Vector Algebra

A quantity having both magnitude AND direction is called a ~~scaler~~ vector quantity.
Examples : displacement, velocity, force, acceleration, momentum, weight.

Scalar : only magnitude (no direction).
Examples : mass, time, distance, speed, temperature, work, energy.

Notation

A vector from A to B is written as $\overrightarrow{AB}$.
Magnitude : $\overrightarrow{AB}$ = AB (a scalar).
A is the initial point (tail), B is the terminal point (head).

$$\boxed{\overrightarrow{OP} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}}$$ <-position vector <-of $P(x,y,z)$

Here i , j , k are unit vectors along the positive x, y, z axes respectively. They are mutually perpendicular and each has unit magnitude (i = j = k

Magnitude of a Vector

If $\vec{a} = a_1 i + a_2 j + a_3 k$ then

$\vec{a}$ = sqrt(a_1 2 + a_2 2 + a_3 2) <-length of <-the vector

Types of Vectors

(1) Zero vector : mag = 0, no direction.
Denoted by $\vec{O}$. Initial pt = terminal pt.

(2) Unit vector : magnitude = 1.
$\hat{a} = \vec{a}$ / $\vec{a}$ <-direction of a

(3) Coinitial vectors : same initial point.

(4) Collinear vectors : parallel to same line (irrespective of magnitude & direction).

(5) Equal vectors : same magnitude & same direction (initial pt may differ).

(6) Negative vector : same magnitude, opposite direction.

Vector Joining Two Points

If A(x1, y1, z1) and B(x2, y2, z2) are any two points in space, then

$$\boxed{\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}}$$

<-triangle law

In component form :

$$\boxed{\overrightarrow{AB} = (x_2 - x)\ i + (y_2 - y)\ j + (z_2 - z)\ k}$$

Magnitude (distance from A to B) :

$$\boxed{\overrightarrow{AB} = sqrt(\ (x_2 - x)\ \ 2 + (y_2 - y)\ \ 2 +}\ (z_2 - z)$$

<-this is exactly the distance formula

Example

A(1, -2, 3) , B(4, 2, -1)

$\overrightarrow{AB} = 3\ i + 4\ j - 4\ k$

AB = sqrt(9 + 16 + 16) = sqrt(41) units.

Direction Cosines & Direction Ratios

Let a vector a make angles alpha, beta, gamma with the positive x, y, z axes respectively.

Then l = cos(alpha) , m = cos(beta) , n = cos(gamma) are the DIRECTION COSINES.

$\boxed{l = a_1 / a \quad , m = a_2 / a \quad , n = a_3 /} \; a$

<-always between -1 and 1

Important identity (used in every CBSE Q):

$\boxed{l \quad 2 + m \quad 2 + n}$ <-Pythagoras in 3D

Direction Ratios (DRs)

Any 3 numbers a, b, c that are proportional to l, m, n are the ~~DCs~~ direction ratios.

i.e. a:b:c = l:m:n .

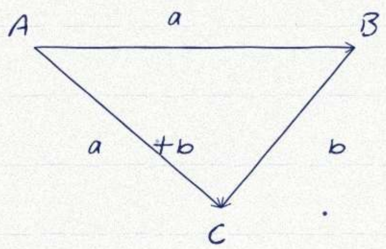
$\boxed{l = a / sqrt(a \quad 2+b \quad 2+c}$ 2)<-normalise the DRs

Similar for m and n.

Addition of Vectors

(1) Triangle Law

If two vectors are represented by two sides of a triangle taken in order, the closing side taken in reverse order gives the resultant.



$$\boxed{\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}}$$

(2) Parallelogram Law

If two vectors are co-initial along adjacent sides of a parallelogram, then their sum is the diagonal through the common point.

$$\boxed{\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}}$$ <-C = opposite vertex

Properties of Vector Addition

① Commutative :

$$\boxed{\vec{a} + \vec{b} = \vec{b} + \vec{a}}$$

② Associative :

$$\boxed{(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})}$$

③ Additive identity :

$$\boxed{\vec{a} + \vec{O} = \vec{a}}$$

④ Additive inverse :

$$\boxed{\vec{a} + (-\vec{a}) = \vec{O}}$$

Multiplication by a Scalar

If k is a scalar, $\vec{a}$ is a vector,

then $k\vec{a}$ is a vector with magnitude $k \cdot \vec{a}$.

Direction : same as a if $k > 0$,

opposite if $k < 0$.

If $k = 0$, $k a = 0$.

Section Formula

Let A and B have position vectors a and b . Let P divide AB in the ratio m : n .

(i) Internal Division

$\boxed{\overrightarrow{OP} = (m \vec{b} + n \vec{a}) / (m + n)}$ <-internal division
<-ratio m : n

(ii) External Division

$\boxed{\overrightarrow{OP} = (m \vec{b} - n \vec{a}) / (m - n)}$ <-external division

(iii) Midpoint (m = n = 1)

$\boxed{\overrightarrow{OP} = (\vec{a} + \vec{b}) / 2}$ <-average of a and b

Note : these are vector versions of CBSE Class 10 section formulas in coordinate geometry.

Scalar (Dot) Product

If $\vec{a}$ and $\vec{b}$ are two non-zero vectors and theta is the angle between them, then their scalar (dot) product is defined as :

$$\boxed{\vec{a} \cdot \vec{b} = \vec{a} \qquad \vec{b}} \; \cos(\text{theta})$$

<- result : scalar
<-(not a vector !)

Component Form

If $\vec{a} = a_1 i + a_2 j + a_3 k$,
$\vec{b} = b_1 i + b_2 j + b_3 k$ then

$$\boxed{\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3}$$

Special Cases

theta = 0 : $\vec{a} \cdot \vec{b} = a \quad b$

theta = pi : $\vec{a} \cdot \vec{b} = - \quad a \quad b$

theta = pi/2 : $\vec{a} \cdot \vec{b} = 0$ ~~(parallel)~~ (perp.)

Self-dot :

$$\boxed{\vec{a} \cdot \vec{a} = \vec{a}} \; 2$$

Properties of Dot Product

① Commutative :

$$\boxed{\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}}$$

② Distributive over addition :

$$\boxed{\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}}$$

③ Scalar multiplication :

$$\boxed{(k \vec{a}) \cdot \vec{b} = k (\vec{a} \cdot \vec{b})}$$

Unit Vector Dot Products

$i \cdot i = j \cdot j = k \cdot k = 1$

$i \cdot j = j \cdot k = k \cdot i = 0$

Angle Between Two Vectors

$$\boxed{\cos(theta) = (\vec{a} \cdot \vec{b}) / (\quad \vec{a}}$$

<- use this in $\vec{b}$
<-every angle Q

Projection of a Vector

The scalar projection of $\vec{a}$ on $\vec{b}$ is the length of the shadow that a casts on the line of b when light is perpendicular.

$$\boxed{proj\_b\ \vec{a} = (\vec{a}\ .\ \vec{b})\ /\ \vec{b}}$$ <-scalar projection

Vector Projection

The full vector along the line of b is :

$$\boxed{(\ (\vec{a}\ .\ \vec{b})\ /\ \vec{b}\ \ 2\ )\ \vec{b}}$$ <-vector projection

Example (CBSE 2022 style)

a = 2 i + 3 j + 2 k , b = i + 2 j

a .b = 2 + 6 + 2 = 10 .

b = sqrt(1 + 4 + 1) = sqrt(6) .

$$\boxed{proj = 10\ /\ sqrt(6)}$$ <-scalar answer

Vector (Cross) Product

The vector (cross) product of $\vec{a}$ and $\vec{b}$.
is a VECTOR (unlike dot product), defined by :

$\boxed{\vec{a} \times \vec{b} = \vec{a} \qquad \vec{b} \qquad \text{sin(theta)}}$ <- $\hat{n}$: unit vector
<-by right-hand rule

n is perpendicular to BOTH a and b ,
chosen so that a , b , n form a right-handed system (point fingers a -> b, thumb = n).

Magnitude

$\boxed{\vec{a} \times \vec{b} \qquad = \qquad a \qquad b}$ sin(theta)

<-theta is in [0, pi] -> sin >= 0 always

Test : a b

$\boxed{\vec{a} \times \vec{b} = \vec{0} \quad \text{iff} \quad \vec{a} \qquad \vec{b}}$

Cross Product (Determinant)

If $\vec{a} = a_1 i + a_2 j + a_3 k$,

$\vec{b} = b_1 i + b_2 j + b_3 k$

then $a \times b$ is given by :

$$\boxed{a \times b} = \begin{array}{ccc l} i & j & k & \\ a1 & a2 & a3 & \text{<-expand along R1} \\ b1 & b2 & b3 & \end{array}$$

Expanding the determinant :

$$\begin{aligned} a \times b &= (a2\ b3 - a3\ b2)\, i \\ &\quad - (a1\ b3 - a3\ b1)\, j \\ &\quad + (a1\ b2 - a2\ b1)\, k \end{aligned}$$

Anti-commutative

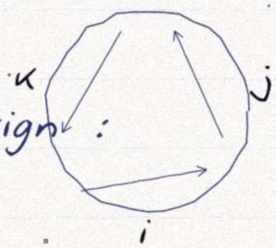
$$\boxed{\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})}$$ <-flip sign on swap!

Common slip : forget the ~~(-)~~ minus sign.

Unit Vector Cross Products

Cyclic rule (right-hand) :

$$i \times j = k$$
$$j \times k = i$$
$$k \times i = j$$



Reverse order -> flip the sign :

$$j \times i = -k$$
$$k \times j = -i$$
$$i \times k = -j$$

Self-cross :

$$i \times i = j \times j = k \times k = 0$$

<- $\sin(0) = 0$

Properties of Cross Product

(a) Distributive : $a \times (b + c) = a \times b + a \times c$

(b) Scalar : $k(a \times b) = (ka) \times b$

(c) $a \times b$ is perpendicular to plane of a, b.

(d) Magnitude = area of parallelogram on a, b.

Geometric Applications

(i) Area of Parallelogram

If sides $\vec{a}$ and $\vec{b}$ are co-initial,

$$\boxed{\text{Area} = \vec{a} \times \vec{b}}$$

<-magnitude of cross

(ii) Area using Diagonals

If diagonals are $\vec{d}_1$ and $\vec{d}_2$,

$$\boxed{\text{Area} = (1/2) \quad \vec{d}_1 \times \vec{d}_2}$$

<-half magnitude

(iii) Area of Triangle

Triangle ABC with vertices A, B, C :

$$\boxed{\text{Area} = (1/2) \quad \overrightarrow{AB} \times \overrightarrow{AC}}$$

<-use ANY 2 sides

Tip : if you already have position vectors of A, B, C - first form AB , AC then cross.

Unit Vector Perpendicular to Two

Often asked : find $\hat{n}$ perpendicular to both $\vec{a}$ and $\vec{b}$ ..

$\boxed{\hat{n} = (\vec{a} \times \vec{b}) / \quad \vec{a} \times \vec{b}}$ <-normalise cross

Worked Out (CBSE 2024)

Find unit vector perpendicular to

$a = i + j + k \quad , \quad b = 2i + 3j$ -

$a \times b = \quad i \quad j \quad k \; ; 1 \; 1 \; 1 \; ; 2 \; 3 \; -1$

$= i \; (-1-3) - j \; (-1-2) + k \; (3-2)$

$= -4i + 3j + k$.

$a \; xb = \text{sqrt}(16 + 9 + 1) = \text{sqrt}(26)$.

$\boxed{n = (-4i + 3j + k) /} \text{sqrt}(26)$

<-this is the standard 5-mark template

Forgetting the ~~(+)~~ (-) signs costs 1-2 marks.

Angle Between Vectors (worked)

Q : Find theta between a = i -j +k and
b = i +j -k .

Step 1 : Dot product
a .b = 1(1) + (-1)(1) + 1(-1) = -1

Step 2 : Magnitudes
a = sqrt(1+1+1) = sqrt(3)
b = sqrt(1+1+1) = sqrt(3)

Step 3 : Apply cos(theta) = a .b / a b
cos(theta) = -1 / (sqrt(3) sqrt(3))
= -1 / 3

$\boxed{\text{theta} = \cos^{-1}(-1/3)}$ <-approx 109.47 deg

Tip

If a .b > 0 -> theta is acute (< 90 deg).
If a .b < 0 -> theta is obtuse (> 90 deg).
If a .b = 0 -> theta = 90 deg (perpendicular).
Use this BEFORE computing to predict the sign.

Lagrange's Identity

An important identity linking the dot and cross products :

$$\boxed{\vec{a} \times \vec{b} \quad 2 + (\vec{a} . \vec{b}) \quad 2 = \quad \vec{a}} \quad 2$$

<-Lagrange's identity

Quick Proof Sketch

$a \times b \quad 2 = a \quad 2 \quad b \quad 2 \sin \quad 2(th$

$(a \; .b \;) \quad 2 = a \quad 2 \quad b \quad 2 \cos \quad 2(theta)$

Adding : $a \quad 2 \quad b \quad 2 (\sin \quad 2 + \cos \quad 2)$

$= a \quad 2 \quad b \quad 2.$

Three Collinear Points

Points A, B, C are collinear iff

$\boxed{\overrightarrow{AB} = lambda . \overrightarrow{AC}}$ (some scalar lambda)

Equivalent test : $\overrightarrow{AB} \times \overrightarrow{AC} = \vec{0}$.

PYQ Templates Summary

Across CBSE Class 12 2021 - 2025 + JEE Main questions cluster into 5 repeating templates :

1. Unit vector perpendicular to two :
   n = (a x b) / a x b

2. Area of triangle by vertices :
   Area = (1/2) AB x AC

3. Angle between two vectors :
   cos(theta) = (a .b) / (a b)

4. Projection of a on b :
   proj = (a .b) / b

5. Direction cosines from two points :
   l = (x2-x1) / d, similarly m, n.
   d = sqrt(sum of squares).

Common Mistakes (cost 1-2 marks)

(a) Writing a .b as a vector (it is scalar !)
(b) Forgetting sin(theta) in a x b .
(c) Reporting DRs when DCs are asked.

Special : Position Vector Formulas

Centroid of Triangle

Triangle ABC with position vectors a , b , c .

$$\boxed{\overrightarrow{OG} = (\vec{a} + \vec{b} + \vec{c}) / 3}$$

<- G = centroid

Linear Combination

P divides AB internally : $\overrightarrow{OP} = (1-t)\vec{a} + t\,\vec{b}$

where $t = m / (m+n)$ in m:n ratio.

If $t = 0$ -> $P = A$.

If $t = 1$ -> $P = B$.

If $t = 1/2$ -> P = midpoint.

Parallelism / Equality

$\vec{a}$ $\vec{b}$ iff $\vec{a}$ = lambda $\vec{b}$

$\vec{a} = \vec{b}$ iff $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$

Use these to test if two vectors point along the same line (frequently asked).

Final Recap : All Formulas

(1) a = sqrt(a1 2 + a2 2 + a3 2)

(2) a . b = a1 b1 + a2 b2 + a3 b3

(3) cos(theta) = a .b / (a b)

(4) proj_b(a) = a .b / b

(5) a x b = a b sin(theta)

(6) n = (a x b) / a x b

(7) Area tri = (1/2) AB x AC

(8) Area par = a x b

(9) Section formula (int) = (mb + na)/(m+n)

(10) Midpoint = (a + b)/2

(11) Centroid = (a + b + c)/3

(12) l 2 + m 2 + n 2 = 1

(13) a xb 2 + (a .b) 2 = a

Memorise THESE FIRST

* Determinant cross product (5-mark answer)
* Angle formula (2-mark short answer)
* Area of triangle (3-mark)
* Direction cosines identity

Good luck for the boards !

- Vector Algebra