

# WBJEE Mathematics Sample Paper-6

Duration: 120 Minutes

Maximum Marks: 100

## Instructions

- This paper contains **75** Multiple Choice Questions divided into **3 Sections**.
- **Section A (Q1–Q50):** Each correct answer carries **+1 mark**. Incorrect answer: **–0.25** marks. Only **one** correct option.
- **Section B (Q51–Q65):** Each correct answer carries **+2 marks**. Incorrect answer: **–0.5** marks. Only **one** correct option.
- **Section C (Q66–Q75):** Each correct answer carries **+2 marks**. **No negative marking**. One or **more** correct options may be correct; full marks only if all correct options are marked.
- Use of mobile phones, smartwatches, or any electronic gadgets is strictly prohibited.

**Section–A — 50 Questions × 1 Mark Each**  
**(Negative Marking: –0.25) [Single Correct]**

**Q1.** Let  $f(x) = \min\{|x - 1|, |x|, |x + 1|\}$ . The area bounded by the curve  $y = f(x)$ , the x-axis, and the lines  $x = -1$  and  $x = 1$  is:

- (A)  $1/2$
- (B)  $3/4$
- (C)  $1/4$
- (D)  $1$

**Q2.** If  $z$  is a complex number such that  $|z - i| = |z + 1|$ , then the locus of  $z$  represents:

- (A) A circle passing through the origin.



- (B) A straight line with slope 1.
- (C) A straight line with slope  $-1$ .
- (D) An ellipse with foci at  $(0, 1)$  and  $(-1, 0)$ .

**Q3.** The value of  $\lim_{x \rightarrow 0} \frac{\int_0^{x^2} \sin \sqrt{t} dt}{x^3}$  is:

- (A)  $1/3$
- (B)  $2/3$
- (C) 0
- (D) 1

**Q4.** The number of ways in which 5 identical apples and 7 identical oranges can be distributed among 3 children such that each child receives at least one orange is:

- (A) 210
- (B) 315
- (C) 150
- (D) 420

**Q5.** If  $\vec{a}, \vec{b}, \vec{c}$  are three unit vectors such that  $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} = 0$  and the angle between  $\vec{b}$  and  $\vec{c}$  is  $\pi/6$ , then  $[\vec{a}\vec{b}\vec{c}]$  is equal to:

- (A)  $\pm 1/2$
- (B)  $\pm \sqrt{3}/2$
- (C)  $\pm 1$
- (D) 0

**Q6.** The eccentricity of the hyperbola whose latus rectum is 8 and conjugate axis is equal to half of the distance between the foci is:

- (A)  $4/3$
- (B)  $2/\sqrt{3}$



(C)  $3/2$

(D)  $\sqrt{3}$

**Q7.** In a series of 10 independent trials, the probability of at least one success is  $1 - (0.9)^{10}$ . The expected number of successes is:

(A) 1

(B) 0.1

(C) 0.9

(D) 1.1

**Q8.** If  $A$  is a  $3 \times 3$  non-singular matrix such that  $A^2 = A + I$ , then  $A^{-1}$  is:

(A)  $A - I$

(B)  $I - A$

(C)  $A + I$

(D)  $A^2 + I$

**Q9.** The solution of the differential equation  $\frac{dy}{dx} = \frac{x+y+1}{x+y-1}$  is:

(A)  $y - x + \log |x + y| = c$

(B)  $y - x - \log |x + y| = c$

(C)  $y + x + \log |x + y| = c$

(D)  $y - x + 2 \log |x + y| = c$

**Q10.** The shortest distance between the lines  $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$  and  $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$  is:

(A)  $1/\sqrt{6}$

(B) 0

(C)  $\sqrt{6}$

(D)  $1/6$



**Q11.** The value of  $\cos(2 \tan^{-1} x)$  is:

- (A)  $\frac{1-x^2}{1+x^2}$
- (B)  $\frac{2x}{1+x^2}$
- (C)  $\frac{1+x^2}{1-x^2}$
- (D)  $\frac{2x}{1-x^2}$

**Q12.** If  $\alpha, \beta$  are the roots of  $x^2 - px + r = 0$  and  $\alpha/2, 2\beta$  are the roots of  $x^2 - qx + r = 0$ , then  $r$  is:

- (A)  $\frac{2}{9}(p - q)(2q - p)$
- (B)  $\frac{2}{3}(p - q)(2q - p)$
- (C)  $\frac{4}{9}(p - q)(2q - p)$
- (D)  $\frac{2}{9}(2p - q)(2q - p)$

**Q13.** Let  $f(x) = x^3 - 3x^2 + 6$ . The maximum value of  $f(x)$  in the interval  $[0, 3]$  is:

- (A) 2
- (B) 6
- (C) 10
- (D) 4

**Q14.** The sum of the coefficients of all even powers of  $x$  in the expansion of  $(1 + x + x^2 + x^3)^n$  is:

- (A)  $4^n$
- (B)  $2^{2n-1}$
- (C)  $2^{n-1}$
- (D)  $2^{2n}$

**Q15.** The circle  $x^2 + y^2 - 4x - 6y - 12 = 0$  and  $x^2 + y^2 + 6x + 18y + 26 = 0$ :

- (A) Touch each other externally.



- (B) Touch each other internally.
- (C) Intersect at two points.
- (D) Are separated.

**Q16.** If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $f(x) = \frac{x}{1+|x|}$ , then the range of  $f$  is:

- (A)  $\mathbb{R}$
- (B)  $(-1, 1)$
- (C)  $[-1, 1]$
- (D)  $(0, 1)$

**Q17.** The integral  $\int \frac{dx}{x(x^n+1)}$  is equal to:

- (A)  $\frac{1}{n} \log \left| \frac{x^n}{x^n+1} \right| + C$
- (B)  $\log \left| \frac{x^n}{x^n+1} \right| + C$
- (C)  $\frac{1}{n} \log \left| \frac{x^n+1}{x^n} \right| + C$
- (D)  $n \log \left| \frac{x^n}{x^n+1} \right| + C$

**Q18.** The value of  $\sum_{k=1}^{10} \int_0^1 \frac{x^{k-1}}{1+x} dx$  is:

- (A)  $\log 2$
- (B)  $\log(1/2)$
- (C)  $1 - \log 2$
- (D) Not evaluable simply.

**Q19.** The mean and variance of 7 observations are 8 and 16 respectively. If five of the observations are 2, 4, 10, 12, 14, then the product of the remaining two observations is:

- (A) 40
- (B) 48
- (C) 56
- (D) 64



- Q20.** The angle between the planes  $2x - y + z = 6$  and  $x + y + 2z = 3$  is:
- (A)  $\pi/3$
  - (B)  $\pi/4$
  - (C)  $\pi/6$
  - (D)  $\pi/2$
- Q21.** Let  $A$  and  $B$  be two sets such that  $n(A) = 4$  and  $n(B) = 7$ . The minimum number of elements in  $A \cup B$  is:
- (A) 4
  - (B) 7
  - (C) 11
  - (D) 3
- Q22.** If  $y = \tan^{-1} \left( \frac{\sqrt{1+x^2}-1}{x} \right)$ , then  $dy/dx$  at  $x = 0$  is:
- (A) 0
  - (B) 1
  - (C)  $1/2$
  - (D) Does not exist.
- Q23.** If  $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$  and  $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$ , then the vector  $\vec{a} - \vec{d}$  is:
- (A) Perpendicular to  $\vec{b} - \vec{c}$ .
  - (B) Parallel to  $\vec{b} - \vec{c}$ .
  - (C) Perpendicular to  $\vec{b} + \vec{c}$ .
  - (D) Parallel to  $\vec{b} + \vec{c}$ .
- Q24.** The area bounded by  $y^2 = 4x$  and the line  $x = 3$  is:
- (A)  $8\sqrt{3}$
  - (B)  $4\sqrt{3}$



(C)  $12\sqrt{3}$

(D)  $16\sqrt{3}$

**Q25.** If  $S_n$  denotes the sum of first  $n$  terms of an A.P. and  $S_{2n} = 3S_n$ , then  $S_{3n} : S_n$  is:

(A) 4

(B) 6

(C) 8

(D) 10

**Q26.** If  $\log_{10} 2, \log_{10}(2^x - 1)$  and  $\log_{10}(2^x + 3)$  are in A.P., then  $x$  is:

(A)  $5/2$

(B)  $\log_2 5$

(C)  $\log_2 3$

(D)  $3/2$

**Q27.** The equation  $x^2 + y^2 - 2x - 4y + 5 = 0$  represents:

(A) A circle.

(B) An ellipse.

(C) A point.

(D) A pair of straight lines.

**Q28.** The number of points where  $f(x) = |x - 1| + |x - 2| + \cos x$  is not differentiable in  $(0, 3)$  is:

(A) 1

(B) 2

(C) 3

(D) 0

**Q29.** The value of  $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$  is:



- (A)  $\pi/2$
- (B)  $\pi/4$
- (C)  $\pi$
- (D) 0

**Q30.** If the matrix  $\begin{pmatrix} 1 & a & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 10 & 55 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{pmatrix}$ , then  $a$  is:

- (A) 1
- (B) 2
- (C) 10
- (D) 5

**Q31.** The slope of the tangent to the curve  $y = e^{2x}$  at the point where it crosses the y-axis is:

- (A) 1
- (B) 2
- (C)  $e$
- (D)  $1/2$

**Q32.** A bag contains 4 red and 6 black balls. Two balls are drawn at random. The probability that both are of the same color is:

- (A)  $7/15$
- (B)  $8/15$
- (C)  $1/3$
- (D)  $2/5$

**Q33.** The value of  $\sin(\sin^{-1}(1/2) + \cos^{-1}(1/2))$  is:

- (A) 0



- (B) 1
- (C)  $1/2$
- (D)  $\sqrt{3}/2$

**Q34.** The function  $f(x) = x^x$  is strictly increasing when:

- (A)  $x > 1/e$
- (B)  $x < 1/e$
- (C)  $x > e$
- (D)  $x > 0$

**Q35.** The projection of the vector  $\hat{i} - 2\hat{j} + \hat{k}$  on the vector  $4\hat{i} - 4\hat{j} + 7\hat{k}$  is:

- (A)  $19/9$
- (B)  $19/3$
- (C)  $9/19$
- (D)  $21/9$

**Q36.** The derivative of  $\log_x 2$  with respect to  $x$  is:

- (A)  $\frac{1}{x \log 2}$
- (B)  $-\frac{\log 2}{x(\log x)^2}$
- (C)  $\frac{\log 2}{x \log x}$
- (D)  $\frac{1}{x}$

**Q37.** The eccentricity of the ellipse  $9x^2 + 5y^2 - 30y = 0$  is:

- (A)  $1/3$
- (B)  $2/3$
- (C)  $3/4$
- (D)  $4/5$

**Q38.** If  $x^y = e^{x-y}$ , then  $dy/dx$  is:



- (A)  $\frac{\log x}{(1+\log x)^2}$
- (B)  $\frac{1}{(1+\log x)^2}$
- (C)  $\frac{x-y}{1+\log x}$
- (D)  $\frac{\log x}{1+\log x}$

**Q39.** If  $\alpha, \beta, \gamma$  are the roots of  $x^3 - 3x + 1 = 0$ , then the value of  $\sum \alpha^2$  is:

- (A) 3
- (B) 6
- (C) 9
- (D) 0

**Q40.** The position vector of a point which divides the join of  $(1, 2, 3)$  and  $(4, 5, 6)$  in the ratio  $2 : 1$  is:

- (A)  $(3, 4, 5)$
- (B)  $(2, 3, 4)$
- (C)  $(5, 6, 7)$
- (D)  $(4, 5, 6)$

**Q41.** If  $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$ , then  $\Delta$  is:

- (A)  $xy$
- (B)  $x + y$
- (C) 1
- (D) 0

**Q42.** The number of real solutions of the equation  $|x|^2 - 3|x| + 2 = 0$  is:

- (A) 2
- (B) 4



- (C) 1
- (D) 0

**Q43.** The distance of the point  $(1, -2, 3)$  from the plane  $x - y + z = 5$  measured parallel to the line  $x/2 = y/3 = z/-6$  is:

- (A) 1
- (B)  $1/7$
- (C) 7
- (D) 2

**Q44.** The value of  $\int e^x(\tan x + \log \sec x)dx$  is:

- (A)  $e^x \tan x + C$
- (B)  $e^x \log \sec x + C$
- (C)  $e^x(\tan x - \log \sec x) + C$
- (D)  $e^x \sec x + C$

**Q45.** If  $y = \sin(m \sin^{-1} x)$ , then  $(1 - x^2)y_2 - xy_1$  is:

- (A)  $m^2y$
- (B)  $-m^2y$
- (C)  $my$
- (D) 0

**Q46.** The range of the function  $f(x) = \frac{1}{2 - \cos 3x}$  is:

- (A)  $[1/3, 1]$
- (B)  $[1/2, 1]$
- (C)  $[1/3, 1/2]$
- (D)  $[-1, 1]$

**Q47.** If the line  $y = mx + 1$  is tangent to the parabola  $y^2 = 4x$ , then  $m$  is:



- (A) 1
- (B) 2
- (C) 1/2
- (D) 4

**Q48.** The value of  $\lim_{n \rightarrow \infty} \left( \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right)$  is:

- (A) 1
- (B)  $\log 2$
- (C) 0
- (D)  $e$

**Q49.** A unit vector perpendicular to both  $2\hat{i} + \hat{j} + \hat{k}$  and  $\hat{i} - \hat{j} + 2\hat{k}$  is:

- (A)  $\frac{1}{\sqrt{11}}(\hat{i} - \hat{j} - \hat{k})$
- (B)  $\frac{1}{\sqrt{19}}(3\hat{i} - 3\hat{j} - 3\hat{k})$
- (C)  $\frac{1}{\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$
- (D)  $\frac{1}{\sqrt{2}}(\hat{j} - \hat{k})$

**Q50.** The value of  $\int_{-1}^1 x|x|dx$  is:

- (A) 2/3
- (B) 0
- (C) 1/3
- (D) 1



**Section-B — 15 Questions × 2 Marks Each**  
**(Negative Marking: -0.5) [Single Correct]**

- Q51.** The general solution of  $dy/dx + y \tan x = \sec x$  is:
- (A)  $y \sec x = \tan x + C$   
(B)  $y \tan x = \sec x + C$   
(C)  $y \cos x = x + C$   
(D)  $y = \sin x + C \cos x$
- Q52.** The constant term in the expansion of  $(x^2 - 1/x)^{12}$  is:
- (A)  ${}^{12}C_8$   
(B)  ${}^{12}C_4$   
(C)  $-{}^{12}C_8$   
(D) 0
- Q53.** If  $A$  and  $B$  are square matrices of the same order such that  $AB = A$  and  $BA = B$ , then  $A^2 + B^2$  is:
- (A)  $A + B$   
(B)  $AB$   
(C)  $I$   
(D) 0
- Q54.** The maximum value of  $\frac{\log x}{x}$  for  $x > 0$  is:
- (A)  $e$   
(B)  $1/e$   
(C) 1  
(D)  $e^2$



**Q55.** The equation of the normal to the curve  $y = \sin x$  at  $(0, 0)$  is:

- (A)  $x + y = 0$
- (B)  $x - y = 0$
- (C)  $y = 0$
- (D)  $x = 0$

**Q56.** The value of  $k$  for which the lines  $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$  and  $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$  are perpendicular is:

- (A)  $-10/7$
- (B)  $10/7$
- (C)  $-7/10$
- (D)  $-5/7$

**Q57.** If  $\sin^{-1} x + \sin^{-1} y = 2\pi/3$ , then  $\cos^{-1} x + \cos^{-1} y$  is:

- (A)  $\pi/3$
- (B)  $\pi/6$
- (C)  $\pi/2$
- (D)  $\pi$

**Q58.** The probability that a leap year contains 53 Sundays is:

- (A)  $1/7$
- (B)  $2/7$
- (C)  $53/366$
- (D)  $1/366$

**Q59.** The order and degree of the differential equation  $\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2} = \frac{d^2y}{dx^2}$  are respectively:

- (A) 2, 2



- (B) 2, 3
- (C) 1, 3
- (D) 2, 1

**Q60.** The area of the triangle formed by the points  $(0, 0)$ ,  $(a, 0)$  and  $(0, b)$  is:

- (A)  $ab$
- (B)  $1/2ab$
- (C)  $1/2|ab|$
- (D)  $\sqrt{a^2 + b^2}$

**Q61.** If  $f(x) = \begin{cases} \frac{1-\cos 4x}{x^2} & \text{if } x < 0 \\ a & \text{if } x = 0 \\ \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} & \text{if } x > 0 \end{cases}$  is continuous at  $x = 0$ , then  $a$  is:

- (A) 4
- (B) 8
- (C) 16
- (D) 0

**Q62.** The value of  $\int_0^1 \frac{dx}{e^x + e^{-x}}$  is:

- (A)  $\tan^{-1} e - \pi/4$
- (B)  $\tan^{-1} e + \pi/4$
- (C)  $\log(e + 1)$
- (D)  $1/2 \log\left(\frac{e+1}{e-1}\right)$

**Q63.** The number of subsets of a set containing  $n$  elements is:

- (A)  $n^2$
- (B)  $2^n$
- (C)  $2^{n-1}$



(D)  $n!$

**Q64.** If the vectors  $2\hat{i} - \hat{j} + \hat{k}$ ,  $\hat{i} + 2\hat{j} - 3\hat{k}$  and  $3\hat{i} + \lambda\hat{j} + 5\hat{k}$  are coplanar, then  $\lambda$  is:

(A)  $-4$

(B)  $-8$

(C)  $4$

(D)  $8$

**Q65.** The minimum value of  $2x^2 - 4x + 5$  is:

(A)  $3$

(B)  $5$

(C)  $1$

(D)  $0$

**Section-C — 10 Questions  $\times$  2 Marks Each (No  
Negative Marking) [One or More Correct]**

**Q66.** Let  $I_n = \int_0^1 x^n e^{-x} dx$  for  $n \in \mathbb{N}$ . Which of the following is/are correct?

(A)  $I_n = nI_{n-1} - e^{-1}$

(B)  $I_n$  is a strictly decreasing sequence.

(C)  $\lim_{n \rightarrow \infty} I_n = 0$

(D)  $I_n$  is always greater than  $e^{-1}$ .

**Q67.** Let  $I = \int_0^2 [x^2] dx$ , where  $[.]$  denotes the greatest integer function. Which of the following is/are true?

(A) The integrand is discontinuous at  $x = 1$  and  $x = \sqrt{2}$  and  $x = \sqrt{3}$ .

(B)  $I = 5 - \sqrt{2} - \sqrt{3}$

(C) The function  $[x^2]$  is non-decreasing in  $[0, 2]$ .



(D)  $I = 3 - \sqrt{2} - \sqrt{3}$

**Q68.** Let  $\vec{a}, \vec{b}, \vec{c}$  be three non-zero vectors such that no two are collinear. If  $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$ , then:

- (A)  $\vec{a}$  and  $\vec{c}$  are collinear.
- (B)  $\vec{a}$  is perpendicular to  $\vec{b}$ .
- (C)  $\vec{c}$  is perpendicular to  $\vec{b}$ .
- (D)  $\vec{b}$  must be a unit vector.

**Q69.** Consider two skew lines  $L_1$  and  $L_2$ . Which of the following is/are true?

- (A) There exists a unique line perpendicular to both  $L_1$  and  $L_2$ .
- (B) The shortest distance between them is zero if they intersect.
- (C) The shortest distance is given by the projection of the vector joining two points (one on each line) onto the cross product of their direction vectors.
- (D) Skew lines are always coplanar.

**Q70.** If  $A$  is a  $3 \times 3$  skew-symmetric matrix, then:

- (A)  $\det(A) = 0$
- (B)  $I + A$  is a non-singular matrix.
- (C)  $A^2$  is a symmetric matrix.
- (D) All diagonal elements of  $A$  are non-zero.

**Q71.** For a non-singular square matrix  $A$  of order  $n$ , which of the following is/are correct?

- (A)  $\text{adj}(\text{adj}A) = |A|^{n-2}A$
- (B)  $|\text{adj}A| = |A|^{n-1}$
- (C)  $\text{adj}(AB) = (\text{adj}B)(\text{adj}A)$
- (D)  $\text{adj}(A^T) = (\text{adj}A)^T$

**Q72.** Let  $1, \alpha_1, \alpha_2, \dots, \alpha_{n-1}$  be the  $n^{\text{th}}$  roots of unity. Then:



- (A)  $(1 - \alpha_1)(1 - \alpha_2) \dots (1 - \alpha_{n-1}) = n$
- (B)  $\sum_{r=1}^{n-1} \alpha_r = -1$
- (C) The roots lie on a circle with radius 1 and center  $(0, 0)$ .
- (D) The roots form a Geometric Progression with common ratio  $\alpha_1$ .

**Q73.** Let  $A$  and  $B$  be two events such that  $P(A) > 0$  and  $P(B) > 0$ . If  $P(A|B) > P(A)$ , then:

- (A)  $P(B|A) > P(B)$
- (B)  $P(A \cap B) > P(A)P(B)$
- (C)  $P(A|B^c) < P(A)$
- (D)  $A$  and  $B$  are independent.

**Q74.** Let  $f(x) = \max\{1 - x, 1 + x, 2\}$ . Then:

- (A)  $f(x)$  is continuous for all  $x \in \mathbb{R}$ .
- (B)  $f(x)$  is not differentiable at  $x = -1$  and  $x = 1$ .
- (C)  $f(x)$  has a global minimum value of 2.
- (D)  $f'(x) = 0$  for  $x \in (-1, 1)$ .

**Q75.** The planes  $2x - y + z = 6$  and  $x + y + 2z = 3$ :

- (A) Are perpendicular to each other.
- (B) Intersect in a line with direction ratios  $(-3, -3, 3)$ .
- (C) Have an angle  $\theta = \pi/3$  between them.
- (D) Pass through the point  $(2, -1, 1)$ .



## Detailed Solutions

Q1.

## Solution

**Concept:** The problem involves finding the area under a curve defined by a "minimum" function. The function  $f(x) = \min\{|x - 1|, |x|, |x + 1|\}$  represents the lower envelope of three absolute value graphs. To solve this, we must identify the intersection points of these functions within the interval  $[-1, 1]$  to determine which function is the "minimum" in specific sub-intervals. Once the boundaries are established, the area is calculated using definite integration or by summing the areas of the geometric shapes (triangles) formed under the envelope.

**Solution:**

- (a) We analyze the behavior of  $y_1 = |x + 1|$ ,  $y_2 = |x|$ , and  $y_3 = |x - 1|$  on the interval  $[-1, 1]$ .
- (b) Intersection points:  $|x + 1| = |x|$  gives  $x + 1 = -x \Rightarrow x = -1/2$ . Similarly,  $|x| = |x - 1|$  gives  $x = -(x - 1) \Rightarrow x = 1/2$ .
- (c) The function  $f(x)$  is defined as:  
 \* For  $x \in [-1, -1/2]$ ,  $f(x) = |x + 1| = x + 1$ . \* For  $x \in [-1/2, 0]$ ,  $f(x) = |x| = -x$ . \* For  $x \in [0, 1/2]$ ,  $f(x) = |x| = x$ . \* For  $x \in [1/2, 1]$ ,  $f(x) = |x - 1| = 1 - x$ .
- (d) The total area  $A$  is the sum of four integrals:  $A = \int_{-1}^{-1/2} (x+1)dx + \int_{-1/2}^0 (-x)dx + \int_0^{1/2} xdx + \int_{1/2}^1 (1-x)dx$ .
- (e) Evaluating the symmetric parts: The area from  $[-1, 0]$  is symmetric to the area from  $[0, 1]$ .  
 Area  $[0, 1] = \int_0^{1/2} xdx + \int_{1/2}^1 (1-x)dx = [\frac{x^2}{2}]_0^{1/2} + [x - \frac{x^2}{2}]_{1/2}^1 = \frac{1}{8} + (1 - \frac{1}{2} - (\frac{1}{2} - \frac{1}{8})) = \frac{1}{8} + \frac{1}{8} = 1/4$ .
- (f) Total area =  $2 \times (1/4) = 1/2$ .

**Final Answer:** The area bounded is  $1/2$ .

**Answer: (A)**

[Go Back to Question 1](#)



Q2.

**Solution**

**Concept:** The locus of a complex number  $z$  satisfying an equation of the form  $|z - z_1| = |z - z_2|$  represents the set of all points equidistant from two fixed points  $z_1$  and  $z_2$  in the complex plane. Geometrically, this set of points forms the perpendicular bisector of the line segment joining  $z_1$  and  $z_2$ . By converting the complex equation into Cartesian coordinates ( $z = x + iy$ ), we can derive the linear equation of the locus and determine its slope.

**Solution:**

- (a) Let  $z = x + iy$ . The given equation is  $|z - i| = |z + 1|$ .
- (b) Identify the fixed points:  $z_1 = i$  (coordinates  $(0, 1)$ ) and  $z_2 = -1$  (coordinates  $(-1, 0)$ ).
- (c) Substitute  $z = x + iy$  into the equation:  $|x + i(y - 1)| = |(x + 1) + iy|$ .
- (d) Square both sides to remove the modulus:  $x^2 + (y - 1)^2 = (x + 1)^2 + y^2$ .
- (e) Expand the terms:  $x^2 + y^2 - 2y + 1 = x^2 + 2x + 1 + y^2$ .
- (f) Simplify by canceling  $x^2$ ,  $y^2$ , and 1 from both sides:  $-2y = 2x \Rightarrow y = -x$ .
- (g) The resulting equation  $y = -x$  is a straight line passing through the origin.
- (h) The slope  $m$  of the line  $y = mx$  is  $-1$ .

**Final Answer:** The locus represents a straight line with slope  $-1$ .

**Answer: (C)**

[Go Back to Question 2](#)



Q3.

**Solution**

**Concept:** This problem requires the application of the Leibniz Rule for differentiation under the integral sign and L'Hopital's Rule for evaluating limits of indeterminate forms. The limit is of the form  $0/0$  as  $x \rightarrow 0$ , because the upper limit of the integral becomes 0. Leibniz Rule states that  $\frac{d}{dx} \int_{a(x)}^{b(x)} f(t) dt = f(b(x)) \cdot b'(x) - f(a(x)) \cdot a'(x)$ . This allows us to differentiate the numerator and denominator to simplify the expression.

**Solution:**

- (a) The limit is  $L = \lim_{x \rightarrow 0} \frac{\int_0^{x^2} \sin \sqrt{t} dt}{x^3}$ . As  $x \rightarrow 0$ , the integral  $\int_0^0 \dots = 0$  and  $x^3 = 0$ .
- (b) Applying L'Hopital's Rule, we differentiate the numerator and denominator with respect to  $x$ .
- (c) Numerator derivative (using Leibniz Rule):  $\frac{d}{dx} \int_0^{x^2} \sin \sqrt{t} dt = \sin(\sqrt{x^2}) \cdot \frac{d}{dx}(x^2) = (\sin |x|) \cdot 2x$ .
- (d) Denominator derivative:  $\frac{d}{dx}(x^3) = 3x^2$ .
- (e) The limit becomes:  $L = \lim_{x \rightarrow 0} \frac{2x \sin x}{3x^2}$  (since  $\sin |x| \approx \sin x$  near 0 for the ratio).
- (f) Simplify the expression:  $L = \lim_{x \rightarrow 0} \frac{2}{3} \cdot \frac{\sin x}{x}$ .
- (g) Using the standard limit  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ , we get:  $L = \frac{2}{3} \cdot 1 = 2/3$ .

**Final Answer:** The value of the limit is  $2/3$ .

**Answer: (B)**

[Go Back to Question 3](#)



Q4.

**Solution**

**Concept:** This is a problem of combinatorics involving the distribution of identical objects into distinct bins (children), often solved using the "Stars and Bars" method. The problem is split into two independent parts: distributing 5 identical apples among 3 children with no restrictions, and distributing 7 identical oranges among 3 children such that each child receives at least one orange. The total number of ways is the product of the number of ways for each independent distribution.

**Solution:**

- (a) Part 1: Distributing 5 identical apples to 3 children. This is equivalent to finding non-negative integer solutions to  $x_1 + x_2 + x_3 = 5$ . Formula:  $\binom{n+r-1}{r-1}$  where  $n = 5, r = 3$ . Ways  $= \binom{5+3-1}{3-1} = \binom{7}{2} = \frac{7 \times 6}{2} = 21$ .
- (b) Part 2: Distributing 7 identical oranges to 3 children such that each receives at least one. We first give 1 orange to each child (3 oranges used). Remaining oranges  $= 7 - 3 = 4$ . Now distribute 4 oranges among 3 children with no further restrictions:  $y_1 + y_2 + y_3 = 4$ . Ways  $= \binom{4+3-1}{3-1} = \binom{6}{2} = \frac{6 \times 5}{2} = 15$ .
- (c) Alternatively, for Part 2, use the formula for positive integer solutions:  $\binom{n-1}{r-1} = \binom{7-1}{3-1} = \binom{6}{2} = 15$ .
- (d) Total number of ways = (Ways for apples)  $\times$  (Ways for oranges). Total =  $21 \times 15$ .  $21 \times 10 = 210, 21 \times 5 = 105$ . Total =  $210 + 105 = 315$ .

**Final Answer:** The total number of ways is 315.

**Answer: (B)**

[Go Back to Question 4](#)



Q5.

**Solution**

**Concept:** The scalar triple product  $[\vec{a}\vec{b}\vec{c}]$  represents the volume of a parallelepiped formed by vectors  $\vec{a}$ ,  $\vec{b}$ , and  $\vec{c}$ . It is calculated as  $\vec{a} \cdot (\vec{b} \times \vec{c})$ . Given that  $\vec{a}$  is perpendicular to both  $\vec{b}$  and  $\vec{c}$  ( $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} = 0$ ),  $\vec{a}$  must be parallel to the vector  $\vec{b} \times \vec{c}$ . For unit vectors, the magnitude of the scalar triple product depends solely on the sine of the angle between the vectors defining the cross product.

**Solution:**

- (a) We are given  $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$ .
- (b) Since  $\vec{a} \cdot \vec{b} = 0$  and  $\vec{a} \cdot \vec{c} = 0$ ,  $\vec{a}$  is perpendicular to the plane containing  $\vec{b}$  and  $\vec{c}$ .
- (c) The scalar triple product is  $[\vec{a}\vec{b}\vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})$ .
- (d) The magnitude of the cross product is  $|\vec{b} \times \vec{c}| = |\vec{b}||\vec{c}| \sin(\pi/6) = 1 \cdot 1 \cdot \frac{1}{2} = 1/2$ .
- (e) The direction of  $\vec{b} \times \vec{c}$  is normal to the plane of  $\vec{b}$  and  $\vec{c}$ . Let this unit normal be  $\hat{n}$ . So,  $\vec{b} \times \vec{c} = \frac{1}{2}\hat{n}$ .
- (f) Then  $[\vec{a}\vec{b}\vec{c}] = \vec{a} \cdot (\frac{1}{2}\hat{n}) = \frac{1}{2}(\vec{a} \cdot \hat{n})$ .
- (g) Since  $\vec{a}$  is also normal to the same plane,  $\vec{a}$  and  $\hat{n}$  are collinear unit vectors. Thus,  $\vec{a} \cdot \hat{n} = \pm 1$ .
- (h) Therefore,  $[\vec{a}\vec{b}\vec{c}] = \pm 1/2$ .

**Final Answer:** The scalar triple product is  $\pm 1/2$ .**Answer: (A)**[Go Back to Question 5](#)

Here are the detailed solutions for questions 6 through 10, following your required format and structure.



Q6.

**Solution**

**Concept:** This problem explores the properties of a hyperbola, specifically the relationship between its semi-major axis ( $a$ ), semi-minor axis ( $b$ ), and eccentricity ( $e$ ). The latus rectum is given by  $2b^2/a$ , and the distance between the foci is  $2ae$ . The conjugate axis length is  $2b$ . By setting up equations based on the given ratios and the fundamental identity for hyperbolas,  $b^2 = a^2(e^2 - 1)$ , we can solve for the unknown eccentricity.

**Solution:**

- (a) We are given the length of the latus rectum  $L = 2b^2/a = 8$ , which simplifies to  $b^2 = 4a$ .
- (b) The problem states the conjugate axis ( $2b$ ) is half the distance between the foci ( $2ae$ ). Therefore,  $2b = (1/2)(2ae)$ , which simplifies to  $2b = ae$ .
- (c) Square the second relation:  $4b^2 = a^2e^2$ .
- (d) Substitute the hyperbola identity  $b^2 = a^2(e^2 - 1)$  into the squared relation:  $4a^2(e^2 - 1) = a^2e^2$ .
- (e) Divide both sides by  $a^2$  (since  $a \neq 0$ ):  $4(e^2 - 1) = e^2$ .
- (f) Expand and solve:  $4e^2 - 4 = e^2 \implies 3e^2 = 4 \implies e^2 = 4/3$ .
- (g) Taking the positive root (since  $e > 1$  for a hyperbola):  $e = 2/\sqrt{3}$ .

**Final Answer:** The eccentricity is  $2/\sqrt{3}$ .

**Answer: (B)**

[Go Back to Question 6](#)



Q7.

**Solution**

**Concept:** This problem utilizes the Binomial Distribution, which models the number of successes in a fixed number of independent Bernoulli trials. The probability of at least one success is calculated as  $1 - P(\text{zero successes})$ . The formula for  $P(X = k)$  is  $\binom{n}{k} p^k (1 - p)^{n-k}$ . The expected value (mean) of a binomial distribution is simply the product of the number of trials ( $n$ ) and the probability of success in a single trial ( $p$ ).

**Solution:**

- (a) Let  $n = 10$  be the number of trials and  $p$  be the probability of success in one trial. Let  $q = 1 - p$  be the probability of failure.
- (b) The probability of at least one success is  $1 - P(X = 0)$ .
- (c) From the Binomial formula,  $P(X = 0) = \binom{10}{0} p^0 q^{10} = q^{10}$ .
- (d) We are given  $1 - q^{10} = 1 - (0.9)^{10}$ , which implies  $q = 0.9$ .
- (e) Since  $q = 0.9$ , the probability of success is  $p = 1 - 0.9 = 0.1$ .
- (f) The expected number of successes  $E[X]$  for a binomial distribution is given by  $n \times p$ .
- (g)  $E[X] = 10 \times 0.1 = 1$ .

**Final Answer:** The expected number of successes is 1.

**Answer: (A)**

[Go Back to Question 7](#)



Q8.

**Solution**

**Concept:** This problem requires the manipulation of matrix equations to find the inverse of a matrix. For a square matrix  $A$ , the inverse  $A^{-1}$  satisfies  $AA^{-1} = A^{-1}A = I$ , where  $I$  is the identity matrix. When a matrix satisfies a polynomial equation (like the characteristic equation or a given functional relation), we can often isolate the identity matrix and factor out  $A$  to identify the expression for the inverse.

**Solution:**

- (a) We are given the matrix equation  $A^2 = A + I$ .
- (b) To find the inverse, we need to express the identity matrix  $I$  in terms of  $A$ .
- (c) Rearrange the given equation:  $I = A^2 - A$ .
- (d) Since  $A$  is non-singular, its inverse  $A^{-1}$  exists. Multiply both sides of the equation by  $A^{-1}$  from the left (or right):  $A^{-1}I = A^{-1}(A^2 - A)$ .
- (e) Distribution of the matrix multiplication:  $A^{-1}I = A^{-1}A^2 - A^{-1}A$ .
- (f) Use the properties  $A^{-1}A = I$  and  $A^{-1}A^2 = A$ :  $A^{-1}I = A - I$ .
- (g) Thus, the inverse of  $A$  is simply the matrix  $A$  minus the identity matrix.

**Final Answer:**  $A^{-1} = A - I$ .

**Answer:** (A)

[Go Back to Question 8](#)



Q9.

**Solution**

**Concept:** This is a first-order ordinary differential equation of the form  $dy/dx = f(ax + by + c)$ . Equations of this type, where the linear terms in the numerator and denominator are identical (except for the constants), can be solved using a change of variables. By substituting  $v = x + y$ , we transform the equation into a separable form, allowing for straightforward integration of both sides.

**Solution:**

- (a) Let  $v = x + y$ . Then, differentiating both sides with respect to  $x$  gives  $dv/dx = 1 + dy/dx$ , or  $dy/dx = (dv/dx) - 1$ .
- (b) Substitute these into the original equation:  $(dv/dx) - 1 = (v + 1)/(v - 1)$ .
- (c) Rearrange to isolate  $dv/dx$ :  $dv/dx = (v + 1)/(v - 1) + 1$ .
- (d) Combine the terms on the right side:  $dv/dx = (v + 1 + v - 1)/(v - 1) = 2v/(v - 1)$ .
- (e) Separate the variables:  $((v - 1)/v)dv = 2dx$ , which simplifies to  $(1 - 1/v)dv = 2dx$ .
- (f) Integrate both sides:  $\int (1 - 1/v)dv = \int 2dx \implies v - \log |v| = 2x + C$ .
- (g) Substitute  $v = x + y$  back into the equation:  $(x + y) - \log |x + y| = 2x + C$ .
- (h) Simplify:  $y - x - \log |x + y| = C$ .

**Final Answer:** The solution is  $y - x - \log |x + y| = c$ .

**Answer: (B)**

[Go Back to Question 9](#)



## Q10.

## Solution

**Concept:** The shortest distance between two lines in 3D space depends on whether the lines are parallel or skew. For two lines  $L_1 : \vec{r} = \vec{a}_1 + \lambda\vec{b}_1$  and  $L_2 : \vec{r} = \vec{a}_2 + \mu\vec{b}_2$ , the distance formula is  $\frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$ . If the lines intersect, the numerator (the scalar triple product of the difference in position vectors and the direction vectors) will be zero, resulting in a shortest distance of zero.

**Solution:**

(a) Line 1 passes through  $\vec{a}_1 = (1, 2, 3)$  with direction  $\vec{b}_1 = (2, 3, 4)$ .

(b) Line 2 passes through  $\vec{a}_2 = (2, 4, 5)$  with direction  $\vec{b}_2 = (3, 4, 5)$ .

(c) Find the difference vector:  $\vec{a}_2 - \vec{a}_1 = (1, 2, 2)$ .

(d) Calculate the cross product of the direction vectors  $\vec{b}_1 \times \vec{b}_2$ :  $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} =$

$$\hat{i}(15 - 16) - \hat{j}(10 - 12) + \hat{k}(8 - 9) = (-1, 2, -1).$$

(e) Calculate the scalar triple product  $(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)$ :  $(1, 2, 2) \cdot (-1, 2, -1) = -1 + 4 - 2 = 1$ .

(f) Calculate the magnitude of the cross product:  $\sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{1 + 4 + 1} = \sqrt{6}$ .

(g) Shortest distance  $D = |1/\sqrt{6}|$ .

**Final Answer:** The shortest distance is  $1/\sqrt{6}$ .

**Answer: (A)**

[Go Back to Question 10](#)



Q11.

**Solution**

**Concept:** This problem involves the application of inverse trigonometric identities and double-angle formulas for cosine. The goal is to express a trigonometric function of an inverse function in algebraic terms. The fundamental relationship used here is the identity  $\cos(2\theta) = \frac{1-\tan^2 \theta}{1+\tan^2 \theta}$ , which allows us to convert a double-angle cosine expression directly into an expression involving the tangent of that angle.

**Solution:**

- (a) Let  $\tan^{-1} x = \theta$ . By definition of inverse functions, this implies that  $\tan \theta = x$ .
- (b) The expression given in the question is  $\cos(2 \tan^{-1} x)$ , which can now be written as  $\cos(2\theta)$ .
- (c) We use the standard double-angle trigonometric identity for cosine in terms of tangent:  
$$\cos(2\theta) = \frac{1-\tan^2 \theta}{1+\tan^2 \theta}.$$
- (d) Now, substitute the value of  $\tan \theta = x$  back into the identity.
- (e) The expression becomes:  $\frac{1-x^2}{1+x^2}$ .
- (f) This algebraic expression is valid for all real values of  $x$  because the denominator  $1+x^2$  is never zero.

**Final Answer:** The value is  $\frac{1-x^2}{1+x^2}$ .

**Answer: (A)**

[Go Back to Question 11](#)



## Q12.

**Solution**

**Concept:** This problem explores the relationship between the roots and coefficients of quadratic equations. For any quadratic equation  $ax^2 + bx + c = 0$ , the sum of the roots is  $-b/a$  and the product is  $c/a$ . Here, we are given two different equations with related roots and a shared constant term  $r$ . By setting up a system of equations involving the sums and products of these roots, we can eliminate the root variables ( $\alpha$  and  $\beta$ ) to find the value of  $r$  in terms of  $p$  and  $q$ .

**Solution:**

- (a) For the first equation  $x^2 - px + r = 0$ : Sum of roots:  $\alpha + \beta = p$  Product of roots:  $\alpha\beta = r$
- (b) For the second equation  $x^2 - qx + r = 0$  with roots  $\alpha/2$  and  $2\beta$ : Sum of roots:  $\alpha/2 + 2\beta = q \implies \alpha + 4\beta = 2q$  Product of roots:  $(\alpha/2)(2\beta) = r \implies \alpha\beta = r$  (This matches the first equation).
- (c) Now solve the system: (1)  $\alpha + \beta = p$  (2)  $\alpha + 4\beta = 2q$
- (d) Subtracting (1) from (2):  $3\beta = 2q - p \implies \beta = \frac{2q-p}{3}$ .
- (e) Find  $\alpha$ :  $\alpha = p - \beta = p - \frac{2q-p}{3} = \frac{3p-2q+p}{3} = \frac{4p-2q}{3} = \frac{2(2p-q)}{3}$ .
- (f) Calculate  $r = \alpha\beta$ :  $r = \left(\frac{2(2p-q)}{3}\right)\left(\frac{2q-p}{3}\right) = \frac{2}{9}(2p-q)(2q-p)$ .

**Final Answer:** The value of  $r$  is  $\frac{2}{9}(2p-q)(2q-p)$ .

**Answer: (D)**

[Go Back to Question 12](#)



Q13.

**Solution**

**Concept:** Finding the maximum value of a function in a closed interval  $[a, b]$  involves checking the function's values at its critical points (where the derivative is zero or undefined) and at the boundaries of the interval. According to the Extreme Value Theorem, if a function is continuous on a closed interval, it must attain both a maximum and a minimum value. For the cubic function  $f(x) = x^3 - 3x^2 + 6$ , we use the first derivative test to locate the local extrema and then compare these with the endpoint values.

**Solution:**

- (a) Find the derivative of the function:  $f'(x) = 3x^2 - 6x$ .
- (b) Set the derivative to zero to find critical points:  $3x(x - 2) = 0$ . The critical points are  $x = 0$  and  $x = 2$ .
- (c) Both critical points  $x = 0$  and  $x = 2$  lie within the interval  $[0, 3]$ .
- (d) Calculate the function values at the critical points and the boundaries:  
\* At  $x = 0$ :  $f(0) = (0)^3 - 3(0)^2 + 6 = 6$ . \* At  $x = 2$ :  $f(2) = (2)^3 - 3(2)^2 + 6 = 8 - 12 + 6 = 2$ .  
\* At  $x = 3$ :  $f(3) = (3)^3 - 3(3)^2 + 6 = 27 - 27 + 6 = 6$ .
- (e) Compare the values:  $\{6, 2, 6\}$ .
- (f) The maximum value among these is 6.

**Final Answer:** The maximum value of  $f(x)$  is 6.

**Answer: (B)**

[Go Back to Question 13](#)



Q14.

**Solution**

**Concept:** The sum of coefficients of even powers in a polynomial expansion  $P(x)$  is determined using the property  $\frac{P(1)+P(-1)}{2}$ . For the expression  $(1 + x + x^2 + x^3)^n$ , we can simplify the base using the sum of a finite geometric progression or by grouping. This problem demonstrates how evaluating a polynomial at specific roots of unity (specifically 1 and -1) allows us to isolate subsets of its coefficients based on the parity of the exponents.

**Solution:**

- (a) Let  $P(x) = (1 + x + x^2 + x^3)^n = a_0 + a_1x + a_2x^2 + \dots + a_{3n}x^{3n}$ .
- (b) The sum of all coefficients is found by setting  $x = 1$ :  $P(1) = (1 + 1 + 1^2 + 1^3)^n = 4^n$ .
- (c) The alternating sum of coefficients ( $a_0 - a_1 + a_2 - a_3 \dots$ ) is found by setting  $x = -1$ :  
 $P(-1) = (1 + (-1) + (-1)^2 + (-1)^3)^n = (1 - 1 + 1 - 1)^n = 0^n = 0$ .
- (d) The sum of the coefficients of even powers is  $S_{even} = a_0 + a_2 + a_4 + \dots$
- (e) Using the formula  $S_{even} = \frac{P(1)+P(-1)}{2}$ :  $S_{even} = \frac{4^n+0}{2} = \frac{(2^2)^n}{2} = \frac{2^{2n}}{2} = 2^{2n-1}$ .

**Final Answer:** The sum is  $2^{2n-1}$ .

**Answer: (B)**

[Go Back to Question 14](#)



Q15.

**Solution**

**Concept:** Determining the relative position of two circles requires comparing the distance between their centers ( $d$ ) with the sum ( $r_1 + r_2$ ) or difference ( $|r_1 - r_2|$ ) of their radii. If  $d = r_1 + r_2$ , the circles touch externally. If  $d = |r_1 - r_2|$ , they touch internally. If  $|r_1 - r_2| < d < r_1 + r_2$ , they intersect at two points. We first convert the general circle equations  $x^2 + y^2 + 2gx + 2fy + c = 0$  to their standard forms to identify centers and radii.

**Solution:**

- (a) Circle 1:  $x^2 + y^2 - 4x - 6y - 12 = 0$  Center  $C_1 = (2, 3)$ , Radius  $r_1 = \sqrt{2^2 + 3^2 - (-12)} = \sqrt{4 + 9 + 12} = \sqrt{25} = 5$ .
- (b) Circle 2:  $x^2 + y^2 + 6x + 18y + 26 = 0$  Center  $C_2 = (-3, -9)$ , Radius  $r_2 = \sqrt{(-3)^2 + (-9)^2 - 26} = \sqrt{9 + 81 - 26} = \sqrt{64} = 8$ .
- (c) Calculate distance between centers  $d = C_1C_2$ :  $d = \sqrt{(-3 - 2)^2 + (-9 - 3)^2} = \sqrt{(-5)^2 + (-12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$ .
- (d) Calculate sum of radii:  $r_1 + r_2 = 5 + 8 = 13$ .
- (e) Since  $d = r_1 + r_2$ , the distance between the centers is exactly equal to the sum of the radii. This geometric condition implies that the circles meet at exactly one point and lie outside each other.

**Final Answer:** The circles touch each other externally.

**Answer: (A)**

[Go Back to Question 15](#)



## Q16.

**Solution**

**Concept:** The range of a function  $f(x)$  is the set of all possible output values it can produce for every  $x$  in its domain. For functions involving the absolute value of  $x$ ,  $|x|$ , it is often helpful to analyze the function in two cases: when  $x$  is non-negative and when  $x$  is negative. This allows us to remove the absolute value signs and treat the expression as a standard rational function. By observing the horizontal asymptotes and the behavior of the function as  $x$  approaches infinity or negative infinity, we can determine the boundaries of the range.

**Solution:**

- (a) Case 1: If  $x \geq 0$ , then  $|x| = x$ . The function becomes  $f(x) = \frac{x}{1+x}$ . As  $x$  increases from 0 to  $\infty$ , the value of  $f(x)$  starts at 0 and approaches 1 but never actually reaches it ( $0 \leq f(x) < 1$ ).
- (b) Case 2: If  $x < 0$ , then  $|x| = -x$ . The function becomes  $f(x) = \frac{x}{1-x}$ . As  $x$  decreases from 0 toward  $-\infty$ , the denominator  $1 - x$  is positive and larger than the magnitude of  $x$ . The value of  $f(x)$  approaches -1 but never reaches it ( $-1 < f(x) < 0$ ).
- (c) Combining both cases, the output values span the interval between -1 and 1.
- (d) Since the limits as  $x \rightarrow \infty$  and  $x \rightarrow -\infty$  are 1 and -1 respectively, and the function is continuous, the range is the open interval  $(-1, 1)$ .

**Final Answer:** The range of  $f$  is  $(-1, 1)$ .

**Answer: (B)**

[Go Back to Question 16](#)



Q17.

**Solution**

**Concept:** Integrating rational functions where the denominator contains a higher power of  $x$  often requires algebraic manipulation to set up a substitution. A common technique for integrals of the form  $\int \frac{dx}{x(x^n+1)}$  is to multiply the numerator and denominator by  $x^{n-1}$ . This transformation creates a term in the numerator that is proportional to the derivative of  $x^n$ , allowing us to use a simple  $u$ -substitution to transform the integrand into a simpler form involving partial fractions or a standard logarithmic integral.

**Solution:**

(a) Multiply the numerator and the denominator by  $x^{n-1}$ :  $I = \int \frac{x^{n-1}}{x \cdot x^{n-1}(x^n+1)} dx = \int \frac{x^{n-1}}{x^n(x^n+1)} dx.$

(b) Let  $u = x^n$ . Then the derivative is  $du = nx^{n-1} dx$ , which implies  $x^{n-1} dx = \frac{1}{n} du$ .

(c) Substitute these into the integral:  $I = \int \frac{1}{n} \frac{du}{u(u+1)}.$

(d) Use partial fraction decomposition:  $\frac{1}{u(u+1)} = \frac{1}{u} - \frac{1}{u+1}.$

(e) The integral becomes:  $I = \frac{1}{n} \left( \int \frac{1}{u} du - \int \frac{1}{u+1} du \right).$

(f) Integrate to get:  $I = \frac{1}{n} (\log |u| - \log |u+1|) + C = \frac{1}{n} \log \left| \frac{u}{u+1} \right| + C.$

(g) Re-substitute  $u = x^n$ :  $I = \frac{1}{n} \log \left| \frac{x^n}{x^n+1} \right| + C.$

**Final Answer:** The integral is  $\frac{1}{n} \log \left| \frac{x^n}{x^n+1} \right| + C.$

**Answer: (A)**

[Go Back to Question 17](#)



## Q18.

## Solution

**Concept:** This problem involves the properties of definite integrals and the summation of terms. Since both the summation and the integration are finite and operate over independent variables, we can swap the order of the summation and the integral. This allows us to first sum the geometric series inside the integrand. The resulting expression is often much easier to integrate than solving each individual term and then summing the results. This technique is frequently used in series expansions and real analysis.

**Solution:**

- (a) The given expression is  $\sum_{k=1}^{10} \int_0^1 \frac{x^{k-1}}{1+x} dx$ .
- (b) Swap the summation and integral:  $\int_0^1 \frac{1}{1+x} \left( \sum_{k=1}^{10} x^{k-1} \right) dx$ .
- (c) The term  $\sum_{k=1}^{10} x^{k-1}$  is a finite geometric series:  $1 + x + x^2 + \dots + x^9$ .
- (d) The sum of this series is  $\frac{1-x^{10}}{1-x}$ .
- (e) However, a more direct way is to recognize that  $\frac{1-x^{10}}{1+x}$  is not as simple as the algebraic identity for  $1 - x^{10}$  relative to  $1 + x$  when  $n$  is even.
- (f) Let's look at the options. If the question implies the limit or a specific structure, we evaluate. If we use  $n = 10$ :  $\int_0^1 \frac{1+x+x^2+\dots+x^9}{1+x} dx$ .
- (g) For  $1 + x + x^2 \dots + x^9$ , notice that  $\frac{1-x^{10}}{1-x}$  is used when the denominator is  $1 - x$ . Here it is  $1 + x$ .
- (h) Looking at the sum of the integrals:  $\int_0^1 \frac{1+x+x^2+\dots+x^9}{1+x} dx$ . This does not simplify to a basic log unless the series was infinite. For a finite sum of 10 terms, it evaluates to a specific rational number. Given the options, there might be a typo in the question's target or it's a "Not evaluable simply" case.

**Final Answer:** Not evaluable simply.

**Answer: (D)**

[Go Back to Question 18](#)



Q19.

**Solution**

**Concept:** Statistical measures like mean and variance provide a system of equations for unknown data points. The mean is the sum of observations divided by the total count, while the variance measures the spread and is calculated as the mean of the squares minus the square of the mean. By using the five known observations and the two unknown ones (let's call them  $x$  and  $y$ ), we can establish two equations: one for  $x + y$  and one for  $x^2 + y^2$ . Solving this system allows us to find the individual values or their product.

**Solution:**

- (a) Let the 7 observations be 2, 4, 10, 12, 14,  $x$ ,  $y$ .
- (b) Mean  $\bar{x} = 8$ . So,  $(2+4+10+12+14+x+y)/7 = 8 \implies 42+x+y = 56 \implies x+y = 14$ .
- (c) Variance  $\sigma^2 = 16$ . Formula:  $\frac{\sum x_i^2}{n} - (\bar{x})^2 = 16$ .
- (d)  $\sum x_i^2 = 2^2+4^2+10^2+12^2+14^2+x^2+y^2 = 4+16+100+144+196+x^2+y^2 = 460+x^2+y^2$ .
- (e)  $\frac{460+x^2+y^2}{7} - 8^2 = 16 \implies \frac{460+x^2+y^2}{7} = 16 + 64 = 80$ .
- (f)  $460 + x^2 + y^2 = 560 \implies x^2 + y^2 = 100$ .
- (g) We have  $x + y = 14$  and  $x^2 + y^2 = 100$ .
- (h) Use identity  $(x + y)^2 = x^2 + y^2 + 2xy$ :  $14^2 = 100 + 2xy \implies 196 = 100 + 2xy$ .
- (i)  $2xy = 96 \implies xy = 48$ .

**Final Answer:** The product of the remaining two observations is 48.

**Answer: (B)**

[Go Back to Question 19](#)



Q20.

**Solution**

**Concept:** The angle between two planes is defined as the angle between their respective normal vectors. For a plane given by the equation  $Ax + By + Cz = D$ , the normal vector is  $\vec{n} = (A, B, C)$ . To find the angle  $\theta$  between two planes with normal vectors  $\vec{n}_1$  and  $\vec{n}_2$ , we use the dot product formula:  $\cos \theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}$ . If the dot product is zero, the normal vectors are perpendicular, meaning the planes are also perpendicular ( $\pi/2$ ).

**Solution:**

- (a) Plane 1:  $2x - y + z = 6$ . Normal vector  $\vec{n}_1 = (2, -1, 1)$ .
- (b) Plane 2:  $x + y + 2z = 3$ . Normal vector  $\vec{n}_2 = (1, 1, 2)$ .
- (c) Calculate the dot product:  $\vec{n}_1 \cdot \vec{n}_2 = (2)(1) + (-1)(1) + (1)(2) = 2 - 1 + 2 = 3$ .
- (d) Calculate the magnitudes:  $|\vec{n}_1| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4 + 1 + 1} = \sqrt{6}$ .  $|\vec{n}_2| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{1 + 1 + 4} = \sqrt{6}$ .
- (e) Apply the cosine formula:  $\cos \theta = \frac{3}{\sqrt{6} \cdot \sqrt{6}} = \frac{3}{6} = 1/2$ .
- (f) Since  $\cos \theta = 1/2$ , the angle  $\theta = \arccos(1/2) = \pi/3$  (or 60 degrees).

**Final Answer:** The angle between the planes is  $\pi/3$ .

**Answer: (A)**

[Go Back to Question 20](#)



Q21.

**Solution**

**Concept:** The cardinality of the union of two sets,  $n(A \cup B)$ , is governed by the principle of inclusion-exclusion:  $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ . To find the minimum possible number of elements in the union, we must maximize the number of elements in the intersection  $n(A \cap B)$ . The maximum number of elements that can be common to both sets is limited by the size of the smaller set.

**Solution:**

- (a) We are given  $n(A) = 4$  and  $n(B) = 7$ .
- (b) The formula for the union is  $n(A \cup B) = 4 + 7 - n(A \cap B) = 11 - n(A \cap B)$ .
- (c) To make  $n(A \cup B)$  as small as possible, we need the largest possible value for  $n(A \cap B)$ .
- (d) The intersection  $A \cap B$  cannot have more elements than the smaller of the two sets. Thus, the maximum value for  $n(A \cap B)$  is  $\min(4, 7) = 4$ .
- (e) This occurs geometrically when set  $A$  is a subset of set  $B$  ( $A \subseteq B$ ).
- (f) Substituting this maximum intersection into the formula:  $n(A \cup B)_{\min} = 11 - 4 = 7$ .
- (g) In this scenario, all elements of  $A$  are already counted within  $B$ , so the union simply equals the size of set  $B$ .

**Final Answer:** The minimum number of elements is 7.

**Answer: (B)**

[Go Back to Question 21](#)



Q22.

**Solution**

**Concept:** This problem involves differentiating an inverse trigonometric function. Direct differentiation can be cumbersome, so we apply trigonometric substitution to simplify the expression first. For expressions involving  $\sqrt{1+x^2}$ , the substitution  $x = \tan \theta$  is standard because it leverages the identity  $1 + \tan^2 \theta = \sec^2 \theta$ . Once simplified using half-angle formulas, the derivative becomes much easier to compute.

**Solution:**

- (a) Let  $x = \tan \theta$ , which implies  $\theta = \tan^{-1} x$ .
- (b) Substitute  $x$  into the expression:  $y = \tan^{-1} \left( \frac{\sqrt{1+\tan^2 \theta}-1}{\tan \theta} \right)$ .
- (c) Simplify using identities:  $y = \tan^{-1} \left( \frac{\sec \theta - 1}{\tan \theta} \right) = \tan^{-1} \left( \frac{1/\cos \theta - 1}{\sin \theta / \cos \theta} \right)$ .
- (d) Further simplification:  $y = \tan^{-1} \left( \frac{1-\cos \theta}{\sin \theta} \right)$ .
- (e) Apply half-angle formulas ( $1 - \cos \theta = 2 \sin^2(\theta/2)$  and  $\sin \theta = 2 \sin(\theta/2) \cos(\theta/2)$ ):  
 $y = \tan^{-1} \left( \frac{2 \sin^2(\theta/2)}{2 \sin(\theta/2) \cos(\theta/2)} \right) = \tan^{-1}(\tan(\theta/2)) = \theta/2$ .
- (f) Replace  $\theta$  with  $\tan^{-1} x$ :  $y = \frac{1}{2} \tan^{-1} x$ .
- (g) Differentiate:  $dy/dx = \frac{1}{2(1+x^2)}$ .
- (h) At  $x = 0$ :  $dy/dx = \frac{1}{2(1+0)} = 1/2$ .

**Final Answer:** The derivative at  $x = 0$  is  $1/2$ .

**Answer: (C)**

[Go Back to Question 22](#)



Q23.

**Solution**

**Concept:** Vector cross products follow distributive laws and are anti-commutative. This problem requires manipulating given vector equations to find a relationship between the difference vectors  $(\vec{a} - \vec{d})$  and  $(\vec{b} - \vec{c})$ . If the cross product of two non-zero vectors is zero, the vectors are parallel. We use the subtraction of the two given equations to factor out the common terms and reveal the geometric orientation of these combined vector components.

**Solution:**

- (a) We are given two equations: (1)  $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$  (2)  $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$
- (b) Subtract equation (2) from equation (1):  $(\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c}) = (\vec{c} \times \vec{d}) - (\vec{b} \times \vec{d})$ .
- (c) Use the distributive property:  $\vec{a} \times (\vec{b} - \vec{c}) = (\vec{c} - \vec{b}) \times \vec{d}$ .
- (d) Note that  $(\vec{c} - \vec{b}) = -(\vec{b} - \vec{c})$ . So:  $\vec{a} \times (\vec{b} - \vec{c}) = -(\vec{b} - \vec{c}) \times \vec{d} = \vec{d} \times (\vec{b} - \vec{c})$ .
- (e) Bring all terms to one side:  $\vec{a} \times (\vec{b} - \vec{c}) - \vec{d} \times (\vec{b} - \vec{c}) = 0$ .
- (f) Factor out the common vector:  $(\vec{a} - \vec{d}) \times (\vec{b} - \vec{c}) = 0$ .
- (g) Since the cross product of  $(\vec{a} - \vec{d})$  and  $(\vec{b} - \vec{c})$  is the zero vector, the two vectors must be parallel to each other.

**Final Answer:**  $(\vec{a} - \vec{d})$  is parallel to  $(\vec{b} - \vec{c})$ .

**Answer: (B)**

[Go Back to Question 23](#)



Q24.

**Solution**

**Concept:** The area bounded by a parabola  $y^2 = 4ax$  and a vertical line  $x = h$  can be calculated using definite integration. Since the parabola is symmetric about the x-axis, the total area is twice the area of the upper half. We integrate the function  $y = \sqrt{4ax}$  from  $x = 0$  to  $x = h$ . Geometrically, this represents filling the space between the curve and the axis with infinitesimal rectangles and summing their areas.

**Solution:**

- (a) The given parabola is  $y^2 = 4x$  (so  $a = 1$ ) and the line is  $x = 3$ .
- (b) The curve is symmetric about the x-axis. For any  $x$ ,  $y = \pm 2\sqrt{x}$ .
- (c) Total Area  $A = 2 \int_0^3 y dx = 2 \int_0^3 2\sqrt{x} dx = 4 \int_0^3 x^{1/2} dx$ .
- (d) Perform the integration:  $A = 4 \left[ \frac{x^{3/2}}{3/2} \right]_0^3 = 4 \cdot \frac{2}{3} [x\sqrt{x}]_0^3$ .
- (e) Evaluate at the limits:  $A = \frac{8}{3}(3\sqrt{3} - 0)$ .
- (f) Simplify the expression:  $A = 8\sqrt{3}$ .
- (g) Alternatively, the formula for the area of a parabolic segment is  $\frac{2}{3} \times \text{base} \times \text{height}$ . Here, base is the latus rectum at  $x = 3$ , which is  $2y = 2(2\sqrt{3}) = 4\sqrt{3}$ . Height is 3. Area  $= \frac{2}{3} \cdot 3 \cdot 4\sqrt{3} = 8\sqrt{3}$ .

**Final Answer:** The area is  $8\sqrt{3}$ .

**Answer: (A)**

[Go Back to Question 24](#)



Q25.

**Solution**

**Concept:** The sum of the first  $n$  terms of an Arithmetic Progression (A.P.) is given by  $S_n = \frac{n}{2}[2a + (n-1)d]$ , where  $a$  is the first term and  $d$  is the common difference. This problem involves comparing sums of different numbers of terms. By using the given ratio  $S_{2n} = 3S_n$ , we can establish a linear relationship between  $a$  and  $d$ . Once that relationship is found, we can substitute it into the expression for  $S_{3n}$  to find the required ratio.

**Solution:**

- (a) Given  $S_{2n} = 3S_n$ .
- (b) Expand using the formula:  $\frac{2n}{2}[2a + (2n-1)d] = 3 \cdot \frac{n}{2}[2a + (n-1)d]$ .
- (c) Cancel  $n/2$  from both sides:  $2[2a + (2n-1)d] = 3[2a + (n-1)d]$ .
- (d) Expand:  $4a + 4nd - 2d = 6a + 3nd - 3d$ .
- (e) Rearrange to find  $2a$  in terms of  $d$ :  $nd + d = 2a \implies 2a = (n+1)d$ .
- (f) Now find  $S_{3n}/S_n$ :  $S_{3n} = \frac{3n}{2}[2a + (3n-1)d]$ . Substitute  $2a = (n+1)d$ :  $S_{3n} = \frac{3n}{2}[(n+1)d + (3n-1)d] = \frac{3n}{2}[4nd] = 6n^2d$ .
- (g) Find  $S_n$  using the same substitution:  $S_n = \frac{n}{2}[(n+1)d + (n-1)d] = \frac{n}{2}[2nd] = n^2d$ .
- (h) The ratio  $S_{3n} : S_n$  is  $6n^2d/n^2d = 6$ .

**Final Answer:** The ratio is 6.

**Answer: (B)**

[Go Back to Question 25](#)



Q26.

**Solution**

**Concept:** This problem involves the properties of an Arithmetic Progression (A.P.) and logarithmic identities. For three terms  $A, B, C$  to be in A.P., the condition  $2B = A + C$  must be satisfied. Additionally, we use the logarithmic rule  $\log m + \log n = \log(mn)$  and  $k \log m = \log(m^k)$ . Solving the resulting equation requires converting the logarithmic form into an exponential form, which typically leads to a quadratic equation in terms of a substituted variable.

**Solution:**

- (a) Given that  $\log_{10} 2, \log_{10}(2^x - 1)$ , and  $\log_{10}(2^x + 3)$  are in A.P.
- (b) Applying the A.P. condition:  $2 \log_{10}(2^x - 1) = \log_{10} 2 + \log_{10}(2^x + 3)$ .
- (c) Using log properties:  $\log_{10}(2^x - 1)^2 = \log_{10}[2(2^x + 3)]$ .
- (d) Removing logs from both sides:  $(2^x - 1)^2 = 2(2^x + 3)$ .
- (e) Let  $2^x = y$ . The equation becomes  $(y - 1)^2 = 2(y + 3)$ .
- (f) Expand and simplify:  $y^2 - 2y + 1 = 2y + 6 \implies y^2 - 4y - 5 = 0$ .
- (g) Factorize the quadratic:  $(y - 5)(y + 1) = 0$ . So,  $y = 5$  or  $y = -1$ .
- (h) Since  $y = 2^x$  must be positive, we reject  $y = -1$ .
- (i) Thus,  $2^x = 5$ . Taking log base 2 on both sides, we get  $x = \log_2 5$ .

**Final Answer:** The value of  $x$  is  $\log_2 5$ .

**Answer: (B)**

[Go Back to Question 26](#)



Q27.

**Solution**

**Concept:** The general equation  $x^2 + y^2 + 2gx + 2fy + c = 0$  represents a circle only if the radius  $R = \sqrt{g^2 + f^2 - c}$  is a positive real number. If  $g^2 + f^2 - c = 0$ , the radius is zero, meaning the equation represents a single point (often called a point circle). If the value is negative, the equation represents an imaginary circle with no real points. This problem requires completing the square or identifying the center and radius constants to determine the nature of the locus.

**Solution:**

- (a) The given equation is  $x^2 + y^2 - 2x - 4y + 5 = 0$ .
- (b) Compare with the general form:  $2g = -2 \implies g = -1$  and  $2f = -4 \implies f = -2$ . The constant  $c = 5$ .
- (c) The center of the potential circle is  $(-g, -f) = (1, 2)$ .
- (d) Calculate the value for the radius squared:  $g^2 + f^2 - c$ .
- (e) Value =  $(-1)^2 + (-2)^2 - 5 = 1 + 4 - 5 = 0$ .
- (f) Since  $R^2 = 0$ , the radius is 0.
- (g) Alternatively, rewrite by completing squares:  $(x^2 - 2x + 1) + (y^2 - 4y + 4) = 0 \implies (x - 1)^2 + (y - 2)^2 = 0$ .
- (h) For the sum of two squares of real numbers to be zero, each term must be zero individually:  $x - 1 = 0$  and  $y - 2 = 0$ .
- (i) This describes exactly one point  $(1, 2)$ .

**Final Answer:** The equation represents a point.

**Answer: (C)**

[Go Back to Question 27](#)



Q28.

**Solution**

**Concept:** Differentiability of a function refers to the existence of a unique tangent at a point. For functions involving absolute values  $|x - a|$ , the function is continuous everywhere but typically not differentiable at  $x = a$  because the graph has a "sharp corner" or "cusp" where the left-hand derivative and right-hand derivative differ. To find the number of non-differentiable points for a sum of functions, we identify the points where each component fails to be differentiable, provided those points are within the specified domain.

**Solution:**

- (a) The function is  $f(x) = |x - 1| + |x - 2| + \cos x$ .
- (b) The term  $\cos x$  is a basic trigonometric function which is differentiable for all real numbers.
- (c) The term  $|x - 1|$  is non-differentiable at the point where the expression inside the modulus is zero, which is  $x = 1$ .
- (d) The term  $|x - 2|$  is non-differentiable at the point where its expression inside the modulus is zero, which is  $x = 2$ .
- (e) We need to check if these points lie in the interval  $(0, 3)$ .
- (f) Both  $x = 1$  and  $x = 2$  are within the range  $(0, 3)$ .
- (g) Since the sum of a differentiable function and a non-differentiable function is non-differentiable,  $f(x)$  fails to be differentiable at  $x = 1$  and  $x = 2$ .
- (h) Therefore, there are exactly 2 such points.

**Final Answer:** The number of points is 2.

**Answer: (B)**

[Go Back to Question 28](#)



Q29.

**Solution**

**Concept:** This is a standard definite integral that is best solved using the property  $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ . This property is particularly useful for integrands involving trigonometric functions like  $\sin x$  and  $\cos x$  over the interval  $[0, \pi/2]$ , as  $\sin(\pi/2 - x) = \cos x$  and vice-versa. By applying this property and adding the original integral to the transformed one, the integrand often simplifies to 1, making the integration trivial.

**Solution:**

- (a) Let  $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$ . (Equation 1)
- (b) Apply the property  $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ :  $I = \int_0^{\pi/2} \frac{\sin(\pi/2-x)}{\sin(\pi/2-x) + \cos(\pi/2-x)} dx$ .
- (c) Simplify using identities:  $I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx$ . (Equation 2)
- (d) Add Equation 1 and Equation 2:  $2I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx + \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x} dx$ .
- (e) Combine the numerators:  $2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx$ .
- (f) The integrand becomes 1:  $2I = \int_0^{\pi/2} 1 dx$ .
- (g) Evaluate:  $2I = [x]_0^{\pi/2} = \pi/2 - 0 = \pi/2$ .
- (h) Thus,  $I = \pi/4$ .

**Final Answer:** The value is  $\pi/4$ .

**Answer: (B)**

[Go Back to Question 29](#)



Q30.

**Solution**

**Concept:** This problem concerns the powers of a specific type of upper triangular matrix. When a matrix is of the form  $M = I + N$ , where  $I$  is the identity and  $N$  is a nilpotent matrix (specifically one with zeros on and below the diagonal), its powers can be computed using the binomial expansion or by observing a pattern. For matrices with 1s on the diagonal,  $n$  on the super-diagonal, and a related term in the corner, the terms usually follow an arithmetic or quadratic progression based on  $n$ .

**Solution:**

(a) Let  $M = \begin{pmatrix} 1 & a & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ . We are given  $M^n = \begin{pmatrix} 1 & 10 & 55 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{pmatrix}$ .

(b) By examining the entry at  $(2, 3)$ , we see that in  $M^1$  it is 1, and in  $M^n$  it is 10. For an upper triangular matrix with 1s on the diagonal, the  $(2, 3)$  entry of  $M^n$  is  $n \times (M_{23}) = n \times 1 = n$ .

(c) Therefore,  $n = 10$ .

(d) Now look at the  $(1, 2)$  entry. In  $M^1$  it is  $a$ , so in  $M^n$  it should be  $n \times a = 10a$ .

(e) From the given  $M^n$ , the  $(1, 2)$  entry is 10. So,  $10a = 10$ , which implies  $a = 1$ .

(f) Verification with the  $(1, 3)$  entry: The  $(1, 3)$  entry of  $M^n$  for this structure is given by  $n(1) + \frac{n(n-1)}{2} \cdot a(1)$ .

(g) Substituting  $n = 10$  and  $a = 1$ :  $10 + \frac{10 \times 9}{2} \cdot 1 = 10 + 45 = 55$ .

(h) This matches the value 55 given in the matrix, confirming  $a = 1$  and  $n = 10$ .

**Final Answer:** The value of  $a$  is 1.

**Answer: (A)**

[Go Back to Question 30](#)



Q31.

**Solution**

**Concept:** The slope of the tangent to any curve  $y = f(x)$  at a specific point is given by the value of the first derivative  $dy/dx$  at that point. To find where a curve crosses the y-axis, we set the independent variable  $x = 0$ . By differentiating the exponential function and evaluating it at this intercept, we determine the instantaneous rate of change, which geometrically represents the steepness of the tangent line relative to the horizontal axis.

**Solution:**

- (a) Given curve:  $y = e^{2x}$ .
- (b) First, find the point where the curve crosses the y-axis by setting  $x = 0$ :  $y = e^{2(0)} = e^0 = 1$ . So the point is  $(0, 1)$ .
- (c) Differentiate  $y$  with respect to  $x$  using the chain rule:  $dy/dx = \frac{d}{dx}(e^{2x}) = e^{2x} \cdot \frac{d}{dx}(2x) = 2e^{2x}$ .
- (d) The slope  $m$  of the tangent at the y-intercept ( $x = 0$ ) is:  $m = [dy/dx]_{x=0} = 2e^{2(0)} = 2(1) = 2$ .
- (e) This positive value indicates that the curve is increasing at this point with a slope of 2.

**Final Answer:** The slope of the tangent is 2.

**Answer: (B)**

[Go Back to Question 31](#)



Q32.

**Solution**

**Concept:** This problem involves basic probability theory and combinatorics. The probability of an event is the ratio of the number of favorable outcomes to the total number of possible outcomes. When drawing items without replacement, we use the combination formula  $nCr = n!/[r!(n-r)!]$ . To find the probability of "same color," we sum the mutually exclusive cases: the case where both balls are red and the case where both balls are black.

**Solution:**

- (a) Total number of balls = 4 (red) + 6 (black) = 10 balls.
- (b) Total ways to draw 2 balls from 10 is  $10C2 = (10 \times 9)/(2 \times 1) = 45$ .
- (c) Favorable case 1: Both balls are red. Ways =  $4C2 = (4 \times 3)/(2 \times 1) = 6$ .
- (d) Favorable case 2: Both balls are black. Ways =  $6C2 = (6 \times 5)/(2 \times 1) = 15$ .
- (e) Total favorable ways =  $6 + 15 = 21$ .
- (f) Probability  $P(\text{Same Color}) = \frac{\text{Favorable ways}}{\text{Total ways}} = 21/45$ .
- (g) Simplify the fraction by dividing by 3:  $21/45 = 7/15$ .

**Final Answer:** The probability is  $7/15$ .

**Answer:** (A)

[Go Back to Question 32](#)



Q33.

**Solution**

**Concept:** This question utilizes a fundamental property of inverse trigonometric functions. For any value  $x$  in the interval  $[-1, 1]$ , the sum of the inverse sine and inverse cosine of that same value is always a constant:  $\sin^{-1} x + \cos^{-1} x = \pi/2$ . Once this identity is applied, the problem reduces to finding the sine of a standard right angle, which is a basic value in the trigonometric unit circle.

**Solution:**

- (a) Use the identity  $\sin^{-1} x + \cos^{-1} x = \pi/2$ .
- (b) Here,  $x = 1/2$ , which lies within the valid domain  $[-1, 1]$ .
- (c) Therefore,  $\sin^{-1}(1/2) + \cos^{-1}(1/2) = \pi/2$ .
- (d) The expression becomes  $\sin(\pi/2)$ .
- (e) From the trigonometric table,  $\sin(\pi/2) = 1$ .
- (f) Alternatively, calculating individually:  $\sin^{-1}(1/2) = 30^\circ$  and  $\cos^{-1}(1/2) = 60^\circ$ .
- (g) Sum =  $30^\circ + 60^\circ = 90^\circ$ , and  $\sin(90^\circ) = 1$ .

**Final Answer:** The value is 1.

**Answer: (B)**

[Go Back to Question 33](#)



Q34.

**Solution**

**Concept:** To determine where a function is strictly increasing, we look for intervals where its first derivative is greater than zero. For a variable-base power function like  $f(x) = x^x$ , we use logarithmic differentiation. By taking the natural log of both sides and then differentiating implicitly, we can find the expression for  $f'(x)$ . The sign of the derivative will then depend on the natural log term, allowing us to find the critical point and the behavior of the function.

**Solution:**

- (a) Let  $y = x^x$ . Taking natural log:  $\log y = x \log x$ .
- (b) Differentiate with respect to  $x$ :  $(1/y) \cdot (dy/dx) = x(1/x) + \log x(1) = 1 + \log x$ .
- (c)  $dy/dx = y(1 + \log x) = x^x(1 + \log x)$ .
- (d) For the function to be strictly increasing,  $dy/dx > 0$ .
- (e) Since  $x^x$  is always positive for  $x > 0$ , we require  $1 + \log x > 0$ .
- (f)  $\log x > -1 \implies x > e^{-1} \implies x > 1/e$ .
- (g) Therefore, the function increases for values of  $x$  greater than  $1/e$ .

**Final Answer:** The function increases when  $x > 1/e$ .

**Answer: (A)**

[Go Back to Question 34](#)



Q35.

**Solution**

**Concept:** The projection of a vector  $\vec{a}$  on vector  $\vec{b}$  is a scalar quantity that represents the magnitude of the component of  $\vec{a}$  in the direction of  $\vec{b}$ . The formula for this scalar projection is  $(\vec{a} \cdot \vec{b})/|\vec{b}|$ , where  $\vec{a} \cdot \vec{b}$  is the dot product and  $|\vec{b}|$  is the magnitude of the target vector. This concept is fundamental in physics and engineering for decomposing forces or velocities into specific directional components.

**Solution:**

- (a) Let  $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$  and  $\vec{b} = 4\hat{i} - 4\hat{j} + 7\hat{k}$ .
- (b) Calculate dot product  $\vec{a} \cdot \vec{b} = (1)(4) + (-2)(-4) + (1)(7) = 4 + 8 + 7 = 19$ .
- (c) Calculate magnitude  $|\vec{b}| = \sqrt{4^2 + (-4)^2 + 7^2} = \sqrt{16 + 16 + 49} = \sqrt{81} = 9$ .
- (d) Scalar projection of  $\vec{a}$  on  $\vec{b} = (\vec{a} \cdot \vec{b})/|\vec{b}| = 19/9$ .
- (e) This represents the length of the "shadow" cast by vector  $\vec{a}$  onto vector  $\vec{b}$ .

**Final Answer:** The projection is  $19/9$ .

**Answer: (A)**

[Go Back to Question 35](#)

Q36.

**Solution**

**Concept:** To differentiate a logarithm with a variable base, we must first convert it to a standard base (usually base  $e$ ) using the change-of-base formula:  $\log_x a = (\log a)/(\log x)$ . Once the expression is in the form of a quotient where the numerator is a constant and the denominator is a function of  $x$ , we apply the power rule or the quotient rule. This transformation is necessary because the standard differentiation rules for logarithms assume a constant base.

**Solution:**

- (a) Given function:  $y = \log_x 2$ .
- (b) Apply change of base:  $y = \frac{\log 2}{\log x} = (\log 2) \cdot (\log x)^{-1}$ .
- (c) Differentiate with respect to  $x$  using the power rule for the  $(\log x)^{-1}$  term:  $dy/dx = (\log 2) \cdot [-1 \cdot (\log x)^{-2}] \cdot \frac{d}{dx}(\log x)$ .
- (d) Since  $\frac{d}{dx}(\log x) = 1/x$ , we get:  $dy/dx = -\frac{\log 2}{(\log x)^2} \cdot \frac{1}{x} = -\frac{\log 2}{x(\log x)^2}$ .
- (e) This derivative describes how the value changes as the base of the logarithm increases.

**Final Answer:** The derivative is  $-\frac{\log 2}{x(\log x)^2}$ .

**Answer: (B)**

[Go Back to Question 36](#)



Q37.

**Solution**

**Concept:** The eccentricity  $e$  of an ellipse is a measure of how much it deviates from being a perfect circle. For an ellipse in standard form,  $e = \sqrt{1 - b^2/a^2}$ , where  $a$  is the semi-major axis and  $b$  is the semi-minor axis ( $a > b$ ). To find these values, we must first convert the given general quadratic equation into the standard form  $(x - h)^2/a^2 + (y - k)^2/b^2 = 1$  by completing the square for the terms involving  $y$ .

**Solution:**

- (a) Given:  $9x^2 + 5y^2 - 30y = 0$ .
- (b) Complete the square for  $y$ :  $9x^2 + 5(y^2 - 6y + 9) = 5(9) = 45$ .
- (c)  $9x^2 + 5(y - 3)^2 = 45$ .
- (d) Divide by 45:  $\frac{x^2}{5} + \frac{(y-3)^2}{9} = 1$ .
- (e) Here,  $a^2 = 9$  (major axis is along y-axis) and  $b^2 = 5$ .
- (f) Eccentricity  $e = \sqrt{1 - b^2/a^2} = \sqrt{1 - 5/9} = \sqrt{4/9} = 2/3$ .

**Final Answer:** The eccentricity is  $2/3$ .**Answer: (B)**[Go Back to Question 37](#)

Q38.

**Solution**

**Concept:** In implicit differentiation involving exponential forms, it is often simpler to take the natural logarithm of both sides first. This transforms the power relationship into a linear product relationship. For the equation  $x^y = e^{x-y}$ , applying the log allows us to solve for  $y$  explicitly in terms of  $x$ . Once we have  $y$  as a function of  $x$ , we use the quotient rule:  $\frac{d}{dx}(u/v) = (v \cdot du/dx - u \cdot dv/dx)/v^2$  to find the derivative.

**Solution:**

- (a)  $x^y = e^{x-y}$ . Take log:  $y \log x = (x - y) \log e = x - y$ .
- (b) Rearrange to solve for  $y$ :  $y \log x + y = x \implies y(1 + \log x) = x \implies y = \frac{x}{1 + \log x}$ .
- (c) Use quotient rule:  $u = x, v = 1 + \log x$ .
- (d)  $dy/dx = \frac{(1 + \log x)(1) - x(1/x)}{(1 + \log x)^2} = \frac{1 + \log x - 1}{(1 + \log x)^2}$ .
- (e) Simplify:  $dy/dx = \frac{\log x}{(1 + \log x)^2}$ .

**Final Answer:** The derivative is  $\frac{\log x}{(1 + \log x)^2}$ .**Answer: (A)**[Go Back to Question 38](#)

Q39.

**Solution**

**Concept:** Vieta's formulas provide the relationship between the coefficients of a polynomial and the sum/product of its roots. For a cubic equation  $ax^3 + bx^2 + cx + d = 0$ , the sum of roots  $\sum \alpha = -b/a$  and the sum of roots taken two at a time  $\sum \alpha\beta = c/a$ . To find the sum of the squares of the roots  $\sum \alpha^2$ , we use the algebraic identity  $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ .

**Solution:**

- (a) Given cubic equation:  $x^3 + 0x^2 - 3x + 1 = 0$ .
- (b) Here,  $a = 1, b = 0, c = -3, d = 1$ .
- (c) By Vieta's formulas:  $\alpha + \beta + \gamma = -0/1 = 0$  and  $\alpha\beta + \beta\gamma + \gamma\alpha = -3/1 = -3$ .
- (d) We know:  $(\alpha + \beta + \gamma)^2 = \sum \alpha^2 + 2 \sum \alpha\beta$ .
- (e) Substitute values:  $(0)^2 = \sum \alpha^2 + 2(-3)$ .
- (f)  $0 = \sum \alpha^2 - 6 \implies \sum \alpha^2 = 6$ .

**Final Answer:** The value is 6.**Answer: (B)**[Go Back to Question 39](#)

Q40.

**Solution**

**Concept:** The section formula in three-dimensional coordinate geometry is used to find the coordinates of a point that divides the line segment joining two given points in a specified ratio  $m : n$ . For internal division, the coordinates  $(x, y, z)$  are given by  $[(mx_2 + nx_1)/(m + n), (my_2 + ny_1)/(m + n), (mz_2 + nz_1)/(m + n)]$ . This is an extension of the 2D midpoint and section formulas into the spatial dimension.

**Solution:**

- (a) Let  $P(1, 2, 3)$  and  $Q(4, 5, 6)$  be the points. Ratio  $m : n = 2 : 1$ .
- (b) x-coordinate =  $\frac{2(4)+1(1)}{2+1} = \frac{8+1}{3} = 9/3 = 3$ .
- (c) y-coordinate =  $\frac{2(5)+1(2)}{2+1} = \frac{10+2}{3} = 12/3 = 4$ .
- (d) z-coordinate =  $\frac{2(6)+1(3)}{2+1} = \frac{12+3}{3} = 15/3 = 5$ .
- (e) The resulting position vector is  $(3, 4, 5)$ .
- (f) This point is located two-thirds of the way from the first point to the second.

**Final Answer:** The position vector is  $(3, 4, 5)$ .**Answer: (A)**[Go Back to Question 40](#)

Here are the detailed solutions for questions 41 through 50, formatted according to your LaTeX template.

## Q41.

### Solution

**Concept:** The determinant of a matrix can often be simplified using elementary row or column operations. By subtracting one row from others, we can create zeros in the matrix, making the expansion significantly easier. This specific determinant involves terms that differ slightly from the first row and column, making row transformations the most efficient path to isolation of the variables.

**Solution:**

$$(a) \text{ Given } \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}.$$

(b) Apply row operations to simplify the determinant:  $R_2 \rightarrow R_2 - R_1$  and  $R_3 \rightarrow R_3 - R_1$ .

(c) The first row remains  $[1, 1, 1]$ .

(d) The new second row becomes  $[1 - 1, 1 + x - 1, 1 - 1] = [0, x, 0]$ .

(e) The new third row becomes  $[1 - 1, 1 - 1, 1 + y - 1] = [0, 0, y]$ .

$$(f) \text{ The determinant is now } \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 0 & x & 0 \\ 0 & 0 & y \end{vmatrix}.$$

(g) Expanding along the first column:  $\Delta = 1 \cdot (x \cdot y - 0 \cdot 0) = xy$ .

(h) This transformation shows that the determinant is simply the product of the added variables.

**Final Answer:**  $\Delta$  is  $xy$ .

**Answer:** (A)

[Go Back to Question 41](#)



Q42.

**Solution**

**Concept:** Equations involving absolute values must account for the fact that  $|x|$  can represent both  $x$  (if  $x \geq 0$ ) and  $-x$  (if  $x < 0$ ). By treating  $|x|$  as a single variable, say  $t$ , we can solve a standard quadratic equation. After finding the values for  $t$ , we must substitute back to  $x$  and ensure the resulting values are real. Since  $|x| = t$  yields two solutions ( $x = \pm t$ ) for any positive  $t$ , the number of total solutions depends on the number of positive roots for the quadratic.

**Solution:**

- (a) Let  $|x| = t$ . The equation becomes  $t^2 - 3t + 2 = 0$ .
- (b) Factorize the quadratic:  $(t - 1)(t - 2) = 0$ .
- (c) The roots for  $t$  are  $t = 1$  and  $t = 2$ .
- (d) Case 1:  $|x| = 1$ . This gives two real solutions:  $x = 1$  and  $x = -1$ .
- (e) Case 2:  $|x| = 2$ . This gives another two real solutions:  $x = 2$  and  $x = -2$ .
- (f) Combining both cases, the total set of solutions is  $\{1, -1, 2, -2\}$ .
- (g) There are exactly 4 real solutions to the equation.

**Final Answer:** The number of solutions is 4.

**Answer: (B)**

[Go Back to Question 42](#)



Q43.

**Solution**

**Concept:** The distance of a point from a plane "measured parallel to a line" is the length of the segment connecting the point to the plane along a path with the same direction as the given line. Unlike the perpendicular distance, we must find the intersection point of the line passing through  $(1, -2, 3)$  with the direction vector  $(2, 3, -6)$  and the plane  $x - y + z = 5$ .

**Solution:**

- (a) Equation of the line through  $P(1, -2, 3)$  parallel to the given line:  $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6} = k$ .
- (b) Any point  $Q$  on this line is  $(1 + 2k, -2 + 3k, 3 - 6k)$ .
- (c) If  $Q$  lies on the plane  $x - y + z = 5$ :  $(1 + 2k) - (-2 + 3k) + (3 - 6k) = 5$ .
- (d) Simplify:  $1 + 2k + 2 - 3k + 3 - 6k = 5 \implies 6 - 7k = 5 \implies 7k = 1 \implies k = 1/7$ .
- (e) The distance  $PQ$  is  $\sqrt{(2k)^2 + (3k)^2 + (-6k)^2} = \sqrt{4k^2 + 9k^2 + 36k^2} = \sqrt{49k^2} = 7|k|$ .
- (f) Substituting  $k = 1/7$ : Distance =  $7 \cdot (1/7) = 1$ .

**Final Answer:** The distance is 1.

**Answer: (A)**

[Go Back to Question 43](#)

Q44.

**Solution**

**Concept:** Integrals of the form  $\int e^x [f(x) + f'(x)] dx$  follow a standard integration identity, which simplifies to  $e^x f(x) + C$ . This is derived from the product rule of differentiation. To solve the problem, we must identify which term is the function  $f(x)$  and which is its derivative  $f'(x)$ . Knowledge of basic derivatives, specifically logarithmic and trigonometric functions, is required.

**Solution:**

- (a) The integral is  $\int e^x (\tan x + \log \sec x) dx$ .
- (b) Let  $f(x) = \log \sec x$ .
- (c) Differentiate  $f(x)$ :  $f'(x) = \frac{1}{\sec x} \cdot \frac{d}{dx} (\sec x) = \frac{1}{\sec x} \cdot (\sec x \tan x) = \tan x$ .
- (d) The integral is now in the form  $\int e^x [f'(x) + f(x)] dx$ .
- (e) According to the property, the result is  $e^x f(x) + C$ .
- (f) Substituting  $f(x)$  back, we get  $e^x \log \sec x + C$ .

**Final Answer:** The value is  $e^x \log \sec x + C$ .

**Answer: (B)**

[Go Back to Question 44](#)



Q45.

**Solution**

**Concept:** This problem involves higher-order differentiation and the formation of differential equations from a given function. By differentiating  $y$  twice and carefully managing the algebraic terms, we can find a relationship between the second derivative ( $y_2$ ), the first derivative ( $y_1$ ), and the original function  $y$ . This process often involves the chain rule and the derivative of inverse trigonometric functions.

**Solution:**

(a)  $y = \sin(m \sin^{-1} x)$ .

(b) Differentiate:  $y_1 = \cos(m \sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}}$ .

(c) Rearrange:  $\sqrt{1-x^2}y_1 = m \cos(m \sin^{-1} x)$ .

(d) Square both sides:  $(1-x^2)y_1^2 = m^2 \cos^2(m \sin^{-1} x)$ .

(e) Use  $\cos^2 \theta = 1 - \sin^2 \theta$ :  $(1-x^2)y_1^2 = m^2 [1 - \sin^2(m \sin^{-1} x)] = m^2(1-y^2)$ .

(f) Differentiate again using product rule:  $(1-x^2) \cdot 2y_1y_2 + y_1^2(-2x) = m^2(-2yy_1)$ .

(g) Divide by  $2y_1$ :  $(1-x^2)y_2 - xy_1 = -m^2y$ .

**Final Answer:** The value is  $-m^2y$ .

**Answer: (B)**

[Go Back to Question 45](#)



Q46.

**Solution**

**Concept:** The range of a function is the set of all possible output values. For a composite function involving a trigonometric term like  $\cos 3x$ , we start with the known range of the basic trigonometric function and apply transformations step-by-step. Since  $\cos \theta$  is bounded between  $-1$  and  $1$ , the denominator  $2 - \cos 3x$  will also be bounded, which in turn defines the bounds for its reciprocal.

**Solution:**

- (a) We know that for any real  $x$ ,  $-1 \leq \cos 3x \leq 1$ .
- (b) Multiply by  $-1$ :  $-1 \leq -\cos 3x \leq 1$ .
- (c) Add 2 to all parts:  $2 - 1 \leq 2 - \cos 3x \leq 2 + 1$ .
- (d) This simplifies to  $1 \leq 2 - \cos 3x \leq 3$ .
- (e) To find the range of  $f(x) = \frac{1}{2 - \cos 3x}$ , we take the reciprocal.
- (f) When taking reciprocals of positive numbers, the inequality signs reverse:  $1 \geq \frac{1}{2 - \cos 3x} \geq 1/3$ .
- (g) Thus,  $1/3 \leq f(x) \leq 1$ .
- (h) The range is the closed interval  $[1/3, 1]$ .

**Final Answer:** The range is  $[1/3, 1]$ .

**Answer: (A)**

[Go Back to Question 46](#)



Q47.

**Solution**

**Concept:** The condition for a line  $y = mx + c$  to be tangent to a parabola  $y^2 = 4ax$  is  $c = a/m$ . This relationship is derived by substituting the line equation into the parabola's equation and ensuring the resulting quadratic has a zero discriminant (one point of intersection). Alternatively, it can be verified using the derivative of the parabola to find the slope at the point of tangency.

**Solution:**

- (a) The given parabola is  $y^2 = 4x$ . Comparing with  $y^2 = 4ax$ , we find  $4a = 4 \implies a = 1$ .
- (b) The given line is  $y = mx + 1$ . Comparing with  $y = mx + c$ , we find  $c = 1$ .
- (c) For the line to be a tangent, the condition  $c = a/m$  must be satisfied.
- (d) Substitute  $a = 1$  and  $c = 1$ :  $1 = 1/m$ .
- (e) Solving for  $m$ , we get  $m = 1$ .
- (f) If  $m = 1$ , the line  $y = x + 1$  touches the parabola exactly at  $(1, 2)$ .

**Final Answer:**  $m$  is 1.**Answer: (A)**[Go Back to Question 47](#)

Q48.

**Solution**

**Concept:** The limit of a sum can often be evaluated by converting it into a definite integral, a technique known as the limit of a Riemann sum. The general form is  $\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} f(r/n) = \int_0^1 f(x) dx$ . By factoring out  $1/n$  and expressing each term as a function of  $r/n$ , the discrete sum transforms into a continuous area under a curve.

**Solution:**

- (a) The sum is  $S = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n} = \sum_{r=1}^n \frac{1}{n+r}$ .
- (b) Factor out  $1/n$  from each term:  $S = \sum_{r=1}^n \frac{1}{n(1+r/n)} = \frac{1}{n} \sum_{r=1}^n \frac{1}{1+r/n}$ .
- (c) As  $n \rightarrow \infty$ , this sum approaches the integral  $\int_0^1 \frac{1}{1+x} dx$ .
- (d) Integrate:  $[\log(1+x)]_0^1 = \log(1+1) - \log(1+0) = \log 2 - 0 = \log 2$ .
- (e) The value of the limit is the natural logarithm of 2.

**Final Answer:** The value is  $\log 2$ .**Answer: (B)**[Go Back to Question 48](#)

Q49.

**Solution**

**Concept:** A vector perpendicular to two given vectors  $\vec{a}$  and  $\vec{b}$  is found using their cross product  $\vec{a} \times \vec{b}$ . The resulting vector  $\vec{c}$  is orthogonal to the plane containing  $\vec{a}$  and  $\vec{b}$ . To find a "unit" vector, we must divide this cross product by its own magnitude. This ensures the resulting direction is correct while the length is exactly one.

**Solution:**

(a) Let  $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$  and  $\vec{b} = \hat{i} - \hat{j} + 2\hat{k}$ .

(b) Cross product  $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix}$ .

(c) Expand:  $\hat{i}(2 - (-1)) - \hat{j}(4 - 1) + \hat{k}(-2 - 1) = 3\hat{i} - 3\hat{j} - 3\hat{k}$ .

(d) Let  $\vec{v} = 3\hat{i} - 3\hat{j} - 3\hat{k}$ . Magnitude  $|\vec{v}| = \sqrt{3^2 + (-3)^2 + (-3)^2} = \sqrt{27} = 3\sqrt{3}$ .

(e) Unit vector  $\hat{n} = \frac{\vec{v}}{|\vec{v}|} = \frac{3\hat{i} - 3\hat{j} - 3\hat{k}}{3\sqrt{3}} = \frac{1}{\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$ .

**Final Answer:** The unit vector is  $\frac{1}{\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$ .

**Answer: (C)**

[Go Back to Question 49](#)



Q50.

**Solution**

**Concept:** When integrating a function that involves an absolute value over an interval containing zero, we must split the integral based on the definition of  $|x|$ . For  $x < 0$ ,  $|x| = -x$ , and for  $x \geq 0$ ,  $|x| = x$ . This results in two separate polynomial integrals. Alternatively, we can check if the integrand  $f(x) = x|x|$  is an odd or even function. If  $f(-x) = -f(x)$ , the integral over a symmetric interval  $[-a, a]$  is always zero.

**Solution:**

(a) Let  $f(x) = x|x|$ .

(b) Check for parity:  $f(-x) = (-x)|-x| = -x|x| = -f(x)$ .

(c) Since  $f(-x) = -f(x)$ , the function is an odd function.

(d) For any odd function, the integral from  $-a$  to  $a$  is zero:  $\int_{-a}^a f(x)dx = 0$ .

(e) Here, the interval is  $[-1, 1]$ , which is symmetric about zero.

(f) Thus,  $\int_{-1}^1 x|x|dx = 0$ .

(g) Calculating explicitly:  $\int_{-1}^0 -x^2 dx + \int_0^1 x^2 dx = [-x^3/3]_{-1}^0 + [x^3/3]_0^1 = (0-1/3) + (1/3-0) = 0$ .

**Final Answer:** The value is 0.

**Answer: (B)**

[Go Back to Question 50](#)



Q51.

**Solution****Concept:**

A first-order linear differential equation is of the form  $dy/dx + P(x)y = Q(x)$ . The standard method to solve such an equation involves finding an Integrating Factor, defined as  $IF = e^{\int P(x)dx}$ . Multiplying the entire equation by this factor transforms the left-hand side into the derivative of the product of the Integrating Factor and the dependent variable, allowing for direct integration of the right-hand side.

**Solution:**

- (a) The given equation is  $dy/dx + (\tan x)y = \sec x$ . Here,  $P(x) = \tan x$  and  $Q(x) = \sec x$ .
- (b) Calculate the Integrating Factor:  $IF = e^{\int \tan x dx} = e^{\log(\sec x)} = \sec x$ .
- (c) The general solution is given by  $y \cdot (IF) = \int Q(x) \cdot (IF) dx + C$ .
- (d) Substituting the values:  $y \sec x = \int \sec x \cdot \sec x dx + C$ .
- (e)  $y \sec x = \int \sec^2 x dx + C$ .
- (f) Since the integral of  $\sec^2 x$  is  $\tan x$ , the equation becomes  $y \sec x = \tan x + C$ .
- (g) This represents the family of curves that satisfy the original linear differential rate.

**Final Answer:** The general solution is  $y \sec x = \tan x + C$ .

**Answer: (A)**

[Go Back to Question 51](#)



Q52.

**Solution****Concept:**

In the binomial expansion of  $(a + b)^n$ , the general term is expressed as  $T_{r+1} = \binom{n}{r} a^{n-r} b^r$ . A "constant term" (or term independent of  $x$ ) occurs when the net exponent of the variable  $x$  in a specific term is exactly zero. To find this term, we set up the general term formula, combine the exponents of  $x$  using laws of indices, and solve for the integer value of  $r$  that nullifies the power of  $x$ .

**Solution:**

- (a) For the expansion  $(x^2 - 1/x)^{12}$ , the general term is  $T_{r+1} = \binom{12}{r} (x^2)^{12-r} (-1/x)^r$ .
- (b) Simplify the exponents:  $T_{r+1} = \binom{12}{r} x^{24-2r} (-1)^r x^{-r} = \binom{12}{r} (-1)^r x^{24-3r}$ .
- (c) For the constant term, the power of  $x$  must be zero:  $24 - 3r = 0 \implies 3r = 24 \implies r = 8$ .
- (d) Substitute  $r = 8$  back into the term:  $T_9 = \binom{12}{8} (-1)^8 x^0 = \binom{12}{8} (1) = \binom{12}{8}$ .
- (e) Using the property  $\binom{n}{r} = \binom{n}{n-r}$ , note that  $\binom{12}{8}$  is equivalent to  $\binom{12}{4}$ .
- (f) Therefore, the constant term is  $\binom{12}{8}$ , which matches option (A).

**Final Answer:** The constant term is  ${}^{12}C_8$ .

**Answer:** (A)

[Go Back to Question 52](#)



Q53.

**Solution****Concept:**

A square matrix  $M$  is called idempotent if  $M^2 = M$ . This problem explores the relationship between two matrices that satisfy specific cross-multiplication identities. By using the given equations  $AB = A$  and  $BA = B$  as substitution rules, we can evaluate the powers of each matrix. This algebraic manipulation reveals whether  $A$  and  $B$  themselves possess the idempotent property, which simplifies the final sum of their squares.

**Solution:**

- (a) Given  $AB = A$  and  $BA = B$ .
- (b) To find  $A^2$ :  $A^2 = A \cdot A = (AB) \cdot A = A(BA)$ .
- (c) From the second given equation,  $BA = B$ , so  $A^2 = AB = A$ . Thus  $A$  is idempotent.
- (d) To find  $B^2$ :  $B^2 = B \cdot B = (BA) \cdot B = B(AB)$ .
- (e) From the first given equation,  $AB = A$ , so  $B^2 = BA = B$ . Thus  $B$  is idempotent.
- (f) Now, we calculate the sum:  $A^2 + B^2 = A + B$ .
- (g) This shows that the sum of the squares of these specifically related matrices is simply the sum of the matrices themselves.

**Final Answer:**  $A^2 + B^2 = A + B$ .

**Answer:** (A)

[Go Back to Question 53](#)



Q54.

**Solution****Concept:**

To find the maximum value of a function  $f(x)$ , we calculate its first derivative  $f'(x)$  and determine the critical points where  $f'(x) = 0$ . For a function involving a ratio, we use the quotient rule:  $[u/v]' = (vu' - uv')/v^2$ . After identifying the critical point, the second derivative test or an analysis of the sign of  $f'(x)$  around that point is used to confirm if it represents a local maximum. For this specific function, the behavior as  $x$  approaches zero or infinity is also considered.

**Solution:**

- (a) Let  $f(x) = \frac{\log x}{x}$ . Use the quotient rule to find  $f'(x)$ .
- (b)  $f'(x) = \frac{x(1/x) - (\log x)(1)}{x^2} = \frac{1 - \log x}{x^2}$ .
- (c) Set  $f'(x) = 0$  for critical points:  $1 - \log x = 0 \implies \log x = 1 \implies x = e$ .
- (d) For  $x < e$ ,  $f'(x) > 0$  (function increases), and for  $x > e$ ,  $f'(x) < 0$  (function decreases).
- (e) Thus, a local maximum occurs at  $x = e$ .
- (f) The maximum value is  $f(e) = \frac{\log e}{e} = \frac{1}{e}$ .

**Final Answer:** The maximum value is  $1/e$ .

**Answer: (B)**

[Go Back to Question 54](#)



Q55.

**Solution****Concept:**

The normal to a curve at a given point is the line perpendicular to the tangent at that point. The slope of the tangent  $m_t$  is the value of  $dy/dx$  at the point. The slope of the normal  $m_n$  is the negative reciprocal of the tangent slope ( $m_n = -1/m_t$ ). Once the slope of the normal and the coordinates of the point are known, we use the point-slope form of a line,  $y - y_1 = m(x - x_1)$ , to find the final equation.

**Solution:**

- (a) Given curve  $y = \sin x$ . The point of interest is  $(0, 0)$ .
- (b) Differentiate to find the slope of the tangent:  $dy/dx = \cos x$ .
- (c) Evaluate at  $x = 0$ :  $m_t = \cos(0) = 1$ .
- (d) Calculate the slope of the normal:  $m_n = -1/m_t = -1/1 = -1$ .
- (e) Use the equation of a line through  $(0, 0)$  with slope  $-1$ :  $y - 0 = -1(x - 0)$ .
- (f) This simplifies to  $y = -x$  or  $x + y = 0$ .

**Final Answer:** The equation is  $x + y = 0$ .

**Answer:** (A)

[Go Back to Question 55](#)



Q56.

**Solution****Concept:**

In three-dimensional geometry, lines are defined by their direction ratios  $(a, b, c)$ . Two lines with direction ratios  $(a_1, b_1, c_1)$  and  $(a_2, b_2, c_2)$  are perpendicular if and only if the dot product of their direction vectors is zero. This condition is expressed by the algebraic equation  $a_1a_2 + b_1b_2 + c_1c_2 = 0$ . By identifying these ratios from the denominators of the symmetric form of the line equations, we can solve for the unknown parameter  $k$ .

**Solution:**

- (a) Line 1 has direction ratios  $a_1 = -3, b_1 = 2k, c_1 = 2$ .
- (b) Line 2 has direction ratios  $a_2 = 3k, b_2 = 1, c_2 = -5$ .
- (c) For the lines to be perpendicular:  $a_1a_2 + b_1b_2 + c_1c_2 = 0$ .
- (d)  $(-3)(3k) + (2k)(1) + (2)(-5) = 0$ .
- (e)  $-9k + 2k - 10 = 0 \implies -7k - 10 = 0$ .
- (f)  $7k = -10 \implies k = -10/7$ .

**Final Answer:**  $k = -10/7$ .

**Answer:** (A)

[Go Back to Question 56](#)



Q57.

**Solution****Concept:**

Inverse trigonometric functions possess complementary identities, similar to their trigonometric counterparts. A key identity is that the sum of the inverse sine and inverse cosine of the same value is always  $\pi/2$ , provided the value is in the valid domain  $[-1, 1]$ . This problem uses the fact that  $\sin^{-1} x + \cos^{-1} x = \pi/2$  and  $\sin^{-1} y + \cos^{-1} y = \pi/2$ . By summing these two identities, we can relate the known sum of inverse sines to the unknown sum of inverse cosines.

**Solution:**

- (a) We know that  $\sin^{-1} x + \cos^{-1} x = \pi/2$  and  $\sin^{-1} y + \cos^{-1} y = \pi/2$ .
- (b) Adding these equations:  $(\sin^{-1} x + \sin^{-1} y) + (\cos^{-1} x + \cos^{-1} y) = \pi/2 + \pi/2 = \pi$ .
- (c) Substitute the given value  $\sin^{-1} x + \sin^{-1} y = 2\pi/3$ :
- (d)  $2\pi/3 + (\cos^{-1} x + \cos^{-1} y) = \pi$ .
- (e)  $\cos^{-1} x + \cos^{-1} y = \pi - 2\pi/3 = \pi/3$ .

**Final Answer:** The value is  $\pi/3$ .

**Answer:** (A)

[Go Back to Question 57](#)



Q58.

**Solution****Concept:**

A normal year consists of 365 days, which equals 52 weeks and 1 extra day. A leap year contains 366 days, which equals 52 weeks and 2 extra days. Every year is guaranteed to have at least 52 of each day of the week. The occurrence of a 53rd day depends entirely on those extra days. In a leap year, the two extra days are consecutive. To find the probability of having 53 Sundays, we identify the sample space of these two days and determine how many pairs include a Sunday.

**Solution:**

- (a) 366 days = 52 weeks + 2 days.
- (b) The 2 extra days can be any of the following 7 pairs: (Mon, Tue), (Tue, Wed), (Wed, Thu), (Thu, Fri), (Fri, Sat), (Sat, Sun), (Sun, Mon).
- (c) Total number of outcomes  $n(S) = 7$ .
- (d) For the year to have 53 Sundays, one of the extra days must be a Sunday.
- (e) Looking at the pairs, Sunday appears in: (Sat, Sun) and (Sun, Mon).
- (f) Number of favorable outcomes  $n(E) = 2$ .
- (g) Probability  $P(E) = n(E)/n(S) = 2/7$ .

**Final Answer:** The probability is  $2/7$ .

**Answer: (B)**

[Go Back to Question 58](#)



Q59.

**Solution****Concept:**

The "order" of a differential equation is the highest order derivative present in the equation. The "degree" is the power of that highest order derivative, provided the equation is written as a polynomial in terms of its derivatives (i.e., there are no fractional or negative powers affecting the derivatives). To find the degree of an equation with fractional exponents, we must first rationalize the equation by raising both sides to an appropriate power to eliminate the radicals.

**Solution:**

- (a) The equation is  $(1 + (y_1)^2)^{3/2} = y_2$ . The highest derivative is  $d^2y/dx^2$ , so the order is 2.
- (b) To find the degree, we must remove the fractional power  $3/2$ .
- (c) Square both sides:  $[(1 + (y_1)^2)^{3/2}]^2 = (y_2)^2$ .
- (d) This becomes  $(1 + (y_1)^2)^3 = (y_2)^2$ .
- (e) Now the equation is a polynomial in derivatives.
- (f) The power of the highest order derivative ( $y_2$ ) is 2.
- (g) Therefore, the order is 2 and the degree is 2.

**Final Answer:** The order and degree are 2, 2.

**Answer: (A)**

[Go Back to Question 59](#)



Q60.

**Solution****Concept:**

The area of a triangle with vertices  $(x_1, y_1)$ ,  $(x_2, y_2)$ , and  $(x_3, y_3)$  can be calculated using the determinant formula:  $\text{Area} = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$ . Alternatively, if the triangle is a right-angled triangle with its legs aligned along the coordinate axes, the area is simply half the product of the base and the height. Since area is a physical quantity, we always take the absolute value to ensure a non-negative result.

**Solution:**

- (a) Vertices are  $A(0, 0)$ ,  $B(a, 0)$ , and  $C(0, b)$ .
- (b) Point  $B$  lies on the x-axis (distance  $a$  from origin).
- (c) Point  $C$  lies on the y-axis (distance  $b$  from origin).
- (d) This forms a right-angled triangle with base length  $|a|$  and height  $|b|$ .
- (e)  $\text{Area} = \frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2} \cdot |a| \cdot |b| = \frac{1}{2}|ab|$ .
- (f) Using the determinant formula:  $\text{Area} = \frac{1}{2}|0(0 - b) + a(b - 0) + 0(0 - 0)| = \frac{1}{2}|ab|$ .

**Final Answer:** The area is  $\frac{1}{2}|ab|$ .

**Answer: (C)**

[Go Back to Question 60](#)



Q61.

**Solution**

**Concept:** For a function to be continuous at a point  $x = c$ , the left-hand limit (LHL), the right-hand limit (RHL), and the functional value  $f(c)$  must all exist and be equal. This problem requires evaluating limits of different functional forms as  $x$  approaches 0. The LHL involves a trigonometric limit that can be simplified using the identity  $1 - \cos \theta = 2 \sin^2(\theta/2)$ , while the RHL involves an algebraic expression that requires rationalization or the application of L'Hopital's Rule.

**Solution:**

- (a) Functional Value:  $f(0) = a$ .
- (b) LHL ( $x \rightarrow 0^-$ ):  $\lim_{x \rightarrow 0} \frac{1 - \cos 4x}{x^2} = \lim_{x \rightarrow 0} \frac{2 \sin^2 2x}{x^2}$ .
- (c) Multiply and divide by 4:  $\lim_{x \rightarrow 0} 2 \cdot 4 \cdot \left(\frac{\sin 2x}{2x}\right)^2 = 8(1)^2 = 8$ .
- (d) RHL ( $x \rightarrow 0^+$ ):  $\lim_{x \rightarrow 0} \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4}$ .
- (e) Rationalize the denominator:  $\lim_{x \rightarrow 0} \frac{\sqrt{x}(\sqrt{16 + \sqrt{x}} + 4)}{(16 + \sqrt{x}) - 16}$ .
- (f) Simplify:  $\lim_{x \rightarrow 0} \frac{\sqrt{x}(\sqrt{16 + \sqrt{x}} + 4)}{\sqrt{x}} = \lim_{x \rightarrow 0} (\sqrt{16 + \sqrt{x}} + 4)$ .
- (g) Substitute  $x = 0$ :  $\sqrt{16} + 4 = 4 + 4 = 8$ .
- (h) Since LHL = RHL =  $f(0)$  for continuity,  $a = 8$ .

**Final Answer:** The value of  $a$  is 8.

**Answer: (B)**

[Go Back to Question 61](#)



Q62.

**Solution**

**Concept:** The integration of a function involving exponential terms in the denominator often requires algebraic manipulation to reach a standard form. By multiplying the numerator and denominator by  $e^x$ , we can perform a substitution  $u = e^x$ . This transforms the integrand into a rational form resembling  $1/(u^2 + 1)$ , the integral of which is the inverse tangent function. This technique is a common strategy in calculus for handling symmetric exponential denominators.

**Solution:**

- (a) Let  $I = \int_0^1 \frac{1}{e^x + e^{-x}} dx$ .
- (b) Multiply numerator and denominator by  $e^x$ :  $I = \int_0^1 \frac{e^x}{e^{2x} + 1} dx$ .
- (c) Let  $u = e^x$ , then  $du = e^x dx$ .
- (d) Change the limits: when  $x = 0, u = 1$ ; when  $x = 1, u = e$ .
- (e) Substitute into the integral:  $I = \int_1^e \frac{1}{u^2 + 1} du$ .
- (f) Evaluate the integral:  $[\tan^{-1} u]_1^e$ .
- (g) Apply the limits:  $\tan^{-1} e - \tan^{-1} 1$ .
- (h) Since  $\tan^{-1} 1 = \pi/4$ , the final result is  $\tan^{-1} e - \pi/4$ .

**Final Answer:** The integral is  $\tan^{-1} e - \pi/4$ .

**Answer: (A)**

[Go Back to Question 62](#)



Q63.

**Solution**

**Concept:** A subset is a collection of elements where every member of the subset is also a member of the original set. For a set with  $n$  elements, every element has two distinct possibilities: it can either be included in a subset or excluded from it. By applying the fundamental principle of counting (multiplication rule), we multiply these two choices for each of the  $n$  elements. This derivation also relates to the sum of binomial coefficients, where  $\sum_{r=0}^n \binom{n}{r} = 2^n$ .

**Solution:**

- (a) Consider a set  $S = \{a_1, a_2, \dots, a_n\}$ .
- (b) To form any subset, we look at each element  $a_i$  individually.
- (c) Element  $a_1$  has 2 choices (In or Out).
- (d) Element  $a_2$  has 2 choices (In or Out).
- (e) This logic continues for all elements up to  $a_n$ .
- (f) Total number of subsets =  $2 \times 2 \times 2 \dots$  ( $n$  times).
- (g) This yields  $2^n$  total subsets.
- (h) This includes the empty set (where all elements are "Out") and the set itself (where all elements are "In").
- (i) If we were looking for "proper" subsets, we would subtract one (the set itself).

**Final Answer:** The number of subsets is  $2^n$ .

**Answer: (B)**

[Go Back to Question 63](#)



Q64.

**Solution**

**Concept:** Three vectors are coplanar if they lie in the same plane. Mathematically, this condition is satisfied when their scalar triple product is zero, meaning the volume of the parallelepiped formed by them is zero. The scalar triple product is most easily calculated as the determinant of a  $3 \times 3$  matrix where each row represents the components of one vector. If the determinant vanishes, the vectors are linearly dependent and thus coplanar.

**Solution:**

(a) Let  $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ ,  $\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}$ , and  $\vec{c} = 3\hat{i} + \lambda\hat{j} + 5\hat{k}$ .

(b) For coplanarity, 
$$\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & \lambda & 5 \end{vmatrix} = 0.$$

(c) Expand the determinant along the first row:

(d)  $2[2(5) - (-3)(\lambda)] - (-1)[1(5) - (-3)(3)] + 1[1(\lambda) - 2(3)] = 0.$

(e)  $2[10 + 3\lambda] + 1[5 + 9] + 1[\lambda - 6] = 0.$

(f)  $20 + 6\lambda + 14 + \lambda - 6 = 0.$

(g) Combine like terms:  $7\lambda + 28 = 0.$

(h) Solve for  $\lambda$ :  $7\lambda = -28 \implies \lambda = -4.$

**Final Answer:** The value of  $\lambda$  is  $-4$ .

**Answer: (A)**

[Go Back to Question 64](#)



Q65.

**Solution**

**Concept:** The minimum or maximum value of a quadratic expression  $ax^2 + bx + c$  can be found using several methods: completing the square, using the vertex formula  $x = -b/2a$ , or using differentiation. For a parabola that opens upwards ( $a > 0$ ), the vertex represents the absolute minimum. Differentiation is a powerful tool here as setting the first derivative to zero identifies the horizontal tangent at the vertex of the parabola.

**Solution:**

(a) Let  $f(x) = 2x^2 - 4x + 5$ .

(b) Method 1 (Differentiation): Find  $f'(x) = 4x - 4$ .

(c) Set  $f'(x) = 0$  to find the critical point:  $4x = 4 \implies x = 1$ .

(d) Since  $f''(x) = 4 > 0$ , the function has a minimum at  $x = 1$ .

(e) Substitute  $x = 1$  into  $f(x)$ :  $f(1) = 2(1)^2 - 4(1) + 5 = 2 - 4 + 5 = 3$ .

(f) Method 2 (Completing the Square):  $2(x^2 - 2x) + 5 = 2(x^2 - 2x + 1 - 1) + 5$ .

(g)  $2(x - 1)^2 - 2 + 5 = 2(x - 1)^2 + 3$ .

(h) Since  $2(x - 1)^2 \geq 0$ , the minimum value is  $0 + 3 = 3$ .

**Final Answer:** The minimum value is 3.

**Answer: (A)**

[Go Back to Question 65](#)



Q66.

**Solution****Concept:**

The reduction formula for integrals involving powers of  $x$  and exponential functions is derived using Integration by Parts. The behavior of sequences of integrals depends on the nature of the integrand within the limits  $[0, 1]$ , specifically how the term  $x^n$  behaves as  $n$  increases.

**Solution:**

Step 1: To find the relation between  $I_n$  and  $I_{n-1}$ , we use integration by parts on  $I_n = \int_0^1 x^n e^{-x} dx$ . Let  $u = x^n$  and  $dv = e^{-x} dx$ . Then  $du = nx^{n-1} dx$  and  $v = -e^{-x}$ .

Step 2: Applying the formula  $\int u dv = uv - \int v du$ , we get  $I_n = [-x^n e^{-x}]_0^1 - \int_0^1 (-e^{-x})(nx^{n-1}) dx$ . Evaluating the boundary terms gives  $I_n = -e^{-1} + n \int_0^1 x^{n-1} e^{-x} dx$ , which simplifies to  $I_n = nI_{n-1} - e^{-1}$ . Thus, (A) is correct.

Step 3: Analyze monotonicity. For  $x \in (0, 1)$ , we know that  $x^n < x^{n-1}$  for all  $n \in \mathbb{N}$ . Since  $e^{-x}$  is positive,  $x^n e^{-x} < x^{n-1} e^{-x}$ . Integrating both sides from 0 to 1 shows  $I_n < I_{n-1}$ , meaning the sequence is strictly decreasing. Thus, (B) is correct.

Step 4: Analyze the limit. For  $x \in [0, 1]$ ,  $e^{-x} \leq 1$ . Thus,  $0 < I_n < \int_0^1 x^n dx = \frac{1}{n+1}$ . By the Squeeze Theorem, as  $n \rightarrow \infty$ ,  $1/(n+1) \rightarrow 0$ , so  $\lim_{n \rightarrow \infty} I_n = 0$ . Thus, (C) is correct.

Step 5: Check (D). Since the limit is 0 and  $e^{-1} \approx 0.36$ ,  $I_n$  eventually becomes smaller than  $e^{-1}$ . Thus, (D) is incorrect.

**Final Answer:**

**Answer:**

[Go Back to Question 66](#)



Q67.

**Solution****Concept:**

The integral of a Step Function (Greatest Integer Function) is solved by partitioning the interval of integration at points where the expression inside the bracket takes integer values. The area under such a function is the sum of the areas of various rectangles.

**Solution:**

Step 1: Identify points of discontinuity for  $[x^2]$  in  $[0, 2]$ . The function changes value when  $x^2 = k$  (where  $k \in \mathbb{Z}$ ). In the interval  $[0, 2]$ ,  $x^2$  ranges from 0 to 4. The transition points are  $x = \sqrt{1} = 1$ ,  $x = \sqrt{2}$ , and  $x = \sqrt{3}$ . Thus, (A) is correct.

Step 2: Note that  $x^2$  is an increasing function for  $x \geq 0$ . Therefore,  $[x^2]$  is a non-decreasing function in  $[0, 2]$ . Thus, (C) is correct.

Step 3: Break the integral  $I = \int_0^2 [x^2] dx$  into sub-intervals based on the transition points:  
 $I = \int_0^1 [x^2] dx + \int_1^{\sqrt{2}} [x^2] dx + \int_{\sqrt{2}}^{\sqrt{3}} [x^2] dx + \int_{\sqrt{3}}^2 [x^2] dx$ .

Step 4: Evaluate each part. For  $0 \leq x < 1$ ,  $[x^2] = 0$ . For  $1 \leq x < \sqrt{2}$ ,  $[x^2] = 1$ . For  $\sqrt{2} \leq x < \sqrt{3}$ ,  $[x^2] = 2$ . For  $\sqrt{3} \leq x < 2$ ,  $[x^2] = 3$ .  $I = 0(1 - 0) + 1(\sqrt{2} - 1) + 2(\sqrt{3} - \sqrt{2}) + 3(2 - \sqrt{3})$ .

Step 5: Simplify the expression:  $I = \sqrt{2} - 1 + 2\sqrt{3} - 2\sqrt{2} + 6 - 3\sqrt{3}$ .  $I = 5 - \sqrt{2} - \sqrt{3}$ . Thus, (B) is correct and (D) is incorrect.

**Final Answer:**

**Answer:**

[Go Back to Question 67](#)



Q68.

**Solution****Concept:**

The Vector Triple Product identity is defined as  $(\vec{u} \times \vec{v}) \times \vec{w} = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{v} \cdot \vec{w})\vec{u}$ . Equality between different associations of the cross product indicates specific geometric alignments of the constituent vectors.

**Solution:**

Step 1: Expand the left hand side (LHS) using the identity:  $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$ .

Step 2: Expand the right hand side (RHS) using the identity:  $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$ .

Step 3: Set LHS equal to RHS as per the problem statement:  $(\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$ .

Step 4: The term  $(\vec{a} \cdot \vec{c})\vec{b}$  cancels from both sides, leaving:  $-(\vec{b} \cdot \vec{c})\vec{a} = -(\vec{a} \cdot \vec{b})\vec{c}$ , which simplifies to  $(\vec{b} \cdot \vec{c})\vec{a} = (\vec{a} \cdot \vec{b})\vec{c}$ .

Step 5: This equation shows that a scalar multiple of  $\vec{a}$  is equal to a scalar multiple of  $\vec{c}$ . For non-zero vectors, this implies that  $\vec{a}$  and  $\vec{c}$  must be collinear. Thus, (A) is correct. Since no two vectors are collinear among  $\vec{a}, \vec{b}, \vec{c}$  was given as a general constraint, this specifically forces the dot products  $(\vec{b} \cdot \vec{c})$  and  $(\vec{a} \cdot \vec{b})$  to be zero to satisfy the equation if  $\vec{a}$  and  $\vec{c}$  aren't collinear. However, the identity forces collinearity to satisfy the equality generally. Options (B), (C), and (D) are not necessary conditions.

**Final Answer:**

**Answer:** (A)

[Go Back to Question 68](#)



Q69.

**Solution****Concept:**

Skew lines are lines in three-dimensional space that are neither parallel nor intersecting. They lie in parallel planes rather than a single common plane. The geometry of skew lines involves unique properties regarding the distance and perpendicularity between them.

**Solution:**

Step 1: Consider perpendicularity. For any two skew lines, there exists exactly one line that intersects both and is perpendicular to both. This line represents the shortest distance path. Thus, (A) is correct.

Step 2: Analyze the Shortest Distance (SD). By definition, skew lines do not intersect. If the distance between two lines were zero, they would intersect or be coincident, making them coplanar. Thus, statement (B) is technically true as a conditional "if they intersect," but skew lines by definition do not. However, in the context of general lines, it's a valid property.

Step 3: Examine the formula for SD. Let the lines be  $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$  and  $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ . The vector joining two points is  $(\vec{a}_2 - \vec{a}_1)$ . The common perpendicular is in the direction of  $(\vec{b}_1 \times \vec{b}_2)$ . The SD is the projection of  $(\vec{a}_2 - \vec{a}_1)$  onto  $(\vec{b}_1 \times \vec{b}_2)$ . Thus, (C) is correct.

Step 4: Check (D). Skew lines are non-coplanar by definition. If they were coplanar, they would either be parallel or intersect. Thus, (D) is incorrect.

Step 5: Consequently, the properties regarding uniqueness of the perpendicular and the projection formula for distance are correct.

**Final Answer:**

**Answer:**

[Go Back to Question 69](#)



Q70.

**Solution****Concept:**

A skew-symmetric matrix  $A$  satisfies  $A^T = -A$ . These matrices have distinct properties regarding their determinants (especially in odd dimensions) and their diagonal entries.

**Solution:**

Step 1: Examine diagonal elements. For  $A$  to be skew-symmetric,  $a_{ij} = -a_{ji}$ . For diagonal elements,  $i = j$ , so  $a_{ii} = -a_{ii} \implies 2a_{ii} = 0 \implies a_{ii} = 0$ . All diagonal elements must be zero. Thus, (D) is incorrect.

Step 2: Analyze the determinant. For any  $n \times n$  matrix,  $\det(A^T) = \det(A)$ . For skew-symmetric  $A$ ,  $\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A)$ . For  $n = 3$  (odd),  $\det(A) = -\det(A)$ , so  $2\det(A) = 0 \implies \det(A) = 0$ . Thus, (A) is correct.

Step 3: Consider  $I + A$ . The eigenvalues of a skew-symmetric matrix are either zero or purely imaginary. Therefore,  $I + A$  has eigenvalues of the form  $1 + i\beta$ . None of these can be zero. Since no eigenvalue is zero,  $\det(I + A) \neq 0$ , meaning it is non-singular. Thus, (B) is correct.

Step 4: Check  $A^2$ .  $(A^2)^T = (A^T)^2 = (-A)^2 = A^2$ . Since  $(A^2)^T = A^2$ ,  $A^2$  is a symmetric matrix. Thus, (C) is correct.

Step 5: Final summary of correct properties: Determinant is zero,  $I + A$  is invertible, and the square is symmetric.

**Final Answer:** A, B, C

**Answer:** (A,B,C)

[Go Back to Question 70](#)



Q71.

**Solution****Concept:**

The adjoint (or adjugate) of a matrix is the transpose of its cofactor matrix. It satisfies the fundamental relation  $A \cdot \text{adj}(A) = |A|I$ . Several higher-order properties can be derived from this relation for non-singular matrices.

**Solution:**

Step 1: Check  $\text{adj}(\text{adj}A)$ . From the property  $M \cdot \text{adj}(M) = |M|I$ , substituting  $M = \text{adj}A$  and using  $|\text{adj}A| = |A|^{n-1}$ , we derive  $\text{adj}(\text{adj}A) = |A|^{n-2}A$ . Thus, (A) is correct.

Step 2: Check  $|\text{adj}A|$ . Taking the determinant of both sides of  $A \cdot \text{adj}A = |A|I$  gives  $|A| \cdot |\text{adj}A| = ||A|I| = |A|^n$ . Dividing by  $|A|$  (since  $A$  is non-singular), we get  $|\text{adj}A| = |A|^{n-1}$ . Thus, (B) is correct.

Step 3: Check  $\text{adj}(AB)$ . Since  $(AB)^{-1} = B^{-1}A^{-1}$  and  $M^{-1} = \frac{1}{|M|}\text{adj}M$ , we can show that  $\text{adj}(AB) = (\text{adj}B)(\text{adj}A)$ . This follows the reverse order law similar to inverses and transposes. Thus, (C) is correct.

Step 4: Check  $\text{adj}(A^T)$ . The cofactor of an element in  $A^T$  is the same as the cofactor of the corresponding element in  $A$  after transposing. Thus  $\text{adj}(A^T) = (\text{adj}A)^T$ . Thus, (D) is correct.

Step 5: All given options are standard and correct properties of the adjoint matrix.

**Final Answer:** A, B, C, D

**Answer:** (A,B,C,D)

[Go Back to Question 71](#)



Q72.

**Solution****Concept:**

The  $n^{\text{th}}$  roots of unity are solutions to  $z^n - 1 = 0$ . These roots are  $e^{i(2\pi k/n)}$  for  $k = 0, 1, \dots, n-1$ . They possess symmetric properties on the complex plane (Argand diagram) and satisfy specific polynomial identities.

**Solution:**

Step 1: Polynomial factorization.  $z^n - 1 = (z - 1)(z - \alpha_1)(z - \alpha_2) \dots (z - \alpha_{n-1})$ . Dividing by  $(z - 1)$ , we get  $z^{n-1} + z^{n-2} + \dots + 1 = (z - \alpha_1)(z - \alpha_2) \dots (z - \alpha_{n-1})$ . Substituting  $z = 1$  gives  $1 + 1 + \dots + 1 = n = (1 - \alpha_1)(1 - \alpha_2) \dots (1 - \alpha_{n-1})$ . Thus, (A) is correct.

Step 2: Sum of roots. For the equation  $z^n - 1 = 0$ , the coefficient of  $z^{n-1}$  is 0. Thus, the sum of all roots  $1 + \alpha_1 + \dots + \alpha_{n-1} = 0$ . This implies  $\sum_{r=1}^{n-1} \alpha_r = -1$ . Thus, (B) is correct.

Step 3: Geometry. Each root  $\alpha_k = \cos(2\pi k/n) + i \sin(2\pi k/n)$ . The modulus  $|\alpha_k| = 1$ . This means all roots lie on a circle centered at the origin with radius 1. Thus, (C) is correct.

Step 4: Sequence properties. Let  $\alpha_1 = \omega = e^{i(2\pi/n)}$ . Then the roots are  $1, \omega, \omega^2, \dots, \omega^{n-1}$ . This is a GP with common ratio  $\omega = \alpha_1$ . Thus, (D) is correct.

**Final Answer:**

**Answer:**

[Go Back to Question 72](#)



Q73.

**Solution****Concept:**

Conditional probability  $P(A|B) = P(A \cap B)/P(B)$  measures the probability of  $A$  given that  $B$  has occurred. The inequality  $P(A|B) > P(A)$  implies a positive correlation between  $A$  and  $B$ , meaning the occurrence of one increases the likelihood of the other.

**Solution:**

Step 1: Analyze  $P(A \cap B)$ . From  $P(A|B) > P(A)$ , we have  $\frac{P(A \cap B)}{P(B)} > P(A)$ . Since  $P(B) > 0$ , this leads to  $P(A \cap B) > P(A)P(B)$ . Thus, (B) is correct.

Step 2: Check symmetry. If  $P(A \cap B) > P(A)P(B)$ , then  $\frac{P(A \cap B)}{P(A)} > P(B)$ , which means  $P(B|A) > P(B)$ . Thus, (A) is correct.

Step 3: Analyze the complement  $B^c$ . We know  $P(A) = P(A \cap B) + P(A \cap B^c)$ .

$$P(A) > P(A)P(B) + P(A \cap B^c) \implies P(A)(1 - P(B)) > P(A \cap B^c).$$

$$P(A)P(B^c) > P(A \cap B^c) \implies P(A) > \frac{P(A \cap B^c)}{P(B^c)}.$$

Therefore,  $P(A|B^c) < P(A)$ . Thus, (C) is correct.

Step 4: Check independence. If  $P(A|B) > P(A)$ , the events are specifically not independent, as independence requires  $P(A|B) = P(A)$ . Thus, (D) is incorrect.

**Final Answer:**

**Answer:**

[Go Back to Question 73](#)



Q74.

**Solution****Concept:**

The function  $f(x) = \max\{g_1(x), g_2(x), g_3(x)\}$  takes the value of the largest of the three functions at any point  $x$ . The continuity and differentiability of such a function are determined by the behavior of the constituent functions at their intersection points.

**Solution:**

Step 1: Evaluate the functions. Let  $y_1 = 1 - x$ ,  $y_2 = 1 + x$ , and  $y_3 = 2$ . Intersection of  $y_1$  and  $y_3$ :  $1 - x = 2 \implies x = -1$ . Intersection of  $y_2$  and  $y_3$ :  $1 + x = 2 \implies x = 1$ . Intersection of  $y_1$  and  $y_2$ :  $1 - x = 1 + x \implies x = 0$ .

Step 2: Define  $f(x)$  piecewise. For  $x \leq -1$ ,  $1 - x \geq 2$ , so  $f(x) = 1 - x$ . For  $-1 < x < 1$ , both  $1 - x < 2$  and  $1 + x < 2$ , so  $f(x) = 2$ . For  $x \geq 1$ ,  $1 + x \geq 2$ , so  $f(x) = 1 + x$ .

Step 3: Check continuity. The pieces  $1 - x$ ,  $2$ , and  $1 + x$  are polynomials and hence continuous. At  $x = -1$ ,  $f(-1) = 2$  and limits are  $2$ . At  $x = 1$ ,  $f(1) = 2$  and limits are  $2$ .  $f(x)$  is continuous for all  $x$ . Thus, (A) is correct.

Step 4: Check differentiability. At  $x = -1$ , the slope changes from  $-1$  to  $0$ . At  $x = 1$ , the slope changes from  $0$  to  $1$ . These "sharp corners" mean  $f(x)$  is not differentiable at  $x = \pm 1$ . Thus, (B) is correct.

Step 5: Check min and derivative. In  $(-1, 1)$ ,  $f(x) = 2$  (constant), so  $f'(x) = 0$ . Thus, (D) is correct. The graph goes down to  $2$ , stays at  $2$ , then goes up. The global minimum is  $2$ . Thus, (C) is correct.

**Final Answer:**

**Answer:**

[Go Back to Question 74](#)



Q75.

**Solution****Concept:**

The relationship between two planes is analyzed using their normal vectors  $\vec{n}_1$  and  $\vec{n}_2$ . The angle  $\theta$  between planes is the angle between their normals, given by  $\cos \theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1| |\vec{n}_2|}$ . The direction of the line of intersection is  $\vec{n}_1 \times \vec{n}_2$ .

**Solution:**

Step 1: Identify normals.  $\vec{n}_1 = (2, -1, 1)$  and  $\vec{n}_2 = (1, 1, 2)$ . Dot product:  $\vec{n}_1 \cdot \vec{n}_2 = 2(1) + (-1)(1) + 1(2) = 2 - 1 + 2 = 3$ . Since this is not zero, the planes are not perpendicular. Thus, (A) is incorrect.

Step 2: Calculate magnitudes:  $|\vec{n}_1| = \sqrt{4+1+1} = \sqrt{6}$  and  $|\vec{n}_2| = \sqrt{1+1+4} = \sqrt{6}$ .  $\cos \theta = \frac{3}{\sqrt{6}\sqrt{6}} = \frac{3}{6} = 1/2$ . Thus,  $\theta = \pi/3$ . Thus, (C) is correct.

Step 3: Find direction ratios of the intersection line.  $\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \hat{i}(-2-1) - \hat{j}(4-1) + \hat{k}(2+1) = (-3, -3, 3)$ . Thus, (B) is correct.

Step 4: Check the point  $(2, -1, 1)$ . Plane 1:  $2(2) - (-1) + 1 = 4 + 1 + 1 = 6$ . (Satisfied) Plane 2:  $2 + (-1) + 2(1) = 2 - 1 + 2 = 3$ . (Satisfied) The point lies on both planes. Thus, (D) is correct.

**Final Answer:** B, C, D

**Answer:** (B,C,D)

[Go Back to Question 75](#)



## Answer Key

Q	Ans	Q	Ans	Q	Ans	Q	Ans	Q	Ans
1	A	2	C	3	B	4	B	5	A
6	B	7	A	8	A	9	B	10	A
11	A	12	D	13	B	14	B	15	A
16	B	17	A	18	D	19	B	20	A
21	B	22	C	23	B	24	A	25	B
26	B	27	C	28	B	29	B	30	A
31	B	32	A	33	B	34	A	35	A
36	B	37	B	38	A	39	B	40	A
41	A	42	B	43	A	44	B	45	B
46	A	47	A	48	B	49	C	50	B
51	A	52	A	53	A	54	B	55	A
56	A	57	A	58	B	59	A	60	C
61	B	62	A	63	B	64	A	65	A
66	A,B,C	67	A,B,C	68	A	69	A,B,C	70	A,B,C
71	A,B,C,D	72	A,B,C,D	73	A,B,C	74	A,B,C,D	75	B,C,D

