Linear Programming

Linear Programming (LP) is a method to find the optimum (max or min) value of a linear function of variables, subject to linear inequalities (constraints).

Standard form

$\boxed{\text{Maximise / Minimise} \quad Z = ax + by}$ <- objective <- function

subject to constraints :

$a_1 x + b_1 y <= c_1$

$a_2 x + b_2 y <= c_2$ ... etc.

$x >= 0, \quad y >= 0$ (non-negativity)

Key terms

1. Decision variables - x, y (unknowns)
2. Objective function - $Z = ax + by$
3. Constraints - linear inequalities
4. Non-negativity - $x >= 0, y >= 0$

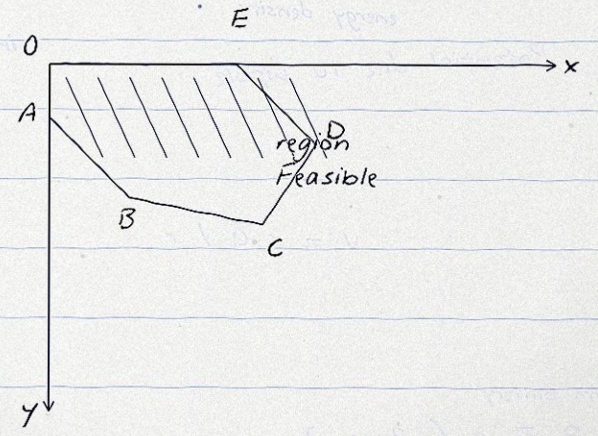
5. Feasible region - set of all points satisfying ALL constraints.
6. Optimal solution - point in feasible region where Z is opt.

Graphical Method

For LP problems in TWO variables, we use the graphical method.

Steps

(1) Formulate the LPP from word problem.
(2) Convert each inequality into eqn. and plot the line on graph.
(3) Shade the half-plane that satisfies the inequality (test origin if 0 notlie).
(4) Intersect all shaded regions -> this is the feasible region.
(5) Find corner points (vertices).
(6) Evaluate Z at each corner.
(7) Pick the max / min value.



Vertices A, B, C, D, E are corner points.

Corner Point Method

The optimum of Z occurs at a corner point

<- Key
<- theorem

Procedure

(i) Find feasible region R (graph).
(ii) Find all corner points of R.
(iii) Compute $Z = ax + by$ at each corner.
(iv) Pick max / min as required.

Theorem 1

If R is bounded, then Z attains BOTH its maximum AND minimum values at corner points of R.

Theorem 2

If R is unbounded, M = max value at a corner is the maximum of Z over R only if the open half-plane $ax + by > M$ has NO point in common with R.

Same rule for min : $ax + by < m$ must have no point in R.

Memory hook Corners decide everything (bounded).

Worked Example (Manufacturing)

A factory makes x chairs & y tables.

Profit $Z = 30x + 50y$.

Constraints :

$x + 2y \le 10$ (wood)

$3x + y \le 12$ (labour)

$x \ge 0$, $y \ge 0$.

Step 1 : Plot lines

$x + 2y = 10$: $(10,0)$, $(0,5)$

$3x + y = 12$: $(4,0)$, $(0,12)$

Step 2 : Corner points

Solve simultaneously :

$x + 2y = 10$, $3x + y = 12$

$\Rightarrow$ from eqn 1 : $x = 10 - 2y$

$\Rightarrow 3(10 - 2y) + y = 12$

$\Rightarrow 30 - 5y = 12 \Rightarrow y = 18/5$

$\Rightarrow x = 10 - 36/5 = 14/5$

Corners : $O(0,0)$, $A(4,0)$, $B(14/5, 18/5)$, $C(0,5)$.

Step 3 : Eval Z

$Z(B) = 30(14/5) + 50(18/5) = 264$

Worked Example (contd.)

- $Z(C) = 50 \times 5 = 250$

Compare : 0, 120, 264, 250

$$\boxed{\text{Maximum profit } Z = 264 \quad \text{at} \quad (14/5 \, , \, 18/5)}$$

Step 4 : Conclusion

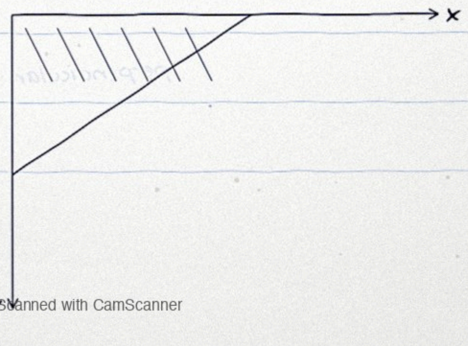
Make 14/5 2.8 chairs and 18/5 3.6 tables (integer rounding if needed).
Maximum profit = Rs. 264.

Reading the inequality direction

Test origin (0,0) in inequality.
 Origin satisfies => origin side
 Origin fails => opposite side

Boundary vs strict

 <=, >= : line included (solid line)
 <, > : line NOT included (dashed)



Types of feasible regions

Bounded region

Closed polygon. Max AND min of Z both exist at corner points.

Unbounded region

Extends to infinity. Z may not have max / min. Check the open half-plane test (see theorem 2 earlier).

Multiple optimal solutions

If Z equals the optimum at TWO adjacent corners, then EVERY point on the edge joining them is also optimum -> infinitely many solutions.

Caused when $Z = ax + by$ is parallel to a constraint line.

Infeasible LPP

Constraints contradict each other.
Feasible region = empty set.
No solution exists.

[illegible] BEFORE

Diet Problem (example)

Minimise diet cost. Two foods F1, F2.

Let x kg of F1 and y kg of F2.

Minimise $C = 4x + 6y$ (cost in Rs.)

Constraints (nutrition) :

$3x + 6y >= 80$ (protein)

$4x + 3y >= 100$ (carbs)

$x >= 0$, $y >= 0$

Plot lines

$3x + 6y = 80$: $(80/3, 0)$, $(0, 80/6)$

$4x + 3y = 100$: $(25, 0)$, $(0, 100/3)$

Find corners

Region is unbounded (open to +inf).

Solve $3x+6y = 80$ and $4x+3y = 100$:

$x = (200-80)/5 = 24$, $y = (80-72)/6 = 4/3$

wait recompute : multiply 1st by 2 :

$6x+12y = 160$, $4x + 3y = 100$

Subtract : $2x + 9y = 60$

Vertices : $A(25, 0)$, $B(24, 4/3)$,
$C(0, 100/3)$.

Eval cost $C = 4x + 6y$ at each corner.

Diet Problem (contd.)

$C(A) = 4(25) = 100$

$C(B) = 4(24) + 6(4/3) = 96 + 8 = 104$

$C(C) = 6(100/3) = 200$

Min so far = 100 at A.

Region unbounded -> check $4x+6y < 100$ intersects R or not.

Origin (0,0) gives $0 < 100$ but origin is NOT in R (fails constraints).

$$\boxed{\text{Min } C = \text{Rs. } 100 \quad \text{at} \quad (25, 0)}$$

Transportation problem

Goods shipped from godowns to shops.

Minimise total transport cost.

Use 2 decision variables (x, y).

Rest follow by demand / supply.

Manufacturing - mixture problems

Mixing two products with limited raw materials. Max profit.

Resources => <= constraints

Demand minimum => >= constraints

Common mistakes (avoid ❗)

(1) Forgetting $x >= 0$, $y >= 0$.

Always mention non-negativity.

(2) Wrong half-plane while shading.

Always do the origin test.

(3) Skipping a corner of feasible reg.

List ALL vertices systematically.

(4) Solving wrong pair of equations.

Each corner = of 2 lines.

(5) Misreading max as min or vice versa.

(6) Forgetting to round in real-life Q.

Integer constraints (objects).

Quick checklist

* Define decision variables clearly.
* Write $Z = ax + by$.
* Write constraints + non-negativity.
* Plot, shade, identify R.
* Corners -> Z values -> compare.
* State final answer in words.

Summary

Definitions

Objective fn $Z = ax + by$ (linear)

Constraints linear inequalities in x, y

Feasible reg. set of (x, y) satisfying all

Corner point vertex of feasible region

Optimal soln point where Z is max / min

Theorems

T1 : Bounded R => both max & min occur at corner points.

T2 : Unbounded R => use half-plane test.

Solving steps (golden)

Formulate -> Plot -> Vertices -> Eval Z

<- the 4
<- steps

Tip

If two corners give the same Z value, every point on the edge between them is optimum (infinite solutions).

End of Chapter 12.