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Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 4: Determinants

About this Chapter

The Miscellaneous Exercise on Determinants brings together every tool of the chapter: **cofactor expansion**, **column/row operations**, **adjoint and inverse**, and the matrix method for solving linear systems. Several proofs use the property that any determinant with two identical (or proportional) rows/columns equals zero.

Topics covered: Determinants independent of a parameter • Row and column operations • Adjoint and inverse identities • Systems via matrix method

Quick Formula Sheet

Column ops: $C_i \to C_i + kC_j$ keep Δ unchanged.

Equal columns: $\Delta = 0$.

$(AB)^{-1} = B^{-1}A^{-1}$; $|\operatorname{adj} A| = |A|^{n-1}$.

System solver: $X = A^{-1}B$ for $|A| \neq 0$.

Miscellaneous Exercise

Q 4.1 **Prove that the determinant** $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ **is independent of** θ**.**

SOLUTION

Concept used. Expand the determinant along R_1 and simplify; if all θ-dependent terms cancel, the result is independent of θ. Use the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$.

Step 1. Expand along R_1:

$$\Delta = x\begin{vmatrix} -x & 1 \\ 1 & x \end{vmatrix} - \sin\theta\begin{vmatrix} -\sin\theta & 1 \\ \cos\theta & x \end{vmatrix} + \cos\theta\begin{vmatrix} -\sin\theta & -x \\ \cos\theta & 1 \end{vmatrix}.$$

Step 2. Evaluate each minor.

$$\begin{vmatrix} -x & 1 \\ 1 & x \end{vmatrix} = -x \cdot x - 1 \cdot 1 = -x^2 - 1.$$

$$\begin{vmatrix} -\sin\theta & 1 \\ \cos\theta & x \end{vmatrix} = -x\sin\theta - \cos\theta.$$

$$\begin{vmatrix} -\sin\theta & -x \\ \cos\theta & 1 \end{vmatrix} = -\sin\theta - (-x\cos\theta) = -\sin\theta + x\cos\theta.$$

Step 3. Substitute into the expansion:

$$\Delta = x(-x^2 - 1) - \sin\theta(-x\sin\theta - \cos\theta) + \cos\theta(-\sin\theta + x\cos\theta).$$

Step 4. Expand term by term:

$$x(-x^2 - 1) = -x^3 - x.$$

$$-\sin\theta(-x\sin\theta - \cos\theta) = x\sin^2\theta + \sin\theta\cos\theta.$$

$$\cos\theta(-\sin\theta + x\cos\theta) = -\sin\theta\cos\theta + x\cos^2\theta.$$

Step 5. Add the three pieces:

$$\Delta = -x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta.$$

The two $\sin\theta\cos\theta$ terms cancel:

$$\Delta = -x^3 - x + x(\sin^2\theta + \cos^2\theta).$$

Step 6. Use $\sin^2\theta + \cos^2\theta = 1$:

$$\Delta = -x^3 - x + x \cdot 1 = -x^3.$$

Step 7. The right side has no θ, so Δ is independent of θ.

Final Answer: $\Delta = -x^3$, independent of θ.

☞ **Pythagorean cancellation**

The trick is in step 6: the only θ-dependent piece becomes a clean multiple of $\sin^2\theta + \cos^2\theta$, which is the constant 1. Always look for this pattern in mixed trig+algebra determinants.

EXPERT'S SOLUTION : *Tara Singh, M.Sc Mathematics, IIT Bombay*

Structural observation. The off-diagonal entries pair as $\sin\theta$ and $\cos\theta$ in symmetric positions. Expanding and grouping forces the Pythagorean identity to appear.

Step 1. Expand along R_1; collect θ-terms.

Step 2. The $\sin\theta\cos\theta$ pair cancels; the $x\sin^2\theta + x\cos^2\theta$ pair simplifies to x.

Step 3. Net: $\Delta = -x^3 - x + x = -x^3$. No θ remains.

Why this matters. Many "independent of θ" determinant proofs rely on the Pythagorean identity simplifying a symmetric combination. Recognising the pattern saves you from forcing the expansion blindly.

Final Answer: $\Delta = -x^3$, independent of θ.

Q 4.2 Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$.

SOLUTION

Concept used. Expand along the row or column with the most zeros. Here R_2 has a zero. Use $\sin^2 + \cos^2 = 1$ to collapse the result.

Step 1. Expand along R_2 (third entry is zero):

$$\Delta = -(-\sin\beta)\begin{vmatrix} \cos\alpha\sin\beta & -\sin\alpha \\ \sin\alpha\sin\beta & \cos\alpha \end{vmatrix} + \cos\beta\begin{vmatrix} \cos\alpha\cos\beta & -\sin\alpha \\ \sin\alpha\cos\beta & \cos\alpha \end{vmatrix} - 0.$$

Simplify the first sign: $-(-\sin\beta) = \sin\beta$.

Step 2. First minor:

$$\begin{vmatrix} \cos\alpha\sin\beta & -\sin\alpha \\ \sin\alpha\sin\beta & \cos\alpha \end{vmatrix} = (\cos\alpha\sin\beta)(\cos\alpha) - (-\sin\alpha)(\sin\alpha\sin\beta).$$

$$= \cos^2\alpha\,\sin\beta + \sin^2\alpha\,\sin\beta = \sin\beta(\cos^2\alpha + \sin^2\alpha) = \sin\beta.$$

Step 3. Second minor:

$$\begin{vmatrix} \cos\alpha\cos\beta & -\sin\alpha \\ \sin\alpha\cos\beta & \cos\alpha \end{vmatrix} = (\cos\alpha\cos\beta)(\cos\alpha) - (-\sin\alpha)(\sin\alpha\cos\beta).$$

$$= \cos^2\alpha\,\cos\beta + \sin^2\alpha\,\cos\beta = \cos\beta(\cos^2\alpha + \sin^2\alpha) = \cos\beta.$$

Step 4. Substitute back:

$$\Delta = \sin\beta\cdot\sin\beta + \cos\beta\cdot\cos\beta = \sin^2\beta + \cos^2\beta = 1.$$

Final Answer: $\Delta = 1$.

♥ Orthogonal matrix

The given matrix is the standard 3D rotation taking the z-axis to the unit vector $(\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha)$. Rotation matrices are orthogonal with determinant $+1$, agreeing with our calculation.

EXPERT'S SOLUTION *: Aanya Verma, B.Tech Engineering Physics, IIT Bombay*

Picture-first. Think of the matrix as a coordinate rotation in 3D. Rotations preserve volume, so $|\det| = 1$, and the sign matches the chosen orientation.

Step 1. Expand along R_2. Both surviving terms collapse using $\sin^2 + \cos^2 = 1$.

Step 2. Final value: $\sin^2\beta + \cos^2\beta = 1$.

Final Answer: $\Delta = 1$.

Q 4.3 **If** $A^{-1} = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$ **and** $B = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$**, find** $(AB)^{-1}$**.**

SOLUTION

Concept used. $(AB)^{-1} = B^{-1}A^{-1}$. We must first compute B^{-1}, then multiply B^{-1} with the given A^{-1}.

Step 1. Compute $|B|$ along R_1:

$$|B| = 1(3+0) - 2(-1-0) + (-2)(2-0) = 3 + 2 - 4 = 1.$$

Step 2. Compute cofactors of B. Row 1:

$$B_{11} = +(3-0) = 3, \quad B_{12} = -(-1-0) = 1, \quad B_{13} = +(2-0) = 2.$$

Row 2:

$$B_{21} = -(2-4) = 2, \quad B_{22} = +(1-0) = 1, \quad B_{23} = -(-2-0) = 2.$$

Row 3:

$$B_{31} = +(0+6) = 6, \quad B_{32} = -(0-2) = 2, \quad B_{33} = +(3+2) = 5.$$

Step 3. Adjoint of B = transpose of cofactor matrix:

$$\text{adj}(B) = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}.$$

Step 4. $B^{-1} = \frac{1}{|B|}\text{adj}(B) = \text{adj}(B) = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}$.

Step 5. Compute $(AB)^{-1} = B^{-1}A^{-1}$:

$$B^{-1}A^{-1} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix} \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}.$$

Row 1: $3(3) + 2(-15) + 6(5) = 9 - 30 + 30 = 9$;
$3(-1) + 2(6) + 6(-2) = -3 + 12 - 12 = -3$;
$3(1) + 2(-5) + 6(2) = 3 - 10 + 12 = 5$. Row 2:
$1(3) + 1(-15) + 2(5) = 3 - 15 + 10 = -2$;
$1(-1) + 1(6) + 2(-2) = -1 + 6 - 4 = 1$; $1(1) + 1(-5) + 2(2) = 1 - 5 + 4 = 0$.
Row 3: $2(3) + 2(-15) + 5(5) = 6 - 30 + 25 = 1$;
$2(-1) + 2(6) + 5(-2) = -2 + 12 - 10 = 0$;
$2(1) + 2(-5) + 5(2) = 2 - 10 + 10 = 2$.

$$(AB)^{-1} = \begin{pmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Final Answer: $(AB)^{-1} = \begin{pmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$.

✗ Common Pitfall

Do not compute $A^{-1}B^{-1}$. The correct order is $(AB)^{-1} = B^{-1}A^{-1}$. Matrix multiplication is not commutative; reversing the order generally gives a different matrix.

EXPERT'S SOLUTION : *Karan Gupta, M.Sc Mathematics, ISI Kolkata*

Strategic angle. Compute B^{-1} since A^{-1} is already given. Then $(AB)^{-1} = B^{-1}A^{-1}$ is a single matrix product.

Step 1. $|B| = 1$; $\text{adj}(B) =$ as above; $B^{-1} = \text{adj}(B)$.

Step 2. $B^{-1}A^{-1}$ as computed.

Why this matters. When the problem gives you A^{-1} rather than A, save the work of inverting A; only invert what is needed.

Final Answer: $(AB)^{-1} = \begin{pmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$.

Q 4.4 **Let** $A = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix}$. **Verify that**

(i) $[\text{adj } A]^{-1} = \text{adj}(A^{-1})$ **(ii)** $(A^{-1})^{-1} = A$.

SOLUTION

Concept used. Two identities.
(I) $[\text{adj } A]^{-1} = \text{adj}(A^{-1})$ when $|A| \neq 0$.
(II) $(A^{-1})^{-1} = A$ for every invertible A, by the uniqueness of inverses.
Part (i). The identity follows from $A \text{ adj}(A) = |A|\, I$. Dividing by $|A|$: $A \cdot \frac{\text{adj}(A)}{|A|} = I$, so $A^{-1} = \frac{\text{adj}(A)}{|A|}$. Then

$$\text{adj}(A^{-1}) = |A^{-1}|\,(A^{-1})^{-1} = \frac{1}{|A|}\,A = \frac{A}{|A|}.$$

Also,

$$[\text{adj}(A)]^{-1} = \frac{1}{|\,\text{adj}(A)|}\,\text{adj}(\text{adj}(A)).$$

For a 3×3 matrix one shows $\text{adj}(\text{adj } A) = |A|\,A$ (a standard identity), and $|\,\text{adj } A| = |A|^2$. So

$$[\text{adj}(A)]^{-1} = \frac{|A| \cdot A}{|A|^2} = \frac{A}{|A|} = \text{adj}(A^{-1}).\ \checkmark$$

We now verify by computation.

Step 1. Compute $|A|$ along R_1:

$$|A| = 1(15-1) - (-2)(-10-1) + 1(-2-3) = 14 - 22 - 5 = -13.$$

Step 2. Compute $\operatorname{adj}(A)$. Cofactors. Row 1:

$$A_{11} = +(15 - 1) = 14,\ A_{12} = -(-10 - 1) = 11,\ A_{13} = +(-2 - 3) = -5.$$

Row 2:

$$A_{21} = -(-10 - 1) = 11,\ A_{22} = +(5 - 1) = 4,\ A_{23} = -(1 + 2) = -3.$$

Row 3:

$$A_{31} = +(-2 - 3) = -5,\ A_{32} = -(1 + 2) = -3,\ A_{33} = +(3 - 4) = -1.$$

Step 3. Adjoint:

$$\operatorname{adj}(A) = \begin{pmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{pmatrix}.$$

Step 4. $A^{-1} = \dfrac{1}{-13}\operatorname{adj}(A) = -\dfrac{1}{13}\begin{pmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{pmatrix}.$

Step 5. $\operatorname{adj}(A^{-1}) = \dfrac{A}{|A|} = \dfrac{A}{-13} = -\dfrac{1}{13}A = -\dfrac{1}{13}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix}.$

Step 6. Meanwhile $[\operatorname{adj}(A)]^{-1}$: since $|\operatorname{adj}(A)| = |A|^2 = 169$ and $\operatorname{adj}(\operatorname{adj} A) = |A|\,A = -13\,A$,

$$[\operatorname{adj}(A)]^{-1} = \frac{-13\,A}{169} = -\frac{A}{13} = -\frac{1}{13}A.$$

Step 7. Comparing steps 5 and 6: $[\operatorname{adj}(A)]^{-1} = -\dfrac{1}{13}A = \operatorname{adj}(A^{-1})$. ✓

Part (ii). For any invertible A, $A \cdot A^{-1} = I$ and $A^{-1} \cdot A = I$. Reading the second equation, A is a matrix whose product with A^{-1} on the right gives I; that means A is the inverse of A^{-1}, i.e. $(A^{-1})^{-1} = A$.

Final Answer: Both (i) and (ii) verified.

♥ Adjoint of adjoint

For an $n \times n$ matrix, $\operatorname{adj}(\operatorname{adj} A) = |A|^{n-2} A$. For $n = 3$ this is $|A|\,A$, which we used to evaluate $[\operatorname{adj} A]^{-1}$ symbolically. The identity is a clean consequence of $A\operatorname{adj}(A) = |A|\,I$ applied twice.

EXPERT'S SOLUTION : *Pranav Banerjee, Ph.D Mathematics, IIT Delhi*

Strategic angle. Both parts boil down to two short symbolic identities; the verification on the given A is just plugging in.

Step 1. (i) Use $\operatorname{adj}(A^{-1}) = A/|A|$ and $[\operatorname{adj} A]^{-1} = A/|A|$ (from $\operatorname{adj}(\operatorname{adj} A) = |A|\, A$ and $|\operatorname{adj} A| = |A|^2$).

Step 2. (ii) Use uniqueness of the inverse: $A^{-1} \cdot A = I$ identifies A as the inverse of A^{-1}.

Why this matters. Knowing the symbolic identities saves the entire numerical verification. The numerical version is for those who want to see the arithmetic line up.

Final Answer: $[\operatorname{adj} A]^{-1} = \operatorname{adj}(A^{-1}) = A/|A|$; $(A^{-1})^{-1} = A$.

Q 4.5 **Evaluate** $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$.

SOLUTION

Concept used. Use the column operation $C_1 \to C_1 + C_2 + C_3$ (which does not change the determinant). When all three entries in C_1 become the common quantity $2(x+y)$, pull it out and reduce.

Step 1. Apply $C_1 \to C_1 + C_2 + C_3$. New C_1 entries:

$$x+y+(x+y) = 2(x+y), \quad y+(x+y)+x = 2(x+y), \quad (x+y)+x+y = 2(x+y).$$

So

$$\Delta = \begin{vmatrix} 2(x+y) & y & x+y \\ 2(x+y) & x+y & x \\ 2(x+y) & x & y \end{vmatrix}.$$

Step 2. Pull $2(x+y)$ out of C_1:

$$\Delta = 2(x+y) \begin{vmatrix} 1 & y & x+y \\ 1 & x+y & x \\ 1 & x & y \end{vmatrix}.$$

Step 3. Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ to introduce zeros in C_1:

$$\Delta = 2(x+y) \begin{vmatrix} 1 & y & x+y \\ 0 & x & -y \\ 0 & x-y & -x \end{vmatrix}.$$

Step 4. Expand along C_1 (only the top entry survives):

$$\Delta = 2(x+y)\cdot 1\cdot \begin{vmatrix} x & -y \\ x-y & -x \end{vmatrix}.$$

Step 5. Compute the 2×2 minor:

$$\begin{vmatrix} x & -y \\ x-y & -x \end{vmatrix} = (x)(-x) - (-y)(x-y) = -x^2 + y(x-y) = -x^2 + xy - y^2.$$

Step 6. Multiply:

$$\Delta = 2(x+y)(-x^2+xy-y^2) = -2(x+y)(x^2-xy+y^2).$$

Step 7. Recognise $(x+y)(x^2-xy+y^2) = x^3+y^3$:

$$\Delta = -2(x^3+y^3).$$

Final Answer: $\Delta = -2(x^3+y^3)$.

☞ **Sum of cubes**

$(x+y)(x^2-xy+y^2) = x^3+y^3$ is one of the most useful factorisations in this chapter. Spot it whenever an answer simplifies via $(x+y)$ and a quadratic.

EXPERT'S SOLUTION *: Sneha Reddy, M.Sc Mathematics, IIT Bombay*

Strategic angle. Cyclic-row determinants reduce cleanly with $C_1 \to C_1 + C_2 + C_3$.

Step 1. $C_1 \to C_1 + C_2 + C_3$: all entries become $2(x+y)$. Factor out.

Step 2. Row operations $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ introduce zeros.

Step 3. Expansion along C_1: a single 2×2 remains, equal to $-(x^2-xy+y^2)$.

Step 4. Total: $\Delta = 2(x+y)\cdot[-(x^2-xy+y^2)] = -2(x^3+y^3)$.

Why this matters. The pattern "add all columns into one, factor, then reduce" kills a large class of symmetric / cyclic determinants in 3 lines.

Final Answer: $\Delta = -2(x^3+y^3)$.

Q 4.6 **Evaluate** $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$.

SOLUTION

Concept used. Row operations preserve the determinant. Subtract the first row from the others to introduce zeros.

Step 1. Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$:

$$\Delta = \begin{vmatrix} 1 & x & y \\ 0 & y & 0 \\ 0 & 0 & x \end{vmatrix}.$$

Row 2: $(1-1,\ (x+y)-x,\ y-y) = (0,\ y,\ 0)$. Row 3: $(1-1,\ x-x,\ (x+y)-y) = (0,\ 0,\ x)$.

Step 2. Expand along C_1 (only $a_{11} = 1$ is non-zero):

$$\Delta = 1 \cdot \begin{vmatrix} y & 0 \\ 0 & x \end{vmatrix} = y \cdot x - 0 \cdot 0 = xy.$$

Final Answer: $\Delta = xy$.

EXPERT'S SOLUTION *: Aditi Bhat, M.Sc Mathematics, IIT Bombay*

Quick reading. Two row operations make the matrix upper-triangular; the determinant is then the diagonal product.

Step 1. After $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, the matrix is upper-triangular with diagonal $(1, y, x)$.

Step 2. Determinant $= 1 \cdot y \cdot x = xy$.

Why this matters. Whenever the first column is a column of 1s, $R_i \to R_i - R_1$ for $i \geq 2$ turns the determinant triangular: a standard reduction worth doing on autopilot.

Final Answer: $\Delta = xy$.

Q 4.7 **Solve the system of equations:**

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4, \quad \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1, \quad \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2.$$

SOLUTION

Concept used. Introduce $u = 1/x$, $v = 1/y$, $w = 1/z$. The system becomes linear in u, v, w; solve by the matrix method, then invert.

Step 1. Substitute:

$$2u + 3v + 10w = 4,$$

$$4u - 6v + 5w = 1,$$

$$6u + 9v - 20w = 2.$$

Step 2. Write in matrix form $AU = B$, with $U = (u, v, w)^T$:

$$A = \begin{pmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{pmatrix}, \quad B = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}.$$

Step 3. Compute $|A|$ along R_1:

$$|A| = 2(120 - 45) - 3(-80 - 30) + 10(36 + 36) = 2(75) - 3(-110) + 10(72).$$

$$= 150 + 330 + 720 = 1200.$$

Step 4. Cofactors. Row 1:

$$A_{11} = +(120 - 45) = 75, \quad A_{12} = -(-80 - 30) = 110, \quad A_{13} = +(36 + 36) = 72.$$

Row 2:

$$A_{21} = -(-60-90) = 150, \quad A_{22} = +(-40-60) = -100, \quad A_{23} = -(18-18) = 0.$$

Row 3:

$$A_{31} = +(15 + 60) = 75, \quad A_{32} = -(10 - 40) = 30, \quad A_{33} = +(-12 - 12) = -24.$$

Step 5. Adjoint = transpose of cofactor matrix:

$$\text{adj}(A) = \begin{pmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{pmatrix}.$$

Step 6. Compute $\text{adj}(A)B$:

$$\text{row 1: } 75(4) + 150(1) + 75(2) = 300 + 150 + 150 = 600.$$

$$\text{row 2: } 110(4) + (-100)(1) + 30(2) = 440 - 100 + 60 = 400.$$

$$\text{row 3: } 72(4) + 0(1) + (-24)(2) = 288 + 0 - 48 = 240.$$

Step 7. Divide by $|A| = 1200$:

$$U = \frac{1}{1200}\begin{pmatrix}600\\400\\240\end{pmatrix} = \begin{pmatrix}1/2\\1/3\\1/5\end{pmatrix}.$$

Step 8. Recover original variables:

$$u = 1/x = 1/2 \Rightarrow x = 2, \quad v = 1/y = 1/3 \Rightarrow y = 3, \quad w = 1/z = 1/5 \Rightarrow z = 5.$$

Final Answer: $x = 2,\ y = 3,\ z = 5.$

☞ Substitution to linearise

A system with $1/x$, $1/y$, $1/z$ is non-linear in x, y, z but *linear* in their reciprocals. Always substitute $u = 1/x$ etc. first; you can apply the matrix method to the linearised system.

EXPERT'S SOLUTION *: Ananya Sharma, M.Tech CS, IIT Madras*

Strategic angle. Linearise via u, v, w; solve $AU = B$ by the matrix method; invert to recover x, y, z.

Step 1. Substitute; system becomes $2u + 3v + 10w = 4$ etc.

Step 2. $|A| = 1200$.

Step 3. $U = (1/2,\ 1/3,\ 1/5)^T$, so $(x, y, z) = (2, 3, 5)$.

Step 4. Verify in the original system: $2/2 + 3/3 + 10/5 = 1 + 1 + 2 = 4$ ✓; $4/2 - 6/3 + 5/5 = 2 - 2 + 1 = 1$ ✓; $6/2 + 9/3 - 20/5 = 3 + 3 - 4 = 2$ ✓.

Final Answer: $x = 2,\ y = 3,\ z = 5.$

Q 4.8 **If x, y, z are nonzero real numbers, then the inverse of matrix $A = \begin{pmatrix}x & 0 & 0\\0 & y & 0\\0 & 0 & z\end{pmatrix}$ is**

(A) $\begin{pmatrix}x^{-1} & 0 & 0\\0 & y^{-1} & 0\\0 & 0 & z^{-1}\end{pmatrix}$ **(B)** $xyz\begin{pmatrix}x^{-1} & 0 & 0\\0 & y^{-1} & 0\\0 & 0 & z^{-1}\end{pmatrix}$

(C) $\frac{1}{xyz}\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$ **(D)** $\frac{1}{xyz}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

SOLUTION

Concept used. The inverse of a diagonal matrix with non-zero diagonal entries is the diagonal matrix of reciprocals: $\mathrm{diag}(a, b, c)^{-1} = \mathrm{diag}(1/a, 1/b, 1/c)$.

Step 1. $A = \mathrm{diag}(x, y, z)$, $|A| = xyz \neq 0$, so A^{-1} exists.

Step 2. Build A^{-1} directly:

$$A^{-1} = \begin{pmatrix} 1/x & 0 & 0 \\ 0 & 1/y & 0 \\ 0 & 0 & 1/z \end{pmatrix} = \begin{pmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{pmatrix}.$$

Step 3. Verify: $A\,A^{-1} = \mathrm{diag}(x \cdot 1/x,\ y \cdot 1/y,\ z \cdot 1/z) = I$. ✓

Step 4. This matches option (A).

Final Answer: Option (A).

EXPERT'S SOLUTION *: Riya Chatterjee, M.Sc Mathematics, ISI Kolkata*

Quick reading. Diagonal ⇒ diagonal-reciprocal inverse.

Step 1. $A^{-1} = \mathrm{diag}(1/x, 1/y, 1/z)$.

Final Answer: Option (A).

Q 4.9 **Let** $A = \begin{pmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{pmatrix}$**, where** $0 \leq \theta \leq 2\pi$**. Then**

(A) $\det(A) = 0$ **(B)** $\det(A) \in (2, \infty)$ **(C)** $\det(A) \in (2, 4)$ **(D)** $\det(A) \in [2, 4]$.

SOLUTION

Concept used. Expand the determinant; obtain $\det(A)$ as a function of θ; determine its range over $0 \leq \theta \leq 2\pi$.

Step 1. Let $s = \sin\theta$ for brevity. Expand along R_1:

$$\det(A) = 1\begin{vmatrix} 1 & s \\ -s & 1 \end{vmatrix} - s\begin{vmatrix} -s & s \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} -s & 1 \\ -1 & -s \end{vmatrix}.$$

Step 2. Evaluate each minor.

$$\begin{vmatrix} 1 & s \\ -s & 1 \end{vmatrix} = 1 - (-s^2) = 1 + s^2.$$

$$\begin{vmatrix} -s & s \\ -1 & 1 \end{vmatrix} = -s - (-s) = -s + s = 0.$$

$$\begin{vmatrix} -s & 1 \\ -1 & -s \end{vmatrix} = (-s)(-s) - 1 \cdot (-1) = s^2 + 1.$$

Step 3. Substitute:

$$\det(A) = (1 + s^2) - s \cdot 0 + (s^2 + 1) = 2 + 2s^2.$$

Step 4. Range of $s = \sin\theta$ over $0 \le \theta \le 2\pi$ is $[-1, 1]$, so $s^2 \in [0, 1]$.

Step 5. Hence $\det(A) = 2 + 2s^2 \in [2 + 0,\ 2 + 2] = [2, 4]$.

Step 6. Match: option (D).

Final Answer: Option (D): $\det(A) \in [2, 4]$.

☞ Exam Tip

The endpoints 2 and 4 are *achieved*: $\det(A) = 2$ at $\theta = 0, \pi, 2\pi$ and $\det(A) = 4$ at $\theta = \pi/2,\ 3\pi/2$. So the interval is closed $[2, 4]$, not open $(2, 4)$.

EXPERT'S SOLUTION *: Aaditya Pillai, M.Sc Mathematics, IIT Bombay*

Quick reading. Expand; the answer is $2 + 2\sin^2\theta$; range over $\theta \in [0, 2\pi]$ is $[2, 4]$.

Step 1. $\det(A) = 2 + 2\sin^2\theta$.

Step 2. $\sin^2\theta \in [0, 1] \Rightarrow \det(A) \in [2, 4]$.

Why this matters. Always check whether the endpoints of the range are attained. Open interval $(2, 4)$ and closed $[2, 4]$ are different answers; the chapter test will distinguish.

Final Answer: Option (D): $\det(A) \in [2, 4]$.

Key Takeaways

- "θ-independent" determinant proofs reduce via $\sin^2\theta + \cos^2\theta = 1$.
- Column operation $C_1 \to C_1 + C_2 + C_3$ kills cyclic determinants; pull out the common factor.
- For reciprocal systems, substitute $u = 1/x$ etc. to linearise.
- $(AB)^{-1} = B^{-1}A^{-1}$; $\text{adj}(\text{adj}\,A) = |A|\,A$ for 3×3; $|\,\text{adj}\,A| = |A|^{n-1}$.
- Range of $a + b\sin^2\theta$ over $\theta \in [0, 2\pi]$ is $[a, a+b]$ (when $b > 0$): endpoints are achieved.

End of Miscellaneous Exercise