NCERT Exemplar Solutions

Solved NCERT Exemplar Problems for Class 12 Mathematics, Chapter 12 — Representative Set

Chapter 12: Linear Programming

About this Chapter

Linear Programming is the technique of optimising a linear **objective function** $Z = ax + by$ subject to a system of linear **constraints** $a_i x + b_i y \leq (\text{or } \geq)\, c_i$, with $x \geq 0,\ y \geq 0$. The Exemplar problems drill the four classical question types you will face in CBSE board, JEE and CUET: identifying the **feasible region**, computing Z at **corner points**, comparing values to read off maximum/minimum, and dealing with the special cases of **unbounded feasible regions** and **multiple optimal solutions**.

Topics covered: Mathematical formulation of an LPP • Objective function • Linear constraints • Feasible region • Bounded vs unbounded • Corner-point method • Manufacturing problems • Diet problems • Transportation problems • Multiple optima

Quick Formula Sheet

General LPP:
Maximise / Minimise $Z = ax + by$
s.t. $a_i x + b_i y \leq c_i,\ x, y \geq 0$

Corner-point theorem:
Optimum occurs at a vertex of the feasible region

Bounded region:
both max and min of Z exist at corner points

Unbounded region:
M is max only if open half-plane $ax+by > M$ has no point in common with the region

Multiple optima:
Z equal at two adjacent corners $\Rightarrow$
every point on that edge is optimal

I. Multiple Choice Questions (MCQ)

Q 12.1 **The corner points of the feasible region determined by the system of linear constraints are $(0, 10),\ (5, 5),\ (15, 15),\ (0, 20)$. Let $Z = px + qy$ where $p, q > 0$. Condition on p and q so that the maximum of Z occurs at both the points $(15, 15)$ and $(0, 20)$ is**

(A) $p = q$ **(B)** $p = 2q$ **(C)** $q = 2p$ **(D)** $q = 3p$

SOLUTION

Correct option: (D) $q = 3p$.

Concept used. The value of the objective function $Z = px + qy$ at two corner points is the same precisely when the line $px + qy =$ const is parallel to (or passes through) the segment joining those corner points. If Z attains its maximum simultaneously at two corners (x_1, y_1) and (x_2, y_2), then $Z(x_1, y_1) = Z(x_2, y_2)$, i.e. $p\,x_1 + q\,y_1 = p\,x_2 + q\,y_2$.

Step 1. Write Z at the two given corner points:

$$Z(15, 15) = 15p + 15q, \qquad Z(0, 20) = 0 \cdot p + 20q = 20q.$$

Step 2. Set them equal (both are required to be the maximum, so they must produce the same value of Z):

$$15p + 15q = 20q.$$

Step 3. Solve for the ratio of p and q:

$$15p = 20q - 15q = 5q \;\Rightarrow\; 3p = q.$$

Step 4. Therefore $q = 3p$, with both $p, q > 0$ as given.

> **Final Answer:** $q = 3p$; option **(D)**.

☞ Equal-Z at two corners $\Rightarrow$ multiple optima

Whenever the same maximum (or minimum) value of Z appears at two adjacent corner points, every point on the segment joining them is also optimal — an LPP with infinitely many optimal solutions.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Equation-of-the-line angle. The objective $Z = px + qy$ is constant on lines parallel to $px + qy = 0$. “Max at both $(15, 15)$ and $(0, 20)$” means the maximum-Z line contains both points. So the slope of Z equals the slope of the segment joining $(15, 15)$ to $(0, 20)$.

Concept used. Slope of the level curve $Z = px + qy = c$ is $-p/q$. Slope of segment from $(15, 15)$ to $(0, 20)$ is $(20 - 15)/(0 - 15) = 5/(-15) = -1/3$.

Step 1. Equate slopes:

$$-\frac{p}{q} = -\frac{1}{3} \;\Rightarrow\; \frac{p}{q} = \frac{1}{3}.$$

Step 2. Therefore $q = 3p$.

Step 3. Plug-back check: at $(15, 15)$, $Z = 15p + 15q = 15p + 15(3p) = 60p$; at $(0, 20)$, $Z = 20q = 20(3p) = 60p$. Equal. ✓

Final Answer: $q = 3p$; option **(D)**.

Q 12.2 **Feasible region (shaded) for a LPP is shown in the figure. The maximum value of $Z = 11x + 7y$ subject to the corner points $(0, 3),\ (3, 2),\ (0, 5)$ is**
(A) 31 **(B)** 33 **(C)** 47 **(D)** 35

SOLUTION

Correct option: (C) 47.
Concept used. The corner-point theorem for an LPP with a bounded feasible region states: if the feasible region is bounded and $Z = ax + by$ is the objective function, then both the maximum and the minimum of Z over the feasible region are attained at a corner point of the region.

Step 1. Evaluate $Z = 11x + 7y$ at each corner point. At $(0, 3)$:

$$Z = 11(0) + 7(3) = 0 + 21 = 21.$$

Step 2. At $(3, 2)$:

$$Z = 11(3) + 7(2) = 33 + 14 = 47.$$

Step 3. At $(0, 5)$:

$$Z = 11(0) + 7(5) = 0 + 35 = 35.$$

Step 4. Compare values: $\{21,\ 47,\ 35\}$. The largest is 47, attained at $(3, 2)$.

Final Answer: Maximum $Z = 47$ at $(3, 2)$; option **(C)**.

☞ Always tabulate Z values

Make a small table with two columns: corner (x, y) and $Z(x, y)$. Then read off the largest (or smallest) row. This avoids the mistake of "eyeballing" which corner looks farthest.

EXPERT'S SOLUTION *: Priya Iyer, Ph.D Mathematics, IISc Bangalore*

Tabulation angle. A small table is faster than checking inequalities or drawing the Z-line.
Concept used. For bounded regions, the corner-point method is mechanical: compute Z at each vertex and pick the extremum.

Step 1. Build the table:

Corner	$Z = 11x + 7y$
$(0, 3)$	21
$(3, 2)$	47
$(0, 5)$	35

Step 2. Maximum $= 47$, attained at $(3, 2)$.

Final Answer: $Z_{\max} = 47$ at $(3, 2)$.

Q 12.3 **Corner points of the feasible region for an LPP are $(0, 2)$, $(3, 0)$, $(6, 0)$, $(6, 8)$ and $(0, 5)$. Let $F = 4x + 6y$ be the objective function. The minimum value of F occurs at**
(A) $(0, 2)$ only (B) $(3, 0)$ only (C) the mid-point of the line segment joining the points $(0, 2)$ and $(3, 0)$ only (D) any point on the line segment joining $(0, 2)$ and $(3, 0)$

SOLUTION

Correct option: (D) any point on the line segment joining the points $(0, 2)$ and $(3, 0)$.
Concept used. When the objective function $Z = ax + by$ attains the same optimum value at two adjacent corner points, then every point on the edge of the feasible polygon joining those two corners is also optimum. This is the **multiple optimal solutions** case — it occurs when the level curve $Z =$ const is parallel to that edge.

Step 1. Compute $F = 4x + 6y$ at each corner:

$$\begin{aligned} F(0, 2) &= 4(0) + 6(2) = 0 + 12 = 12, \\ F(3, 0) &= 4(3) + 6(0) = 12 + 0 = 12, \\ F(6, 0) &= 4(6) + 6(0) = 24, \\ F(6, 8) &= 4(6) + 6(8) = 24 + 48 = 72, \\ F(0, 5) &= 4(0) + 6(5) = 30. \end{aligned}$$

Step 2. Minimum value is 12, attained at *both* $(0, 2)$ and $(3, 0)$.

Step 3. Since F is linear and equal at the two endpoints, F takes the same value 12 on every point of the segment joining $(0, 2)$ and $(3, 0)$ as well.

Step 4. Hence the minimum $F = 12$ is attained on the entire edge between $(0, 2)$ and $(3, 0)$.

Final Answer: Minimum $F = 12$ attained on the whole segment $(0, 2)$ to $(3, 0)$; option **(D)**.

☞ **Why this happens**

The objective $F = 4x + 6y$ has slope $-4/6 = -2/3$. The segment from $(0, 2)$ to $(3, 0)$ has slope $(0 - 2)/(3 - 0) = -2/3$ also. Same slope — the level curve and the edge coincide, so every interior point of the edge is also optimal.

EXPERT'S SOLUTION *: Vikram Rao, M.Sc Mathematics, Delhi University*

Slope-equal angle. Spot it in one line: $\dfrac{-4}{6} = \dfrac{0-2}{3-0} = -\dfrac{2}{3}$. The level line $F = 12$ is the very edge. Infinitely many minima.

Final Answer: Option **(D)**.

Q 12.4 **In an LPP, if the objective function $Z = ax + by$ has the same maximum value on two corner points of the feasible region, then the number of points at which $Z_{\max}$ occurs is**

(A) 0 **(B)** 2 **(C) finite** **(D) infinite**

SOLUTION

Correct option: (D) infinite.

Concept used. If a linear function attains the same value at two different points P, Q, then by linearity it attains the same value at every point on the line segment PQ. In an LPP, when two corner points give the same Z, the entire edge connecting them yields the same Z, so there are infinitely many optimal points.

Step 1. Let $Z(P) = Z(Q) = M$ for two adjacent corner points P and Q.

Step 2. For any point $R = \lambda P + (1 - \lambda)Q$ on the segment ($\lambda \in [0, 1]$), linearity gives

$$Z(R) = \lambda Z(P) + (1 - \lambda)Z(Q) = \lambda M + (1 - \lambda)M = M.$$

Step 3. So every $R \in \overline{PQ}$ has $Z(R) = M$.

Step 4. The segment is uncountable, so infinitely many points attain $Z_{\max}$.

Final Answer: Infinitely many; option **(D)**.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Convex-combination angle. Linearity of Z on a convex set: if two points give the same value, every convex combination does too.

Step 1. For any $\lambda \in [0,1]$, the convex combination $\lambda P + (1-\lambda)Q$ lies on segment PQ, hence in the feasible region.

Step 2. Z being linear: $Z(\lambda P + (1-\lambda)Q) = \lambda Z(P) + (1-\lambda)Z(Q) = M$.

Step 3. So $Z_{\text{max}} = M$ occurs at every such point — infinitely many.

Final Answer: Option **(D)**.

Q 12.5 **The corner points of the feasible region determined by the system of linear inequalities are $(0,0)$, $(4,0)$, $(2,4)$ and $(0,5)$. If the maximum value of $Z = ax + by$, where $a, b > 0$ occurs at both $(2,4)$ and $(4,0)$, then**

(A) $a = 2b$ **(B)** $2a = b$ **(C)** $a = b$ **(D)** $3a = b$

SOLUTION

Correct option: (A) $a = 2b$.

Concept used. The maximum of $Z = ax + by$ at two corners means both corners satisfy $Z = Z_{\text{max}}$. Setting $Z(2,4) = Z(4,0)$ gives a single equation relating a and b.

Step 1. Compute Z at the two corners:

$$Z(2,4) = a(2) + b(4) = 2a + 4b, \qquad Z(4,0) = a(4) + b(0) = 4a.$$

Step 2. Equate (both are the maximum):

$$2a + 4b = 4a.$$

Step 3. Solve:

$$4b = 4a - 2a = 2a \;\Rightarrow\; 4b = 2a \;\Rightarrow\; 2b = a.$$

Step 4. Rewrite as $a = 2b$, which matches option (A).

Final Answer: $a = 2b$; option **(A)**.

☞ Equating Z at twin corners

Whenever the question says "maximum occurs at two corners (x_1, y_1) and (x_2, y_2)", the magic step is $Z(x_1, y_1) = Z(x_2, y_2)$, which gives a single linear equation in the unknowns

a, b.

EXPERT'S SOLUTION : *Priya Iyer, Ph.D Mathematics, IISc Bangalore*

Slope-match angle. The level line $ax + by = Z_{\max}$ passes through $(2, 4)$ and $(4, 0)$, so its slope is $-a/b$. From the two points, slope $= (0 - 4)/(4 - 2) = -2$. Equate:

$$-\frac{a}{b} = -2 \Rightarrow a = 2b.$$

Final Answer: $a = 2b$; option **(A)**.

Q 12.6 **Corner points of the feasible region of a LPP are $(0, 2)$, $(3, 0)$, $(6, 0)$, $(6, 8)$ and $(0, 5)$. Let $F = 4x + 6y$ be the objective function. (Maximum of F) − (Minimum of F) is**

(A) 60 **(B)** 48 **(C)** 42 **(D)** 18

SOLUTION

Correct option: (A) 60.

Concept used. For a *bounded* feasible region, both $F_{\max}$ and $F_{\min}$ are attained at corner points (by the corner-point theorem). Compute F at each corner; difference is $F_{\max} - F_{\min}$.

Step 1. Tabulate $F = 4x + 6y$ at each given corner:

Corner	$F = 4x + 6y$
$(0, 2)$	$0 + 12 = 12$
$(3, 0)$	$12 + 0 = 12$
$(6, 0)$	$24 + 0 = 24$
$(6, 8)$	$24 + 48 = 72$
$(0, 5)$	$0 + 30 = 30$

Step 2. Read off $F_{\max} = 72$ at $(6, 8)$ and $F_{\min} = 12$ at both $(0, 2)$ and $(3, 0)$.

Step 3. Difference: $F_{\max} - F_{\min} = 72 - 12 = 60$.

Final Answer: $F_{\max} - F_{\min} = 60$; option **(A)**.

EXPERT'S SOLUTION : *Vikram Rao, M.Sc Mathematics, Delhi University*

Direct evaluation. For 5 small corners, the corner-point method takes well under a minute — no need to look at slopes or geometry.

Step 1. Max is clearly at the "upper-right" corner $(6, 8)$: $F = 72$.

Step 2. Min is at the corners with smallest weighted sum: both $(0, 2)$ and $(3, 0)$ give 12.

Step 3. Difference $= 72 - 12 = 60$.

> **Final Answer:** 60; option **(A)**.

Q 12.7 **In a LPP, the objective function is always**

(A) linear **(B) quadratic** **(C) cubic** **(D) exponential**

SOLUTION

Correct option: (A) linear.

Concept used. By the very name *Linear* Programming, every defining ingredient must be linear: the **objective function** $Z = ax + by$ (or $\sum a_i x_i$ in higher dimensions) and every **constraint** $a_i x + b_i y \leq c_i$ (or $\geq$, or $=$). Non-linear objectives belong to *non-linear* optimisation (quadratic, convex, etc.), not LPP.

Step 1. Definition of an LPP (Chapter 12, Section 12.1): a problem in which a *linear* function (objective) is maximised or minimised subject to *linear* inequalities/equalities (constraints), and non-negativity restrictions.

Step 2. "Linear" means each variable appears to the first power; no products xy, no x^2, no $\sin x$, no e^x.

Step 3. Hence the objective must be linear, option (A).

> **Final Answer:** Linear; option **(A)**.

☞ The 4 "L" rules

LPP $=$ (i) Linear objective, (ii) Linear constraints, (iii) variables appear to power 1 only (Linear), (iv) feasible region is a Linear (polygonal/polyhedral) set.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Eliminate-by-name angle. The word "Linear" in "Linear Programming" nails it — everything in the formulation, including the objective, is linear in the decision variables.

> **Final Answer:** Option **(A)**.

Q 12.8 **A constraint in an LPP is modelled as**
(A) a linear inequality **(B) a quadratic inequality**
(C) an equation **(D) both (A) and (C)**

SOLUTION

Correct option: (D) both (A) and (C).
Concept used. **Constraints** of an LPP are restrictions that the decision variables must satisfy. They can be of two algebraic shapes: linear *inequalities* ($\leq$ or $\geq$) and linear *equations* ($=$). Strict inequalities ($<$, $>$) are not allowed because optima are then not attained.

Step 1. Resource constraints are usually of the form "usage $\leq$ availability", a linear inequality $\leq$. Example: $2x + 3y \leq 12$ (hours of labour).

Step 2. Demand/minimum constraints are usually "output $\geq$ minimum requirement", a linear inequality $\geq$. Example: $x + y \geq 5$ (minimum batch).

Step 3. Balance/budget constraints can be *exact*, hence linear equations. Example: $x + y = 100$ (sell all 100 units).

Step 4. Hence both (A) inequality *and* (C) equation are valid constraint forms in LPP.

Final Answer: Both linear inequality and linear equation; option **(D)**.

EXPERT'S SOLUTION *: Priya Iyer, Ph.D Mathematics, IISc Bangalore*

Classification. Standard LPP textbook taxonomy:

Step 1. Inequality constraints — by far the most common (resources, demands).

Step 2. Equality constraints — balance/budget conditions, or slack-variable form.

Step 3. Non-negativity $x, y \geq 0$ is itself a linear inequality.

Final Answer: Option **(D)**.

Q 12.9 **The feasible region for an LPP is always a**
(A) concave polygon **(B) convex polygon**
(C) bounded triangle **(D) unbounded region**

SOLUTION

Correct option: (B) convex polygon.

Concept used. A set $S \subseteq \mathbb{R}^2$ is **convex** if for any two points $P, Q \in S$, the entire segment $\overline{PQ}$ also lies in S. Each linear inequality $ax + by \leq c$ defines a closed half-plane, which is convex. The intersection of any collection of convex sets is convex. Therefore the feasible region (an intersection of half-planes) is convex; in two variables it is a convex polygonal region (bounded or unbounded).

Step 1. Each linear inequality $a_i x + b_i y \leq c_i$ defines a half-plane.

Step 2. Each half-plane is a convex set (take any two points in it, the segment between them is also in the half-plane by linearity).

Step 3. Feasible region = intersection of all half-planes (one per constraint, plus $x \geq 0,\ y \geq 0$).

Step 4. Intersection of convex sets is convex.

Step 5. So the feasible region is a convex polygonal region.

> **Final Answer:** Convex polygon; option **(B)**.

☞ Why convexity matters

Convexity guarantees the corner-point theorem. On a convex polygonal region, a linear function achieves its extremum at a vertex; this would not be true on a concave region.

EXPERT'S SOLUTION *: Vikram Rao, M.Sc Mathematics, Delhi University*

Half-plane angle. Every constraint cuts the plane into two parts; you keep one. After all cuts, what remains is convex (intersect of convex sets).

> **Final Answer:** Option **(B)**.

Q 12.10 **Of the LPP: Maximise $Z = 4x+y$ subject to $x+y \leq 50,\ 3x+y \leq 90,\ x \geq 0,\ y \geq 0$. The maximum value of Z is**

(A) 110 **(B)** 120 **(C)** 130 **(D)** 140

SOLUTION

Correct option: (B) 120.

Concept used. For a bounded LPP, solve by the corner-point method: (i) plot the constraint lines and identify the feasible region; (ii) find each vertex (corner) by intersecting pairs of binding constraint lines and the axes; (iii) evaluate Z at every corner; (iv) pick the largest value for a maximisation problem.

Step 1. Re-write the constraint lines:

$$\ell_1: \; x + y = 50, \qquad \ell_2: \; 3x + y = 90.$$

Step 2. x- and y-intercepts: ℓ_1 cuts the axes at $(50, 0)$ and $(0, 50)$; ℓ_2 cuts at $(30, 0)$ and $(0, 90)$.

Step 3. Intersection of ℓ_1 and ℓ_2:

$$x + y = 50, \quad 3x + y = 90 \;\Rightarrow\; \text{subtract}: \; 2x = 40 \;\Rightarrow\; x = 20, \; y = 30.$$

Step 4. The feasible region is bounded by $x \geq 0$, $y \geq 0$, and both lines (with $\leq$). Corners are:

$$O(0,0), \; A(30,0), \; B(20,30), \; C(0,50).$$

(We use $A = (30, 0)$ not $(50, 0)$ because the binding constraint on the x-axis is $3x + y \leq 90$.)

Step 5. Evaluate $Z = 4x + y$:

$$\begin{aligned} Z(O) &= 0, \\ Z(A) &= 4(30) + 0 = 120, \\ Z(B) &= 4(20) + 30 = 80 + 30 = 110, \\ Z(C) &= 0 + 50 = 50. \end{aligned}$$

Step 6. Maximum value is 120 at $(30, 0)$.

Final Answer: $Z_{\max} = 120$ at $(30, 0)$; option **(B)**.

☞ Which axis-intercept to use

For each axis-binding constraint, take the *smaller* intercept — that's the binding one. Here ℓ_2 gives $(30, 0)$ which is closer to the origin than ℓ_1's $(50, 0)$, so $A = (30, 0)$.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Direction-of-coefficients angle. Since $Z = 4x + y$ rewards x heavily, push to the x-axis corner. The corner on the x-axis closest to origin satisfying both inequalities is $(30, 0)$ (ℓ_2 binds). There, $Z = 120$.

Concept used. Linear objective with a much larger coefficient on x is maximised by going as far as possible in the x-direction.

Step 1. Compare Z at the corners: $0,\ 120,\ 110,\ 50$.

Step 2. Max $= 120$.

> **Final Answer:** Option **(B)**.

II. Short Answer Questions (SA)

Q 12.1 **Determine the maximum value of $Z = 11x + 7y$ subject to the constraints $2x + y \le 6,\ x \le 2,\ x \ge 0,\ y \ge 0$.**

SOLUTION

Concept used. *Corner-point method for a bounded LPP.* Plot the lines $2x + y = 6$ and $x = 2$, find the feasible region in the first quadrant, list the corner points, evaluate Z at each, and pick the maximum.

Step 1. *Plot constraint lines.* $\ell_1 :\ 2x + y = 6$ goes through $(3, 0)$ and $(0, 6)$. $\ell_2 :\ x = 2$ is a vertical line.

Step 2. *Find vertices of the feasible region.* Intersect ℓ_1 and ℓ_2: substitute $x = 2$ into ℓ_1

$$2(2) + y = 6 \ \Rightarrow\ y = 2.$$

So $(2, 2)$ is a corner. Other corners are on the axes: $(0, 0)$, $(2, 0)$ (where $x = 2$ meets the x-axis), and $(0, 6)$ (where ℓ_1 meets the y-axis).

Step 3. *Sanity check* $(0, 6)$ *in* $x \le 2$*:* $0 \le 2$, holds. So $(0, 6)$ is a vertex.

Step 4. *List corners:* $O(0, 0),\ A(2, 0),\ B(2, 2),\ C(0, 6)$.

Step 5. *Evaluate* $Z = 11x + 7y$ *at each:*

$$\begin{aligned} Z(O) &= 11(0) + 7(0) = 0, \\ Z(A) &= 11(2) + 7(0) = 22, \\ Z(B) &= 11(2) + 7(2) = 22 + 14 = 36, \\ Z(C) &= 11(0) + 7(6) = 0 + 42 = 42. \end{aligned}$$

Step 6. *Compare:* $\{0,\ 22,\ 36,\ 42\}$. Maximum is 42 at $(0, 6)$.

> **Final Answer:** $Z_{\text{max}} = 42$ at $(x, y) = (0, 6)$.

☞ **Don't forget axis-cap corners**

A constraint like $x \le 2$ adds a vertical cap. The corners are formed by intersections of *any two* binding lines (constraints or axes), not just the slanted ones.

EXPERT'S SOLUTION : *Priya Iyer, Ph.D Mathematics, IISc Bangalore*

Strategy-first angle. The coefficient on y (which is 7) is smaller than the coefficient on x (which is 11), but the constraint $x \le 2$ caps x tightly, while y can grow to 6. So pushing y to its maximum wins.

Concept used. Compare the maximum allowable contribution of each variable: $11 \cdot 2 = 22$ from x, vs. $7 \cdot 6 = 42$ from y. The y-direction wins.

Step 1. Check the corner with y maximal: $(0, 6)$. Feasible ($2 \cdot 0 + 6 = 6 \le 6$ ✓; $0 \le 2$ ✓).

Step 2. $Z(0, 6) = 0 + 42 = 42$.

Step 3. Verify no other corner beats it: $(2, 2)$ gives 36, smaller.

Final Answer: $Z_{\max} = 42$ at $(0, 6)$.

Q 12.2 **Maximise** $Z = 3x + 4y$ **subject to the constraints** $x + y \le 1,\ x \ge 0,\ y \ge 0$.

SOLUTION

Concept used. Classic two-variable LPP with a single slant constraint plus non-negativity. The feasible region is a triangle with vertices on the axes and at the slant line.

Step 1. *Constraint line:* $x + y = 1$ goes through $(1, 0)$ and $(0, 1)$.

Step 2. *Feasible region:* the triangle $O(0, 0),\ A(1, 0),\ B(0, 1)$.

Step 3. *Evaluate* $Z = 3x + 4y$ *at each corner:*

$$Z(O) = 3(0) + 4(0) = 0,$$
$$Z(A) = 3(1) + 4(0) = 3,$$
$$Z(B) = 3(0) + 4(1) = 4.$$

Step 4. *Maximum:* compare $\{0,\ 3,\ 4\}$. The largest is 4, attained at $(0, 1)$.

Final Answer: $Z_{\max} = 4$ at $(x, y) = (0, 1)$.

☞ **Heaviest coefficient wins on simple regions**

On the standard triangle $x + y \le c,\ x, y \ge 0$, the optimum of $Z = ax + by$ is at $(c, 0)$ if $a > b$ and at $(0, c)$ if $b > a$. Memorise this for fast elimination.

EXPERT'S SOLUTION *: Vikram Rao, M.Sc Mathematics, Delhi University*

Heavy-coefficient angle. $b = 4 > a = 3$, so the y-corner of the simplex wins.
Concept used. On a triangle bounded by $x + y \le c$ and the axes, $Z = ax + by$ attains its max at $(c, 0)$ if $a > b$, else at $(0, c)$.

Step 1. Identify the slant line: $x + y = 1$, so $c = 1$.

Step 2. Compare $a = 3$ vs. $b = 4$. Since $b > a$, the y-corner $(0, c) = (0, 1)$ wins.

Step 3. Read off the max: $Z = 3(0) + 4(1) = 4$ at $(0, 1)$.

Final Answer: $Z_{\text{max}} = 4$ at $(0, 1)$.

Q 12.3 **Minimise** $Z = 3x + 2y$ **subject to the constraints** $x + y \ge 8,\ 3x + 5y \le 15,\ x \ge 0,\ y \ge 0$.

SOLUTION

Concept used. Before applying the corner-point method, always *check whether the feasible region is non-empty*. Conflicting constraints can yield an empty feasible region, in which case the LPP is **infeasible** and there is no minimum to report.

Step 1. *Plot the lines.* $\ell_1:\ x + y = 8$ through $(8, 0)$ and $(0, 8)$ — region $x + y \ge 8$ is the half-plane *above* this line. $\ell_2:\ 3x + 5y = 15$ through $(5, 0)$ and $(0, 3)$ — region $3x + 5y \le 15$ is the half-plane *below* this line.

Step 2. *Test a sample point in both regions.* Try $(0, 8)$: check $x + y \ge 8$: $0 + 8 = 8 \ge 8$ ✓. check $3x + 5y \le 15$: $0 + 40 = 40 \not\le 15$. **Fails.**

Step 3. Try $(0, 3)$: $0 + 3 = 3 \not\ge 8\ \times$.

Step 4. Try any point with $x + y \ge 8$ and $3x + 5y \le 15$. From the first, $y \ge 8 - x$. Substitute in the second: $3x + 5(8 - x) \le 15$, i.e. $3x + 40 - 5x \le 15$, $-2x \le -25$, $x \ge 12.5$. But the first constraint $x + y \ge 8$ with $x \ge 12.5$ and $y \ge 0$ is fine. Check the second again at $x = 12.5,\ y = 0$: $3(12.5) + 0 = 37.5 \not\le 15$. So no (x, y) in the first quadrant simultaneously satisfies both.

Step 5. *Conclusion:* the feasible region is **empty**; the LPP is infeasible.

Final Answer: The feasible region is empty; the LPP has *no* feasible solution.

✗ Don't just plug in corners blindly

If you plough straight into the corner-point method without first sketching the region, you can end up computing Z at the *intersection of* ℓ_1 *and* ℓ_2 — a point that is not in the feasible region. Always sanity-check with a sample point.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Algebra-of-half-planes angle. Add the constraints to see if they are compatible.
Concept used. Two half-planes can intersect in the empty set if their boundary lines are "out of order" relative to the feasible direction.

Step 1. ℓ_1: $x + y \geq 8$ means stay above the diagonal $x + y = 8$. Lowest point on this boundary is $\sim (8, 0)$ or $(0, 8)$, both at "distance ≥ 8 from origin".

Step 2. ℓ_2: $3x + 5y \leq 15$ means stay below the line through $(5, 0)$ and $(0, 3)$. Highest point is on the boundary, at ~ 5 on x-axis or ~ 3 on y-axis.

Step 3. These two regions don't overlap in the first quadrant: the "stay above $x + y = 8$" region is north-east of the line, while "stay below $3x + 5y = 15$" is south-west of a line which is south-west of $x + y = 8$. Disjoint.

Final Answer: Infeasible LPP: no feasible solution exists.

Q 12.4 **Maximise** $Z = x + y$ **subject to** $x - y \leq -1,\ -x + y \leq 0,\ x, y \geq 0$.

SOLUTION

Concept used. Check feasibility first by sketching the two half-planes, then run the corner-point method only if the region is non-empty.

Step 1. *Re-write the constraints:* $\ell_1:\ x - y = -1$ i.e. $y = x + 1$. Constraint $x - y \leq -1$ means $y \geq x + 1$ (above the line). $\ell_2:\ -x + y = 0$ i.e. $y = x$. Constraint $-x + y \leq 0$ means $y \leq x$ (below the line).

Step 2. *Combine:* we need both $y \geq x + 1$ and $y \leq x$, i.e. $x + 1 \leq y \leq x$, which forces $x + 1 \leq x$, i.e. $1 \leq 0$. Impossible.

Step 3. Therefore the feasible region is empty.

Final Answer: Infeasible LPP: no feasible solution exists.

☞ Sandwich test for two inequalities in y

If two constraints give you "$y \geq A$" and "$y \leq B$", check $A \leq B$. If not, the region is empty.

EXPERT'S SOLUTION *: Priya Iyer, Ph.D Mathematics, IISc Bangalore*

Pure-algebra angle. Subtract the two constraints: from $y \geq x+1$ and $y \leq x$, $x+1 \leq y \leq x$, impossible.

Final Answer: No feasible region.

Q 12.5 **Maximise $Z = 3x + 9y$ subject to the constraints $x + 3y \leq 60,\ x + y \geq 10,\ x \leq y,\ x \geq 0,\ y \geq 0$.**

SOLUTION

Concept used. Identify all binding pairs of constraints (lines) to find the corners of the feasible region, then evaluate Z.

Step 1. *Constraint lines:* $\ell_1:\ x+3y=60$ through $(60,0),\ (0,20)$. $\ell_2:\ x+y=10$ through $(10,0),\ (0,10)$. $\ell_3:\ x=y$ through $(0,0)$ at $45°$.

Step 2. *Half-plane directions* (test origin where applicable): $x+3y \leq 60$: origin gives $0 \leq 60$ ✓, so origin side. $x+y \geq 10$: origin gives $0 \ngeq 10$, so opposite side. $x \leq y$: origin gives $0 \leq 0$ ✓, so origin side / above the line $y = x$.

Step 3. *Find vertices.* Solve binding pairs:

- $\ell_2 \cap \ell_3$: $x+y=10,\ x=y \Rightarrow 2x=10,\ x=y=5.\ (5,5)$.
- $\ell_1 \cap \ell_3$: $x+3y=60,\ x=y \Rightarrow y+3y=60,\ y=x=15.\ (15,15)$.
- $\ell_1 \cap \{x=0\}$: $(0,20)$. Check $x \leq y$: $0 \leq 20$ ✓. Check $x+y \geq 10$: $20 \geq 10$ ✓. Feasible.
- $\ell_2 \cap \{x=0\}$: $(0,10)$. Check $x+3y \leq 60$: $30 \leq 60$ ✓. Check $x \leq y$: $0 \leq 10$ ✓. Feasible.

Step 4. *Corner list:* $A(0,10),\ B(5,5),\ C(15,15),\ D(0,20)$.

Step 5. *Evaluate $Z = 3x + 9y$:*

$$Z(A) = 0 + 90 = 90,$$
$$Z(B) = 15 + 45 = 60,$$
$$Z(C) = 45 + 135 = 180,$$
$$Z(D) = 0 + 180 = 180.$$

Step 6. *Maximum:* $Z_{\max} = 180$, attained at *both* $C(15, 15)$ and $D(0, 20)$. Since Z equal at two corners, every point on the edge CD also gives $Z = 180$ — infinitely many optima.

Final Answer: $Z_{\max} = 180$ on the entire segment from $(15, 15)$ to $(0, 20)$.

☞ **Detect multiple optima by slope**

$Z = 3x + 9y$ has slope $-3/9 = -1/3$. The edge CD joining $(15, 15)$ and $(0, 20)$ has slope $(20 - 15)/(0 - 15) = -1/3$. Same slope ⇒ parallel ⇒ multiple optima.

EXPERT'S SOLUTION *: Vikram Rao, M.Sc Mathematics, Delhi University*

Parallel-edge angle. Spot in advance that $Z = 3x + 9y = 3(x + 3y)$, which is constant on every line $x + 3y = k$. So the contour of Z is parallel to constraint line $\ell_1 : x + 3y = 60$. Pushing k as high as possible takes us to $k = 60$, the line ℓ_1 itself. Every point on the segment of ℓ_1 that is feasible (from $C(15, 15)$ to $D(0, 20)$) is optimal.
Concept used. If the objective is a scalar multiple of a constraint, the max/min is attained on the entire boundary segment of that constraint.

Final Answer: $Z_{\max} = 180$ on the edge from $(15, 15)$ to $(0, 20)$.

III. Long Answer Questions (LA)

Q 12.1 **A furniture trader deals in only two items — tables and chairs. He has Rs. 50000 to invest and a space to store at most 60 pieces. A table costs him Rs. 2500 and a chair Rs. 500. He estimates that he can sell a table at a profit of Rs. 250 and a chair at a profit of Rs. 75. Assuming that he can sell all the items he buys, how should he invest his money so that he maximises his profit? Formulate this as an LPP and solve it graphically.**

SOLUTION

Concept used. Formulate a real-world LPP in four steps: (i) identify decision variables; (ii) write the objective function; (iii) write all resource and demand constraints, including non-negativity; (iv) solve graphically by the corner-point method.

Step 1. *Decision variables.* Let x = number of tables bought, y = number of chairs bought. Both must be non-negative integers, though for the LP-relaxation we treat them as reals in $[0, \infty)$.

Step 2. *Objective function.* Profit per table = Rs. 250; profit per chair = Rs. 75. So

$$Z = 250\,x + 75\,y \quad \text{(to be maximised)}.$$

Step 3. *Constraints.*

(1) Capital constraint: a table costs 2500, a chair 500, total spend ≤ 50000:

$$2500\,x + 500\,y \le 50000.$$

Divide by 500:

$$5x + y \le 100.$$

(2) Storage constraint: total pieces ≤ 60:

$$x + y \le 60.$$

(3) Non-negativity: $x \ge 0,\ y \ge 0$.

Step 4. *Plot the constraint lines.*

- $\ell_1 :\ 5x + y = 100$. x-intercept $(20, 0)$, y-intercept $(0, 100)$.
- $\ell_2 :\ x + y = 60$. x-intercept $(60, 0)$, y-intercept $(0, 60)$.

Step 5. *Find vertices of the feasible region.*

- $\ell_1 \cap \ell_2$: $5x + y = 100,\ x + y = 60$. Subtract: $4x = 40 \Rightarrow x = 10$, then $y = 50$. So $(10, 50)$. Check non-negativity: ✓.
- $\ell_1 \cap \{y = 0\}$: $5x = 100 \Rightarrow x = 20$. So $(20, 0)$. Check $x + y \le 60$: $20 \le 60$ ✓.
- $\ell_2 \cap \{x = 0\}$: $y = 60$. So $(0, 60)$. Check $5x + y \le 100$: $60 \le 100$ ✓.
- Origin $O(0, 0)$.

Step 6. *Corner list:* $O(0,0),\ A(20,0),\ B(10,50),\ C(0,60)$.

Step 7. *Evaluate $Z = 250\,x + 75\,y$:*

$$\begin{aligned} Z(O) &= 0, \\ Z(A) &= 250(20) + 75(0) = 5000, \\ Z(B) &= 250(10) + 75(50) = 2500 + 3750 = 6250, \\ Z(C) &= 250(0) + 75(60) = 4500. \end{aligned}$$

Step 8. *Maximum:* $Z_{\max} = 6250$ at $(x, y) = (10, 50)$, i.e. buy 10 tables and 50 chairs.

Step 9. *Sanity check.* Capital used: $2500(10) + 500(50) = 25000 + 25000 = 50000$ ✓. Storage used: $10 + 50 = 60$ ✓. Both constraints active — the optimum sits at the intersection of both binding lines, as expected.

Final Answer: Buy 10 tables and 50 chairs for maximum profit of Rs. 6250.

☞ Integer answers from continuous LPP

This question has integer-valued vertices, so no rounding is needed. If the optimal corner were non-integer (e.g. $(10.5, 49.5)$), you would have to either re-state the answer in fractional form, mention that the strictly integer optimum may differ, or invoke "Integer Programming" beyond the scope of Class 12.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Profit-per-rupee angle. Compute the profit return per rupee for each item:

$$\text{Table: } \frac{250}{2500} = 0.10 = 10\%, \qquad \text{Chair: } \frac{75}{500} = 0.15 = 15\%.$$

Chairs return more per rupee, so the trader leans towards chairs. However, the storage cap of 60 pieces means he can't fill the entire budget with chairs alone (that would need 100 chairs).

Concept used. When two resources are both fully binding, the optimum sits at the corner where both bind. Here both "capital $= 50000$" and "space $= 60$" are tight, giving $(10, 50)$.

Step 1. Use up all storage with chairs: 60 chairs cost 30000, leaving 20000 to spend on tables. Tables cost 2500 each, so $20000/2500 = 8$ tables — but this gives $60 + 8 = 68$ pieces, violates storage. So can't max-out chairs.

Step 2. Use up all capital with tables: $50000/2500 = 20$ tables, 0 chairs. Pieces $= 20$, well within 60. Profit $= 20 \cdot 250 = 5000$.

Step 3. Better corner: bind both constraints.
$5x + y = 100,\ x + y = 60 \Rightarrow x = 10,\ y = 50$. Profit $= 6250$. **Best.**

Final Answer: 10 tables, 50 chairs $\Rightarrow$ profit $=$ Rs. 6250.

Q 12.2 **A company produces two products P and Q. Product P requires 4 hours of labour and 2 units of raw material per unit; product Q requires 3 hours of labour and 5**

units of raw material per unit. The company has 200 labour-hours and 150 units of raw material available. Profit per unit of P is Rs. 40 and per unit of Q is Rs. 30. Formulate this LPP and find the maximum profit.

SOLUTION

Concept used. Resource-allocation LPP. Decision variables are production quantities; objective is total profit; constraints are resource limits and non-negativity.

Step 1. *Decision variables.* Let x = units of P produced, y = units of Q produced.

Step 2. *Objective function.*

$$Z = 40\,x + 30\,y \quad \text{(maximise)}.$$

Step 3. *Constraints.*

(1) Labour: $4x + 3y \le 200$.

(2) Raw material: $2x + 5y \le 150$.

(3) Non-negativity: $x, y \ge 0$.

Step 4. *Plot constraint lines.* $\ell_1: \ 4x + 3y = 200$ through $(50, 0), \ (0, 200/3) \approx (0, 66.67)$. $\ell_2: \ 2x + 5y = 150$ through $(75, 0), \ (0, 30)$.

Step 5. *Vertices of feasible region.*

- $\ell_1 \cap \ell_2$: solve

$$4x + 3y = 200, \qquad 2x + 5y = 150.$$

  Multiply 2nd by 2: $4x + 10y = 300$. Subtract 1st: $7y = 100 \Rightarrow y = 100/7$. Then $4x = 200 - 3(100/7) = 200 - 300/7 = (1400 - 300)/7 = 1100/7$, so $x = 275/7$. Vertex $\left(\frac{275}{7}, \frac{100}{7}\right) \approx (39.29, \ 14.29)$.
- $\ell_1 \cap \{y = 0\}$: $(50, 0)$. Check ℓ_2: $2(50) + 0 = 100 \le 150$ ✓.
- $\ell_2 \cap \{x = 0\}$: $(0, 30)$. Check ℓ_1: $0 + 3(30) = 90 \le 200$ ✓.
- Origin $O(0, 0)$.

Step 6. *Corner list:*

$$O(0,0), \ A(50,0), \ B\left(\frac{275}{7}, \frac{100}{7}\right), \ C(0,30).$$

Step 7. *Evaluate $Z = 40x + 30y$:*

$$\begin{aligned} Z(O) &= 0, \\ Z(A) &= 40(50) + 30(0) = 2000, \\ Z(B) &= 40 \cdot \tfrac{275}{7} + 30 \cdot \tfrac{100}{7} = \tfrac{11000+3000}{7} = \tfrac{14000}{7} = 2000, \\ Z(C) &= 40(0) + 30(30) = 900. \end{aligned}$$

Step 8. *Comparison:* $Z_{\max} = 2000$, attained at both $A(50, 0)$ and $B(275/7, 100/7)$ — and hence on the entire segment AB (multiple optima).

Final Answer: $Z_{\max} = $ Rs. 2000, on the edge from $(50,0)$ to $(275/7, 100/7)$.

☞ Quick equal-Z test

$Z = 40x + 30y$ at $(50,0)$ is 2000. At $(275/7, 100/7)$ is also 2000. So Z is constant on the segment AB. This is the multiple-optima case again.

EXPERT'S SOLUTION *: Priya Iyer, Ph.D Mathematics, IISc Bangalore*

Parallel-detection angle. Slope of $Z = 40x + 30y$ level curves: $-40/30 = -4/3$. Slope of $\ell_1 :\ 4x + 3y = 200$ is also $-4/3$. Same slope $\Rightarrow$ objective is parallel to $\ell_1 \Rightarrow$ multiple optima on the segment of ℓ_1 that bounds the feasible region.

Concept used. When the level lines of Z are parallel to a binding constraint, the entire binding-constraint edge is optimal.

Step 1. Compute slope of Z: $-40/30 = -4/3$.

Step 2. Compute slope of ℓ_1: $-4/3$. *Match!*

Step 3. Conclude: the segment of ℓ_1 from $(50,0)$ to $(275/7, 100/7)$ is the optimal set.

Step 4. Optimal Z: $40(50) = 2000$ (or equivalently $40(275/7) + 30(100/7) = 14000/7 = 2000$).

Final Answer: $Z_{\max} = 2000$ on segment from $(50,0)$ to $(275/7, 100/7)$.

Q 12.3 **Minimise** $Z = 200x + 500y$ **subject to** $x + 2y \geq 10,\ 3x + 4y \leq 24,\ x \geq 0,\ y \geq 0$.

SOLUTION

Concept used. *Mixed-direction* LPP — one constraint is $\geq$ (a "floor"), another is $\leq$ (a "ceiling"). The feasible region is bounded between them; check both for binding.

Step 1. *Plot lines.* $\ell_1 :\ x + 2y = 10$ through $(10,0),\ (0,5)$. Region $x + 2y \geq 10$ is *above* this line. $\ell_2 :\ 3x + 4y = 24$ through $(8,0),\ (0,6)$. Region $3x + 4y \leq 24$ is *below* this line.

Step 2. *Vertices of feasible region.*

- $\ell_1 \cap \ell_2$: solve $x + 2y = 10$, $3x + 4y = 24$. Multiply 1st by 2: $2x + 4y = 20$. Subtract from 2nd: $x = 4$. Back-sub: $4 + 2y = 10 \Rightarrow y = 3$. Vertex $(4,3)$.
- $\ell_1 \cap \{x = 0\}$: $(0,5)$. Check ℓ_2: $0 + 20 = 20 \leq 24$ ✓.
- $\ell_2 \cap \{x = 0\}$: $(0,6)$. Check ℓ_1: $0 + 12 = 12 \geq 10$ ✓.

- $\ell_1 \cap \{y = 0\}$: $(10, 0)$. Check ℓ_2: $30 + 0 = 30 \not\leq 24$. $\times$
- $\ell_2 \cap \{y = 0\}$: $(8, 0)$. Check ℓ_1: $8 + 0 = 8 \not\geq 10$. $\times$

So vertices are $A(0, 5),\ B(4, 3),\ C(0, 6)$.

Step 3. *Evaluate* $Z = 200x + 500y$*:*

$$Z(A) = 200(0) + 500(5) = 2500,$$
$$Z(B) = 200(4) + 500(3) = 800 + 1500 = 2300,$$
$$Z(C) = 200(0) + 500(6) = 3000.$$

Step 4. *Minimum:* $Z_{\text{min}} = 2300$ at $(4, 3)$.

Step 5. *Verify both constraints are binding at* $(4, 3)$*:* $x + 2y = 4 + 6 = 10$ ✓ (eq.); $3x + 4y = 12 + 12 = 24$ ✓ (eq.).

Final Answer: $Z_{\text{min}} = 2300$ at $(x, y) = (4, 3)$.

☞ Region between two lines

A region defined by "above ℓ_1" AND "below ℓ_2" is a band between them. Its corners come from $\ell_1 \cap \ell_2$ and the intersections of each line with the axes (whichever are feasible).

EXPERT'S SOLUTION : *Vikram Rao, M.Sc Mathematics, Delhi University*

Coefficient-imbalance angle. Z punishes y (500) heavily compared to x (200). So minimising pushes us to reduce y as much as possible while staying feasible. The $\geq$ constraint $x + 2y \geq 10$ forbids y from being too small unless x is large.
Concept used. Trade-off between rising x-cost $(200$ per unit$)$ and falling y-cost $(500$ per unit saved$)$. Optimum is at the binding intersection $(4, 3)$.

Step 1. At $(0, 5)$ (cheapest pure-y corner): $Z = 2500$.

Step 2. Push x up to reduce y: at $(4, 3)$, $Z = 2300$ $(200$ less$)$.

Step 3. Going further would exit the "$\leq$" region. So min is at $(4, 3)$, $Z = 2300$.

Final Answer: Min $Z = 2300$ at $(4, 3)$.