Continuity & Differentiability

Continuity at a Point

A function f is continuous at x = c if the ~~value~~ limit exists at c and equals the value of the function there.

$$\boxed{\lim_{x \to c} \quad f(x) \quad = \quad f(c)}$$

<-all 3 must agree
<-L = R = f(c)

Three conditions :

(i) f(c) is defined
(ii) lim f(x) exists (LHL = RHL)
(iii) LHL = RHL = f(c)

Continuity on an Interval

f is continuous on (a,b) if it is continuous at every point of (a,b). On [a,b] add :

RHL at a = f(a), LHL at b = f(b).

Types of Discontinuity

(a) Removable : limit exists but != f(c)
(b) Jump : LHL exists, RHL exists, LHL != RHL
(c) Infinite : one or both limits are infty
(d) Oscillatory : e.g. sin(1/x) near 0

Polynomials & sin, cos, e x are continuous everywhere.

Algebra of Continuous Functions

If f and g are continuous at x = c, then :

$$\boxed{(f +/- g)(x) \text{ is continuous at } c}$$ <-sum/diff rule

$$\boxed{(f . g)(x) \text{ is continuous at } c}$$ <-product rule

$$\boxed{(f / g)(x) \text{ cts at } c \text{ if } g(c) \ != }\ 0$$ <-quotient rule

Composition Rule

If g is cts at c and f is cts at g(c),
then ~~g . f~~ $(f \circ g)(x) = f(g(x))$
is continuous at x = c.

$$\boxed{(f \circ g)(c) = f(g(c))}$$ <-chain of cts fns
<-is itself cts

Differentiability at a Point

f is differentiable at x = c if the limit

$$\boxed{f(c) = \lim_{h \to 0} [f(c+h) - f(c)] / h}$$ <-slope of
<-tangent line

exists and is finite. LHD = RHD required.

Diff at c => cts at c but NOT ~~vice versa~~ converse.

Derivatives — Standard Results

d/dx (x n) = n x. (←power rule

Trigonometric

d/dx (sin x) = cos x

d/dx (cos x) = - sin x

d/dx (tan x) = sec 2 x

d/dx (cot x) = - cosec 2 x

d/dx (sec x) = sec x . tan x

d/dx (cosec x) = - cosec x . cot x

Inverse Trigonometric

d/dx (sin -1 x) = 1 / sqrt(1 - x 2)

d/dx (cos -1 x) = - 1 / sqrt(1 - x 2)

d/dx (tan -1 x) = 1 / (1 + x 2)

d/dx (cot -1 x) = - 1 / (1 + x 2)

d/dx (sec -1 x) = 1 / (x sqrt(x 2-1))

Exponential & Logarithmic

d/dx (e x) = e x

d/dx (a x) = a x . log a

d/dx (log x) = 1 / x (x > 0)

d/dx ($\log_a$ x) = 1 / (x log a)

Memory tip : ~~memorize~~ practise these daily !

Chain Rule & Implicit Diff

Chain Rule

If $y = f(u)$ and $u = g(x)$, then

$$\boxed{dy/dx = dy/du \cdot du/dx}$$ <-compose, <-then multiply

Ex : $y = \sin(x\ \ 2) \Rightarrow dy/dx = \cos(x\ \ 2) \cdot 2x$

Sum, Product & Quotient

$(u +/- v)' = u' +/- v'$

$(u v)' = u' v + u v'$ (Leibnitz)

$(u/v)' = (u' v - u v') / v\ \ 2$ $v\ != 0$

Implicit Differentiation

When y is given implicitly via $F(x,y) = 0$: differentiate term-by-term w.r.t. x, treat y as a function of x and use ~~d/dy~~ chain rule.

Ex : $x\ \ 2 + y\ \ 2 = r\ \ 2$

Differentiate : $2x + 2y \cdot dy/dx = 0$

$$\boxed{dy/dx = -x / y}$$ <-slope on circle

All inverse trig results come from this idea !

E.g. $y = \sin\ -1\ x \Rightarrow \sin y = x \Rightarrow$ ...

Logarithmic & Parametric

Logarithmic Differentiation

Useful when y = [f(x)] g(x) or products/ quotients of many factors. Take log first.

Steps :

1. Take log on both sides
2. Use log laws to expand
3. Differentiate w.r.t. x
4. Multiply both sides by y

Ex : y = x x => log y = x log x

1/y . dy/dx = log x + 1

$$\boxed{dy/dx = x \quad x \, (\log x + 1)}$$

<-variable base
<-AND exponent

Parametric Form

If x = f(t), y = g(t), parameter t :

$$\boxed{dy/dx = (dy/dt) \, / \, (dx/dt)}$$

<-divide rates

Ex : x = a cos t, y = a sin t.

dx/dt = - a sin t, dy/dt = a cos t

dy/dx = - cot t (slope of circle)

Need dx/dt != 0 for dy/dx to be defined.

Second Derivative & MVT

Second Order Derivative

Differentiate dy/dx once more to get :

$$\boxed{d^2y/dx^2 = d/dx\,(dy/dx)}$$ y'' also <- written f''(x)

Ex : y = sin x

y' = cos x , y'' = - sin x

So y'' + y = 0 (SHM-type relation)

For parametric x=f(t), y=g(t) :

$$d^2y/dx^2 = d/dt\,(dy/dx) \,/\, (dx/dt)$$

Rolle's Theorem

If f is cts on [a,b], diff on (a,b), and f(a) = f(b), then Exists c in (a,b) such that f'(c) = 0. (horizontal tangent)

Mean Value Theorem

If f is cts on [a,b], diff on (a,b) :

$$\boxed{f'(c) = [\,f(b) - f(a)\,] / (b - a)}$$ <- Lagrange's <- MVT

Geometric : slope of tangent at c equals slope of chord joining (a,f(a)) and (b,f(b)).