**. Using** A^{-1} **solve the following system of equations:**
2x - 3y + 5z = 11
3x + 2y - 4z = -5
x + y - 2z = -3.

Solution:

Step 1: Understanding the Concept:
This problem has two parts. First, we need to find the inverse of a 3x3 matrix. Second, we use this inverse to solve a system of linear equations. The system of equations is given in the form AX = B, where A is the same matrix for which we need to find the inverse. The solution is given by X = A^{-1}B.

Step 2: Key Formula or Approach:
Part 1: Finding A^{-1}
1. Calculate the determinant of A, $|A|$.
2. Find the matrix of cofactors, C.
3. Find the adjugate of A, adj(A), which is the transpose of C.
4. The inverse is $A^{-1} = \frac{1}{|A|}\text{adj}(A)$.
Part 2: Solving the system AX = B
1. Identify the matrices A, X, and B from the system of equations.
2. Calculate the solution using X = A^{-1}B.

Step 3: Detailed Explanation:
Part 1: Finding A^{-1}

Given $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$.

1. Determinant of A:

$$|A| = 2(2(-2) - (-4)(1)) - (-3)(3(-2) - (-4)(1)) + 5(3(1) - 2(1))$$

$$|A| = 2(-4+4) + 3(-6+4) + 5(3-2) = 2(0) + 3(-2) + 5(1) = 0 - 6 + 5 = -1$$

Since $|A| \neq 0$, the inverse exists.

2. Adjugate of A: The cofactor matrix C is:

$$C_{11} = 0, \quad C_{12} = -(-6+4) = 2, \quad C_{13} = 3 - 2 = 1$$

$$C_{21} = -(6-5) = -1, \quad C_{22} = -4 - 5 = -9, \quad C_{23} = -(2 - (-3)) = -5$$

$$C_{31} = 12 - 10 = 2, \quad C_{32} = -(-8 - 15) = 23, \quad C_{33} = 4 - (-9) = 13$$

So, $C = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}$.

adj(A) $= C^T = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$.

3. Inverse of A:

$$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

Part 2: Solving the System of Equations

The system is:

$$2x - 3y + 5z = 11$$

$$3x + 2y - 4z = -5$$

$$x + y - 2z = -3$$

This can be written as AX = B, where:

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

The solution is X = A^{-1}B.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0(11) + 1(-5) + (-2)(-3) \\ -2(11) + 9(-5) + (-23)(-3) \\ -1(11) + 5(-5) + (-13)(-3) \end{bmatrix} = \begin{bmatrix} 0 - 5 + 6 \\ -22 - 45 + 69 \\ -11 - 25 + 39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Step 4: Final Answer:

The inverse is $A^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$.

The solution to the system of equations is x = 1, y = 2, z = 3.

> **Quick Tip**
>
> When a question asks you to find an inverse and then solve a system of equations, always check if the coefficient matrix of the system is the same as the matrix whose inverse you just found. This saves a lot of time as you can directly use the result.

---

b.) If $A = \begin{bmatrix} 0 & -\tan(\alpha/2) \\ \tan(\alpha/2) & 0 \end{bmatrix}$ and I is the identity matrix of order 2, prove that $I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

Solution:

Step 1: Understanding the Concept:

This problem involves proving a matrix identity. The strategy is to calculate the Left-Hand Side (LHS) and the Right-Hand Side (RHS) separately and show that they are equal. The proof will involve matrix addition, subtraction, multiplication, and the use of half-angle formulas for sine and cosine in terms of tangent.

Step 2: Key Formula or Approach:

1. Calculate LHS = I + A.
2. Calculate I - A.
3. Use the half-angle formulas: $\cos\alpha = \frac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)}$ and $\sin\alpha = \frac{2\tan(\alpha/2)}{1+\tan^2(\alpha/2)}$.
4. Let $t = \tan(\alpha/2)$ for simplicity.
5. Calculate RHS = $(I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ by substituting the half-angle formulas and performing matrix multiplication.
6. Compare LHS and RHS.

Step 3: Detailed Explanation:

Let $t = \tan(\alpha/2)$. Then $A = \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Calculate LHS:

$$\text{LHS} = I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix} = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}$$

Calculate RHS: First, find I - A:

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix} = \begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}$$

Now, use the half-angle formulas in terms of t:

$$\cos\alpha = \frac{1-t^2}{1+t^2}, \quad \sin\alpha = \frac{2t}{1+t^2}$$

The matrix becomes $\begin{bmatrix} \frac{1-t^2}{1+t^2} & -\frac{2t}{1+t^2} \\ \frac{2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{bmatrix} = \frac{1}{1+t^2}\begin{bmatrix} 1-t^2 & -2t \\ 2t & 1-t^2 \end{bmatrix}$.
Now, calculate the product for the RHS:

$$\begin{aligned} \text{RHS} = (I-A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} &= \begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix} \frac{1}{1+t^2}\begin{bmatrix} 1-t^2 & -2t \\ 2t & 1-t^2 \end{bmatrix} \\ &= \frac{1}{1+t^2}\begin{bmatrix} 1(1-t^2)+t(2t) & 1(-2t)+t(1-t^2) \\ -t(1-t^2)+1(2t) & -t(-2t)+1(1-t^2) \end{bmatrix} \\ &= \frac{1}{1+t^2}\begin{bmatrix} 1-t^2+2t^2 & -2t+t-t^3 \\ -t+t^3+2t & 2t^2+1-t^2 \end{bmatrix} \\ &= \frac{1}{1+t^2}\begin{bmatrix} 1+t^2 & -t-t^3 \\ t+t^3 & 1+t^2 \end{bmatrix} = \frac{1}{1+t^2}\begin{bmatrix} 1+t^2 & -t(1+t^2) \\ t(1+t^2) & 1+t^2 \end{bmatrix} \end{aligned}$$

Divide each element by $1+t^2$:

$$\text{RHS} = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}$$

Compare LHS and RHS: We see that LHS $= \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}$ and RHS $= \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}$.
Since LHS = RHS, the identity is proven.

Step 4: Final Answer:
By calculating both sides of the equation separately and using the half-angle tangent identities, we have shown that LHS = RHS, thus proving the identity.

Quick Tip

When dealing with trigonometric identities in matrices, using the half-angle tangent substitutions ($t = \tan(\alpha/2)$) is a very powerful technique that converts trigonometric expressions into rational algebraic expressions, which are often easier to handle in matrix multiplication.

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8. Do any one part.

a. Prove that the semi-vertical angle of a right circular cone of given surface and maximum volume is $\sin^{-1}(1/3)$.

Solution:

Step 1: Understanding the Concept:
This is an optimization problem using calculus (applications of derivatives). We are given a cone with a fixed total surface area and we need to find the semi-vertical angle that maximizes its volume. The key is to express the volume as a function of a single variable (either radius, height, or the semi-vertical angle) and then use differentiation to find the maximum.

Step 2: Key Formula or Approach:
1. Let r be the radius, h the height, l the slant height, and α the semi-vertical angle of the cone.
2. Write the formulas for total surface area $S = \pi r^2 + \pi r l$ and volume $V = \frac{1}{3}\pi r^2 h$.
3. Use the geometric relations: $l^2 = r^2 + h^2$, $r = l\sin\alpha$, $h = l\cos\alpha$.
4. Since S is constant, express one variable (e.g., l) in terms of another (e.g., r) from the surface area formula.
5. Substitute this into the volume formula to get V as a function of a single variable. To make differentiation easier, it's better to work with V^2.
6. Differentiate V^2 with respect to the chosen variable, set the derivative to zero to find the condition for maximum volume.
7. Relate this condition back to the semi-vertical angle α.

Step 3: Detailed Explanation:
Let S be the constant total surface area. $S = \pi r l + \pi r^2$.
From this, we can express the slant height l in terms of r:

$$l = \frac{S - \pi r^2}{\pi r}$$

The volume of the cone is $V = \frac{1}{3}\pi r^2 h$. We also know $h = \sqrt{l^2 - r^2}$.
To simplify the differentiation, we will maximize V^2 instead of V.

$$V^2 = \frac{1}{9}\pi^2 r^4 h^2 = \frac{1}{9}\pi^2 r^4 (l^2 - r^2)$$

Substitute the expression for l:

$$V^2 = \frac{1}{9}\pi^2 r^4 \left[\left(\frac{S - \pi r^2}{\pi r}\right)^2 - r^2\right]$$

$$V^2 = \frac{1}{9}\pi^2 r^4 \left[\frac{(S - \pi r^2)^2 - (\pi r^2)^2}{\pi^2 r^2}\right] = \frac{1}{9} r^2 [(S - \pi r^2)^2 - (\pi r^2)^2]$$

Using the difference of squares $a^2 - b^2 = (a - b)(a + b)$:

$$V^2 = \frac{1}{9} r^2 (S - \pi r^2 - \pi r^2)(S - \pi r^2 + \pi r^2) = \frac{1}{9} r^2 (S - 2\pi r^2)(S)$$

Let $Z = V^2$. Then $Z = \frac{S}{9}(Sr^2 - 2\pi r^4)$.
To find the maximum, we differentiate Z with respect to r and set it to zero:

$$\frac{dZ}{dr} = \frac{S}{9}(2Sr - 8\pi r^3)$$

Setting $\frac{dZ}{dr} = 0$:

$$\frac{S}{9}(2Sr - 8\pi r^3) = 0 \implies 2Sr = 8\pi r^3$$

Since $r \neq 0$, we can divide by $2r$:

$$S = 4\pi r^2$$

This is the condition for maximum volume. Now we relate this back to the semi-vertical angle α.
Substitute $S = \pi rl + \pi r^2$ back into this condition:

$$\pi rl + \pi r^2 = 4\pi r^2$$

$$\pi rl = 3\pi r^2$$

$$l = 3r$$

The semi-vertical angle α is defined by $\sin\alpha = \frac{r}{l}$.
Substituting $l = 3r$:

$$\sin\alpha = \frac{r}{3r} = \frac{1}{3}$$

$$\alpha = \sin^{-1}\left(\frac{1}{3}\right)$$

(One can verify this is a maximum by checking the second derivative, which will be negative).

Step 4: Final Answer:
The semi-vertical angle of the cone with a given surface area and maximum volume is $\alpha = \sin^{-1}(1/3)$.

> **Quick Tip**
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> In optimization problems, maximizing or minimizing a function $f(x)$ is equivalent to maximizing or minimizing $(f(x))^2$, provided $f(x)$ is non-negative (which volume is). This trick often eliminates square roots and makes differentiation much simpler.

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b. Prove that $\int_0^{\pi/2} \log(\cos x)dx = -\frac{\pi}{2}\log 2$.

Solution:

Step 1: Understanding the Concept:
This is a classic problem in definite integrals. It requires the use of the properties of definite

integrals, particularly the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. This allows us to create a second related integral which, when combined with the first, leads to a simpler expression that can be evaluated.

Step 2: Key Formula or Approach:

1. Let the given integral be I. So, $I = \int_0^{\pi/2} \log(\cos x)dx$.
2. Apply the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ with $a = \pi/2$.
3. This will create a second equation for I in terms of $\log(\sin x)$.
4. Add the two equations for I. This will combine the logarithms into $\log(\sin x \cos x)$.
5. Use the identity $\sin(2x) = 2\sin x \cos x$ to simplify the logarithm.
6. Split the resulting integral and solve for I.

Step 3: Detailed Explanation:

Let $I = \int_0^{\pi/2} \log(\cos x)dx \quad \cdots (1)$.

Using the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$:

$$I = \int_0^{\pi/2} \log(\cos(\pi/2 - x))dx$$

Since $\cos(\pi/2 - x) = \sin x$, we have:

$$I = \int_0^{\pi/2} \log(\sin x)dx \quad \cdots (2)$$

Now, add equations (1) and (2):

$$2I = \int_0^{\pi/2} \log(\cos x)dx + \int_0^{\pi/2} \log(\sin x)dx$$

$$2I = \int_0^{\pi/2} (\log(\cos x) + \log(\sin x))dx$$

Using the logarithm property $\log A + \log B = \log(AB)$:

$$2I = \int_0^{\pi/2} \log(\sin x \cos x)dx$$

To use the double angle formula, multiply and divide by 2 inside the logarithm:

$$2I = \int_0^{\pi/2} \log\left(\frac{2\sin x \cos x}{2}\right) dx = \int_0^{\pi/2} \log\left(\frac{\sin(2x)}{2}\right) dx$$

Using the property $\log(A/B) = \log A - \log B$:

$$2I = \int_0^{\pi/2} \log(\sin(2x))dx - \int_0^{\pi/2} \log 2 dx$$

Let's evaluate the second integral first:

$$\int_0^{\pi/2} \log 2 dx = \log 2 [x]_0^{\pi/2} = \frac{\pi}{2}\log 2$$

Now consider the first integral, $\int_0^{\pi/2} \log(\sin(2x))dx$. Let $t = 2x$, so $dt = 2dx$ or $dx = dt/2$. The limits of integration change from $x = 0 \to t = 0$ and $x = \pi/2 \to t = \pi$.

$$\int_0^{\pi/2} \log(\sin(2x))dx = \int_0^{\pi} \log(\sin t)\frac{dt}{2} = \frac{1}{2}\int_0^{\pi} \log(\sin t)dt$$

Using the property $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$ if $f(2a - x) = f(x)$. Here $2a = \pi$, $a = \pi/2$. $\sin(\pi - t) = \sin t$, so the property applies.

$$\frac{1}{2}\int_0^{\pi} \log(\sin t)dt = \frac{1}{2} \cdot 2\int_0^{\pi/2} \log(\sin t)dt = \int_0^{\pi/2} \log(\sin x)dx = I$$

Substituting everything back into the equation for $2I$:

$$2I = I - \frac{\pi}{2}\log 2$$

Solving for I:

$$2I - I = -\frac{\pi}{2}\log 2$$
$$I = -\frac{\pi}{2}\log 2$$

Step 4: Final Answer:
We have proven that $\int_0^{\pi/2} \log(\cos x)dx = -\frac{\pi}{2}\log 2$.

Quick Tip

This is a standard result in definite integration and is very useful to remember. Both $\int_0^{\pi/2} \log(\cos x)dx$ and $\int_0^{\pi/2} \log(\sin x)dx$ are equal to $-\frac{\pi}{2}\log 2$. The property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ is extremely powerful for integrals with limits 0 to a.

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9. Do any one part.

a. Find the area of the region bounded by the ellipse $\frac{x^2}{9^2} + \frac{y^2}{4^2} = 1$.

Solution:

Step 1: Understanding the Concept:
The problem asks for the total area enclosed by an ellipse given in its standard form. This can be found directly using the standard formula for the area of an ellipse.

Step 2: Key Formula or Approach:
The standard equation of an ellipse centered at the origin is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
The area (A) enclosed by such an ellipse is given by the formula:

$$A = \pi ab$$

where a and b are the lengths of the semi-major and semi-minor axes, respectively.

Step 3: Detailed Explanation:
The given equation of the ellipse is:

$$\frac{x^2}{9^2} + \frac{y^2}{4^2} = 1$$

By comparing this with the standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we can identify the values of a and b.

$$a^2 = 9^2 \implies a = 9$$
$$b^2 = 4^2 \implies b = 4$$

Now, we apply the formula for the area of the ellipse:

$$A = \pi ab$$

Substituting the values of a and b:

$$A = \pi(9)(4) = 36\pi$$

Step 4: Final Answer:
The area of the region bounded by the ellipse is 36π square units.

> **Quick Tip**
>
> The area formula for an ellipse, $A = \pi ab$, is a direct generalization of the area of a circle. For a circle, the semi-major and semi-minor axes are both equal to the radius ($a = b = r$), and the formula becomes $A = \pi(r)(r) = \pi r^2$. Memorizing this formula is essential for competitive exams.

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b. Solve: $(\tan^{-1} y - x)dy = (1 + y^2)dx$**.**

Solution:

Step 1: Understanding the Concept:
The given equation is a first-order differential equation. It is not easily separable. We can try to rearrange it into the form of a linear differential equation. By rewriting it with x as the dependent variable and y as the independent variable, we can see that it fits the standard linear form $\frac{dx}{dy} + P(y)x = Q(y)$.

Step 2: Key Formula or Approach:
1. Rearrange the equation into the standard linear form.
2. Identify the functions $P(y)$ and $Q(y)$.

3. Calculate the integrating factor (I.F.) using the formula I.F. $= e^{\int P(y)dy}$.
4. The general solution is given by the formula: $x \cdot (\text{I.F.}) = \int Q(y) \cdot (\text{I.F.})dy + C$.
5. Solve the resulting integral, which may require techniques like integration by parts.

Step 3: Detailed Explanation:

The given differential equation is:

$$(\tan^{-1} y - x)dy = (1 + y^2)dx$$

Rearrange to get $\frac{dx}{dy}$:

$$\frac{dx}{dy} = \frac{\tan^{-1} y - x}{1 + y^2}$$

$$\frac{dx}{dy} = \frac{\tan^{-1} y}{1 + y^2} - \frac{x}{1 + y^2}$$

Bring the term with x to the LHS to match the standard linear form:

$$\frac{dx}{dy} + \frac{1}{1 + y^2}x = \frac{\tan^{-1} y}{1 + y^2}$$

This is a linear differential equation in x.

1. Identify P(y) and Q(y):

$$P(y) = \frac{1}{1 + y^2}, \quad Q(y) = \frac{\tan^{-1} y}{1 + y^2}$$

2. Calculate the Integrating Factor (I.F.):

$$\text{I.F.} = e^{\int P(y)dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y}$$

3. Apply the solution formula:

$$x \cdot e^{\tan^{-1} y} = \int \left(\frac{\tan^{-1} y}{1 + y^2} \right) e^{\tan^{-1} y} dy + C$$

To solve the integral on the RHS, let $t = \tan^{-1} y$. Then $dt = \frac{1}{1+y^2}dy$. The integral becomes:

$$\int te^t dt$$

We solve this using integration by parts, $\int udv = uv - \int vdu$.
Let $u = t$ and $dv = e^t dt$. Then $du = dt$ and $v = e^t$.

$$\int te^t dt = te^t - \int e^t dt = te^t - e^t = e^t(t - 1)$$

Substituting back $t = \tan^{-1} y$, the integral is $e^{\tan^{-1} y}(\tan^{-1} y - 1)$.
The general solution is:

$$x \cdot e^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + C$$

4. Isolate x: Divide the entire equation by $e^{\tan^{-1} y}$:

$$x = (\tan^{-1} y - 1) + Ce^{-\tan^{-1} y}$$

Step 4: Final Answer:
The solution to the differential equation is $x = \tan^{-1} y - 1 + Ce^{-\tan^{-1} y}$, where C is an arbitrary constant.

Quick Tip

If a first-order differential equation doesn't seem to be separable or linear in the form $\frac{dy}{dx}$, always check if it becomes linear by treating x as the function of y, i.e., by rearranging it into the form $\frac{dx}{dy} + P(y)x = Q(y)$. This is a common pattern in exam questions.
