

Integrals - Antiderivatives

Integration is the inverse process of differentiation. If $d/dx [F(x)] = f(x)$, then $F(x)$ is called an antiderivative of $f(x)$.

$$\int f(x) dx = F(x) + C$$

$\leftarrow C$: constant
 \leftarrow of integration

Geometrically every ~~derivative~~ antiderivative represents a family of parallel curves, shifted vertically by the constant C .

Standard Integrals

$$x^n dx = x^{n+1} / (n+1) + C, \quad n \neq -1$$

$$1/x dx = \log x + C$$

$$e^x dx = e^x + C$$

$$a^x dx = a^x / \log a + C$$

$$\sin x dx = -\cos x + C$$

$$\cos x dx = \sin x + C$$

$$\sec^2 x dx = \tan x + C$$

$$\operatorname{cosec}^2 x dx = -\cot x + C$$

$$\sec x \tan x dx = \sec x + C$$

$$1/\sqrt{1-x^2} dx = \sin^{-1} x + C$$

$$1/(1+x^2) dx = \tan^{-1} x + C$$

Methods of Integration

Linearity: for constants a, b :

$$\int [a f(x) + b g(x)] dx \quad \leftarrow = a \int f + b \int g$$

1. Substitution Method

Put $t = \phi(x)$ so that $dt = \phi'(x) dx$:

$$\int f(\phi(x)) \phi'(x) dx = \int f(t) dt$$

\leftarrow factor
 \leftarrow choose t to
 \leftarrow kill the messy

e.g. $I = \int 2x \cos(x^2) dx$; put $t = x^2$
 x^2 , so $dt = 2x dx$: $I = \int \cos t dt$
 $= \sin t + C = \sin(x^2) + C$

2. Integration by Partial Fractions

For $P(x)/Q(x)$, $\deg(P) < \deg(Q)$, split into simpler fractions using standard forms:

$$A/(x-a), \quad A/(x-a)^2, \quad (Ax+B)/(x^2+bx+c)$$

Then integrate each piece using standard formulas (log, arctan, $1/x$ rules above).

Tip: cover-up rule speeds up $A, B, C \dots$

Integration by Parts

Comes from the product rule of derivatives.

If u and v are functions of x :

$$\int u v' dx = u v - \int u' v dx$$

$$\leftarrow \text{or : } \int u du \\ \leftarrow = uv - \int v du$$

ILATE rule (choose u in this order):

I - Inverse trig L - Logarithm

A - Algebraic T - Trigonometric

E - Exponential

Take first letter as u , rest as dv .

Useful Result

$$\int e^x [f(x) + f'(x)] dx = e^x f(x) + C$$

Special Integrals

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x-a}{x+a} + C$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log \frac{x + \sqrt{x^2 + a^2}}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \frac{x + \sqrt{x^2 - a^2}}{a} + C$$

Memorise these - they appear in JEE often.

Definite Integrals

Defined as limit of a sum (Riemann sum)

geometrically = area under curve $y=f(x)$,

x-axis, $x=a$, $x=b$.

Fundamental Theorem of Calculus

$$\int_a^b f(x) dx = F(b) - F(a)$$

← F is any anti-
derivative of f

Key Properties

P1. $\int_a^b f(x) dx = - \int_b^a f(x) dx$

P2. $\int_a^a f(x) dx = 0$

P3. $\int_a^b f(x) dx = \int_a^c f + \int_c^b f$

P4. $\int_0^a f(x) dx = \int_0^a f(a-x) dx$

P5. $\int_{-a}^a f(x) dx :$

$= 2 \int_0^a f(x) dx$ if f even

$= 0$ if f odd

P6. $\int_0^{2a} f(x) dx = \int_0^a [f(x) + f(2a-x)] dx$