Probability

Probability measures how likely an event is to happen.

$$\boxed{P(A) = n(A) / n(S)}$$ <- classical <- definition

S = sample space (all possible outcomes)
A = favourable outcomes (subset of S)

Basic axioms

(i) $0 <= P(A) <= 1$
(ii) $P(S) = 1$, $P(\text{phi}) = 0$
(iii) $P(A') = 1 - P(A)$

Compound events

$$\boxed{P(A \quad B) = P(A) + P(B) - P(A \quad B)}$$ <- addn rule

If A, B are mutually exclusive ($A \quad B$ = phi)
$P(A \quad B) = P(A) + P(B)$

Recall (Class 11)

Equally likely outcomes -> count favourable / total.

Conditional Probability

P(A B) means probability of A, given that B has already occurred.

$$\boxed{P(A \quad B) = P(AB) \,/\, P(B) \;, \quad P(B) > 0}$$

<- Key
<- formula

Properties

(i) $0 <= P(A \quad B) <= 1$

(ii) $P(S \quad B) = P(B \quad B) = 1$

(iii) $P((A \quad C) \quad B) = P(A \quad B) + P(C \quad B)$
$- P((A \quad C) \quad B)$

(iv) $P(A' \quad B) = 1 - P(A \quad B)$

Example

Toss 2 dice. Given sum > 8 (event B), find P(both 4 B).

Pairs with sum > 8 :
(3,6)(4,5)(4,6)(5,4)(5,5)(5,6)
(6,3)(6,4)(6,5)(6,6) ... 10 in all

Both >= 4 : (4,5)(4,6)(5,4)(5,5)(5,6)
(6,4)(6,5)(6,6) = 8 pairs

$$\boxed{P(\text{both} \quad 4 \quad B) = 8/10 = 4/5}$$

Multiplication Theorem

$$\boxed{P(A \quad B) = P(A) \,.\, P(B \quad A) = P(B) \,.\, P(A \quad B)}$$

If $P(A) > 0$ and $P(B) > 0$.

Three events (chain rule)

$$\boxed{P(ABC) = P(A)\, P(B \quad A)\, P(C \quad AB)}$$

Independent events

A and B are independent if occurrence of one does NOT affect the other.

*

$$\boxed{A\,,\,B \text{ indep.} \iff P(A \quad B) = P(A)\,.\,P(B)}$$

Equivalently $P(A \quad B) = P(A)$

and $P(B \quad A) = P(B)$.

Mutually exclusive vs independent

M.E. : $A \quad B = \text{phi}$

Indep. : $P(A \quad B) = P(A)\, P(B)$

Two diff. concepts ! Don't confuse.

Scanned with CamScanner

Total Probability Theorem

Partition of S : events $E1, E2, \ldots En$
pairwise disjoint and $Ei = S$.
(each $P(Ei) > 0$).

$$\boxed{P(A) = P(Ei) . P(A \quad Ei)}$$

<-total
<-prob.

Example . (two-urn)

Urn-1 : 3 red, 2 black
Urn-2 : 4 red, 1 black
Pick urn at random, then a ball.
$P(red) = ?$

$P(U1) = P(U2) = 1/2$
$P(R \quad U1) = 3/5, \quad P(R \quad U2) = 4/5$

$$P(R) = 1/2 . 3/5 + 1/2 . 4/5$$
$$= 3/10 + 4/10 = 7/10$$

$$\boxed{P(red) = 7/10}$$

Always check $P(Ei) = 1$ first.

Bayes' Theorem

Reverses the conditioning.

If $E_1, \ldots, E_n$ partition S and A is an event,

$$\boxed{P(E_i \;\; A) = P(E_i)\, P(A \;\; E_i) \;/\; P(E_j)\, P(A \;\; E_j)}$$ <-B

Terms

- $P(E_i)$ - prior probability
- $P(A \;\; E_i)$ - likelihood
- $P(E_i \;\; A)$ - posterior probability

Mnemonic

Prior * Likelihood / Evidence

Quick proof

From mult. theorem :

$$P(E_i \;\; A) = P(E_i)\, P(A \;\; E_i)$$
$$= P(A)\, P(E_i \;\; A)$$

Equate : $P(E_i \;\; A) = P(E_i)\, P(A \;\; E_i) \;/\; P(A)$

$P(A)$ by total prob. theorem.

Bayes is the most common 4/5-mark Q.

Master the setup of the prior table.

Bayes Example (defective bolt)

Factory has 3 machines A, B, C.

Output rates : 50%, 30%, 20%.

Defect rates : 1%, 4%, 3%.

A bolt is defective. Find P(from A).

Step 1 : Priors

$P(A) = 0.5, \quad P(B) = 0.3, \quad P(C) = 0.2$

Step 2 : Likelihoods

$P(D \quad A) \overset{*}{=} 0.01, \quad P(D \quad B) = 0.04,$

$P(D \quad C) \overset{*}{=} 0.03$

Step 3 : Evidence (Total P.)

$P(D) = 0.5(0.01) + 0.3(0.04) + 0.2(0.03)$

$P(D) = 0.005 + 0.012 + 0.006 = 0.023$

Step 4 : Posterior

$P(A \quad D) = (0.5)(0.01) \,/\, 0.023$

$P(A \quad D) = 0.005 \,/\, 0.023$

$P(A \quad D) \quad 5/23$

$\boxed{\text{Answer} \;:\; P(A \quad D) = 5/23}$

Random Variable

A random variable X is a real-valued function defined on the sample space.

$$\boxed{X : S \to R}$$

Discrete random variable

X takes a finite (or countable) number of values x1, x2, ... xn.

Probability distribution

Table of values xi and probabilities pi :

pi = P(X = xi)

0 <= pi <= 1

pi = 1 (mandatory check)

X :	0	1	2	3
P :	1/8	3/8	3/8	1/8

(eg. # heads in 3 tosses)

Mean / Expectation

$$\boxed{E(X) = \; = \; xi \cdot pi}$$ <- average

Variance & SD

$\boxed{Var(X) = \quad = E(X) - [E(X)]}$ <- shortcut

OR equivalently :

$\boxed{Var(X) = \quad (xi - \quad) pi}$

Standard deviation = sqrt(Var(X))

= sqrt()

Example

X = # heads in 2 fair tosses :

X : 0, 1, 2

P : 1/4, 2/4, 1/4

E(X) = 0 + 2/4 + 2/4 = 1

E(X) = 0 + 1.(2/4) + 4.(1/4) = 6/4 = 1.5

Var(X) = 1.5 - 1 = 0.5

= sqrt(0.5) 0.707

measures spread about the mean.

= 0 => X is constant a.s.

Bernoulli & Binomial

Bernoulli trial

A trial with exactly 2 outcomes :

success (prob $= p$) and

failure (prob $= q = 1 - p$)

Binomial distribution

n independent Bernoulli trials.

X = # successes in n trials.

$$\boxed{P(X = r) = nCr \;.\; p \quad r \;.\; q \quad (n-r)}$$ <- Binomial <- PMF

$r = 0, 1, 2, \ldots, n$

$q = 1 - p$

Mean and Variance

$$\boxed{= np \quad , \quad = npq \quad , \quad = sqrt(npq)}$$

Conditions (BINS)

B : Binary outcomes (S / F)

I : Independent trials

N : fixed N , S : same p for all trials

Binomial Example

A die is rolled 5 times. Find prob. of exactly 2 sixes.

Success = roll a 6 ; $p = 1/6$, $q = 5/6$

$n = 5$, $r = 2$

$$P(X=2) = 5C2 \ . \ (1/6) \quad 2 \ . \ (5/6) \quad 3$$
$$= 10 \ . \ (1/36) \ . \ (125/216)$$
$$= 1250 \ / \ 7776 \quad 0.1608$$

$\boxed{P(2 \text{ sixes in } 5 \text{ rolls}) \quad 0.161}$

Tip

For 'at least one', use $1 - P(0)$

$$P(X >= 1) = 1 - P(X = 0) = 1 - q \quad n$$

Sum / cumulative form

$$P(X <= r) = \text{ from } k=0 \text{ to } r \text{ of } P(X=k)$$

Independence reminder

All n trials must be independent.

If sampling 'without replacement',

X is NOT binomial (use hypergeom).

Common mistakes (avoid !)

(1) Confusing P(A B) with P(B A).

Order MATTERS in conditional.

(2) Forgetting pi = 1 in distribution.

(3) Adding probabilities of indep. events.

Indep. => MULTIPLY, not add.

(4) Treating mutually exclusive as indep.

Diff. concepts ; rarely both.

(5) Wrong substitution in Bayes.

Always write prior table first.

(6) Binomial w/o checking n independence.

(7) Forgetting 0 P 1 as sanity check.

Quick checklist

* Define sample space S clearly.
* Note dependencies of events.
* Use tree / Venn diagram if helpful.
* pi = 1 , 0 pi 1 : check.
* State answer in fraction or decimal.

Summary

Core formulas

P(AB) = P(A) + P(B) - P(AB)

P(A B) = P(AB)/P(B)

P(AB) = P(A) P(B A)

Indep. : P(AB) = P(A) P(B)

Total P : P(A) = P(Ei) P(A Ei)

Bayes : P(Ei A) = P(Ei) P(A Ei) / P(A)

Distribution

E(X) = xi pi

Var = E(X) - [E(X)]

= sqrt(Var)

Binomial

P(X=r) = nCr p r q (n-r)

= np , = npq

Strategy

1. Identify type : basic / cond. / Bayes / binomial / distribution.
2. Write down given data clearly.
3. Pick correct formula + substitute.

End of Chapter 13.