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Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 4: Determinants

About this Chapter

This exercise builds the **adjoint** of a square matrix and uses it to write the **inverse**: if A is non-singular, $A^{-1} = \dfrac{1}{|A|}\,\text{adj}(A)$. We verify $A(\text{adj}\,A) = (\text{adj}\,A)A = |A|\,I$, the matrix-product determinant rule $|AB| = |A|\,|B|$, and standard MCQ identities such as $|\,\text{adj}\,A| = |A|^{n-1}$ and $\det(A^{-1}) = 1/\det(A)$.

Topics covered: Adjoint of a matrix • $A \cdot \text{adj}(A) = |A|\,I$ • Inverse $A^{-1} = \frac{1}{|A|}\,\text{adj}(A)$ • $(AB)^{-1} = B^{-1}A^{-1}$ • Cayley-Hamilton-style identities

Quick Formula Sheet

Adjoint: $\text{adj}(A) = [A_{ij}]^T$.

Inverse: $A^{-1} = \dfrac{1}{|A|}\,\text{adj}(A)$, valid when $|A| \neq 0$.

Product rule: $|AB| = |A|\,|B|$.

$|\,\text{adj}\,A|$: $|\,\text{adj}\,A| = |A|^{n-1}$.

$\det(A^{-1})$: $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Exercise 4.4

Q 4.1 **Find the adjoint of the matrix** $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

SOLUTION

Concept used. For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the adjoint is the transpose of its cofactor matrix:

$$\text{adj}(A) = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}.$$

Equivalently, swap the diagonal entries and flip the sign of the off-diagonals:

$$\operatorname{adj}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Step 1. Read off entries: $a = 1, b = 2, c = 3, d = 4$.

Step 2. Compute cofactors:

$$A_{11} = +4, \qquad A_{12} = -3,$$
$$A_{21} = -2, \qquad A_{22} = +1.$$

Step 3. Transpose the 2×2 cofactor matrix $\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}$ to obtain

$$\operatorname{adj}(A) = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}.$$

Final Answer: $\operatorname{adj}(A) = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$.

☞ **Two-by-two shortcut**

The "swap diagonals, flip off-diagonals" rule is just a re-statement of the chessboard signs for a 2×2 matrix. Whenever the matrix is 2×2, skip the cofactor table and use this rule directly.

EXPERT'S SOLUTION : *Sneha Banerjee, M.Sc Mathematics, IIT Bombay*

Quick reading. The 2×2 adjoint is one swap and two sign flips.

Step 1. Diagonal swap: $(1, 4) \to (4, 1)$.

Step 2. Off-diagonal sign flip: $(2, 3) \to (-2, -3)$.

Step 3. Place them back: $\operatorname{adj}(A) = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$.

Step 4. Verification:

$$A \cdot \operatorname{adj}(A) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 4-6 & -2+2 \\ 12-12 & -6+4 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = -2\,I.$$

Since $|A| = 4 - 6 = -2$, this matches $|A|\,I$. ✓

Why this matters. The verification step $A \cdot \operatorname{adj}(A) = |A|\,I$ is the most reliable sanity check on the adjoint. Make a habit of doing it for 2×2 matrices in 15 seconds.

Final Answer: $\text{adj}(A) = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$.

Q 4.2 **Find the adjoint of the matrix** $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{pmatrix}$.

SOLUTION

Concept used. For an order-3 matrix, $\text{adj}(A) = [A_{ij}]^T$ where $A_{ij} = (-1)^{i+j} M_{ij}$ is the (i,j) cofactor.

Step 1. Compute the nine minors.

$$M_{11} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3, \qquad M_{12} = \begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} = 2 - (-10) = 12,$$
$$M_{13} = \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 0 - (-6) = 6, \qquad M_{21} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1 - 0 = -1,$$
$$M_{22} = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1 - (-4) = 5, \qquad M_{23} = \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = 0 - 2 = -2,$$
$$M_{31} = \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -5 - 6 = -11, \qquad M_{32} = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5 - 4 = 1,$$
$$M_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3 - (-2) = 5.$$

Step 2. Apply chessboard signs $(-1)^{i+j}$:

$$A_{11} = 3, \quad A_{12} = -12, \quad A_{13} = 6,$$
$$A_{21} = 1, \quad A_{22} = 5, \quad A_{23} = 2,$$
$$A_{31} = -11, \quad A_{32} = -1, \quad A_{33} = 5.$$

Step 3. Form the cofactor matrix and transpose:

$$\text{Cof}(A) = \begin{pmatrix} 3 & -12 & 6 \\ 1 & 5 & 2 \\ -11 & -1 & 5 \end{pmatrix} \implies \text{adj}(A) = \begin{pmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{pmatrix}.$$

Final Answer: $\text{adj}(A) = \begin{pmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{pmatrix}$.

☞ Exam Tip

Always lay out a 3×3 table of minors first, then a 3×3 table of cofactors with the chessboard sign flips, then transpose. Doing it in one pass invites sign errors.

EXPERT'S SOLUTION *: Aanya Patel, Ph.D Mathematics, IIT Delhi*

Strategic angle. Compute the minor table, sign-flip to get cofactors, transpose. Cross-check with $A \cdot \text{adj}(A) = |A|\, I$.

Step 1. Minors as in main solution.

Step 2. Cofactor matrix $\begin{pmatrix} 3 & -12 & 6 \\ 1 & 5 & 2 \\ -11 & -1 & 5 \end{pmatrix}$; its transpose is the adjoint.

Step 3. Compute $|A|$ for the sanity check, along R_1:

$$|A| = 1 \cdot 3 - (-1)(12) + 2 \cdot 6 = 3 + 12 + 12 =?\,.$$

Wait: by the cofactor expansion rule,
$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 1(3) + (-1)(-12) + 2(6) = 3 + 12 + 12 = 27\cdot\,.$
Hmm let us redo carefully: $1 \cdot 3 + (-1) \cdot (-12) + 2 \cdot 6 = 3 + 12 + 12 = 27$? That seems large. Recompute via R_1 directly:
$1(3 \cdot 1 - 5 \cdot 0) - (-1)(2 \cdot 1 - 5 \cdot (-2)) + 2(2 \cdot 0 - 3 \cdot (-2)) =$
$1(3) + 1(2 + 10) + 2(0 + 6) = 3 + 12 + 12 = 27$. So $|A| = 27$.

Step 4. Then verify $A \cdot \text{adj}(A) = 27\, I$ by direct multiplication of one diagonal entry, say row 1 of A times column 1 of $\text{adj}(A)$:
$1(3) + (-1)(-12) + 2(6) = 3 + 12 + 12 = 27$. ✓

Why this matters. The cross-check ties the cofactor table to $|A|$ automatically: the entry on row 1 column 1 of $A \cdot \text{adj}(A)$ is exactly the expansion of $|A|$ along row 1.

Final Answer: $\text{adj}(A) = \begin{pmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{pmatrix}$.

Q 4.3 **Verify** $A(\text{adj } A) = (\text{adj } A)A = |A|\, I$ **for** $A = \begin{pmatrix} 2 & 3 \\ -4 & -6 \end{pmatrix}$.

SOLUTION

Concept used. For any square matrix A,

$$A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = |A|\, I.$$

For a 2×2 matrix, $\text{adj}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Step 1. Compute $|A| = (2)(-6) - (3)(-4) = -12 + 12 = 0$.

Step 2. Compute $\text{adj}(A)$: swap diagonals and flip off-diagonals:

$$\text{adj}(A) = \begin{pmatrix} -6 & -3 \\ 4 & 2 \end{pmatrix}.$$

Step 3. Multiply $A \cdot \text{adj}(A)$:

$$\begin{pmatrix} 2 & 3 \\ -4 & -6 \end{pmatrix}\begin{pmatrix} -6 & -3 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 2(-6) + 3(4) & 2(-3) + 3(2) \\ -4(-6) + (-6)(4) & -4(-3) + (-6)(2) \end{pmatrix}.$$

$$= \begin{pmatrix} -12 + 12 & -6 + 6 \\ 24 - 24 & 12 - 12 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Step 4. Multiply $\text{adj}(A) \cdot A$:

$$\begin{pmatrix} -6 & -3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -4 & -6 \end{pmatrix} = \begin{pmatrix} -12 + 12 & -18 + 18 \\ 8 - 8 & 12 - 12 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Step 5. Compute $|A|\, I = 0\, I = O$ (zero matrix).

Step 6. All three quantities equal O, so the identity is verified.

Final Answer: $A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = O = |A|\, I$.

♥ Singular case

$|A| = 0$ here means A is singular and its rows/columns are proportional: $R_2 = -2R_1$. The identity $A \cdot \text{adj}(A) = |A|\, I$ still holds, but it collapses to $A \cdot \text{adj}(A) = O$, and there is no inverse.

EXPERT'S SOLUTION : *Pooja Joshi, B.Tech Engineering Physics, IIT Bombay*

Structural observation. Row 2 is -2 times row 1, so the rows are dependent and $|A| = 0$. Both $A \cdot \text{adj}\, A$ and $\text{adj}\, A \cdot A$ must therefore be the zero matrix.

Step 1. $|A| = -12 - (-12) = 0$.

Step 2. $\text{adj}(A) = \begin{pmatrix} -6 & -3 \\ 4 & 2 \end{pmatrix}$.

Step 3. Direct multiplication confirms $A \cdot \text{adj}(A) = O$ and $\text{adj}(A) \cdot A = O$.

Step 4. All three sides are O.

Why this matters. The general identity $A\,\text{adj}\,A = |A|\,I$ does not need A to be invertible. Even when $|A| = 0$, the equation still talks, just both sides are zero.

> **Final Answer:** $A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = |A|\,I = O$.

Q 4.4 **Verify** $A(\text{adj}\,A) = (\text{adj}\,A)A = |A|\,I$ **for** $A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{pmatrix}$.

SOLUTION

Concept used. Identity: $A\,\text{adj}(A) = \text{adj}(A)A = |A|\,I$.

Step 1. Compute $|A|$ by expansion along C_2 (two zeros):

$$|A| = -(-1)\begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} + 0 - 0 = (9+2) = 11.$$

Cross-check along R_1:

$$|A| = 1(0+0) - (-1)(9+2) + 2(0-0) = 0 + 11 + 0 = 11. \checkmark$$

Step 2. Compute the cofactor matrix. Minors:

$$M_{11} = \begin{vmatrix} 0 & -2 \\ 0 & 3 \end{vmatrix} = 0, \qquad M_{12} = \begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} = 11, \qquad M_{13} = \begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} = 0,$$

$$M_{21} = \begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} = -3, \qquad M_{22} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1, \qquad M_{23} = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 1,$$

$$M_{31} = \begin{vmatrix} -1 & 2 \\ 0 & -2 \end{vmatrix} = 2, \qquad M_{32} = \begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = -8, \qquad M_{33} = \begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} = 3.$$

Cofactors (apply signs):

$$\begin{aligned} &A_{11} = 0, && A_{12} = -11, && A_{13} = 0, \\ &A_{21} = 3, && A_{22} = 1, && A_{23} = -1, \\ &A_{31} = 2, && A_{32} = 8, && A_{33} = 3. \end{aligned}$$

Step 3. Transpose to get the adjoint:

$$\text{adj}(A) = \begin{pmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{pmatrix}.$$

Step 4. Compute $A \cdot \text{adj}(A)$ entry by entry. Row 1 of A is $(1, -1, 2)$.

$$(R_1 A) \cdot (C_1 \text{adj}) = 1 \cdot 0 + (-1)(-11) + 2 \cdot 0 = 11.$$

$$(R_1 A) \cdot (C_2 \text{adj}) = 1 \cdot 3 + (-1) \cdot 1 + 2 \cdot (-1) = 3 - 1 - 2 = 0.$$

$$(R_1 A) \cdot (C_3 \text{adj}) = 1 \cdot 2 + (-1) \cdot 8 + 2 \cdot 3 = 2 - 8 + 6 = 0.$$

Row 2 of A is $(3, 0, -2)$.

$$3{\cdot}0+0+(-2)(0) = 0;\; 3{\cdot}3+0+(-2)(-1) = 9+2 = 11;\; 3{\cdot}2+0+(-2)(3) = 6-6 = 0.$$

Row 3 of A is $(1, 0, 3)$.

$$1 \cdot 0 + 0 + 3 \cdot 0 = 0;\; 1 \cdot 3 + 0 + 3(-1) = 0;\; 1 \cdot 2 + 0 + 3 \cdot 3 = 11.$$

Putting these together:

$$A \cdot \text{adj}(A) = \begin{pmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix} = 11\, I.$$

Step 5. Compute $\text{adj}(A) \cdot A$ similarly. The diagonal entries again come out to 11 and the off-diagonal entries to 0:

$$\text{adj}(A) \cdot A = 11\, I.$$

Step 6. Both products equal $|A|\, I = 11\, I$. Verified.

Final Answer: $A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = 11\, I = |A|\, I.$

EXPERT'S SOLUTION : *Ankit Mehta, M.Sc Applied Mathematics, IIT Kanpur*

Strategic angle. For verification it is enough to multiply $A \cdot \operatorname{adj}(A)$ and observe that diagonal entries are $|A|$ and off-diagonal entries vanish; the second product $\operatorname{adj}(A) \cdot A$ follows by symmetry.

Step 1. $|A| = 11$ (verified via two expansions).

Step 2. $\operatorname{adj}(A) = \begin{pmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{pmatrix}$.

Step 3. Diagonal entries of $A \cdot \operatorname{adj}(A)$: each is the expansion of $|A|$ along the corresponding row of A, so each equals 11.

Step 4. Off-diagonal entries: each is a row of A times cofactors of a *different* row, which is 0 by the cross-row rule.

Step 5. Hence $A \cdot \operatorname{adj}(A) = 11\,I$. By the same logic with rows and columns swapped, $\operatorname{adj}(A) \cdot A = 11\,I$.

Why this matters. The same-row/different-row dichotomy is exactly why $A \cdot \operatorname{adj}(A) = |A|\,I$ in general. It is not an accident; it is the statement of the cofactor expansion.

Final Answer: $A \cdot \operatorname{adj}(A) = \operatorname{adj}(A) \cdot A = 11\,I$.

Q 4.5 **Find the inverse of the matrix** $A = \begin{pmatrix} 2 & -2 \\ 4 & 3 \end{pmatrix}$ **(if it exists).**

SOLUTION

Concept used. For a 2×2 matrix, $A^{-1} = \dfrac{1}{|A|} \operatorname{adj}(A)$ whenever $|A| \neq 0$.

Step 1. $|A| = (2)(3) - (-2)(4) = 6 + 8 = 14 \neq 0$. So A^{-1} exists.

Step 2. $\operatorname{adj}(A) = \begin{pmatrix} 3 & 2 \\ -4 & 2 \end{pmatrix}$.

Step 3. Divide by $|A|$:

$$A^{-1} = \frac{1}{14} \begin{pmatrix} 3 & 2 \\ -4 & 2 \end{pmatrix}.$$

Final Answer: $A^{-1} = \frac{1}{14}\begin{pmatrix} 3 & 2 \\ -4 & 2 \end{pmatrix}$.

EXPERT'S SOLUTION : *Neha Iyer, M.Sc Mathematics, IIT Bombay*

Quick reading. For 2×2: swap diagonals, flip off-diagonals, divide by $|A|$.

Step 1. $|A| = 14$.

Step 2. Adjoint $= \begin{pmatrix} 3 & 2 \\ -4 & 2 \end{pmatrix}$.

Step 3. Verify:

$$A \cdot A^{-1} = \tfrac{1}{14}\begin{pmatrix} 2 & -2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -4 & 2 \end{pmatrix} = \tfrac{1}{14}\begin{pmatrix} 6+8 & 4-4 \\ 12-12 & 8+6 \end{pmatrix} = \tfrac{1}{14}\begin{pmatrix} 14 & 0 \\ 0 & 14 \end{pmatrix} = I. \checkmark$$

Final Answer: $A^{-1} = \frac{1}{14}\begin{pmatrix} 3 & 2 \\ -4 & 2 \end{pmatrix}$.

Q 4.6 **Find the inverse of** $A = \begin{pmatrix} -1 & 5 \\ -3 & 2 \end{pmatrix}$ **(if it exists).**

SOLUTION

Concept used. For 2×2, $A^{-1} = \frac{1}{|A|}\,\mathrm{adj}(A)$ when $|A| \neq 0$.

Step 1. $|A| = (-1)(2) - (5)(-3) = -2 + 15 = 13 \neq 0$, so A^{-1} exists.

Step 2. $\mathrm{adj}(A) = \begin{pmatrix} 2 & -5 \\ 3 & -1 \end{pmatrix}$.

Step 3. $A^{-1} = \frac{1}{13}\begin{pmatrix} 2 & -5 \\ 3 & -1 \end{pmatrix}$.

Final Answer: $A^{-1} = \frac{1}{13}\begin{pmatrix} 2 & -5 \\ 3 & -1 \end{pmatrix}$.

EXPERT'S SOLUTION : *Aaditya Sharma, B.Tech CSE, IIT Roorkee*

Quick reading. Compute $|A|$, write adj by the swap-and-flip rule, divide.

Step 1. $|A| = -2 - (-15) = 13$.

Step 2. $\text{adj}(A) = \begin{pmatrix} 2 & -5 \\ 3 & -1 \end{pmatrix}$.

Step 3. $A^{-1} = \frac{1}{13}\begin{pmatrix} 2 & -5 \\ 3 & -1 \end{pmatrix}$. Quick check:

$$AA^{-1} = \frac{1}{13}\begin{pmatrix} -2+15 & 5-5 \\ -6+6 & 15-2 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix} = I. \checkmark$$

Final Answer: $A^{-1} = \dfrac{1}{13}\begin{pmatrix} 2 & -5 \\ 3 & -1 \end{pmatrix}$.

Q 4.7 **Find the inverse of** $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{pmatrix}$ **(if it exists).**

SOLUTION

Concept used. For a 3×3 matrix, $A^{-1} = \frac{1}{|A|}\text{adj}(A)$. The determinant of an upper-triangular matrix is the product of its diagonal entries.

Step 1. $|A| = 1 \cdot 2 \cdot 5 = 10 \neq 0$, so A^{-1} exists.

Step 2. Minors:

$$M_{11} = \begin{vmatrix} 2 & 4 \\ 0 & 5 \end{vmatrix} = 10, \qquad M_{12} = \begin{vmatrix} 0 & 4 \\ 0 & 5 \end{vmatrix} = 0, \qquad M_{13} = \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 0,$$

$$M_{21} = \begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = 10, \qquad M_{22} = \begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = 5, \qquad M_{23} = \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0,$$

$$M_{31} = \begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} = 2, \qquad M_{32} = \begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = 4, \qquad M_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2.$$

Step 3. Cofactors with chessboard signs:

$$\text{Cof}(A) = \begin{pmatrix} 10 & 0 & 0 \\ -10 & 5 & 0 \\ 2 & -4 & 2 \end{pmatrix}.$$

Step 4. Transpose to get the adjoint:

$$\text{adj}(A) = \begin{pmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{pmatrix}.$$

Step 5. Divide by $|A| = 10$:

$$A^{-1} = \frac{1}{10}\begin{pmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{pmatrix}.$$

Final Answer: $A^{-1} = \dfrac{1}{10}\begin{pmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{pmatrix}$.

☞ **Triangular inverse stays triangular**

The inverse of an upper-triangular matrix is also upper-triangular. The diagonal of A^{-1} is the reciprocal of the diagonal of A: $1, 1/2, 1/5$, which agrees with our result (top-left $= 10/10 = 1$, middle $= 5/10 = 1/2$, bottom-right $= 2/10 = 1/5$).

EXPERT'S SOLUTION : *Riya Nair, Ph.D Mathematics, IIT Delhi*

Structural observation. A is upper-triangular, so $|A|$ is just the diagonal product and $\text{adj}(A)$ inherits the triangular structure.

Step 1. $|A| = 10$.

Step 2. Build adj as in main solution.

Step 3. Final: $A^{-1} = \frac{1}{10}\,\text{adj}(A)$.

Step 4. Sanity check on the diagonal: $A^{-1}_{ii} = 1/A_{ii}$ gives $1, 1/2, 1/5$, matching the diagonal of our answer.

Final Answer: $A^{-1} = \frac{1}{10}\begin{pmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{pmatrix}$.

Q 4.8 **Find the inverse of** $A = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{pmatrix}$ **(if it exists).**

SOLUTION

Concept used. For a 3×3 matrix, $A^{-1} = \frac{1}{|A|}\,\mathrm{adj}(A)$. Lower-triangular gives $|A| =$ diagonal product.

Step 1. $|A| = (1)(3)(-1) = -3 \neq 0$, so A^{-1} exists.

Step 2. Compute minors.

$$M_{11} = \begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} = -3, \quad M_{12} = \begin{vmatrix} 3 & 0 \\ 5 & -1 \end{vmatrix} = -3, \quad M_{13} = \begin{vmatrix} 3 & 3 \\ 5 & 2 \end{vmatrix} = -9,$$

$$M_{21} = \begin{vmatrix} 0 & 0 \\ 2 & -1 \end{vmatrix} = 0, \quad M_{22} = \begin{vmatrix} 1 & 0 \\ 5 & -1 \end{vmatrix} = -1, \quad M_{23} = \begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} = 2,$$

$$M_{31} = \begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix} = 0, \quad M_{32} = \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} = 0, \quad M_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 3 \end{vmatrix} = 3.$$

Step 3. Cofactors after chessboard signs:

$$\mathrm{Cof}(A) = \begin{pmatrix} -3 & 3 & -9 \\ 0 & -1 & -2 \\ 0 & 0 & 3 \end{pmatrix}.$$

Step 4. Transpose:

$$\mathrm{adj}(A) = \begin{pmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{pmatrix}.$$

Step 5. Divide by $|A| = -3$:

$$A^{-1} = -\frac{1}{3}\begin{pmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1/3 & 0 \\ 3 & 2/3 & -1 \end{pmatrix}.$$

Final Answer: $A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1/3 & 0 \\ 3 & 2/3 & -1 \end{pmatrix}$.

EXPERT'S SOLUTION *: Karan Reddy, M.Sc Mathematics, ISI Kolkata*

Strategic angle. Lower-triangular → lower-triangular inverse with reciprocal diagonal.

Step 1. $|A| = -3$.

Step 2. Compute adj as above. The transpose is lower-triangular.

Step 3. Divide by -3 to get a clean inverse with diagonal $\{1, 1/3, -1\} = \{1/1, 1/3, 1/(-1)\}$, matching the reciprocal-diagonal pattern.

Why this matters. For triangular matrices the inverse can be read off by forward (or back) substitution in seconds, no cofactor table needed. The cofactor route is included here only to match the NCERT method.

Final Answer: $A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1/3 & 0 \\ 3 & 2/3 & -1 \end{pmatrix}$.

Q 4.9 **Find the inverse of** $A = \begin{pmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{pmatrix}$ **(if it exists).**

SOLUTION

Concept used. $A^{-1} = \frac{1}{|A|}\,\text{adj}(A)$.

Step 1. Compute $|A|$ along R_1:

$$|A| = 2\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 4 & 0 \\ -7 & 1 \end{vmatrix} + 3\begin{vmatrix} 4 & -1 \\ -7 & 2 \end{vmatrix}.$$

$$= 2(-1-0) - 1(4-0) + 3(8-7) = -2 - 4 + 3 = -3.$$

So $|A| = -3 \neq 0$.

Step 2. Minors:

$$M_{11} = \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = -1, \quad M_{12} = \begin{vmatrix} 4 & 0 \\ -7 & 1 \end{vmatrix} = 4, \quad M_{13} = \begin{vmatrix} 4 & -1 \\ -7 & 2 \end{vmatrix} = 1,$$

$$M_{21} = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = -5, \quad M_{22} = \begin{vmatrix} 2 & 3 \\ -7 & 1 \end{vmatrix} = 23, \quad M_{23} = \begin{vmatrix} 2 & 1 \\ -7 & 2 \end{vmatrix} = 11,$$

$$M_{31} = \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} = 3, \quad M_{32} = \begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix} = -12, \quad M_{33} = \begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix} = -6.$$

Step 3. Cofactors (chessboard signs):

$$\text{Cof}(A) = \begin{pmatrix} -1 & -4 & 1 \\ 5 & 23 & -11 \\ 3 & 12 & -6 \end{pmatrix}.$$

Step 4. Transpose:

$$\text{adj}(A) = \begin{pmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{pmatrix}.$$

Step 5. Divide by $|A| = -3$:

$$A^{-1} = -\frac{1}{3}\begin{pmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & -5 & -3 \\ 4 & -23 & -12 \\ -1 & 11 & 6 \end{pmatrix}.$$

Final Answer: $A^{-1} = \dfrac{1}{3}\begin{pmatrix} 1 & -5 & -3 \\ 4 & -23 & -12 \\ -1 & 11 & 6 \end{pmatrix}$.

✗ Common Pitfall

After dividing by a negative $|A|$, every sign in $\text{adj}(A)$ flips. Forgetting this is the single biggest source of arithmetic errors in this exercise.

EXPERT'S SOLUTION *: Vivaan Singh, M.Tech CS, IIT Madras*

Strategic angle. Build the cofactor table, transpose, divide. The only delicate bit is the sign of $|A|$.

Step 1. $|A| = -3$.

Step 2. Cofactor matrix, transpose, then divide by -3 (equivalently, multiply by $-1/3$).

Step 3. Verification: compute $A \cdot A^{-1}$ row by column for the $(1,1)$ entry:

$$\tfrac{1}{3}[2(1) + 1(4) + 3(-1)] = \tfrac{1}{3}(2 + 4 - 3) = 1. \checkmark$$

Final Answer: $A^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -5 & -3 \\ 4 & -23 & -12 \\ -1 & 11 & 6 \end{pmatrix}$.

Q 4.10 **Find the inverse of** $A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{pmatrix}$ **(if it exists).**

SOLUTION

Concept used. $A^{-1} = \frac{1}{|A|}\operatorname{adj}(A)$.

Step 1. Compute $|A|$ along R_1:

$$|A| = 1(2 \cdot 4 - (-3)(-2)) - (-1)(0 \cdot 4 - (-3)(3)) + 2(0 \cdot (-2) - 2 \cdot 3).$$

$$= 1(8-6) + 1(0+9) + 2(0-6) = 2 + 9 - 12 = -1.$$

So $|A| = -1 \neq 0$.

Step 2. Minors:

$$M_{11} = \begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} = 2, \quad M_{12} = \begin{vmatrix} 0 & -3 \\ 3 & 4 \end{vmatrix} = 9, \quad M_{13} = \begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix} = -6,$$

$$M_{21} = \begin{vmatrix} -1 & 2 \\ -2 & 4 \end{vmatrix} = 0, \quad M_{22} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2, \quad M_{23} = \begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} = 1,$$

$$M_{31} = \begin{vmatrix} -1 & 2 \\ 2 & -3 \end{vmatrix} = -1, \quad M_{32} = \begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} = -3, \quad M_{33} = \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = 2.$$

Step 3. Cofactors:

$$\text{Cof}(A) = \begin{pmatrix} 2 & -9 & -6 \\ 0 & -2 & -1 \\ -1 & 3 & 2 \end{pmatrix}.$$

Step 4. Transpose:

$$\operatorname{adj}(A) = \begin{pmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{pmatrix}.$$

Step 5. Divide by $|A| = -1$:

$$A^{-1} = -\operatorname{adj}(A) = \begin{pmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{pmatrix}.$$

Final Answer: $A^{-1} = \begin{pmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{pmatrix}$.

EXPERT'S SOLUTION : *Aditi Bhat, M.Sc Mathematics, IIT Bombay*

Quick reading. $|A| = -1$, so the inverse is the negative of the adjoint.

Step 1. $|A| = -1$.

Step 2. $\text{adj}(A) = \begin{pmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{pmatrix}$.

Step 3. $A^{-1} = -\,\text{adj}(A)$.

Step 4. Spot-check the $(1,1)$ entry of AA^{-1}: $1(-2) + (-1)(9) + 2(6) = -2 - 9 + 12 = 1$. ✓

Final Answer: $A^{-1} = \begin{pmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{pmatrix}$.

Q 4.11 **Find the inverse of** $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{pmatrix}$ **(if it exists).**

SOLUTION

Concept used. $A^{-1} = \frac{1}{|A|}\,\text{adj}(A)$.

Step 1. $|A|$ along R_1 (two zeros in R_1):

$$|A| = 1\cdot\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{vmatrix} = (\cos\alpha)(-\cos\alpha) - (\sin\alpha)(\sin\alpha) = -\cos^2\alpha - \sin^2\alpha = -1.$$

So $|A| = -1 \neq 0$.

Step 2. Minors. Row 1:

$$M_{11} = -1, \quad M_{12} = \begin{vmatrix} 0 & \sin\alpha \\ 0 & -\cos\alpha \end{vmatrix} = 0, \quad M_{13} = \begin{vmatrix} 0 & \cos\alpha \\ 0 & \sin\alpha \end{vmatrix} = 0.$$

Row 2:

$$M_{21} = 0, \quad M_{22} = \begin{vmatrix} 1 & 0 \\ 0 & -\cos\alpha \end{vmatrix} = -\cos\alpha, \quad M_{23} = \begin{vmatrix} 1 & 0 \\ 0 & \sin\alpha \end{vmatrix} = \sin\alpha.$$

Row 3:

$$M_{31} = 0, \quad M_{32} = \begin{vmatrix} 1 & 0 \\ 0 & \sin\alpha \end{vmatrix} = \sin\alpha, \quad M_{33} = \begin{vmatrix} 1 & 0 \\ 0 & \cos\alpha \end{vmatrix} = \cos\alpha.$$

Step 3. Cofactors (chessboard):

$$\text{Cof}(A) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}.$$

Step 4. Transpose (it is symmetric here):

$$\text{adj}(A) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}.$$

Step 5. Divide by $|A| = -1$:

$$A^{-1} = -\text{adj}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{pmatrix} = A.$$

Final Answer: $A^{-1} = A$. (The matrix is its own inverse.)

♥ Self-inverse matrices

$A^{-1} = A$ means $A^2 = I$. Such matrices are called **involutions**; they show up in reflections (geometric) and in many parity-like operations. Here A is essentially a 2×2 reflection embedded in a 3×3 identity-extended block, and reflections are always involutions.

EXPERT'S SOLUTION : *Ishaan Pillai, B.Tech Engineering Physics, IIT Bombay*

Picture-first. The lower-right 2×2 block, $\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$, is the matrix of reflection about the line $y = \tan(\alpha/2)\,x$. Reflections are involutions, so squaring gives I. The first row/column of A is the trivial identity on a third dimension.

Step 1. $|A| = -1$. (Reflections always have determinant -1.)

Step 2. Adjoint via cofactor table as above.

Step 3. $A^{-1} = -\text{adj}(A) = A$.

Why this matters. Whenever the question gives a geometric description (rotation, reflection, projection), look for $A^2 = I$ before grinding through cofactors.

Final Answer: $A^{-1} = A$.

Q 4.12 **Let** $A = \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}$ **and** $B = \begin{pmatrix} 6 & 8 \\ 7 & 9 \end{pmatrix}$. **Verify that** $(AB)^{-1} = B^{-1}A^{-1}$.

SOLUTION

Concept used. For any two non-singular matrices of the same order, $(AB)^{-1} = B^{-1}A^{-1}$. We will compute both sides explicitly and compare.

Step 1. Compute $|A|$ and $|B|$:

$$|A| = 15 - 14 = 1, \quad |B| = 54 - 56 = -2.$$

Both non-zero, so both inverses exist.

Step 2. Inverses (use 2×2 shortcut). With $|A| = 1$:

$$A^{-1} = \begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix}.$$

With $|B| = -2$:

$$B^{-1} = -\frac{1}{2}\begin{pmatrix} 9 & -8 \\ -7 & 6 \end{pmatrix}.$$

Step 3. Compute $B^{-1}A^{-1}$. First form the product matrix:

$$\begin{pmatrix} 9 & -8 \\ -7 & 6 \end{pmatrix}\begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 45+16 & -63-24 \\ -35-12 & 49+18 \end{pmatrix} = \begin{pmatrix} 61 & -87 \\ -47 & 67 \end{pmatrix}.$$

Then

$$B^{-1}A^{-1} = -\frac{1}{2}\begin{pmatrix} 61 & -87 \\ -47 & 67 \end{pmatrix}.$$

Step 4. Compute AB:

$$AB = \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 6 & 8 \\ 7 & 9 \end{pmatrix} = \begin{pmatrix} 18+49 & 24+63 \\ 12+35 & 16+45 \end{pmatrix} = \begin{pmatrix} 67 & 87 \\ 47 & 61 \end{pmatrix}.$$

Step 5. $|AB| = 67 \cdot 61 - 87 \cdot 47 = 4087 - 4089 = -2$. (Consistent with $|A||B| = 1 \cdot (-2) = -2$.)

Step 6. Compute $(AB)^{-1}$:

$$(AB)^{-1} = -\frac{1}{2}\begin{pmatrix} 61 & -87 \\ -47 & 67 \end{pmatrix}.$$

Step 7. Compare with step 3:

$$(AB)^{-1} = -\frac{1}{2}\begin{pmatrix} 61 & -87 \\ -47 & 67 \end{pmatrix} = B^{-1}A^{-1}. \checkmark$$

Final Answer: $(AB)^{-1} = B^{-1}A^{-1} = -\frac{1}{2}\begin{pmatrix} 61 & -87 \\ -47 & 67 \end{pmatrix}$.

✗ Order matters

The reverse-order rule $(AB)^{-1} = B^{-1}A^{-1}$ is not optional. $A^{-1}B^{-1}$ in general does *not* equal $(AB)^{-1}$. Multiplication of matrices is not commutative.

EXPERT'S SOLUTION *: Tara Verma, M.Sc Mathematics, ISI Kolkata*

Structural observation. The identity $(AB)^{-1} = B^{-1}A^{-1}$ follows from one line: $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = I$. The verification step is direct computation.

Step 1. $|A| = 1$, $|B| = -2$, so both invertible.

Step 2. Compute A^{-1} and B^{-1} via the 2×2 formula.

Step 3. Compute AB and $(AB)^{-1}$ from the formula.

Step 4. Compute $B^{-1}A^{-1}$ from the product.

Step 5. Observe equality of the two final matrices.

Why this matters. The reverse-order rule appears in every chapter that involves matrix products, including transposes ($(AB)^T = B^TA^T$) and adjoints ($(AB)^* = B^*A^*$). Once internalised it never trips you again.

Final Answer: $(AB)^{-1} = B^{-1}A^{-1}$, both equal $-\frac{1}{2}\begin{pmatrix} 61 & -87 \\ -47 & 67 \end{pmatrix}$.

Q 4.13 **If** $A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$**, show that** $A^2 - 5A + 7I = O$**. Hence find** A^{-1}**.**

SOLUTION

Concept used. A polynomial identity of the form $A^2 + pA + qI = O$ (with $q \neq 0$) can be rearranged to $A(A + pI) = -qI$, giving $A^{-1} = -\frac{1}{q}(A + pI)$. This avoids the cofactor computation entirely (the underlying idea is the **Cayley-Hamilton theorem**).

Step 1. Compute A^2:

$$A^2 = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ -5 & 3 \end{pmatrix}.$$

Step 2. Compute $5A$ and $7I$:

$$5A = \begin{pmatrix} 15 & 5 \\ -5 & 10 \end{pmatrix}, \qquad 7I = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}.$$

Step 3. Form $A^2 - 5A + 7I$:

$$\begin{pmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O.$$

The identity is verified.

Step 4. Rearrange to find A^{-1}. From $A^2 - 5A + 7I = O$,

$$A^2 - 5A = -7I.$$

Factor A:

$$A(A - 5I) = -7I.$$

Premultiply both sides by A^{-1} (it exists since $|A| = 6 + 1 = 7 \neq 0$):

$$A - 5I = -7\,A^{-1} \implies A^{-1} = \frac{1}{7}(5I - A).$$

Step 5. Compute $5I - A$:

$$5I - A = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}.$$

Step 6. Hence

$$A^{-1} = \frac{1}{7}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}.$$

Final Answer: $A^{-1} = \frac{1}{7}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$.

♥ Cayley-Hamilton in disguise

Cayley-Hamilton says every 2×2 matrix satisfies its own characteristic polynomial $\lambda^2 - (\text{tr}\, A)\lambda + |A| = 0$. Here $\text{tr}\, A = 3 + 2 = 5$ and $|A| = 7$, exactly matching $A^2 - 5A + 7I = O$. The polynomial identity is automatic; computing it just verifies the theorem.

EXPERT'S SOLUTION : *Aarav Desai, Ph.D Pure Mathematics, IISc Bangalore*

Strategic angle. Verify the polynomial identity, then use it to express A^{-1} as a polynomial in A. No cofactors needed.

Step 1. Show $A^2 = \begin{pmatrix} 8 & 5 \\ -5 & 3 \end{pmatrix}$, $5A = \begin{pmatrix} 15 & 5 \\ -5 & 10 \end{pmatrix}$, $7I = 7I$. Sum: O.

Step 2. From $A^2 - 5A + 7I = O$: $A(A - 5I) = -7I$, so $A^{-1} = \frac{1}{7}(5I - A)$.

Step 3. $5I - A = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$.

Step 4. Cross-check with the direct 2×2 formula: $|A| = 7$, $\text{adj}(A) = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$, so

$A^{-1} = \frac{1}{7}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$. ✓

Why this matters. The polynomial-identity route generalises: for a 3×3 matrix you can build A^{-1} from A^2, A and I using the Cayley-Hamilton polynomial. This is how the next two questions are tackled.

Final Answer: $A^{-1} = \frac{1}{7}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$.

Q 4.14 **For the matrix $A = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$, find the numbers a and b such that $A^2 + aA + bI = O$.**

SOLUTION

Concept used. Cayley-Hamilton for a 2×2 matrix: $A^2 - (\text{tr}\, A)\, A + |A|\, I = O$. So $a = -\,\text{tr}(A)$ and $b = |A|$.

Step 1. Trace: $\text{tr}(A) = 3 + 1 = 4$.

Step 2. Determinant: $|A| = 3 \cdot 1 - 2 \cdot 1 = 1$.

Step 3. By Cayley-Hamilton:

$$A^2 - 4A + 1 \cdot I = O.$$

So $a = -4$ and $b = 1$.

Step 4. Verify by direct computation. First A^2:

$$A^2 = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 9+2 & 6+2 \\ 3+1 & 2+1 \end{pmatrix} = \begin{pmatrix} 11 & 8 \\ 4 & 3 \end{pmatrix}.$$

Then

$$A^2 - 4A + I = \begin{pmatrix} 11-12+1 & 8-8+0 \\ 4-4+0 & 3-4+1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O. \checkmark$$

Final Answer: $a = -4$ and $b = 1$.

EXPERT'S SOLUTION : *Pranav Chatterjee, M.Sc Mathematics, IIT Bombay*

Structural observation. Read off trace and determinant; Cayley-Hamilton instantly gives the polynomial.

Step 1. $\mathrm{tr}(A) = 4,\ |A| = 1.$

Step 2. $A^2 - 4A + I = O \Rightarrow a = -4,\ b = 1.$

Step 3. Verify by direct calculation.

Why this matters. Whenever a 2×2 matrix question asks "find a, b such that $A^2 + aA + bI = O$", the answer is always $(a, b) = (-\mathrm{tr}\,A,\ |A|)$. No multiplication required.

Final Answer: $a = -4,\ b = 1.$

Q 4.15 **For the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$, show that $A^3 - 6A^2 + 5A + 11I = O$. Hence find A^{-1}.**

SOLUTION

Concept used. Cayley-Hamilton for a 3×3 matrix: $A^3 - (\mathrm{tr}\,A)\,A^2 + c_1 A - |A|\,I = O$, where c_1 is the sum of 2×2 principal minors. Once verified, rearrange to extract A^{-1} as a polynomial in A.

Step 1. Compute A^2:

$$A^2 = AA = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{pmatrix}.$$

Step 2. Compute $A^3 = A \cdot A^2$:

$$A^3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{pmatrix}.$$

Row 1: $(4-3+7,\ 2+8-3,\ 1-14+14)=(8,\ 7,\ 1)$. Row 2: $(4-6-21,\ 2+16+9,\ 1-28-42)=(-23,\ 27,\ -69)$. Row 3: $(8+3+21,\ 4-8-9,\ 2+14+42)=(32,\ -13,\ 58)$.

$$A^3=\begin{pmatrix}8&7&1\\-23&27&-69\\32&-13&58\end{pmatrix}.$$

Step 3. Compute $6A^2$:

$$6A^2=\begin{pmatrix}24&12&6\\-18&48&-84\\42&-18&84\end{pmatrix}.$$

Step 4. Compute $5A$:

$$5A=\begin{pmatrix}5&5&5\\5&10&-15\\10&-5&15\end{pmatrix}.$$

Step 5. Compute $A^3-6A^2+5A+11I$ entry by entry. Row 1: $(8-24+5+11,\ 7-12+5+0,\ 1-6+5+0)=(0,\ 0,\ 0)$. Row 2: $(-23+18+5+0,\ 27-48+10+11,\ -69+84-15+0)=(0,\ 0,\ 0)$. Row 3: $(32-42+10+0,\ -13+18-5+0,\ 58-84+15+11)=(0,\ 0,\ 0)$. So $A^3-6A^2+5A+11I=O$. ✓

Step 6. Solve for A^{-1}. Rearrange:

$$A^3-6A^2+5A=-11I.$$

Factor A:

$$A(A^2-6A+5I)=-11I.$$

Premultiply by A^{-1} (it exists since A is non-singular — the identity has constant term $11I\neq 0$):

$$A^2-6A+5I=-11\,A^{-1}\implies A^{-1}=-\frac{1}{11}(A^2-6A+5I).$$

Step 7. Compute $A^2-6A+5I$:

$$A^2-6A=\begin{pmatrix}4&2&1\\-3&8&-14\\7&-3&14\end{pmatrix}-\begin{pmatrix}6&6&6\\6&12&-18\\12&-6&18\end{pmatrix}=\begin{pmatrix}-2&-4&-5\\-9&-4&4\\-5&3&-4\end{pmatrix}.$$

Add $5I$:

$$A^2-6A+5I=\begin{pmatrix}3&-4&-5\\-9&1&4\\-5&3&1\end{pmatrix}.$$

Step 8. Therefore

$$A^{-1} = -\frac{1}{11}\begin{pmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{pmatrix}.$$

Final Answer: $A^{-1} = \frac{1}{11}\begin{pmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{pmatrix}$.

☞ Exam Tip

For a 3×3 Cayley-Hamilton question, compute A^2 once, then both A^3 and A^{-1} flow from products of already-computed matrices. Build the entries in a column rather than a row, and double-check the arithmetic at each row of A^3.

EXPERT'S SOLUTION *: Aanya Banerjee, M.Sc Mathematics, ISI Kolkata*

Strategic angle. Verify the polynomial identity by computing A^2 and A^3; then use the polynomial to write A^{-1} as $-\frac{1}{11}(A^2 - 6A + 5I)$.

Step 1. A^2 and A^3 as in main solution.

Step 2. $A^3 - 6A^2 + 5A + 11I = O$: verified entry by entry.

Step 3. $A^{-1} = -\frac{1}{11}(A^2 - 6A + 5I) = \frac{1}{11}\begin{pmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{pmatrix}$.

Step 4. Sanity check $|A|$: compute along R_1:

$$|A| = 1(6-3) - 1(3+6) + 1(-1-4) = 3 - 9 - 5 = -11.$$

Sign agrees with Cayley-Hamilton's $-|A|\,I = +11I$ (i.e. $|A| = -11$). The factor $1/11$ in A^{-1} is therefore expected (up to overall sign).

Why this matters. Cayley-Hamilton is the conceptual back-bone of the inverse formula. Many higher problems either explicitly invoke it or hide it inside the wording.

Final Answer: $A^{-1} = \frac{1}{11}\begin{pmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{pmatrix}$.

Q 4.16 If $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$, **verify that** $A^3 - 6A^2 + 9A - 4I = O$ **and hence find** A^{-1}.

SOLUTION

Concept used. Same as Q15: verify a Cayley-Hamilton-style polynomial, then rearrange for A^{-1}.

Step 1. Compute A^2. Each entry $\sum_k A_{ik}A_{kj}$. Row 1: $(4+1+1,\ -2-2-1,\ 2+1+2) = (6,\ -5,\ 5)$. Row 2: $(-2-2-1,\ 1+4+1,\ -1-2-2) = (-5,\ 6,\ -5)$. Row 3: $(2+1+2,\ -1-2-2,\ 1+1+4) = (5,\ -5,\ 6)$.

$$A^2 = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}.$$

Step 2. Compute $A^3 = A \cdot A^2$. Row 1: $(12+5+5,\ -10-6-5,\ 10+5+6) = (22,\ -21,\ 21)$. Row 2: $(-6-10-5,\ 5+12+5,\ -5-10-6) = (-21,\ 22,\ -21)$. Row 3: $(6+5+10,\ -5-6-10,\ 5+5+12) = (21,\ -21,\ 22)$.

$$A^3 = \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix}.$$

Step 3. Compute $6A^2 = \begin{pmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{pmatrix}$ and $9A = \begin{pmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{pmatrix}$.

Step 4. Compute $A^3 - 6A^2 + 9A - 4I$ entry by entry. $(1,1)$: $22 - 36 + 18 - 4 = 0$. ✓ $(1,2)$: $-21 + 30 - 9 - 0 = 0$. ✓ $(1,3)$: $21 - 30 + 9 - 0 = 0$. ✓ $(2,1)$: $-21 + 30 - 9 = 0$. $(2,2)$: $22 - 36 + 18 - 4 = 0$. $(2,3)$: $-21 + 30 - 9 = 0$. $(3,1)$: $21 - 30 + 9 = 0$. $(3,2)$: $-21 + 30 - 9 = 0$. $(3,3)$: $22 - 36 + 18 - 4 = 0$. So the identity is verified.

Step 5. Solve for A^{-1}. From $A^3 - 6A^2 + 9A - 4I = O$:

$$A(A^2 - 6A + 9I) = 4I \implies A^{-1} = \frac{1}{4}(A^2 - 6A + 9I).$$

Step 6. Compute $A^2 - 6A + 9I$:

$$A^2 - 6A = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - \begin{pmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{pmatrix} = \begin{pmatrix} -6 & 1 & -1 \\ 1 & -6 & 1 \\ -1 & 1 & -6 \end{pmatrix}.$$

Add $9I$:

$$A^2 - 6A + 9I = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}.$$

Step 7. Therefore

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}.$$

Final Answer: $A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$.

EXPERT'S SOLUTION *: Aaditya Joshi, B.Tech CSE, IIT Roorkee*

Symmetry observation. A is symmetric, so A^2 and A^3 are symmetric, and so is A^{-1}. The structure preserves down the line.

Step 1. A^2, A^3 as in main solution.

Step 2. Verify Cayley-Hamilton.

Step 3. $A^{-1} = \frac{1}{4}(A^2 - 6A + 9I) = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$.

Step 4. Independent check via $|A|$: expand along R_1:
$2(4-1) - (-1)(-2+1) + 1(1-2) = 6 - 1 - 1 = 4$. Cayley-Hamilton gave the constant term $4I$, matching $|A| = 4$.

Why this matters. Symmetric matrices retain symmetry under polynomial operations, so verifying just one off-diagonal entry of A^{-1} matches its mirror image is enough to gain confidence in the rest.

Final Answer: $A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$.

Q 4.17 **Let A be a nonsingular square matrix of order 3×3. Then $|\operatorname{adj} A|$ is equal to**
(A) $|A|$ **(B)** $|A|^2$ **(C)** $|A|^3$ **(D)** $3|A|$.

SOLUTION

Concept used. The identity $A \cdot \text{adj}(A) = |A|\, I$, plus the product rule for determinants $|XY| = |X|\,|Y|$, plus $|kI_n| = k^n$.

Step 1. Start with $A \cdot \text{adj}(A) = |A|\, I$ (order 3 matrix, so $I = I_3$).

Step 2. Take determinants on both sides:

$$|A \cdot \text{adj}(A)| = \big|\, |A|\, I_3 \,\big|.$$

Step 3. LHS: by the product rule, $|A \cdot \text{adj}(A)| = |A| \cdot |\,\text{adj}(A)|$.

Step 4. RHS: $|kI_3| = k^3$ for any scalar k, so $\big|\, |A|\, I_3 \,\big| = |A|^3$.

Step 5. Equate:

$$|A| \cdot |\,\text{adj}(A)| = |A|^3.$$

Since A is non-singular, $|A| \neq 0$; divide by $|A|$:

$$|\,\text{adj}(A)| = |A|^2.$$

Step 6. Match with options: option (B).

> **Final Answer:** Option (B): $|\,\text{adj}(A)| = |A|^2$.

♥ General formula

For an $n \times n$ matrix, the same argument gives $|\,\text{adj}(A)| = |A|^{n-1}$. For $n = 3$ that is $|A|^2$; for $n = 2$ it is $|A|^1 = |A|$; for $n = 1$ it is $|A|^0 = 1$.

EXPERT'S SOLUTION *: Priya Singh, M.Tech CS, IIT Madras*

Quick reading. Use $A \text{ adj } A = |A|\, I$; take determinants; the answer for $n = 3$ is $|A|^2$.

Step 1. Determinant of both sides: $|A| \cdot |\,\text{adj}(A)| = |A|^3$.

Step 2. Divide by $|A|$: $|\,\text{adj}(A)| = |A|^2$.

> **Final Answer:** Option (B).

Q 4.18 **If A is an invertible matrix of order 2, then $\det(A^{-1})$ is equal to**

(A) $\det(A)$ **(B)** $\dfrac{1}{\det(A)}$ **(C)** 1 **(D)** 0.

SOLUTION

Concept used. For any invertible matrix, $AA^{-1} = I$, and taking determinants gives $\det(A) \cdot \det(A^{-1}) = \det(I) = 1$.

Step 1. Start with $A\,A^{-1} = I$.

Step 2. Take determinants:

$$\det(A\,A^{-1}) = \det(I) = 1.$$

Step 3. Use the product rule:

$$\det(A) \cdot \det(A^{-1}) = 1.$$

Step 4. Since A is invertible, $\det(A) \neq 0$. Divide:

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Step 5. Match: option (B).

Final Answer: Option (B): $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

EXPERT'S SOLUTION *: Sanya Sharma, M.Sc Mathematics, IIT Bombay*

One-line argument. $\det(A)\det(A^{-1}) = \det(I) = 1$, so $\det(A^{-1}) = 1/\det(A)$.
Why this matters. This is independent of the order n: the relation holds for every invertible matrix, real or complex. NCERT singles out $n = 2$ only to keep the statement small.

Final Answer: Option (B): $\det(A^{-1}) = 1/\det(A)$.

Key Takeaways

- $\text{adj}(A)$ is the transpose of the cofactor matrix; for 2×2, swap diagonals and flip off-diagonals.
- Identity: $A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = |A|\,I$. Inverse: $A^{-1} = \frac{1}{|A|}\,\text{adj}(A)$ when $|A| \neq 0$.
- $(AB)^{-1} = B^{-1}A^{-1}$ (reverse order).
- Cayley-Hamilton: every $n \times n$ matrix satisfies its own characteristic polynomial. For 2×2, $A^2 - (\text{tr}\,A)A + |A|\,I = O$.
- $|\,\text{adj}(A)| = |A|^{n-1}$ and $\det(A^{-1}) = 1/\det(A)$.

End of Exercise 4.4