Chapter 6

Application of Derivatives

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What is this chapter about?

The derivative dy/dx of a function tells how y changes with x. Here we apply this idea to :

(1) Rate of change of one quantity w.r.t. another (related rates).

(2) Whether a function is increasing or decreasing on an interval.

(3) Finding maximum / minimum values of a function ~~on a graph~~ (optimisation).

(4) Behaviour of curves : tangents, normals, concavity, inflection points.

Pre-requisites

- * Differentiation rules from Ch 5.
- * Chain rule : dy/dx = (dy/du)(du/dx).
- * Standard derivatives :

d/dx(x n) = n x (n-1) ; d/dx(sin x) = cos recall x

d/dx(e x) = e x ; d/dx(ln x) = 1/x

Session 2026-27 . Class XII Maths .

6.1 Rate of change

If $y = f(x)$, then dy/dx at $x = x0$ is the instantaneous rate of change of y w.r.t. x at that point.

$\boxed{\text{rate of change} = dy/dx}$ <-key formula

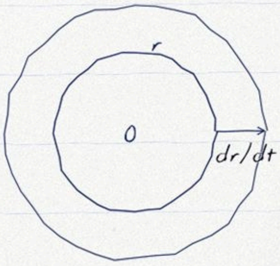
If x and y are both functions of t :

$\boxed{dy/dt = (dy/dx) \cdot (dx/dt)}$ <-chain rule

Eg. expanding circle

Radius r of a circle grows at 2 cm/s.

Find rate of change of area when $r = 5$.



$A = A = \pi \ r \quad 2$

$dA/dt = 2 \pi \ r \ (dr/dt)$

At $r = 5$, $dr/dt = 2$:

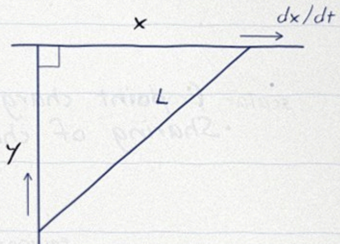
$dA/dt = 2 \pi \ (5)(2) = 20 \pi$ cm 2/s .

Common quantities & their rates

- dx/dt -> linear speed
- dV/dt -> volume rate (filling/draining)
- dA/dt -> area rate (e.g. spilled oil)
- $d\theta/dt$ -> angular speed (rad/s) ,

Sliding ladder problem

A 5 m ladder rests on a wall .
Foot is pulled away at 0.5 m/s .
Find rate at which top slides down
when foot is 3 m from wall .



Let x = foot distance , y = wall height .

$$\boxed{x \quad 2 + y \quad 2 = 25}$$ <-Pythagoras

Differentiate w.r.t. t :
$2x(dx/dt) + 2y(dy/dt) = 0$
$dy/dt = -(x/y)(dx/dt)$

At $x = 3$; $y = \text{sqrt}(25 - 9) = 4$.
$dy/dt = -(3/4)(0.5) = -0.375$ m/s
Negative -> top is ~~rising~~ falling .

Marginal cost & revenue

If $C(x)$ = total cost of producing x units :

$$\boxed{MC = dC/dx \; ; \quad MR = dR/dx}$$ <-economics

Marginal ≐ rate of change of cost (or revenue) per extra unit produced .
It is an important application in CBSE !

6.2 Increasing & decreasing functions

f is increasing on I if

$x1 < x2$ in I $=>$ $f(x1) <= f(x2)$:

f is strictly increasing if '<' instead .

Similarly for decreasing (replace <= by >=).

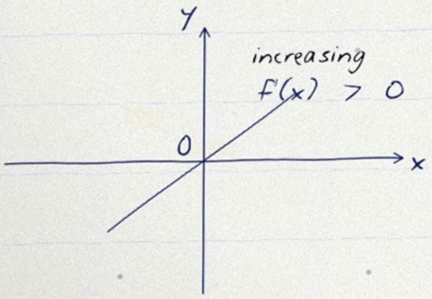
Derivative test

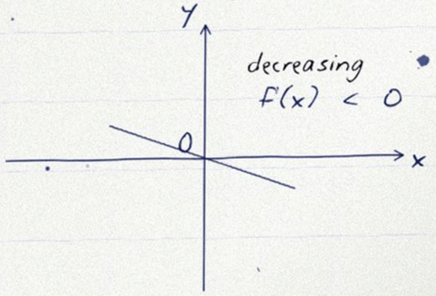
$f'(x) > 0$ on I $=>$ f strictly increasing <- main test

$f'(x) < 0$ on I $=>$ f strictly decreasing

$f'(x) = 0$ on I $=>$ f is constant

These hold on any open interval (a, b).
On a closed interval $[a, b]$ we additionally
require f continuous on $[a, b]$.





Slope > 0 -> rising . Slope < 0 -> falling .

Method to find intervals

Step 1 : find $f'(x)$.

Step 2 : solve $f'(x) = 0$ -> critical pts.

Step 3 : mark them on a number line .

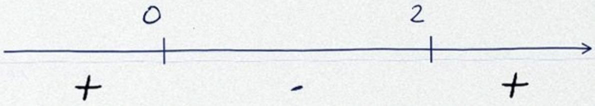
Step 4 : test sign of f' in each interval .

Step 5 : + -> increasing , - -> decreasing .

Eg. $f(x) = x \quad 3 - 3x \quad 2 + 4$

$f'(x) = 3x \quad 2 - 6x = 3x(x - 2)$

$f'(x) = 0 \Rightarrow x = 0 , \quad x = 2$.



Reading the chart :

- $f' > 0$ on $(-\mathrm{inf}, 0)$ -> increasing
- $f' < 0$ on $(0, 2)$ -> decreasing
- $f' > 0$ on $(2, \mathrm{inf})$ -> increasing

Common mistakes

- Sign of f' , not f itself !
- Domain matters - exclude pts where f undefined.
- Endpoints : use $[a, b]$ with continuity .

More worked examples

Eg 1 . $f(x) = \sin x$ on $[0, 2\pi]$.

$f'(x) = \cos x$; $\cos x = 0$

=> $x = \pi/2$, $3\pi/2$

Sign of cos x :

* + on (0 , pi/2) -> increasing

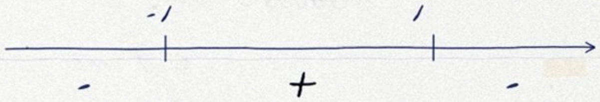
* - on (pi/2, 3pi/2) -> decreasing

* + on (3pi/2, 2pi) -> increasing

Eg 2 . f(x) = x / (1 + x 2)

f'(x) = (1 - x 2) / (1 + x 2) 2

f'(x) = 0 => x = +/- 1 .



* increasing on (-1, 1)

* decreasing on (-inf, -1) and (1, inf)

Note : numerator decides sign here ; denominator is always > 0 .

6.3 Tangents & normals

Tangent at $P(x0, y0)$ on $y = f(x)$:

Slope $m = f'(x0)$.

$$\boxed{y - y0 = f'(x0)\ (x - x0)}$$ <-tangent line

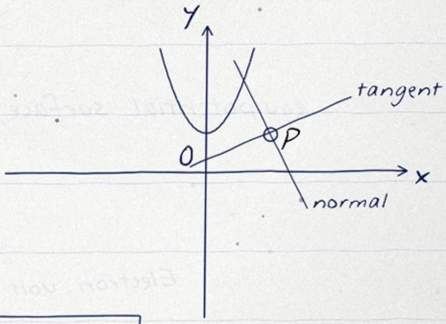
Normal at P is perpendicular to tangent .

$$\boxed{y - y0 = -(1 / f'(x0))\ (x - x0)}$$ <-normal line

Special cases :

* $f'(x0) = 0$ -> tangent is horizontal .
normal is $x = x0$ (vertical).

* $f'(x0)$ infinite -> tangent $x = x0$.
normal is $y = y0$.



Angle between curves

$$\boxed{\tan(t) = (m1 - m2)/(1 + m1\ m2)}$$

$m1$, $m2$ = slopes of the two tangents at the intersection point .

Worked example

Find eq. of tangent & normal to
$y = x \ 2 - 2x + 1$ at $x = 2$.

At $x = 2$: $y = 4 - 4 + 1 = 1$.

$f'(x) = 2x - 2$

Slope $m = f'(2) = 2$.

Tangent : $y - 1 = 2(x - 2)$

$$\boxed{\text{i.e. } y = 2x - 3}$$

Normal : $y - 1 = -(1/2)(x - 2)$

$$\boxed{\text{i.e. } x + 2y - 4 = 0}$$ <-answer

Useful tip

If problem says tangent is parallel to a given line of slope k, set $f'(x) = k$.
If perpendicular, set $f'(x) = -1/k$.

Tangent at a point on a circle

For $x \ 2 + y \ 2 = a \ 2$ at point $(x0, y0)$:

$$\boxed{x \ x0 + y \ y0 = a \ 2}$$ <-T = 0 trick

6.4 Approximations (linear)

Small change formula :

$$\boxed{dy \qquad f'(x) \;.\; dx}$$ <-linearisation

Used to estimate $f(x + dx)$ from $f(x)$:

$$\boxed{f(x+dx) \qquad f(x) + f'(x)\, dx}$$

Eg. approx sqrt(25.3)

Let $f(x) = \text{sqrt}(x)$, $x = 25$, $dx = 0.3$.

$f'(x) = 1 / (2\ \text{sqrt}(x)) = 1/10$

$dy = 0.3 / 10 = 0.03$

$\text{sqrt}(25.3) \qquad 5 + 0.03 = 5.03$

Error in measurement

If $V = f(r)$ and r has error dr :

$$\boxed{dV \qquad f'(r)\, dr}$$

% error = $(dV / V) \times 100$ % .

Eg. sphere : $V = (4/3)\, \Pi \ r \quad 3$.

$dV/V = 3\ dr/r$ => % error in V

= 3 (% error in r)

* note : ~~2~~ 3 because $V \qquad r \quad 3$.

6.5 Maxima & minima — intro

Let $f : I \to R$. A point c in I gives :

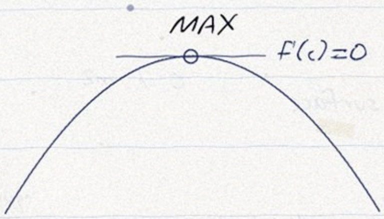
- local max if $f(c) >= f(x)$ near c .
- local min if $f(c) <= f(x)$ near c .

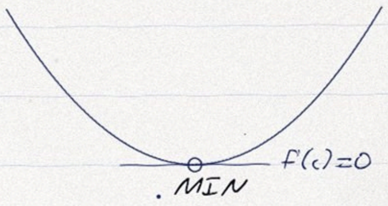
- abs (global) max on I if $f(c) >= f(x)$ all x in I.
- abs (global) min on I if $f(c) <= f(x)$ all x in I.

Critical (stationary) points

$$\boxed{f'(c) = 0 \quad \text{or} \quad f'(c) \text{ doesn't exist}}$$ <-candidates

Fermat's theorem : every interior local extremum is a critical point .
(Converse is NOT true - e.g. x 3 at 0 .)





Both vertices have $f'(c) = 0$ but very different shapes around c . We need a test to tell apart .

First-derivative test

Let c be a critical point of f .

Check the sign of f' on either side :

(a) f' changes + to - at c -> local MAX

(b) f' changes - to + at c -> local MIN

(c) no sign change -> NOT extremum

(point of inflecti

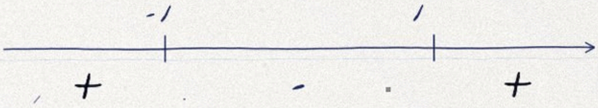
+ -> - : MAX	- -> + : MIN

<- memorise

Eg. f(x) = x 3 - 3x

f'(x) = 3x 2 - 3 = 3(x - 1)(x + 1)

Critical pts : x = +/- 1 .



At x = -1 : + -> - -> local MAX

f(-1) = -1 + 3 = 2 .

At x = +1 : - -> + -> local MIN

f(1) = 1 - 3 = -2 .

Visualise on the cubic curve next page .

Second-derivative test

Let c be a critical point with $f'(c) = 0$.

Compute $f''(c)$:

$\boxed{f''(c) > 0 \quad \Rightarrow \quad \text{local MIN at } c}$ <-shortcut !

$\boxed{f''(c) < 0 \quad \Rightarrow \quad \text{local MAX at } c}$

$\boxed{f''(c) = 0 \quad \Rightarrow \quad \text{test fails ; use 1st test}}$

Mnemonic : +ve f'' -> bowl (min) ;

-ve f'' -> ~~frown~~ arch (max) .

Same eg revisited

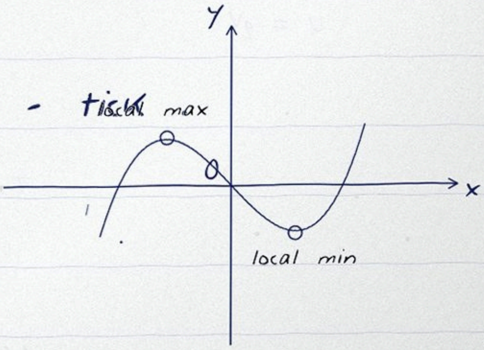
$f(x) = x \quad 3 - 3x$

$f''(x) = 6x$

At $x = -1$: $f''(-1) = -6 < 0$ -> MAX .

At $x = 1$: $f''(1) = +6 > 0$ -> MIN .

Matches the 1st-derivative test - tick



Always quote f'' value while writing answer .

Absolute max & min on [a, b]

If f is continuous on closed [a, b]
then f attains its max and min on [a, b]
(Extreme value theorem).

Procedure

(1) Find all critical pts in (a, b).
(2) Evaluate f at : critical pts + a + b .
(3) The largest value = abs MAX .
The smallest value = abs MIN .

$$\boxed{\text{abs extrema in } (CP) \cup \{a, b\}}$$ <-checklist

Eg. $f(x) = x \; 3 - 12x$ on $[-3, 3]$

$f'(x) = 3x \; 2 - 12 = 3(x-2)(x+2)$
Critical pts in $(-3, 3)$: $x = -2 , 2$.

Now evaluate :
$f(-3) = -27 + 36 = 9$
$f(-2) = -8 + 24 = 16$ <- abs MAX
$f(2) = 8 - 24 = -16$ <- abs MIN
$f(3) = 27 - 36 = -9$

$$\boxed{MAX = 16 \text{ at } x = -2 \; ; \quad MIN = -16 \text{ at } x = 2}$$

Concavity & inflection (extra)

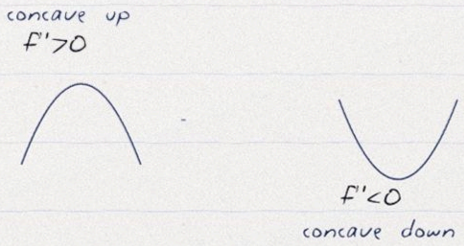
f is concave up on I if $f''(x) > 0$ there .

f is concave down on I if $f''(x) < 0$ there .

An inflection point is where concavity changes (f'' changes sign).

inflection : $f''(c) = 0$ AND sign changes

<-defini



Why useful ?

Curve shape : up vs down bowl .

Helps draw graphs accurately .

Confirms 2nd-derivative test result .

Quick example

$f(x) = x \quad 3$: $f''(x) = 6x$.

$f'' = 0$ at 0 ; sign changes -> $x = 0$ is an inflection point . (Not an extremum !)

6.6 Optimisation - method

Word-problem recipe :

(1) Read carefully ; draw a figure .
(2) Name variables ; mark them on figure .
(3) Write the quantity Q to be max / min
in terms of variables .
(4) Use a constraint to reduce Q to one var .
(5) Find dQ/dx ; set = 0 ; solve .
(6) Use 2nd-deriv test to confirm max/min .
(7) State the answer with units !

model -> reduce -> diff -> test

<-key chain

Common pitfalls

- Don't forget to ~~ignore~~ state units (cm , cm 2 , cm 3)
- Variables must be > 0 in physical problems .
- If 2nd-deriv test fails , use 1st-deriv test .
- Always verify the critical pt is admissible .

Useful identities

- Cylinder : V = pi r 2 h ; S = 2pi r(r+h)
- Cone : V = (1/3) pi r 2 h ; L 2 = r 2 +
- Sphere : V = (4/3) pi r 3 ; S = 4 pi r 2
- Box : V = lbh ; S = 2(lb + bh + hl)
- Circle : A = pi r 2 ; C = 2 pi r

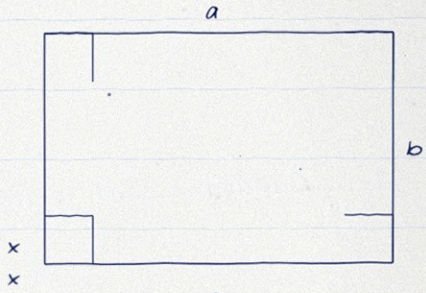
Eg. open box from sheet

Sheet : a x b rectangle .

Squares of side x cut from each corner ,

sides folded up to form open box .

Find x that maximises volume .



cut x from each corner

$$V(x) = x\,(a - 2x)(b - 2x)$$

Take particular case $a = b = L$:

$$V = x\,(L - 2x) \quad 2$$

Differentiate :

$$V'(x) = (L - 2x) \quad 2 + x \,.\, 2(L - 2x)(-2)$$

$$= (L - 2x)(L - 6x)$$

$V'(x) = 0 \Rightarrow x = L/2$ (rejected) or $x = L/6$.

$$V''(x) = -8L + 24x$$

At $x = L/6$: $V'' = -8L + 4L = -4L < 0$.

$$\boxed{V\_max \quad \text{at} \quad x = L/6}$$

$$\boxed{V\_max = (L/6)(2L/3) \quad 2 = 2 \; L \quad 3 \; /}$$ <2a7nswer

Eg. closest pt on a parabola

Find pt on y = x 2 nearest to (0, 1).

Let P(x, x 2) be a point on the curve.

d 2 = x 2 + (x 2 - 1) 2

= x 4 - x 2 + 1

Minimise d 2 (same x as d, easier algebra).

(d 2)' = 4 x 3 - 2 x = 2 x (2 x 2 - 1)

= 0 => x = 0 or x = +/- 1/sqrt(2).

Check 2nd derivative :

(d 2)'' = 12 x 2 - 2

At x = +/- 1/sqrt(2) : 12 . 1/2 - 2 = 4 > 0 ->

At x = 0 : -2 < 0 -> MAX (skip)

So, nearest points : (+/- 1/sqrt(2), 1/2).

d_min 2 = 1/4 - 1/4 + 1 = 3/4

$$\boxed{\text{d\_min = sqrt(3)/2}}$$ <-answer

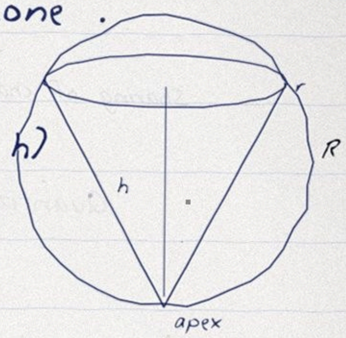
Tip : minimise d 2, not d (avoids sqrt).

Critical pts of d 2 = critical pts of d (d > 0).

Eg. cone inscribed in a sphere

Sphere of radius R. Find max volume of a right circular cone inscribed in it.

Let r = base-radius , h = height of cone .



From geometry : $r \quad 2 = h(2R - h)$

$V = (1/3) \pi r \quad 2 h$

$\quad = (\pi/3) h \quad 2 (2R - h)$

$dV/dh = (\pi/3)(4 R h - 3 h \quad 2)$

$dV/dh = 0 \Rightarrow h(4R - 3h) = 0$

$\Rightarrow h = 4R/3$ (other root $h=0$ rejected)

$d \quad 2V/dh \quad 2 = (\pi/3)(4R - 6h)$

At $h = 4R/3$: $= (pi/3)(4R - 8R) = -4 pi R/3 < 0$

$r \quad 2 = (4R/3)(2R - 4R/3) = 8 R \quad 2 / 9$

$$\boxed{V\_max = (32/81) pi R \quad 3}$$ <-key result

Ratio V_cone : V_sphere = (32/81)/(4/3) = 8/27 .

i.e. cone takes 8/27 (30 %) of sphere's volume .

Eg. perimeter / area trade-off

A wire of length 28 cm is cut into 2 pieces .
One becomes a square ; other becomes a circle .
Find cuts that minimise the total area .

Let x = piece for square , $(28 - x)$ for circle .

$A_{sq} = (x/4) \quad 2 = x \quad 2 / 16$

$A_{cir} = \pi \, ((28-x)/(2\pi)) \quad 2$

$\quad = (28-x) \quad 2 / (4\pi)$

$A_{total} = x \quad 2/16 + (28-x) \quad 2 / (4\pi)$

$A'(x) = x/8 - (28-x)/(2\pi)$

$A'(x) = 0 \Rightarrow \text{pi}\ x = 4(28 - x)$

$\Rightarrow x = 112 / (\pi + 4)$

$A''(x) = 1/8 + 1/(2\pi) > 0$ -> MIN .

$\boxed{x_{min} = 112 / (\text{pi} + 4) \qquad 15.68 \text{ cm}}$ <-answer

circle piece = 28 - 15.68 12.32 cm .
Note : to MAXIMISE area , just make a single shape (don't cut) - boundary case .

Eg. cylinder of max volume

Cylinder inscribed in sphere of radius R.
Find max-volume cylinder.

Let r, h be the cylinder's radius, height.
constraint : $r^2 + (h/2)^2 = R^2$

$V = \pi r^2 h = \pi (R^2 - h^2/4) h$.

$dV/dh = \pi (R^2 - 3h^2/4) = 0$
$\Rightarrow h = 2R/\mathrm{sqrt}(3)$

$d^2V/dh^2 = -\pi(3h/2) < 0$ -> MAX

$r^2 = R^2 - R^2/3 = 2R^2/3$

$$\boxed{V\_max = (4 pi / 3 \mathrm{sqrt}(3)) R^3}$$

Eg. rect of max area in circle

Rect inscribed in circle radius R :
$A = 2x \cdot 2y$ with $x^2 + y^2 = R^2$
Max area -> $x = y = R/\mathrm{sqrt}(2)$ -> square !

$$\boxed{A\_max = 2R^2 \text{ (a square)}}$$ <-intuition

Pro tips & common mistakes

- * Always SKETCH a small figure even if ~~not~~ not specifically asked - clarifies vars .
- * Identify the constraint (sphere , wire , ...).
- * Reduce to one variable before differentiating .
- * After finding critical pts , CONFIRM max/min with f'' test or 1st-deriv sign change .
- * Reject roots that make a length / area ≤ 0 .
- * For closed intervals , check endpoints too .

Quick reference table

Test	Condition	Conclusion
1st deriv	$f'(c)=0$, + -> -	local MAX
1st deriv	$f'(c)=0$, - -> +	local MIN
2nd deriv	$f'(c)=0$, $f'<0$	local MAX
2nd deriv	$f'(c)=0$, $f'>0$	local MIN

Mnemonic - F.D.I.D.

F - Figure first
D - Define variables
I - Identify quantity to max / min
D - Differentiate , set = 0 , test

* Don't get scared by long word problems !
Tackle one sentence at a time .

Final summary - Ch 6

- dy/dx = rate of change of y w.r.t. x .
- Related rates : chain rule on time t .
- $f' > 0$ -> increasing ; $f' < 0$ -> decreasing.
- Tangent slope at $P = f'(x0)$; normal $= -1/f'(x0)$.
- dy $f'(x)\,dx$ for linear approximations .
- Critical pt : $f' = 0$ or f' undefined .
- 1st-deriv test : + -> - MAX ; - -> + MIN .
- 2nd-deriv test : $f'' < 0$ MAX ; $f'' > 0$ MIN .
- Absolute extrema on $[a, b]$: CP ∪ $\{a, b\}$.
- Optimisation : model -> reduce -> diff -> test .

Must-remember results

open box (square sheet) : $x = L/6$ <-famous

cone in sphere R : $h = 4R/3$

cylinder in sphere R : $h = 2R/\text{sqrt}(3)$

rect in circle R : square , side $R\,\text{sqrt}(2)$

Golden rule

always confirm extremum with f'' or sign test <- or

End of Chapter 6 . Session 2026-27 .