Collegedunia NCERT Solutions

Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 4: Determinants

About this Chapter

This exercise uses determinants and matrix inverses to solve systems of linear equations. A system $AX = B$ has a **unique solution** $X = A^{-1}B$ whenever $|A| \neq 0$. When $|A| = 0$ we use $(\text{adj } A)B$ to decide **consistency**: if $(\text{adj } A)B \neq O$ the system is inconsistent (no solution); if $(\text{adj } A)B = O$ the system has infinitely many or no solutions.

Topics covered: Matrix form $AX = B$ • Consistency criterion • Unique solution $X = A^{-1}B$ • Word problems via matrix method

Quick Formula Sheet

Matrix form: write $a_i x + b_i y + c_i z = d_i$ as $AX = B$.

If $|A| \neq 0$: unique solution $X = A^{-1}B$.

If $|A| = 0$:
$(\text{adj } A)B \neq O \Rightarrow$ no solution.
$(\text{adj } A)B = O \Rightarrow$ infinitely many or none.

Exercise 4.5

Q 4.1 **Examine the consistency of the system of equations:**

$x + 2y = 2$

$2x + 3y = 3$.

SOLUTION

Concept used. Write the system as $AX = B$. If $|A| \neq 0$ the system is consistent with a unique solution. If $|A| = 0$, then compute $(\text{adj } A)B$: if it is non-zero the system is inconsistent; if it is zero the system has infinitely many solutions (or sometimes no solution, depending on consistency).

Step 1. Write in matrix form. Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Step 2. Compute $|A|$:

$$|A| = (1)(3) - (2)(2) = 3 - 4 = -1.$$

Step 3. Since $|A| = -1 \neq 0$, A is non-singular. The system has a unique solution $X = A^{-1}B$, so the system is **consistent**.

> **Final Answer:** $|A| = -1 \neq 0$, so the system is consistent with a unique solution.

EXPERT'S SOLUTION *: Aarav Patel, M.Sc Mathematics, IIT Bombay*

Quick reading. For a 2×2 system, compute $|A|$. Non-zero $\Rightarrow$ consistent.

Step 1. $|A| = 3 - 4 = -1 \neq 0$.

Step 2. Hence the system has a unique solution; it is consistent.

> **Final Answer:** Consistent ($|A| = -1$).

Q 4.2 **Examine the consistency of the system:** $2x - y = 5, \; x + y = 4$.

SOLUTION

Concept used. Same as Q1.

Step 1. $A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

Step 2. $|A| = (2)(1) - (-1)(1) = 2 + 1 = 3$.

Step 3. $|A| = 3 \neq 0$, so the system has a unique solution. Consistent.

> **Final Answer:** Consistent ($|A| = 3$).

EXPERT'S SOLUTION : *Pranav Sharma, M.Sc Mathematics, IIT Bombay*

Quick reading. $|A| = 3 \neq 0 \Rightarrow$ consistent.

Final Answer: Consistent.

Q 4.3 **Examine the consistency of the system:** $x + 3y = 5,\ \ 2x + 6y = 8$.

SOLUTION

Concept used. If $|A| = 0$, check $(\text{adj } A)B$. Non-zero $\Rightarrow$ no solution (inconsistent); zero $\Rightarrow$ infinitely many solutions.

Step 1. $A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$.

Step 2. $|A| = (1)(6) - (3)(2) = 6 - 6 = 0$. So A is singular.

Step 3. Compute $\text{adj}(A)$:

$$\text{adj}(A) = \begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}.$$

Step 4. Compute $(\text{adj } A)B$:

$$(\text{adj } A)B = \begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 30 - 24 \\ -10 + 8 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix} \neq O.$$

Step 5. Hence the system has **no solution**; it is inconsistent.

Final Answer: Inconsistent ($|A| = 0$ and $(\text{adj } A)B \neq O$).

♥ Geometric reading

The two lines are $x+3y = 5$ and $2x+6y = 8$. The second has the same slope (i.e. $x+3y = 4$ after dividing by 2) but a different intercept. They are parallel, never intersect, hence no solution.

EXPERT'S SOLUTION : *Aditi Iyer, Ph.D Mathematics, IIT Delhi*

Structural observation. The coefficient ratios $1/2 = 3/6$ but $5/8$ differs, so the lines are parallel and distinct.

Step 1. $|A| = 0$.

Step 2. $(\text{adj } A)B = \binom{6}{-2} \neq O$.

Step 3. Inconsistent.

Final Answer: Inconsistent.

Q 4.4 **Examine the consistency of the system:** $x + y + z = 1$, $2x + 3y + 2z = 2$, $ax + ay + 2az = 4$.

SOLUTION

Concept used. Write as $AX = B$. Use $|A|$ and (if $|A| = 0$) $(\text{adj } A)B$.

Step 1. $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$.

Step 2. Compute $|A|$. Factor a from R_3:

$$|A| = a \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix}.$$

Expand the inner determinant along R_1:

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} = 1(6-2) - 1(4-2) + 1(2-3) = 4 - 2 - 1 = 1.$$

Hence $|A| = a \cdot 1 = a$.

Step 3. Case 1: $a \neq 0$. Then $|A| \neq 0$, system has a unique solution and is **consistent**.

Step 4. Case 2: $a = 0$. Then the third equation becomes $0 = 4$, which is false. So the system has *no solution* and is **inconsistent**.

Final Answer: Consistent (unique solution) if $a \neq 0$; inconsistent if $a = 0$.

☞ **Exam Tip**

When a coefficient parameter a appears across an entire row, always inspect both cases $a = 0$ and $a \neq 0$. NCERT problems involving a free parameter often hide a degenerate case at $a = 0$.

EXPERT'S SOLUTION : *Diya Reddy, M.Sc Mathematics, ISI Kolkata*

Strategic angle. Factor out the row containing the parameter.

Step 1. $|A| = a \cdot \left|\begin{smallmatrix}1&1&1\\2&3&2\\1&1&2\end{smallmatrix}\right| = a.$

Step 2. $a \neq 0$: unique solution; consistent.

Step 3. $a = 0$: the third equation reads $0 = 4$, impossible; inconsistent.

Why this matters. A vanishing determinant alone is not enough to decide inconsistency vs. infinitely many. Always inspect the RHS too. Here $a = 0$ kills the LHS but leaves a non-zero RHS, so the system is inconsistent rather than dependent.

> **Final Answer:** Consistent iff $a \neq 0$.

Q 4.5 **Examine the consistency of the system:** $3x - y - 2z = 2,\ 2y - z = -1,\ 3x - 5y = 3.$

SOLUTION

Concept used. Check $|A|$; if zero, look at $(\text{adj } A)B$.

Step 1. $A = \begin{pmatrix}3&-1&-2\\0&2&-1\\3&-5&0\end{pmatrix},\ B = \begin{pmatrix}2\\-1\\3\end{pmatrix}.$

Step 2. Compute $|A|$ along R_1:

$$|A| = 3\begin{vmatrix}2&-1\\-5&0\end{vmatrix} - (-1)\begin{vmatrix}0&-1\\3&0\end{vmatrix} + (-2)\begin{vmatrix}0&2\\3&-5\end{vmatrix}.$$

$$= 3(0-5) + 1(0+3) - 2(0-6) = -15 + 3 + 12 = 0.$$

Step 3. Since $|A| = 0$, compute $(\text{adj } A)B$. We need the cofactors:

$$A_{11} = +(0-5) = -5,\quad A_{12} = -(0+3) = -3,\quad A_{13} = +(0-6) = -6,$$
$$A_{21} = -(0-10) = 10,\quad A_{22} = +(0+6) = 6,\quad A_{23} = -(-15+3) = 12,$$
$$A_{31} = +(1+4) = 5,\quad A_{32} = -(-3-0) = 3,\quad A_{33} = +(6-0) = 6.$$

Adjoint = transpose of cofactor matrix:

$$\text{adj}(A) = \begin{pmatrix}-5&10&5\\-3&6&3\\-6&12&6\end{pmatrix}.$$

Step 4. Multiply with B:

$$(\text{adj } A)B = \begin{pmatrix}-5&10&5\\-3&6&3\\-6&12&6\end{pmatrix}\begin{pmatrix}2\\-1\\3\end{pmatrix} = \begin{pmatrix}-10-10+15\\-6-6+9\\-12-12+18\end{pmatrix} = \begin{pmatrix}-5\\-3\\-6\end{pmatrix} \neq O.$$

Step 5. Hence the system is inconsistent.

> **Final Answer:** Inconsistent ($|A| = 0$ and $(\text{adj } A)B \neq O$).

EXPERT'S SOLUTION : *Tara Mehta, M.Sc Applied Mathematics, IIT Kanpur*

Quick reading. Compute $|A|$; it is zero. Compute $(\text{adj } A)B$; it is non-zero. Inconsistent.

> **Final Answer:** Inconsistent.

Q 4.6 **Examine the consistency of the system:** $5x - y + 4z = 5$, $2x + 3y + 5z = 2$, $5x - 2y + 6z = -1$.

SOLUTION

Concept used. $|A| \neq 0 \Rightarrow$ unique solution, consistent.

Step 1. $A = \begin{pmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{pmatrix}$.

Step 2. Compute $|A|$ along R_1:

$$|A| = 5\begin{vmatrix} 3 & 5 \\ -2 & 6 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 5 \\ 5 & 6 \end{vmatrix} + 4\begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix}.$$

$$= 5(18 + 10) + 1(12 - 25) + 4(-4 - 15) = 5(28) + 1(-13) + 4(-19).$$

$$= 140 - 13 - 76 = 51.$$

Step 3. $|A| = 51 \neq 0$, so the system has a unique solution and is consistent.

> **Final Answer:** Consistent ($|A| = 51$).

EXPERT'S SOLUTION : *Aditya Pillai, B.Tech Engineering Physics, IIT Bombay*

Quick reading. Non-zero determinant $\Rightarrow$ unique solution.

> **Final Answer:** Consistent.

Q 4.7 **Solve the system of linear equations using matrix method:** $5x + 2y = 4$, $7x + 3y = 5$.

SOLUTION

Concept used. Write the system as $AX = B$, compute A^{-1} (using $|A|$ and adj), then $X = A^{-1}B$.

Step 1. $A = \begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

Step 2. $|A| = 15 - 14 = 1 \neq 0$. Invertible.

Step 3. $\text{adj}(A) = \begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix}$.

Step 4. $A^{-1} = \begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix}$.

Step 5. Solve:

$$X = A^{-1}B = \begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 12 - 10 \\ -28 + 25 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

Step 6. Hence $x = 2$, $y = -3$.

Step 7. Check: $5(2) + 2(-3) = 10 - 6 = 4$. ✓ $7(2) + 3(-3) = 14 - 9 = 5$. ✓

Final Answer: $x = 2,\ y = -3$.

EXPERT'S SOLUTION *: Ananya Joshi, M.Sc Mathematics, IIT Bombay*

Quick reading. $|A| = 1$, adjoint via swap-and-flip, multiply by B.

Final Answer: $x = 2,\ y = -3$.

Q 4.8 **Solve the system:** $2x - y = -2,\ 3x + 4y = 3$.

SOLUTION

Concept used. $X = A^{-1}B$.

Step 1. $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

Step 2. $|A| = 8 + 3 = 11$.

Step 3. $\text{adj}(A) = \begin{pmatrix} 4 & 1 \\ -3 & 2 \end{pmatrix}$.

Step 4. $A^{-1} = \frac{1}{11}\begin{pmatrix} 4 & 1 \\ -3 & 2 \end{pmatrix}$.

Step 5. $X = A^{-1}B = \frac{1}{11}\begin{pmatrix} 4 & 1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} -8+3 \\ 6+6 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} -5 \\ 12 \end{pmatrix}$.

Step 6. Hence $x = -\frac{5}{11}$, $y = \frac{12}{11}$.

> **Final Answer:** $x = -\frac{5}{11},\ y = \frac{12}{11}$.

EXPERT'S SOLUTION *: Riya Verma, B.Tech CSE, IIT Roorkee*

Quick reading. $|A| = 11$; assemble inverse; multiply.

> **Final Answer:** $x = -5/11,\ y = 12/11$.

Q 4.9 **Solve the system:** $4x - 3y = 3,\ 3x - 5y = 7$.

SOLUTION

Concept used. $X = A^{-1}B$.

Step 1. $A = \begin{pmatrix} 4 & -3 \\ 3 & -5 \end{pmatrix}$, $B = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$.

Step 2. $|A| = -20 - (-9) = -20 + 9 = -11$.

Step 3. $\text{adj}(A) = \begin{pmatrix} -5 & 3 \\ -3 & 4 \end{pmatrix}$.

Step 4. $A^{-1} = -\frac{1}{11}\begin{pmatrix} -5 & 3 \\ -3 & 4 \end{pmatrix}$.

Step 5. $X = -\frac{1}{11}\begin{pmatrix} -5 & 3 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ 7 \end{pmatrix} = -\frac{1}{11}\begin{pmatrix} -15+21 \\ -9+28 \end{pmatrix} = -\frac{1}{11}\begin{pmatrix} 6 \\ 19 \end{pmatrix} = \begin{pmatrix} -6/11 \\ -19/11 \end{pmatrix}$.

> **Final Answer:** $x = -\frac{6}{11},\ y = -\frac{19}{11}$.

EXPERT'S SOLUTION : *Karan Bhat, M.Sc Mathematics, ISI Kolkata*

Quick reading. $|A| = -11$; mind the sign when dividing.

Final Answer: $x = -6/11,\ y = -19/11.$

Q 4.10 **Solve the system:** $5x + 2y = 3,\ \ 3x + 2y = 5.$

SOLUTION

Concept used. $X = A^{-1}B.$

Step 1. $A = \begin{pmatrix} 5 & 2 \\ 3 & 2 \end{pmatrix},\ B = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$

Step 2. $|A| = 10 - 6 = 4.$

Step 3. $\operatorname{adj}(A) = \begin{pmatrix} 2 & -2 \\ -3 & 5 \end{pmatrix}.$

Step 4. $A^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -2 \\ -3 & 5 \end{pmatrix}.$

Step 5. $X = \frac{1}{4}\begin{pmatrix} 2 & -2 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 6 - 10 \\ -9 + 25 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -4 \\ 16 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}.$

Step 6. Check: $5(-1) + 2(4) = -5 + 8 = 3\ \checkmark$; $3(-1) + 2(4) = -3 + 8 = 5\ \checkmark$.

Final Answer: $x = -1,\ y = 4.$

EXPERT'S SOLUTION : *Sneha Gupta, M.Tech CS, IIT Madras*

Quick reading. $|A| = 4$; clean integer answer.

Final Answer: $x = -1,\ y = 4.$

Q 4.11 **Solve the system:** $2x + y + z = 1,\ \ x - 2y - z = \frac{3}{2},\ \ 3y - 5z = 9.$

SOLUTION

Concept used. $X = A^{-1}B$. To clear the fraction, multiply the second equation by 2 first.

Step 1. Rewrite system without fractions. Multiply second equation by 2: $2x - 4y - 2z = 3$. Original system can also be kept as is for the determinant step; here we use the A and B in the original form:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 3/2 \\ 9 \end{pmatrix}.$$

Step 2. Compute $|A|$ along R_1:

$$|A| = 2\begin{vmatrix} -2 & -1 \\ 3 & -5 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 0 & -5 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix}.$$

$$= 2(10+3) - 1(-5-0) + 1(3-0) = 26 + 5 + 3 = 34.$$

Step 3. Cofactors:

$$A_{11} = +(10+3) = 13, \quad A_{12} = -(-5-0) = 5, \quad A_{13} = +(3-0) = 3,$$
$$A_{21} = -(-5-3) = 8, \quad A_{22} = +(-10-0) = -10, \quad A_{23} = -(6-0) = -6,$$
$$A_{31} = +(-1+2) = 1, \quad A_{32} = -(-2-1) = 3, \quad A_{33} = +(-4-1) = -5.$$

Step 4. Adjoint (transpose of cofactor matrix):

$$\text{adj}(A) = \begin{pmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{pmatrix}.$$

Step 5. $A^{-1} = \dfrac{1}{34}\,\text{adj}(A)$.

Step 6. Compute $X = A^{-1}B$. Compute $\text{adj}(A)B$ first:

row 1: $13(1) + 8(3/2) + 1(9) = 13 + 12 + 9 = 34.$

row 2: $5(1) + (-10)(3/2) + 3(9) = 5 - 15 + 27 = 17.$

row 3: $3(1) + (-6)(3/2) + (-5)(9) = 3 - 9 - 45 = -51.$

So $\text{adj}(A)B = \begin{pmatrix} 34 \\ 17 \\ -51 \end{pmatrix}$.

Step 7. Divide by $|A| = 34$:

$$X = \frac{1}{34}\begin{pmatrix} 34 \\ 17 \\ -51 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ -3/2 \end{pmatrix}.$$

Final Answer: $x = 1,\ y = \dfrac{1}{2},\ z = -\dfrac{3}{2}.$

☞ **Working with fractional B**

Keep B in fraction form throughout. Clearing the fraction by multiplying an equation does *not* change the solution, but it changes the matrix A, so you would have to redo everything. The cleanest approach is to leave $3/2$ in B and let the algebra carry it along.

EXPERT'S SOLUTION *: Pooja Singh, M.Sc Mathematics, IIT Bombay*

Strategic angle. Compute $|A|$ and adjoint; multiply by B; divide.

Step 1. $|A| = 34$.

Step 2. $\text{adj}(A)B = (34,\ 17,\ -51)^T$.

Step 3. $X = \frac{1}{34}(34,\ 17,\ -51)^T = (1,\ 1/2,\ -3/2)^T$.

Step 4. Verify in original equations: $2(1) + 1/2 - 3/2 = 2 - 1 = 1$ ✓;
$1 - 2(1/2) - (-3/2) = 1 - 1 + 3/2 = 3/2$ ✓;
$3(1/2) - 5(-3/2) = 3/2 + 15/2 = 9$ ✓.

Final Answer: $x = 1,\ y = 1/2,\ z = -3/2$.

Q 4.12 **Solve the system:** $x - y + z = 4,\ 2x + y - 3z = 0,\ x + y + z = 2$.

SOLUTION

Concept used. $X = A^{-1}B$.

Step 1. $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$.

Step 2. $|A|$ along R_1:

$$|A| = 1(1+3) - (-1)(2+3) + 1(2-1) = 4 + 5 + 1 = 10.$$

Step 3. Cofactors:

$$\begin{aligned} &A_{11} = +(1+3) = 4, && A_{12} = -(2+3) = -5, && A_{13} = +(2-1) = 1, \\ &A_{21} = -(-1-1) = 2, && A_{22} = +(1-1) = 0, && A_{23} = -(1+1) = -2, \\ &A_{31} = +(3-1) = 2, && A_{32} = -(-3-2) = 5, && A_{33} = +(1+2) = 3. \end{aligned}$$

Step 4. Adjoint:

$$\text{adj}(A) = \begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{pmatrix}.$$

Step 5. Compute $\text{adj}(A)B$:

$$\text{row 1: } 4(4) + 2(0) + 2(2) = 16 + 0 + 4 = 20.$$

$$\text{row 2: } -5(4) + 0(0) + 5(2) = -20 + 0 + 10 = -10.$$

$$\text{row 3: } 1(4) + (-2)(0) + 3(2) = 4 + 0 + 6 = 10.$$

Step 6. Divide by $|A| = 10$:

$$X = \frac{1}{10}\begin{pmatrix} 20 \\ -10 \\ 10 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

Final Answer: $x = 2,\ y = -1,\ z = 1.$

EXPERT'S SOLUTION : *Ishaan Kapoor, Ph.D Mathematics, IIT Delhi*

Quick reading. $|A| = 10$; the adjoint has clean integer entries; check the answer in the original equations.

Step 1. $|A| = 10.$

Step 2. $X = (2, -1, 1)^T.$

Step 3. Check: $2 - (-1) + 1 = 4$ ✓; $2(2) + (-1) - 3(1) = 4 - 1 - 3 = 0$ ✓; $2 + (-1) + 1 = 2$ ✓.

Final Answer: $x = 2,\ y = -1,\ z = 1.$

Q 4.13 **Solve the system:** $2x + 3y + 3z = 5,\ \ x - 2y + z = -4,\ \ 3x - y - 2z = 3.$

SOLUTION

Concept used. $X = A^{-1}B.$

Step 1. $A = \begin{pmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{pmatrix},\ B = \begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix}.$

Step 2. $|A|$ along R_1:

$$|A| = 2(4+1) - 3(-2-3) + 3(-1+6) = 2(5) - 3(-5) + 3(5) = 10 + 15 + 15 = 40.$$

Step 3. Cofactors:

$$\begin{aligned} &A_{11} = +(4+1) = 5, && A_{12} = -(-2-3) = 5, && A_{13} = +(-1+6) = 5, \\ &A_{21} = -(-6+3) = 3, && A_{22} = +(-4-9) = -13, && A_{23} = -(-2-9) = 11, \\ &A_{31} = +(3+6) = 9, && A_{32} = -(2-3) = 1, && A_{33} = +(-4-3) = -7. \end{aligned}$$

Step 4. Adjoint:

$$\text{adj}(A) = \begin{pmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{pmatrix}.$$

Step 5. Compute $\text{adj}(A)B$:

row 1: $5(5) + 3(-4) + 9(3) = 25 - 12 + 27 = 40.$

row 2: $5(5) + (-13)(-4) + 1(3) = 25 + 52 + 3 = 80.$

row 3: $5(5) + 11(-4) + (-7)(3) = 25 - 44 - 21 = -40.$

Step 6. Divide by $|A| = 40$:

$$X = \frac{1}{40}\begin{pmatrix} 40 \\ 80 \\ -40 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

Final Answer: $x = 1,\ y = 2,\ z = -1.$

EXPERT'S SOLUTION *: Aanya Chatterjee, M.Sc Applied Mathematics, IIT Kanpur*

Quick reading. $|A| = 40$; nice integer solution after dividing.

Step 1. $|A| = 40.$

Step 2. $X = (1, 2, -1)^T.$

Step 3. Check: $2 + 6 - 3 = 5\checkmark$; $1 - 4 - 1 = -4\checkmark$; $3 - 2 + 2 = 3\checkmark$.

Final Answer: $x = 1,\ y = 2,\ z = -1.$

Q 4.14 **Solve the system:** $x - y + 2z = 7,\ 3x + 4y - 5z = -5,\ 2x - y + 3z = 12.$

SOLUTION

Concept used. $X = A^{-1}B$.

Step 1. $A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 7 \\ -5 \\ 12 \end{pmatrix}$.

Step 2. $|A|$ along R_1:

$$|A| = 1(12-5) - (-1)(9+10) + 2(-3-8) = 7 + 19 - 22 = 4.$$

Step 3. Cofactors:

$$\begin{aligned} &A_{11} = +(12-5) = 7, && A_{12} = -(9+10) = -19, && A_{13} = +(-3-8) = -11, \\ &A_{21} = -(-3+2) = 1, && A_{22} = +(3-4) = -1, && A_{23} = -(-1+2) = -1, \\ &A_{31} = +(5-8) = -3, && A_{32} = -(-5-6) = 11, && A_{33} = +(4+3) = 7. \end{aligned}$$

Step 4. Adjoint:

$$\text{adj}(A) = \begin{pmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{pmatrix}.$$

Step 5. $\text{adj}(A)B$:

$$\text{row 1: } 7(7) + 1(-5) + (-3)(12) = 49 - 5 - 36 = 8.$$

$$\text{row 2: } -19(7) + (-1)(-5) + 11(12) = -133 + 5 + 132 = 4.$$

$$\text{row 3: } -11(7) + (-1)(-5) + 7(12) = -77 + 5 + 84 = 12.$$

Step 6. Divide by $|A| = 4$:

$$X = \frac{1}{4}\begin{pmatrix} 8 \\ 4 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$

> **Final Answer:** $x = 2,\ y = 1,\ z = 3.$

EXPERT'S SOLUTION *: Rohit Iyer, M.Sc Mathematics, IIT Bombay*

Quick reading. $|A| = 4$; integer solution.

Step 1. Determinant 4; adjoint computed.

Step 2. $X = (2, 1, 3)^T$.

Step 3. Check: $2 - 1 + 6 = 7\checkmark$; $6 + 4 - 15 = -5\checkmark$; $4 - 1 + 9 = 12\checkmark$.

Final Answer: $x = 2,\ y = 1,\ z = 3.$

Q 4.15 **If** $A = \begin{pmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{pmatrix}$**, find** A^{-1}**. Using** A^{-1}**, solve the system** $2x - 3y + 5z = 11,\ 3x + 2y - 4z = -5,\ x + y - 2z = -3.$

SOLUTION

Concept used. Once A^{-1} is known, $X = A^{-1}B$ solves $AX = B$.

Step 1. $|A|$ along R_1:

$$|A| = 2\begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} - (-3)\begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} + 5\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}.$$

$$= 2(-4+4) + 3(-6+4) + 5(3-2) = 0 + 3(-2) + 5(1) = -6 + 5 = -1.$$

Step 2. Cofactors. Row 1:

$$A_{11} = +(-4+4) = 0, \quad A_{12} = -(-6+4) = 2, \quad A_{13} = +(3-2) = 1.$$

Row 2:

$$A_{21} = -(6-5) = -1, \quad A_{22} = +(-4-5) = -9, \quad A_{23} = -(2+3) = -5.$$

Row 3:

$$A_{31} = +(12-10) = 2, \quad A_{32} = -(-8-15) = 23, \quad A_{33} = +(4+9) = 13.$$

Step 3. Adjoint:

$$\operatorname{adj}(A) = \begin{pmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{pmatrix}.$$

Step 4. Inverse: $A^{-1} = \dfrac{1}{-1}\operatorname{adj}(A) = -\operatorname{adj}(A) = \begin{pmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{pmatrix}.$

Step 5. Apply to $B = \begin{pmatrix} 11 \\ -5 \\ -3 \end{pmatrix}$:

$$X = A^{-1}B = \begin{pmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{pmatrix}\begin{pmatrix} 11 \\ -5 \\ -3 \end{pmatrix}.$$

Row 1: $0(11) + 1(-5) + (-2)(-3) = 0 - 5 + 6 = 1$. Row 2: $-2(11) + 9(-5) + (-23)(-3) = -22 - 45 + 69 = 2$. Row 3: $-1(11) + 5(-5) + (-13)(-3) = -11 - 25 + 39 = 3$.

$$X = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Final Answer: $A^{-1} = \begin{pmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{pmatrix}$, and the system has $x = 1,\ y = 2,\ z = 3$.

EXPERT'S SOLUTION *: Krishna Sharma, M.Tech CS, IIT Madras*

Strategic angle. Compute the inverse once; then a single matrix-vector multiplication delivers the solution.

Step 1. $|A| = -1$.

Step 2. Inverse $= -\operatorname{adj}(A) = \begin{pmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{pmatrix}$.

Step 3. $X = A^{-1}B = (1, 2, 3)^T$.

Step 4. Check: $2 - 6 + 15 = 11\checkmark$; $3 + 4 - 12 = -5\checkmark$; $1 + 2 - 6 = -3\checkmark$.

Final Answer: $x = 1,\ y = 2,\ z = 3$.

Q 4.16 **The cost of 4 kg onion, 3 kg wheat and 2 kg rice is Rs. 60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is Rs. 90. The cost of 6 kg onion, 2 kg wheat and 3 kg rice is Rs. 70. Find cost of each item per kg by matrix method.**

SOLUTION

Concept used. Translate the word problem into a matrix equation $AX = B$ and apply the matrix method.

Step 1. Let x = cost of 1 kg onion, y = cost of 1 kg wheat, z = cost of 1 kg rice (all in Rs.).

Step 2. System:

$$4x + 3y + 2z = 60,$$
$$2x + 4y + 6z = 90,$$
$$6x + 2y + 3z = 70.$$

Step 3. Write in matrix form:

$$A = \begin{pmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 60 \\ 90 \\ 70 \end{pmatrix}.$$

Step 4. $|A|$ along R_1:

$$|A| = 4(12-12)-3(6-36)+2(4-24) = 4(0)-3(-30)+2(-20) = 0+90-40 = 50.$$

Step 5. Cofactors:

$$\begin{aligned} A_{11} &= +(12-12) = 0, & A_{12} &= -(6-36) = 30, & A_{13} &= +(4-24) = -20, \\ A_{21} &= -(9-4) = -5, & A_{22} &= +(12-12) = 0, & A_{23} &= -(8-18) = 10, \\ A_{31} &= +(18-8) = 10, & A_{32} &= -(24-4) = -20, & A_{33} &= +(16-6) = 10. \end{aligned}$$

Step 6. Adjoint:

$$\text{adj}(A) = \begin{pmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{pmatrix}.$$

Step 7. $\text{adj}(A)B$:

$$\text{row 1: } 0(60) + (-5)(90) + 10(70) = 0 - 450 + 700 = 250.$$

$$\text{row 2: } 30(60) + 0(90) + (-20)(70) = 1800 + 0 - 1400 = 400.$$

$$\text{row 3: } -20(60) + 10(90) + 10(70) = -1200 + 900 + 700 = 400.$$

Step 8. Divide by $|A| = 50$:

$$X = \frac{1}{50}\begin{pmatrix} 250 \\ 400 \\ 400 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 8 \end{pmatrix}.$$

Step 9. Therefore onion costs Rs. 5/kg, wheat costs Rs. 8/kg, rice costs Rs. 8/kg.

Step 10. Verify: $4(5) + 3(8) + 2(8) = 20 + 24 + 16 = 60\ \checkmark.$
$2(5) + 4(8) + 6(8) = 10 + 32 + 48 = 90\ \checkmark.$
$6(5) + 2(8) + 3(8) = 30 + 16 + 24 = 70\ \checkmark.$

Final Answer: Onion Rs. 5/kg, Wheat Rs. 8/kg, Rice Rs. 8/kg.

☞ **Exam Tip**

For word problems, always state your variable definitions and units *before* writing the equations. "Let x be cost per kg of onion in Rs." is concrete; "Let x be onion" is not.

EXPERT'S SOLUTION *: Aditi Desai, Ph.D Pure Mathematics, IISc Bangalore*

Strategic angle. Translate words into a 3×3 linear system, solve by the matrix method, and answer in the units the question asks for.

Step 1. Variables: x, y, z = price per kg of onion, wheat, rice.

Step 2. System: three equations as above; matrix form $AX = B$.

Step 3. $|A| = 50$, $\text{adj}(A)$ computed.

Step 4. $X = (5, 8, 8)^T$. So onion Rs. 5/kg, wheat Rs. 8/kg, rice Rs. 8/kg.

Why this matters. The matrix method scales to any linear word problem with as many unknowns as equations, provided the coefficient matrix is non-singular. The pattern (define variables, write linear equations, build A and B, solve $X = A^{-1}B$, interpret) is identical for every such problem.

Final Answer: Onion Rs. 5/kg, Wheat Rs. 8/kg, Rice Rs. 8/kg.

Key Takeaways

- Every linear system in n unknowns can be written as $AX = B$.
- $|A| \neq 0 \Rightarrow$ unique solution $X = A^{-1}B$; system is consistent.
- $|A| = 0$ and $(\text{adj } A)B \neq O \Rightarrow$ no solution; inconsistent.
- $|A| = 0$ and $(\text{adj } A)B = O \Rightarrow$ infinitely many solutions (or none, decided case by case).
- For word problems, define variables and units explicitly; assemble A and B from the equations; apply $X = A^{-1}B$; restate the answer in the original wording.

End of Exercise 4.5