Three Dimensional Geometry

A line in space is fixed by a point + a direction.
Direction is captured by Direction Cosines (DCs)
or Direction Ratios (DRs).

Direction Cosines

If line L makes angles a, b, c with +ve x, y, z axes :

$$\boxed{l = \cos a, \quad m = \cos b, \quad n = \cos}$$ <- DCs

Fundamental identity :

$$\boxed{l^2 + m^2 + n^2 = 1 \quad .}$$ <- MUST
<- remember

*

Direction Ratios

Any 3 numbers a, b, c proportional to l, m, n
are called direction ratios (DRs).

From DRs to DCs :

$$\boxed{l = a/k, \quad m = b/k, \quad n = c/k \quad \text{where} \quad k = \text{sqrt}(a^2 +}$$

*

Note : DCs are unique (up to sign).
DRs are unique only up to a non-zero scalar.

DCs of Line through Two Points

Let $P(x\, y_1, z_1)$, $Q(x_2, y_2, z_2)$.

Distance $PQ = \text{sqrt}((x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2)$

DRs : $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

$$\boxed{l = (x_2-x_1)/PQ, \quad m = (y_2-y_1)/PQ, \quad n = (z_2-z_1)}/PQ$$

DCs <- o

Worked Example

Find DCs of line joining $A(2, 3, -6)$ and $B(3, -4, 5)$.

Sol : DRs $= (1, -7, 11)$.

$k = \text{sqrt}(1 + 49 + 121) = \text{sqrt}(171)$.

$$\boxed{\text{DCs} = (1/\text{sqrt}(171), -7/\text{sqrt}(171), 11/\text{sqrt}(17}1))$$

Sign Convention

Reversing direction of line flips signs of ALL three DCs simultaneously - never just one.

~~Common error~~: l, m, n change one at a time. WRONG.

Equation of a Line in Space

A line passing through point with position vector a, parallel to vector b :

$$\boxed{r = a + l\,b, \quad l \text{ in } R}$$

<- Vector
<- form

r = position vector of any point on the line.
l = real parameter.

Cartesian Form

If $a = x1\,i + y1\,j + z1\,k$, $b = a\,i + b\,j + c\,k$:

$$\boxed{(x - x1)/a = (y - y1)/b = (z - z1)/c}$$

<- symmetric
<- form

Here (a, b, c) are DRs of the line.

Care - Zero DR

If any of a, b, c is zero, do NOT divide by it.
Write that coordinate equation separately.

Eg. if $b = 0 \Rightarrow y = y1$ and $(x-x1)/a = (z-z1)/c$.

Line through Two Points

Line through $A(a)$ and $B(b)$ has direction $(b - a)$.

$$\boxed{r = a + 1(b - a)}$$ <- Vector

$$(x - x1)/(x2 - x1) = (y - y1)/(y2 - y1) = (z$$

Worked Example

Find eqn of line through $P(1, 2, 3)$ and $Q(4, 5, 6)$.

Sol : $a = i + 2j + 3k$, $b = 4i + 5j + 6k$.

$b - a = 3i + 3j + 3k$ (or DRs $= 3, 3, 3$).

Vector eqn :

$$\boxed{r = (i + 2j + 3k) + 1(3i + 3j + 3k)}$$

Cartesian eqn :

$$\boxed{(x-1)/3 = (y-2)/3 = (z-3)/3}$$

Note : same line can be written with any DR multiple.

Angle between Two Lines

Lines with direction vectors b1 and b2 .

Acute angle q between them :

$\boxed{\cos q = b1 . b2 / (b1}$ <- Vector b2)
<- form

In terms of DRs (a1, b1, c1) and (a2, b2, c2) :

$\boxed{\cos q = a1\ a2 + b1\ b2 + c1\ c2 /}$ sqrt(.)sqrt

Special Cases

* PARALLEL : a1/a2 = b1/b2 = c1/c2

* PERPENDICULAR : a1 a2 + b1 b2 + c1 c2 = 0

MOD bars are MANDATORY

CBSE expects the acute angle.

Forgetting . gives obtuse angle => -1 mark.

~~cos q = b1.b2 / b1 b2~~ WRONG (no mod)

Worked Example - Angle

Find acute angle between lines :

L1 : (x-1)/2 = (y-2)/3 = (z-3)/6

L2 : (x+1)/1 = (y-3)/2 = (z+5)/2

Sol :

DRs of L1 = (2, 3, 6), DRs of L2 = (1, 2, 2).

Dot product = 2(1) + 3(2) + 6(2) = 2 + 6 + 12 = 20.

b1 = sqrt(4 + 9 + 36) = sqrt(49) = 7

b2 = sqrt(1 + 4 + 4) = sqrt(9) = 3

$\boxed{\cos q = 20 / (7 . 3) =}$ <- 20 / 21

Therefore q = cos -1 (20/21).

Check : 20/21 < 1 , so valid cosine value.

Skew Lines

Two lines in space are skew if they are :

(i) NOT parallel

(ii) NOT intersecting

i.e. they lie in different planes & never meet.

Shortest Distance (vector form)

L1 : $r = a_1 + l\, b_1$, L2 : $r = a_2 + m\, b_2$

$$\boxed{d = (b_1 \times b_2) \cdot (a_2 - a_1) \;/\; b_1 \times b_2}$$

<- STAR

<- formu

Numerator = scalar triple product (absolute value).

Denominator = magnitude of cross product $b_1 \times b_2$.

Check First !

If $b_1 \times b_2 = 0$ => lines are PARALLEL =>
different formula (next page).

Shortest Distance - Worked Eg.

L1 : r = (i + 2j + 3k) + l(2i + 3j + 4k)

L2 : r = (2i + 4j + 5k) + m(3i + 4j + 5k)

Sol :

a1 = i+2j+3k, a2 = 2i+4j+5k, a2-a1 = i+2j+2k.

$$b1 \times b2 = \begin{vmatrix} i & j & k \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = -i + 2j - k.$$

|b1 x b2| = sqrt(1+4+1) = sqrt(6).

(b1 x b2) . (a2 - a1) = (-1)(1) + 2(2) + (-1)(2)

= -1 + 4 - 2 = 1.

$\boxed{d = 1 / \text{sqrt}(6) = 1 / \text{sqrt}(6)}$ ans units.

Cartesian Form (determinant)

$$\text{Numerator} = \begin{vmatrix} x2-x1 & y2-y1 & z2-z1 \\ a1 & b1 & c1 \\ a2 & b2 & c2 \end{vmatrix}$$

Distance between Parallel Lines

L1 : $r = a1 + l\, b$ L2 : $r = a2 + m\, b$

(same direction vector b)

$$\boxed{d = b \times (a2 - a1) \quad /} \; b$$

<- parallel
<- case

Three-Step Decision Tree

Given two lines, compute $b1 \times b2$:

Step 1. If $b1 \times b2 = 0$ => PARALLEL.

Use $d = b \times (a2-a1) \;/\; b$.

Step 2. If $(b1 \times b2).(a2 - a1) = 0$ => INTERSECT.

Distance = 0 (coplanar lines).

Step 3. Otherwise => SKEW.

Use $d = (b1 \times b2).(a2-a1) \;/\; b1 \times b2$.

* 95 % of CBSE Ch 11 questions match this tree.

Master Formula Recall

$\boxed{l^2 + m^2 + n^2 = 1}$ <- DC identity *

$\boxed{r = a + l b}$ <- line - vector

$\boxed{(x-x1)/a = (y-y1)/b = (z-z1)/c}$ <- Cartesian

$\boxed{\cos q = b1 . b2 \ / (\ b1}$ <- angle)

$\boxed{d = (b1 \times b2).(a2-a1) \ / \ b1 \times b2}$ * <- shortest <- distance

Exam-day Tips

1. Always check parallel before applying SD formula.
2. Keep . bars - don't lose 1 mark on sign.
3. Convert vector <-> Cartesian fluently.
4. 5-mark Q is almost always Shortest Distance.

ALL THE BEST - Practice 6 questions of each type !