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Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 4: Determinants

About this Chapter

This exercise builds the basic vocabulary of $|A|$: the **minor** M_{ij} is the determinant left after deleting row i and column j, and the **cofactor** is $A_{ij} = (-1)^{i+j} M_{ij}$. The full determinant equals the sum, along any row or column, of each entry times its cofactor: $|A| = \sum_j a_{ij} A_{ij}$.

Topics covered: Minor of an element • Cofactor of an element • Cofactor expansion of a determinant • Sign chessboard

Quick Formula Sheet

Minor: M_{ij} = determinant after deleting row i, column j.

Cofactor: $A_{ij} = (-1)^{i+j} M_{ij}$.

Expansion along row i**:** $|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3}$.

Sign pattern: $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$.

Exercise 4.3

Q 4.1 **Write Minors and Cofactors of the elements of the following determinants:**

(i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$ **(ii)** $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$.

SOLUTION

Concept used. For a determinant of order 2, the minor M_{ij} of element a_{ij} is the (single-entry) determinant obtained by deleting row i and column j. Hence each minor is just *the remaining diagonally-opposite entry*. The cofactor is $A_{ij} = (-1)^{i+j} M_{ij}$.

Part (i).

Step 1. Entries: $a_{11} = 2,\ a_{12} = -4,\ a_{21} = 0,\ a_{22} = 3$.

Step 2. Minors. Delete the relevant row and column of the 2×2 matrix:

$$M_{11} = 3, \quad M_{12} = 0, \quad M_{21} = -4, \quad M_{22} = 2.$$

Step 3. Cofactors using $A_{ij} = (-1)^{i+j} M_{ij}$:

$$\begin{aligned} A_{11} &= (-1)^2 \cdot 3 = 3, \\ A_{12} &= (-1)^3 \cdot 0 = 0, \\ A_{21} &= (-1)^3 \cdot (-4) = 4, \\ A_{22} &= (-1)^4 \cdot 2 = 2. \end{aligned}$$

> **Final Answer:** (i) $M_{11} = 3,\ M_{12} = 0,\ M_{21} = -4,\ M_{22} = 2;$
> $A_{11} = 3,\ A_{12} = 0,\ A_{21} = 4,\ A_{22} = 2.$

Part (ii).

Step 1. Entries: $a_{11} = a,\ a_{12} = c,\ a_{21} = b,\ a_{22} = d.$

Step 2. Minors:

$$M_{11} = d, \quad M_{12} = b, \quad M_{21} = c, \quad M_{22} = a.$$

Step 3. Cofactors:

$$\begin{aligned} A_{11} &= +d, \quad A_{12} = -b, \\ A_{21} &= -c, \quad A_{22} = +a. \end{aligned}$$

> **Final Answer:** (ii) $M_{11} = d,\ M_{12} = b,\ M_{21} = c,\ M_{22} = a;$
> $A_{11} = d,\ A_{12} = -b,\ A_{21} = -c,\ A_{22} = a.$

☞ **The sign chessboard**

$(-1)^{i+j}$ produces the alternating pattern $\begin{smallmatrix} + & - \\ - & + \end{smallmatrix}$ for order 2 and $\begin{smallmatrix} + & - & + \\ - & + & - \\ + & - & + \end{smallmatrix}$ for order 3. Read off the sign from the chessboard rather than computing $(-1)^{i+j}$ each time.

EXPERT'S SOLUTION : *Tara Bhat, M.Sc Mathematics, ISI Kolkata*

Quick reading. For a 2×2 matrix, the minor of any entry is simply the diagonally-opposite entry. The cofactor flips the sign of the off-diagonal entries' minors.

Step 1. **(i)** Read off: $M_{11} = 3, M_{12} = 0, M_{21} = -4, M_{22} = 2$. Apply chessboard signs: $A_{11} = 3, A_{12} = 0, A_{21} = 4, A_{22} = 2$.

Step 2. **(ii)** Read off: $M_{11} = d, M_{12} = b, M_{21} = c, M_{22} = a$. Apply signs: $A_{11} = d, A_{12} = -b, A_{21} = -c, A_{22} = a.$

Why this matters. The cofactor matrix is the transpose of the adjugate, which we use throughout Exercise 4.4 to compute inverses. Getting the signs right here saves time later.

> **Final Answer:** See the minors/cofactors lists above.

Q 4.2 **Write Minors and Cofactors of the elements of the following determinants:**

(i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ **(ii)** $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$.

SOLUTION

Concept used. For an order-3 determinant, the minor M_{ij} is the 2×2 determinant left after deleting row i and column j. The cofactor is $A_{ij} = (-1)^{i+j} M_{ij}$.

Part (i): the identity I_3.

Step 1. Each minor is the determinant of an order-2 sub-matrix. With the identity, deleting any row and column leaves either $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (minor $= 1$) when the deleted row and column are equal-indexed, or a matrix with a row of zeros (minor $= 0$) otherwise.

Step 2. Compute each minor explicitly:

$$M_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \quad M_{12} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0, \quad M_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0,$$

$$M_{21} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0, \quad M_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \quad M_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0,$$

$$M_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0, \quad M_{32} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0, \quad M_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Step 3. Cofactors. Each minor is multiplied by $(-1)^{i+j}$ from the chessboard:

$$A_{11} = 1,\ A_{22} = 1,\ A_{33} = 1;\ \text{all other } A_{ij} = 0.$$

(Off-diagonal minors are already 0, so signs do not matter there.)

> **Final Answer:** (i) All diagonal M's and A's equal 1; all off-diagonal M's and A's equal 0.

Part (ii). $A = \begin{pmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{pmatrix}$.

Step 1. Row 1 minors:

$$M_{11} = \begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = (5)(2) - (-1)(1) = 10 + 1 = 11,$$

$$M_{12} = \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = (3)(2) - (-1)(0) = 6 - 0 = 6,$$

$$M_{13} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = (3)(1) - (5)(0) = 3 - 0 = 3.$$

Step 2. Row 2 minors:

$$M_{21} = \begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = 0 - 4 = -4,$$

$$M_{22} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2,$$

$$M_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1.$$

Step 3. Row 3 minors:

$$M_{31} = \begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix} = 0 - 20 = -20,$$

$$M_{32} = \begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} = -1 - 12 = -13,$$

$$M_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 5 - 0 = 5.$$

Step 4. Cofactors (apply chessboard signs):

$$\begin{array}{lll} A_{11} = +11, & A_{12} = -6, & A_{13} = +3, \\ A_{21} = +4, & A_{22} = +2, & A_{23} = -1, \\ A_{31} = -20, & A_{32} = +13, & A_{33} = +5. \end{array}$$

(Notice A_{12} flipped to -6, A_{21} flipped to $+4$, A_{23} to -1, A_{31} stays -20, A_{32} to $+13$.)

Final Answer: (ii) Minors $\{11, 6, 3, -4, 2, 1, -20, -13, 5\}$; Cofactors $\{11, -6, 3, 4, 2, -1, -20, 13, 5\}$.

EXPERT'S SOLUTION *: Krishna Rao, Ph.D Pure Mathematics, IISc Bangalore*

Structural observation. For (i), the identity matrix has a clean structure: the minor at position (i, j) is δ_{ij} (Kronecker delta), so $M_{ij} = A_{ij} = \delta_{ij}$.

Step 1. **(i)** Diagonal cofactors $A_{11} = A_{22} = A_{33} = 1$; everything else 0.

Step 2. **(ii)** Compute the nine 2×2 minors as in the main solution, then apply the chessboard signs. Sanity check the cofactor calculation by expanding $|A|$ along R_1:

$$|A| = 1 \cdot 11 + 0 \cdot (-6) + 4 \cdot 3 = 11 + 0 + 12 = 23.$$

Also expand along C_1: $|A| = 1 \cdot 11 - 3 \cdot (-4) + 0 = 11 + 12 = 23$. ✓ The cofactors are mutually consistent.

Why this matters. Cross-checking by computing $|A|$ along two different rows/columns is the single best way to catch an arithmetic slip in a long cofactor table.

Final Answer: (i) $A = I_3$; (ii) Cofactor matrix $= \begin{pmatrix} 11 & -6 & 3 \\ 4 & 2 & -1 \\ -20 & 13 & 5 \end{pmatrix}$.

Q 4.3 **Using Cofactors of elements of second row, evaluate** $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

SOLUTION

Concept used. A determinant can be expanded along any row or column by the rule

$$|A| \;=\; a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3},$$

where A_{ij} are the cofactors. Here we are told to expand along row 2.

Step 1. Compute the row-2 cofactors. The minors are 2×2 determinants obtained by deleting R_2 and the relevant column.

$$M_{21} = \begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} = 9 - 16 = -7,$$

$$M_{22} = \begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} = 15 - 8 = 7,$$

$$M_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10 - 3 = 7.$$

Step 2. Apply chessboard signs for row 2 $(-, +, -)$:

$$A_{21} = -M_{21} = 7, \quad A_{22} = +M_{22} = 7, \quad A_{23} = -M_{23} = -7.$$

Step 3. Row-2 entries are $a_{21} = 2,\ a_{22} = 0,\ a_{23} = 1$. Expand:

$$\Delta = 2 \cdot 7 + 0 \cdot 7 + 1 \cdot (-7) = 14 + 0 - 7 = 7.$$

Final Answer: $\Delta = 7$.

☞ **Exam Tip**

The middle entry of row 2 is 0, so the middle cofactor never gets used. Whenever a row or column has a zero, expanding along it kills that term outright.

EXPERT'S SOLUTION *: Ananya Gupta, M.Sc Mathematics, IIT Bombay*

Quick reading. Three minors, three signs, three multiplications, one sum.

Step 1. Three minors: $M_{21} = -7,\ M_{22} = 7,\ M_{23} = 7$.

Step 2. Sign flip on the outer two: $A_{21} = 7,\ A_{22} = 7,\ A_{23} = -7$.

Step 3. Multiply by the row-2 entries $(2, 0, 1)$:

$$\Delta = 2(7) + 0(7) + 1(-7) = 14 + 0 - 7 = 7.$$

Step 4. Cross-check by expanding along R_1 instead:

$$\Delta = 5(0 \cdot 3 - 1 \cdot 2) - 3(2 \cdot 3 - 1 \cdot 1) + 8(2 \cdot 2 - 0 \cdot 1)$$

$$= 5(-2) - 3(5) + 8(4) = -10 - 15 + 32 = 7. \checkmark$$

Why this matters. The cross-check is the practical proof that expansion along any row or column gives the same value. Use it once a chapter and it becomes muscle memory.

Final Answer: $\Delta = 7$.

Q 4.4 **Using Cofactors of elements of third column, evaluate** $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$.

SOLUTION

Concept used. Expand along column 3:

$$\Delta = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33},$$

with $A_{i3} = (-1)^{i+3}M_{i3}$.

Step 1. Column-3 entries: $a_{13} = yz$, $a_{23} = zx$, $a_{33} = xy$.

Step 2. Minors (delete column 3 and the row of the entry):

$$M_{13} = \begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix} = z - y,$$
$$M_{23} = \begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix} = z - x,$$
$$M_{33} = \begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix} = y - x.$$

Step 3. Apply signs $(+, -, +)$ for column 3:

$$A_{13} = +(z-y), \quad A_{23} = -(z-x) = x - z, \quad A_{33} = +(y-x).$$

Step 4. Expand:

$$\Delta = yz(z-y) + zx(x-z) + xy(y-x).$$

Step 5. Expand each product:

$$yz(z-y) = yz^2 - y^2z,$$
$$zx(x-z) = x^2z - xz^2,$$
$$xy(y-x) = xy^2 - x^2y.$$

Step 6. Sum:

$$\Delta = yz^2 - y^2z + x^2z - xz^2 + xy^2 - x^2y.$$

Step 7. Group and factor. Pair $yz^2 - xz^2 = z^2(y-x)$ and $-x^2y + x^2z = x^2(z-y)$ and $xy^2 - y^2z = y^2(x-z)$. So

$$\Delta = z^2(y-x) + x^2(z-y) + y^2(x-z).$$

This is the well-known Vandermonde-style factor: $(x-y)(y-z)(z-x)$ up to a sign.

Step 8. Verify by checking a specific case. With $x = y$ we expect $\Delta = 0$ since columns 1 and 2 become identical after... actually with $x = y$ the original determinant has $R_1 - R_2 = (0, x-y, yz-zx) = (0, 0, z(y-x)) = (0,0,0)$, so $\Delta = 0$. Our formula gives $z^2(y-x) + x^2(z-y) + y^2(x-z)\Big|_{x=y} = z^2 \cdot 0 + x^2(z-x) + x^2(x-z) = 0$. ✓

Final Answer: $\Delta = (x-y)(y-z)(z-x)$, equivalently $-(x^2y - x^2z - xy^2 + xz^2 + y^2z - yz^2)$.

☞ Verbose form vs factored form

Both $yz(z-y) + zx(x-z) + xy(y-x)$ and $-(x-y)(y-z)(z-x)$ are acceptable final answers. NCERT-style solutions usually stop at the expanded form; competitive exams prefer the factored form.

EXPERT'S SOLUTION *: Meera Kumar, M.Sc Applied Mathematics, IIT Kanpur*

Structural observation. The matrix has the Vandermonde-like structure (columns $1, x, yz$). Such determinants always factor as a product of differences once expanded.

Step 1. Cofactors: $A_{13} = z - y$, $A_{23} = x - z$, $A_{33} = y - x$.

Step 2. Cofactor expansion:

$$\Delta = yz(z-y) + zx(x-z) + xy(y-x).$$

Step 3. Alternative route: apply the row operation $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$ on the original matrix:

$$\Delta = \begin{vmatrix} 0 & x-y & yz-zx \\ 0 & y-z & zx-xy \\ 1 & z & xy \end{vmatrix} = \begin{vmatrix} 0 & x-y & z(y-x) \\ 0 & y-z & x(z-y) \\ 1 & z & xy \end{vmatrix}.$$

Expand along C_1:

$$\Delta = 1 \cdot \begin{vmatrix} x-y & z(y-x) \\ y-z & x(z-y) \end{vmatrix} = (x-y)\,x(z-y) - z(y-x)(y-z).$$

Factor $(x-y)$ from both terms: $(y-x) = -(x-y)$, so

$$\Delta = (x-y)\big[x(z-y) + z(y-z)\big] = (x-y)\big[xz - xy + zy - z^2\big].$$

Group: $xz - z^2 = z(x-z)$ and $-xy + zy = y(z-x) = -y(x-z)$. So

$$\Delta = (x-y)(x-z)(z-y).$$

Up to a sign reordering this matches the main solution's $-(x-y)(y-z)(z-x)$.

Why this matters. Row operations before expansion often expose hidden factors. Vandermonde determinants are the prototype, but the same trick works wherever rows differ by simple polynomial offsets.

Final Answer: $\Delta = (x-y)(y-z)(z-x)$ (up to sign).

Q 4.5 **If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is the cofactor of a_{ij}, then the value of Δ is given by**

(A) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$ **(B)** $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$
(C) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$ **(D)** $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$.

SOLUTION

Concept used. Two parallel facts about cofactors.
(I) Same-row/column rule. The sum of products of elements of a row (or column) with their *own* cofactors equals $|A|$:

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3} \quad \text{(row } i\text{)},$$

$$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j} \quad \text{(column } j\text{)}.$$

(II) Cross-row/column rule. The sum of products of elements of one row (or column) with cofactors of *another* row (or column) is zero. Symbolically, $\sum_j a_{ij}A_{kj} = 0$ when $i \neq k$.

Step 1. Check option (A): $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$. Row 1 elements times row 3 cofactors. By rule (II) this is 0, not Δ. Reject.

Step 2. Check option (B): $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$. Row 1 elements times column 1 cofactors (mismatched indices). This is neither a row expansion nor a column expansion of Δ. Reject.

Step 3. Check option (C): $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$. Row 2 elements times row 1 cofactors. By rule (II), $= 0$. Reject.

Step 4. Check option (D): $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$. Column 1 elements times column 1 cofactors. This is exactly the column-1 expansion of $|A|$. Accept.

Final Answer: Option (D): $\Delta = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$.

♥ Why the cross-row rule

$a_{i1}A_{k1} + a_{i2}A_{k2} + a_{i3}A_{k3}$ for $i \neq k$ is the determinant of the matrix obtained by *replacing* row k with row i. That new matrix has two equal rows, so its determinant is 0. This is the essence of the cross-row identity and the foundation for the proof of $A \cdot (\text{adj } A) = |A|\, I$.

EXPERT'S SOLUTION : *Siddharth Verma, B.Tech CSE, IIT Roorkee*

Strategic angle. Translate each option to a phrase "row/column i entries $\times$ row/column k cofactors". Only the matching-index case gives $|A|$.

Step 1. (A) Row-1 entries $\times$ Row-3 cofactors $\to 0$.

Step 2. (B) Row-1 entries $\times$ Column-1 cofactors with shuffled column indices $\to$ not a clean expansion.

Step 3. (C) Row-2 entries $\times$ Row-1 cofactors $\to 0$.

Step 4. (D) Column-1 entries $\times$ Column-1 cofactors $\to |A|$. ✓

Why this matters. The same-versus-cross distinction underlies the entire $\mathrm{adj}(A)$ theory and the formula $A^{-1} = \frac{1}{|A|}\,\mathrm{adj}(A)$ we will use in Exercise 4.4.

> **Final Answer:** Option (D).

Key Takeaways

- Minor M_{ij}: determinant after deleting row i and column j.
- Cofactor: $A_{ij} = (-1)^{i+j} M_{ij}$; the signs form a chessboard.
- $|A| = \sum_j a_{ij}A_{ij}$ (same row); $|A| = \sum_i a_{ij}A_{ij}$ (same column).
- Cross-row sum: $\sum_j a_{ij}A_{kj} = 0$ when $i \neq k$ (two rows would be equal in the implied determinant).
- Verify a cofactor computation by expanding along a second row or column; both give $|A|$.

End of Exercise 4.3