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Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 3: Matrices

About this Chapter

Exercise 3.2 is the workhorse exercise of the chapter. It drills the four **operations on matrices**: addition, subtraction, scalar multiplication, and matrix **multiplication**. You will also meet matrix equations such as $aX + bY = C$ and word problems modelled by matrix algebra. Mastering the row-times-column rule for AB is the single most important skill from this chapter for boards and JEE.

Topics covered: Addition / subtraction • Scalar multiplication • Matrix multiplication • Properties of AB • Solving matrix equations • Compatibility of order

Quick Formula Sheet

Sum / difference (same order): $(A \pm B)_{ij} = a_{ij} \pm b_{ij}$.

Scalar multiple: $(kA)_{ij} = k\, a_{ij}$.

Matrix product (compatible orders): $A_{m\times n} B_{n\times p} = C_{m\times p}$, where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$.

Non-commutativity: $AB \neq BA$ in general.

Exercise 3.2

Q 3.1 **Let** $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$.

Find each of the following:

(i) $A + B$, **(ii)** $A - B$, **(iii)** $3A - C$, **(iv)** AB, **(v)** BA.

SOLUTION

Concept used. For matrices of the same order, addition and subtraction are done **entry-wise**: $(A \pm B)_{ij} = a_{ij} \pm b_{ij}$. The **scalar multiple** kA multiplies every entry by k: $(kA)_{ij} = k\, a_{ij}$. For multiplication of compatible matrices $A_{m\times n}B_{n\times p} = C_{m\times p}$, the entry $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$ is the *dot product* of row i of A with column j of B.

☞ **Row-by-column**
$(AB)_{ij} =$ (row i of A) $\cdot$ (column j of B). Always pair row of the *left* factor with column of the *right* factor.

Step 1. **(i)** $A+B$. Add entry-wise:

$$A+B = \begin{bmatrix} 2+1 & 4+3 \\ 3+(-2) & 2+5 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 1 & 7 \end{bmatrix}.$$

Step 2. **(ii)** $A-B$. Subtract entry-wise:

$$A-B = \begin{bmatrix} 2-1 & 4-3 \\ 3-(-2) & 2-5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 5 & -3 \end{bmatrix}.$$

Step 3. **(iii)** $3A-C$. First scale A by 3:

$$3A = \begin{bmatrix} 6 & 12 \\ 9 & 6 \end{bmatrix}.$$

Then subtract C:

$$3A-C = \begin{bmatrix} 6-(-2) & 12-5 \\ 9-3 & 6-4 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 6 & 2 \end{bmatrix}.$$

Step 4. **(iv)** AB. Both are 2×2; product is 2×2. Compute each entry as a row $\cdot$ column dot product:

$$(AB)_{11} = (2,4)\cdot(1,-2) = 2\cdot 1 + 4\cdot(-2) = 2-8 = -6.$$

$$(AB)_{12} = (2,4)\cdot(3,5) = 2\cdot 3 + 4\cdot 5 = 6+20 = 26.$$
$$(AB)_{21} = (3,2)\cdot(1,-2) = 3-4 = -1.$$
$$(AB)_{22} = (3,2)\cdot(3,5) = 9+10 = 19.$$
$$AB = \begin{bmatrix} -6 & 26 \\ -1 & 19 \end{bmatrix}.$$

Step 5. **(v)** BA. Same way, but row i of B, column j of A:

$$(BA)_{11} = (1,3)\cdot(2,3) = 2+9 = 11.$$

$$(BA)_{12} = (1,3)\cdot(4,2) = 4+6 = 10.$$
$$(BA)_{21} = (-2,5)\cdot(2,3) = -4+15 = 11.$$
$$(BA)_{22} = (-2,5)\cdot(4,2) = -8+10 = 2.$$
$$BA = \begin{bmatrix} 11 & 10 \\ 11 & 2 \end{bmatrix}.$$

Step 6. Notice $AB \neq BA$, illustrating that matrix multiplication is *not* commutative.

Final Answer: (i) $\begin{bmatrix} 3 & 7 \\ 1 & 7 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & 1 \\ 5 & -3 \end{bmatrix}$, (iii) $\begin{bmatrix} 8 & 7 \\ 6 & 2 \end{bmatrix}$, (iv) $\begin{bmatrix} -6 & 26 \\ -1 & 19 \end{bmatrix}$, (v) $\begin{bmatrix} 11 & 10 \\ 11 & 2 \end{bmatrix}$.

✗ Common Mistake

A frequent slip is to multiply AB by multiplying corresponding entries (Hadamard style). *Don't.* AB is the row-by-column dot product, not entry-wise multiplication.

EXPERT'S SOLUTION *: Aarav Gupta, M.Sc Mathematics, IIT Bombay*

Strategic angle. Treat $A + B,\ A - B,\ 3A - C$ as pure entry-wise arithmetic (no row-column dance). For AB and BA, list each row of the left matrix and each column of the right matrix as ordered pairs and dot them.

Step 1. Pre-list the data we need.
Rows of A: $R_1^A = (2,4),\ R_2^A = (3,2)$.
Cols of A: $C_1^A = (2,3),\ C_2^A = (4,2)$.
Rows of B: $R_1^B = (1,3),\ R_2^B = (-2,5)$.
Cols of B: $C_1^B = (1,-2),\ C_2^B = (3,5)$.

Step 2. Addition/subtraction: $A + B = \begin{bmatrix} 3 & 7 \\ 1 & 7 \end{bmatrix}$, $A - B = \begin{bmatrix} 1 & 1 \\ 5 & -3 \end{bmatrix}$.

Step 3. Scale: $3A = \begin{bmatrix} 6 & 12 \\ 9 & 6 \end{bmatrix}$, so $3A - C = \begin{bmatrix} 8 & 7 \\ 6 & 2 \end{bmatrix}$.

Step 4. AB: pair R_i^A with C_j^B. $(AB)_{11} = 2 - 8 = -6$; $(AB)_{12} = 6 + 20 = 26$; $(AB)_{21} = 3 - 4 = -1$; $(AB)_{22} = 9 + 10 = 19$.

Step 5. BA: pair R_i^B with C_j^A. $(BA)_{11} = 2 + 9 = 11$; $(BA)_{12} = 4 + 6 = 10$; $(BA)_{21} = -4 + 15 = 11$; $(BA)_{22} = -8 + 10 = 2$.

Why this matters. Listing rows and columns first prevents the most common bug: mixing up which factor's row you're using.

Final Answer: Same answers as the main solution.

Q 3.2 Compute the following:

(i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & b \\ b & a \end{bmatrix}$, **(ii)** $\begin{bmatrix} a^2+b^2 & b^2+c^2 \\ a^2+c^2 & a^2+b^2 \end{bmatrix} + \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix}$,

(iii) $\begin{bmatrix} -1 & 4 & -6 \\ 8 & 5 & 16 \\ 2 & 8 & 5 \end{bmatrix} + \begin{bmatrix} 12 & 7 & 6 \\ 8 & 0 & 5 \\ 3 & 2 & 4 \end{bmatrix}$, **(iv)** $\begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$.

SOLUTION

Concept used. Matrix addition is entry-wise. For the trig-identity part, recall $\sin^2\theta + \cos^2\theta = 1$.

Step 1. **(i)** Add entries:

$$\begin{bmatrix} a+a & b+b \\ -b+b & a+a \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 0 & 2a \end{bmatrix}.$$

Step 2. **(ii)** Add entries:

$$\begin{bmatrix} a^2+b^2+2ab & b^2+c^2+2bc \\ a^2+c^2-2ac & a^2+b^2-2ab \end{bmatrix}.$$

Use $(p+q)^2 = p^2+2pq+q^2$ and $(p-q)^2 = p^2-2pq+q^2$:

$$= \begin{bmatrix} (a+b)^2 & (b+c)^2 \\ (a-c)^2 & (a-b)^2 \end{bmatrix}.$$

Step 3. **(iii)** Add the corresponding entries: Row 1: $-1+12 = 11$, $4+7 = 11$, $-6+6 = 0$.
Row 2: $8+8 = 16$, $5+0 = 5$, $16+5 = 21$.
Row 3: $2+3 = 5$, $8+2 = 10$, $5+4 = 9$.

$$\begin{bmatrix} 11 & 11 & 0 \\ 16 & 5 & 21 \\ 5 & 10 & 9 \end{bmatrix}.$$

Step 4. **(iv)** Add entries and use $\sin^2 x + \cos^2 x = 1$:

$$\begin{bmatrix} \cos^2 x + \sin^2 x & \sin^2 x + \cos^2 x \\ \sin^2 x + \cos^2 x & \cos^2 x + \sin^2 x \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Final Answer: (i) $\begin{bmatrix} 2a & 2b \\ 0 & 2a \end{bmatrix}$, (ii) $\begin{bmatrix} (a+b)^2 & (b+c)^2 \\ (a-c)^2 & (a-b)^2 \end{bmatrix}$, (iii) $\begin{bmatrix} 11 & 11 & 0 \\ 16 & 5 & 21 \\ 5 & 10 & 9 \end{bmatrix}$, (iv) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

EXPERT'S SOLUTION : *Yash Desai, M.Sc Applied Mathematics, IIT Kanpur*

Structural observation. Look for algebraic identities before computing: the second sum is a binomial-square mosaic; the trig sum is the Pythagorean identity, four times.

Step 1. (i) Routine entry sum; the $-b$ and $+b$ cancel out at $(2, 1)$.

Step 2. (ii) Each entry has the shape $p^2 \pm 2pq + q^2$, i.e. $(p \pm q)^2$. So the sum is the matrix of those squares.

$$\begin{bmatrix} (a+b)^2 & (b+c)^2 \\ (a-c)^2 & (a-b)^2 \end{bmatrix}.$$

Step 3. (iii) Plain addition. Quick verify: $(2, 2)$-entry $5 + 0 = 5$ matches.

Step 4. (iv) Each entry is $\sin^2 x + \cos^2 x = 1$.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Why this matters. Spotting identities before number-crunching saves time and keeps the answer in closed algebraic form.

Final Answer: Same as main solution.

Q 3.3 Compute the indicated products:

(i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, **(ii)** $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$,

(iii) $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$, **(iv)** $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}$,

(v) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$, **(vi)** $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$.

SOLUTION

Concept used. For a product $A_{m \times n} B_{n \times p}$, each entry $(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$ is the dot product of row i of A with column j of B. The inner dimensions n must match; the result has order $m \times p$.

Step 1. **(i)** Product is 2×2. Entries:

$$(1, 1): \; a \cdot a + b \cdot b = a^2 + b^2.$$

$$(1,2):\ a(-b)+b\cdot a=-ab+ab=0.$$

$$(2,1):\ -b\cdot a+a\cdot b=0.$$

$$(2,2):\ -b(-b)+a\cdot a=b^2+a^2.$$

$$\Rightarrow \begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix}=(a^2+b^2)I.$$

Step 2. **(ii)** $(3\times1)(1\times3)=3\times3$. Each entry (i,j) is the single product $(\text{col})_i\cdot(\text{row})_j$:

$$\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}2 & 3 & 4\end{bmatrix}=\begin{bmatrix}1\cdot2 & 1\cdot3 & 1\cdot4\\ 2\cdot2 & 2\cdot3 & 2\cdot4\\ 3\cdot2 & 3\cdot3 & 3\cdot4\end{bmatrix}=\begin{bmatrix}2 & 3 & 4\\ 4 & 6 & 8\\ 6 & 9 & 12\end{bmatrix}.$$

Step 3. **(iii)** $(2\times2)(2\times3)=2\times3$. Compute:

$$(1,1):\ 1\cdot1+(-2)\cdot2=1-4=-3.$$

$$(1,2):\ 1\cdot2+(-2)\cdot3=2-6=-4.$$

$$(1,3):\ 1\cdot3+(-2)\cdot1=3-2=1.$$

$$(2,1):\ 2\cdot1+3\cdot2=2+6=8.$$

$$(2,2):\ 2\cdot2+3\cdot3=4+9=13.$$

$$(2,3):\ 2\cdot3+3\cdot1=6+3=9.$$

$$\Rightarrow \begin{bmatrix}-3 & -4 & 1\\ 8 & 13 & 9\end{bmatrix}.$$

Step 4. **(iv)** $(3\times3)(3\times3)=3\times3$. Each entry is a 3-term dot product.

Row 1 of $A=(2,3,4)$:

- $(1,1):\ 2(1)+3(0)+4(3)=2+0+12=14.$
- $(1,2):\ 2(-3)+3(2)+4(0)=-6+6+0=0.$
- $(1,3):\ 2(5)+3(4)+4(5)=10+12+20=42.$

Row 2 of $A=(3,4,5)$:

- $(2,1):\ 3(1)+4(0)+5(3)=3+0+15=18.$
- $(2,2):\ 3(-3)+4(2)+5(0)=-9+8+0=-1.$
- $(2,3):\ 3(5)+4(4)+5(5)=15+16+25=56.$

Row 3 of $A=(4,5,6)$:

- $(3,1):\ 4(1)+5(0)+6(3)=4+0+18=22.$
- $(3,2):\ 4(-3)+5(2)+6(0)=-12+10+0=-2.$
- $(3,3):\ 4(5)+5(4)+6(5)=20+20+30=70.$

$$\Rightarrow \begin{bmatrix} 14 & 0 & 42 \\ 18 & -1 & 56 \\ 22 & -2 & 70 \end{bmatrix}.$$

Step 5. **(v)** $(3\times 2)(2\times 3) = 3\times 3$.

Row 1 of $A = (2,1)$: $(1,1): \; 2(1)+1(-1)=1$; $(1,2): \; 2(0)+1(2)=2$; $(1,3): \; 2(1)+1(1)=3$.

Row 2 of $A = (3,2)$: $(2,1): \; 3(1)+2(-1)=1$; $(2,2): \; 3(0)+2(2)=4$; $(2,3): \; 3(1)+2(1)=5$.

Row 3 of $A = (-1,1)$: $(3,1): \; -1(1)+1(-1)=-2$; $(3,2): \; -1(0)+1(2)=2$; $(3,3): \; -1(1)+1(1)=0$.

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ -2 & 2 & 0 \end{bmatrix}.$$

Step 6. **(vi)** $(2\times 3)(3\times 2) = 2\times 2$.

Row 1 of $A = (3,-1,3)$: $(1,1): \; 3(2)+(-1)(1)+3(3) = 6-1+9 = 14$; $(1,2): \; 3(-3)+(-1)(0)+3(1) = -9+0+3 = -6$.

Row 2 of $A = (-1,0,2)$: $(2,1): \; -1(2)+0(1)+2(3) = -2+0+6 = 4$; $(2,2): \; -1(-3)+0(0)+2(1) = 3+0+2 = 5$.

$$\Rightarrow \begin{bmatrix} 14 & -6 \\ 4 & 5 \end{bmatrix}.$$

Final Answer: (i)$(a^2+b^2)I$, (ii) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}$, (iii) $\begin{bmatrix} -3 & -4 & 1 \\ 8 & 13 & 9 \end{bmatrix}$, (iv) $\begin{bmatrix} 14 & 0 & 42 \\ 18 & -1 & 56 \\ 22 & -2 & 70 \end{bmatrix}$, (v) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ -2 & 2 & 0 \end{bmatrix}$, (vi) $\begin{bmatrix} 14 & -6 \\ 4 & 5 \end{bmatrix}$.

☞ Exam Tip

Before multiplying, write down the orders: $(m\times n)(n\times p) = (m\times p)$. If the inner n's don't match, AB is undefined. This saves you from wasted arithmetic.

EXPERT'S SOLUTION *: Krishna Bhat, Ph.D Pure Mathematics, IISc Bangalore*

Strategic angle. For each pair, first check order compatibility, then carry out the dot products. Watch the structure: (i) is the classic "rotation-times-rotation" pattern that yields a scalar multiple of the identity.

Step 1. (i) The cross-terms cancel because of the sign pattern; only the diagonal survives: $(a^2+b^2)I_2$.

Step 2. (ii) Outer product of $(1,2,3)^T$ and $(2,3,4)$. Each entry is a product of one element from each. Yields the rank-1 matrix $\begin{bmatrix}2&3&4\\4&6&8\\6&9&12\end{bmatrix}$.

Step 3. (iii) Pre-pair row entries with column entries and add: $(1,1)=-3, (1,2)=-4, (1,3)=1, (2,1)=8, (2,2)=13, (2,3)=9$.

Step 4. (iv) Same routine on $3\times3\cdot3\times3$; nine entries each a three-term dot product. Group by row to stay organised. Each row of the result uses the same three rows of B paired with one row of A.

Step 5. (v), (vi) Same routine; track orders carefully. (v) gives a 3×3, (vi) gives a 2×2.

Why this matters. "Order compatibility" is the single biggest pitfall in matrix multiplication problems. Always size-check before multiplying.

> **Final Answer:** See main boxed answer.

Q 3.4 **If** $A=\begin{bmatrix}1&2&-3\\5&0&2\\1&-1&1\end{bmatrix}$, $B=\begin{bmatrix}3&-1&2\\4&2&5\\2&0&3\end{bmatrix}$, $C=\begin{bmatrix}4&1&2\\0&3&2\\1&-2&3\end{bmatrix}$, **compute** $A+B$ **and** $B-C$**. Also verify that** $A+(B-C)=(A+B)-C$**.**

SOLUTION

Concept used. Addition and subtraction of matrices of the same order is entry-wise. The **associative property** $(A+B)+(-C)=A+(B+(-C))$ for matrices is just associativity of addition in $\mathbb{R}$, applied component-wise.

Step 1. Compute $A+B$**.** Row 1: $1+3=4,\ 2+(-1)=1,\ -3+2=-1$.
Row 2: $5+4=9,\ 0+2=2,\ 2+5=7$.
Row 3: $1+2=3,\ -1+0=-1,\ 1+3=4$.

$$A+B=\begin{bmatrix}4&1&-1\\9&2&7\\3&-1&4\end{bmatrix}.$$

Step 2. Compute $B-C$**.** Row 1: $3-4=-1,\ -1-1=-2,\ 2-2=0$.
Row 2: $4-0=4,\ 2-3=-1,\ 5-2=3$.

Row 3: $2-1=1,\ 0-(-2)=2,\ 3-3=0.$

$$B-C=\begin{bmatrix}-1 & -2 & 0\\ 4 & -1 & 3\\ 1 & 2 & 0\end{bmatrix}.$$

Step 3. **Compute $A+(B-C)$.** Row 1: $1+(-1)=0,\ 2+(-2)=0,\ -3+0=-3.$
Row 2: $5+4=9,\ 0+(-1)=-1,\ 2+3=5.$
Row 3: $1+1=2,\ -1+2=1,\ 1+0=1.$

$$A+(B-C)=\begin{bmatrix}0 & 0 & -3\\ 9 & -1 & 5\\ 2 & 1 & 1\end{bmatrix}.$$

Step 4. **Compute $(A+B)-C$.** Use the $A+B$ already found: Row 1: $4-4=0,\ 1-1=0,\ -1-2=-3.$
Row 2: $9-0=9,\ 2-3=-1,\ 7-2=5.$
Row 3: $3-1=2,\ -1-(-2)=1,\ 4-3=1.$

$$(A+B)-C=\begin{bmatrix}0 & 0 & -3\\ 9 & -1 & 5\\ 2 & 1 & 1\end{bmatrix}.$$

Step 5. Both results match entry-by-entry, so $A+(B-C)=(A+B)-C$. ✓

Final Answer: $A+B=\begin{bmatrix}4 & 1 & -1\\ 9 & 2 & 7\\ 3 & -1 & 4\end{bmatrix}$, $B-C=\begin{bmatrix}-1 & -2 & 0\\ 4 & -1 & 3\\ 1 & 2 & 0\end{bmatrix}$, and $A+(B-C)=(A+B)-C$ verified.

EXPERT'S SOLUTION : *Aanya Pillai, M.Sc Mathematics, IIT Bombay*

Quick reading. The verification is automatic because matrix addition inherits associativity from $\mathbb{R}$. Both computations give the same nine numbers because they are evaluating the same expression $A+B-C$.

Step 1. Compute $A+B$ row-by-row; result as above.

Step 2. Compute $B-C$ row-by-row; result as above.

Step 3. For $A+(B-C)$: add A and $B-C$ entry-wise.

Step 4. For $(A+B)-C$: subtract C from $A+B$ entry-wise.

Step 5. Both sequences end at $\begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix}$.

Why this matters. Associativity lets us drop the parentheses in $A + B - C$; this is heavily used when collecting many matrix terms.

Final Answer: Both sides equal $\begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix}$.

Q 3.5 **If** $A = \begin{bmatrix} \frac{2}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{7}{3} & 2 & \frac{2}{3} \end{bmatrix}$ **and** $B = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} & 1 \\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{6}{5} & \frac{2}{5} \end{bmatrix}$**, compute** $3A - 5B$**.**

SOLUTION

Concept used. Scale each matrix by its scalar, then subtract entry-wise. Common-denominator trick: multiplying A by 3 and B by 5 clears the denominators 3 and 5 inside each matrix.

Step 1. **Compute** $3A$**.** Multiplying every entry of A by 3 clears the denominator 3:

$$3A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 4 \\ 7 & 6 & 2 \end{bmatrix}.$$

Step 2. **Compute** $5B$**.** Multiplying every entry of B by 5 clears the denominator 5:

$$5B = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 4 \\ 7 & 6 & 2 \end{bmatrix}.$$

Step 3. **Subtract entry-wise.** $3A - 5B$ has every entry equal to 0:

$$3A - 5B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O.$$

Final Answer: $3A - 5B = O$ (the 3×3 zero matrix).

EXPERT'S SOLUTION : *Tara Singh, M.Sc Mathematics, IIT Bombay*

Structural observation. The matrix B is exactly $\frac{3}{5}$ of A (entry-by-entry). So $5B = 3A$, giving $3A - 5B = 0$.

Step 1. Test: $B_{ij}/A_{ij} = \frac{2/5}{2/3} = \frac{2}{5} \cdot \frac{3}{2} = \frac{3}{5}$ for $(1,1)$; check $(2,2)$: $\frac{2/5}{2/3} = \frac{3}{5}$. Pattern holds for every entry.

Step 2. So $B = \frac{3}{5}A$, hence $5B = 3A$.

Step 3. Therefore $3A - 5B = 3A - 3A = O$.

Why this matters. Spotting a scalar relationship between two matrices instantly collapses a long arithmetic problem to a one-liner.

Final Answer: O (the zero matrix).

Q 3.6 **Simplify** $\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$.

SOLUTION

Concept used. Distribute the scalar over each entry of the matrix, then add the two matrices entry-wise. Finally use $\sin^2\theta + \cos^2\theta = 1$.

Step 1. Scale the first matrix by $\cos\theta$:

$$\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix}.$$

Step 2. Scale the second matrix by $\sin\theta$:

$$\sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} = \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}.$$

Step 3. Add entry-wise:

$$\begin{bmatrix} \cos^2\theta + \sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{bmatrix}.$$

Step 4. Apply $\sin^2\theta + \cos^2\theta = 1$ on the diagonal and note the off-diagonals cancel:

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Final Answer: The simplified matrix is the 2×2 identity, I.

♥ Why This Matters

The matrix $R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is a *rotation matrix* (it rotates plane vectors by $-\theta$). The expression here is $\cos\theta \cdot R(\theta) + \sin\theta \cdot R(\theta - 90°)$, which collapses to the identity by trigonometric balance.

EXPERT'S SOLUTION : *Meera Chatterjee, Ph.D Mathematics, IIT Delhi*

Structural observation. The off-diagonal terms cancel by sign; the diagonal terms combine into the Pythagorean identity.

Step 1. Entry $(1,1)$: $\cos^2\theta + \sin^2\theta = 1$.

Step 2. Entry $(2,2)$: $\cos^2\theta + \sin^2\theta = 1$.

Step 3. Entry $(1,2)$: $\sin\theta\cos\theta - \sin\theta\cos\theta = 0$.

Step 4. Entry $(2,1)$: $-\sin\theta\cos\theta + \sin\theta\cos\theta = 0$.

Step 5. Combine into $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Why this matters. Trig identities embedded in matrix entries turn what looks like a long calculation into I.

Final Answer: I.

Q 3.7 **Find X and Y, if:**

(i) $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ **and** $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$,

(ii) $2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$ **and** $3X + 2Y = \begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix}$.

SOLUTION

Concept used. Matrix equations behave like scalar linear systems: add or subtract the two given equations to eliminate one unknown matrix.

Step 1. **(i)** Add and subtract the two equations:

$$(X+Y) + (X-Y) = 2X \;\Rightarrow\; 2X = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}.$$

Divide by 2:

$$X = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}.$$

$$(X+Y)-(X-Y) = 2Y \;\Rightarrow\; 2Y = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}.$$

Divide by 2:

$$Y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}.$$

Step 2. **(ii)** Let

$$P = 2X+3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}, \quad Q = 3X+2Y = \begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix}.$$

Eliminate Y: compute $3P - 2Q$.

$$3P = \begin{bmatrix} 6 & 9 \\ 12 & 0 \end{bmatrix}, \quad 2Q = \begin{bmatrix} 4 & -4 \\ -2 & 10 \end{bmatrix}.$$

$$3P-2Q = \begin{bmatrix} 6-4 & 9-(-4) \\ 12-(-2) & 0-10 \end{bmatrix} = \begin{bmatrix} 2 & 13 \\ 14 & -10 \end{bmatrix}.$$

Now $3P - 2Q = 3(2X+3Y) - 2(3X+2Y) = 6X + 9Y - 6X - 4Y = 5Y$. So

$$5Y = \begin{bmatrix} 2 & 13 \\ 14 & -10 \end{bmatrix} \Rightarrow Y = \frac{1}{5}\begin{bmatrix} 2 & 13 \\ 14 & -10 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{13}{5} \\ \frac{14}{5} & -2 \end{bmatrix}.$$

Step 3. Eliminate Y the other way:
$2Q - 3P = 2(3X+2Y) - 3(2X+3Y) = 6X + 4Y - 6X - 9Y = -5Y$. Or eliminate X:
$3Q - 2P = 3(3X+2Y) - 2(2X+3Y) = 9X + 6Y - 4X - 6Y = 5X$.

$$3Q = \begin{bmatrix} 6 & -6 \\ -3 & 15 \end{bmatrix}, \quad 2P = \begin{bmatrix} 4 & 6 \\ 8 & 0 \end{bmatrix}.$$

$$3Q - 2P = \begin{bmatrix} 2 & -12 \\ -11 & 15 \end{bmatrix} = 5X.$$

$$X = \frac{1}{5}\begin{bmatrix} 2 & -12 \\ -11 & 15 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & -\frac{12}{5} \\ -\frac{11}{5} & 3 \end{bmatrix}.$$

Step 4. Verify (ii) by plugging into $2X + 3Y$: each entry, $2 \cdot \frac{2}{5} + 3 \cdot \frac{2}{5} = \frac{10}{5} = 2$ ✓.

Final Answer: (i) $X = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$, $Y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$. (ii) $X = \begin{bmatrix} 2/5 & -12/5 \\ -11/5 & 3 \end{bmatrix}$, $Y = \begin{bmatrix} 2/5 & 13/5 \\ 14/5 & -2 \end{bmatrix}$.

EXPERT'S SOLUTION : *Siddharth Rao, M.Sc Mathematics, IIT Kanpur*

Strategic angle. Treat the unknown matrices like scalar unknowns. The right linear combination of the two given equations isolates either X or Y.

Step 1. (i) Sum of the two equations gives $2X$; difference gives $2Y$. Halve both.

Step 2. (ii) To kill Y, scale equations so the Y-coefficients agree: $3P$ has $9Y$, $2Q$ has $4Y$; $3P - 2Q$ leaves $5Y$. To kill X, $3Q - 2P$ leaves $5X$.

Step 3. Divide each result by 5 to extract the final X, Y.

Why this matters. The elimination idea transfers directly to $AX = B$-style linear systems later: combine equations to drop variables.

Final Answer: Same as main solution.

Q 3.8 **Find X, if $Y = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ and $2X + Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$.**

SOLUTION

Concept used. Isolate $2X$ by subtracting Y, then divide by 2.

Step 1. Rearrange: $2X = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} - Y$.

$$2X = \begin{bmatrix} 1-3 & 0-2 \\ -3-1 & 2-4 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix}.$$

Step 2. Halve every entry:

$$X = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}.$$

Step 3. Verify: $2X + Y = \begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$ ✓.

Final Answer: $X = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$.

EXPERT'S SOLUTION : *Neha Iyer, M.Sc Mathematics, IIT Madras*

Quick reading. "Solve for X" just means subtract Y and divide by 2, treating matrices the same way you'd treat numbers in a 1-D linear equation.

Step 1. $2X = B - Y$ where $B = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$.

Step 2. Compute $B - Y$ entry-wise: $\begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix}$.

Step 3. $X = \frac{1}{2}(B - Y) = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$.

Why this matters. Linear matrix equations of the form $aX + B = C$ always solve via $X = (C - B)/a$ when $a \neq 0$.

Final Answer: $X = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$.

Q 3.9 **Find x and y, if** $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$.

SOLUTION

Concept used. Carry out the scalar multiplication and addition, then equate corresponding entries.

Step 1. Scale: $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 0 & 2x \end{bmatrix}$.

Step 2. Add: $\begin{bmatrix} 2 & 6 \\ 0 & 2x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2+y & 6 \\ 1 & 2x+2 \end{bmatrix}$.

Step 3. Equate with RHS $\begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$:

$$2 + y = 5, \quad 6 = 6, \quad 1 = 1, \quad 2x + 2 = 8.$$

Step 4. Solve: $y = 3$ and $2x = 6 \Rightarrow x = 3$.

Final Answer: $x = 3,\ y = 3$.

EXPERT'S SOLUTION : *Ankit Verma, M.Sc Mathematics, IIT Bombay*

Quick reading. Compute LHS as a single matrix, then match entries with the RHS.

Step 1. LHS $= \begin{bmatrix} 2+y & 6 \\ 1 & 2x+2 \end{bmatrix}$.

Step 2. Match $(1,1)$: $2+y=5 \Rightarrow y=3$.

Step 3. Match $(2,2)$: $2x+2=8 \Rightarrow x=3$.

Step 4. Other entries already match.

Why this matters. A matrix equation collapses cleanly to a small scalar system once you've executed the algebraic operations on the LHS.

> **Final Answer:** $x=3,\ y=3.$

Q 3.10 **Solve the equation for** x, y, z, t**, if** $2\begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$**.**

SOLUTION

Concept used. Distribute the scalars, combine into one matrix on the LHS, equate with the RHS entry-wise.

Step 1. Compute $2\begin{bmatrix} x & z \\ y & t \end{bmatrix} = \begin{bmatrix} 2x & 2z \\ 2y & 2t \end{bmatrix}$.

Step 2. Compute $3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix}$.

Step 3. LHS $= \begin{bmatrix} 2x+3 & 2z-3 \\ 2y+0 & 2t+6 \end{bmatrix} = \begin{bmatrix} 2x+3 & 2z-3 \\ 2y & 2t+6 \end{bmatrix}$.

Step 4. RHS $= 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix}$.

Step 5. Match entries:

$$2x+3=9 \Rightarrow x=3.$$

$$2z-3=15 \Rightarrow 2z=18 \Rightarrow z=9.$$

$$2y=12 \Rightarrow y=6.$$

$$2t+6=18 \Rightarrow 2t=12 \Rightarrow t=6.$$

> **Final Answer:** $x=3,\ y=6,\ z=9,\ t=6.$

EXPERT'S SOLUTION *: Riya Joshi, M.Sc Applied Mathematics, IIT Kanpur*

Structural observation. Every scalar equation has the form $2(\text{unknown}) + \text{constant} = \text{RHS constant}$. Solve them in parallel.

Step 1. Entry $(1, 1)$: $2x + 3 = 9 \Rightarrow x = 3$.

Step 2. Entry $(1, 2)$: $2z - 3 = 15 \Rightarrow z = 9$.

Step 3. Entry $(2, 1)$: $2y + 0 = 12 \Rightarrow y = 6$.

Step 4. Entry $(2, 2)$: $2t + 6 = 18 \Rightarrow t = 6$.

Why this matters. Scalar systems hidden inside matrix equations are the norm in this exercise.

Final Answer: $(x, y, z, t) = (3, 6, 9, 6)$.

Q 3.11 **If** $x \begin{bmatrix} 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$**, find** x **and** y**.**

SOLUTION

Concept used. The LHS is a linear combination of two column vectors; setting it equal to a given column vector gives a linear system in x, y.

Step 1. Carry out the scalar multiplications and add columns:

$$x \begin{bmatrix} 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2x - y \\ 3x + y \end{bmatrix}.$$

Step 2. Equate with $\begin{bmatrix} 10 \\ 5 \end{bmatrix}$:

$$2x - y = 10 \ \ (1), \qquad 3x + y = 5 \ \ (2).$$

Step 3. Add (1) and (2) to eliminate y:

$$5x = 15 \ \Rightarrow \ x = 3.$$

Step 4. Substitute $x = 3$ in (1): $6 - y = 10 \Rightarrow y = -4$.

Step 5. Verify in (2): $3(3) + (-4) = 9 - 4 = 5$ ✓.

Final Answer: $x = 3, \ y = -4$.

EXPERT'S SOLUTION : *Pranav Kumar, M.Tech CS, IIT Madras*

Quick reading. A 2-equation, 2-unknown linear system. Pick the elimination that cancels the easiest unknown.

Step 1. Equations: $2x - y = 10$ and $3x + y = 5$.

Step 2. Adding cancels y: $5x = 15 \Rightarrow x = 3$.

Step 3. Back-substitute: $y = 2x - 10 = 6 - 10 = -4$.

Why this matters. Writing the matrix combination as $AX = B$ sets you up for the matrix-method solution of linear systems in Ch 4.

> **Final Answer:** $(x, y) = (3, -4)$.

Q 3.12 **Given** $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$**, find** x, y, z, w**.**

SOLUTION

Concept used. Add the two matrices on the RHS, then equate entry-wise with the LHS.

Step 1. Combine the RHS:

$$\begin{bmatrix} x+4 & 6+(x+y) \\ -1+(z+w) & 2w+3 \end{bmatrix}.$$

Step 2. LHS is $\begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix}$.

Step 3. Match entries:

$$(1,1):\ 3x = x + 4 \Rightarrow 2x = 4 \Rightarrow x = 2.$$

$$(1,2):\ 3y = 6 + x + y \Rightarrow 2y = 6 + x.$$

Substitute $x = 2$: $2y = 8 \Rightarrow y = 4$.

$$(2,2):\ 3w = 2w + 3 \Rightarrow w = 3.$$

$$(2,1):\ 3z = -1 + z + w \Rightarrow 2z = -1 + w.$$

Substitute $w = 3$: $2z = -1 + 3 = 2 \Rightarrow z = 1$.

Step 4. Verify $(1, 2)$: $3y = 3 \cdot 4 = 12$ and $6 + x + y = 6 + 2 + 4 = 12$ ✓.

> **Final Answer:** $x = 2,\ y = 4,\ z = 1,\ w = 3$.

EXPERT'S SOLUTION : *Aditya Mehta, Ph.D Pure Mathematics, IISc Bangalore*

Strategic angle. Pull all unknowns to LHS in each scalar equation. The diagonal entries decouple immediately because the $(1,1)$ and $(2,2)$ equations don't involve other unknowns.

Step 1. $(1,1)$: $3x - x = 4 \Rightarrow x = 2$.

Step 2. $(2,2)$: $3w - 2w = 3 \Rightarrow w = 3$.

Step 3. $(1,2)$: $3y - y = 6 + x = 8 \Rightarrow y = 4$.

Step 4. $(2,1)$: $3z - z = -1 + w = 2 \Rightarrow z = 1$.

Why this matters. Decoupling — solving the standalone equations first — is the standard tactic in 2x2 matrix systems with mild dependencies.

Final Answer: $(x, y, z, w) = (2, 4, 1, 3)$.

Q 3.13 If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$, **show that** $F(x)\,F(y) = F(x+y)$.

SOLUTION

Concept used. Compute $F(x)F(y)$ entry-by-entry and simplify using the cosine/sine addition formulas $\cos(x+y) = \cos x \cos y - \sin x \sin y$ and $\sin(x+y) = \sin x \cos y + \cos x \sin y$.

Step 1. Write $F(x)$ rows as $R_1 = (\cos x, -\sin x, 0)$, $R_2 = (\sin x, \cos x, 0)$, $R_3 = (0,0,1)$, and $F(y)$ columns as $C_1 = (\cos y, \sin y, 0)^T$, $C_2 = (-\sin y, \cos y, 0)^T$, $C_3 = (0,0,1)^T$.

Step 2. Compute the nine entries of $F(x)F(y)$. The last row and last column are trivial (any pairing with the $(0,0,1)$ axis gives either 0 or 1):

$$(3,3) : 1; \quad (3,1), (3,2), (1,3), (2,3) \text{ all } 0.$$

Step 3. $(1,1):\ \cos x \cos y + (-\sin x)\sin y + 0 = \cos x \cos y - \sin x \sin y = \cos(x+y)$.

Step 4.

$(1,2):\ \cos x(-\sin y) + (-\sin x)\cos y + 0 = -\cos x \sin y - \sin x \cos y = -\sin(x+y)$.

Step 5. $(2,1):\ \sin x \cos y + \cos x \sin y + 0 = \sin(x+y)$.

Step 6. $(2,2):\ \sin x(-\sin y) + \cos x \cos y + 0 = \cos x \cos y - \sin x \sin y = \cos(x+y)$.

Step 7. Collect into a matrix:

$$F(x)F(y) = \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix} = F(x+y).$$

Final Answer: $F(x)\,F(y) = F(x+y)$, proved by direct computation using the sine and cosine addition identities.

♥ Why This Matters

$F(\theta)$ is a rotation about the z-axis by $-\theta$. The identity $F(x)F(y) = F(x+y)$ says that composing two such rotations is another rotation through the sum of the angles. This is the matrix incarnation of the addition formulas.

EXPERT'S SOLUTION *: Kavya Pillai, Ph.D Mathematics, IIT Delhi*

Structural observation. Multiplying two block-diagonal matrices is block-wise, so the bottom-right 1×1 block stays as $1 \cdot 1 = 1$ and the top-left 2×2 block multiplies on its own. Treat it like a 2×2 rotation problem.

Step 1. Reduce to the 2×2 block: $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Step 2. Compute $R(x)R(y)$: each entry uses the relevant addition formula.
$(1,1) = \cos(x+y)$, $(1,2) = -\sin(x+y)$, $(2,1) = \sin(x+y)$, $(2,2) = \cos(x+y)$.

Step 3. Reassemble with the trivial 1×1 block to get $F(x+y)$.

Why this matters. "Block-diagonal $\times$ block-diagonal $=$ block-diagonal," so a 3×3 rotation problem reduces to a 2×2 one. The same trick scales to $n \times n$ matrices with block structure.

Final Answer: $F(x)F(y) = F(x+y)$.

Q 3.14 **Show that:**

(i) $\begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$.

(ii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \neq \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$.

SOLUTION

Concept used. Compute each side and show entries differ. This demonstrates that matrix multiplication is **not commutative** in general: $AB \neq BA$.

Step 1. **(i)** Let $A = \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$.

Compute AB:

$$(AB)_{11} = 5(2) + (-1)(3) = 10 - 3 = 7.$$

$$(AB)_{12} = 5(1) + (-1)(4) = 5 - 4 = 1.$$

$$(AB)_{21} = 6(2) + 7(3) = 12 + 21 = 33.$$

$$(AB)_{22} = 6(1) + 7(4) = 6 + 28 = 34.$$

$$AB = \begin{bmatrix} 7 & 1 \\ 33 & 34 \end{bmatrix}.$$

Compute BA:

$$(BA)_{11} = 2(5) + 1(6) = 10 + 6 = 16.$$

$$(BA)_{12} = 2(-1) + 1(7) = -2 + 7 = 5.$$

$$(BA)_{21} = 3(5) + 4(6) = 15 + 24 = 39.$$

$$(BA)_{22} = 3(-1) + 4(7) = -3 + 28 = 25.$$

$$BA = \begin{bmatrix} 16 & 5 \\ 39 & 25 \end{bmatrix}.$$

Since $AB \neq BA$ (e.g. $(1,1)$: $7 \neq 16$), the inequality holds.

Step 2. **(ii)** Let P, Q be the 3×3 matrices on left and right. Compute PQ row-by-row.

Row 1 of $P = (1, 2, 3)$, columns of Q:

- $(1,1): \ 1(-1) + 2(0) + 3(2) = -1 + 0 + 6 = 5.$
- $(1,2): \ 1(1) + 2(-1) + 3(3) = 1 - 2 + 9 = 8.$
- $(1,3): \ 1(0) + 2(1) + 3(4) = 0 + 2 + 12 = 14.$

Row 2 of $P = (0, 1, 0)$:

- $(2,1): \ 0(-1) + 1(0) + 0(2) = 0.$
- $(2,2): \ 0(1) + 1(-1) + 0(3) = -1.$
- $(2,3): \ 0(0) + 1(1) + 0(4) = 1.$

Row 3 of $P = (1, 1, 0)$:

- $(3,1): \ 1(-1) + 1(0) + 0(2) = -1.$
- $(3,2): \ 1(1) + 1(-1) + 0(3) = 0.$
- $(3,3): \ 1(0) + 1(1) + 0(4) = 1.$

$$PQ = \begin{bmatrix} 5 & 8 & 14 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}.$$

Compute QP row-by-row.

Row 1 of $Q = (-1, 1, 0)$:

- $(1,1): \ -1(1) + 1(0) + 0(1) = -1.$
- $(1,2): \ -1(2) + 1(1) + 0(1) = -1.$
- $(1,3): \ -1(3) + 1(0) + 0(0) = -3.$

Row 2 of $Q = (0, -1, 1)$:

- $(2,1): \ 0(1) + (-1)(0) + 1(1) = 1.$
- $(2,2): \ 0(2) + (-1)(1) + 1(1) = 0.$
- $(2,3): \ 0(3) + (-1)(0) + 1(0) = 0.$

Row 3 of $Q = (2, 3, 4)$:

- $(3,1): \ 2(1) + 3(0) + 4(1) = 6.$
- $(3,2): \ 2(2) + 3(1) + 4(1) = 11.$
- $(3,3): \ 2(3) + 3(0) + 4(0) = 6.$

$$QP = \begin{bmatrix} -1 & -1 & -3 \\ 1 & 0 & 0 \\ 6 & 11 & 6 \end{bmatrix}.$$

Clearly $PQ \neq QP$.

Final Answer: (i) and (ii): $AB \neq BA$ (specific entries shown above differ).

♥ Why This Matters

$AB = BA$ holds only in special cases (e.g. $B = I$, or A, B diagonal, or $B = A^{-1}$). In general matrix multiplication is non-commutative, which is fundamental to how matrices encode geometry and transformations.

EXPERT'S SOLUTION *: Vivaan Singh, M.Sc Mathematics, IIT Bombay*

Strategic angle. Just compute one entry of each side and show they differ — you don't need the full product if a single mismatch suffices.

Step 1. (i) Compute $(AB)_{11}$ and $(BA)_{11}$: $(AB)_{11} = 5(2) - 1(3) = 7$; $(BA)_{11} = 2(5) + 1(6) = 16$. $7 \neq 16$, done. (Full products shown for completeness in the main solution.)

Step 2. (ii) Compute $(PQ)_{11}$ and $(QP)_{11}$: $(PQ)_{11} = 1(-1) + 2(0) + 3(2) = 5$; $(QP)_{11} = -1(1) + 1(0) + 0(1) = -1$. $5 \neq -1$, done.

Why this matters. Spotting a single mismatched entry is enough to prove non-equality. Compute the rest only if asked.

Final Answer: Both products differ; $AB \neq BA$ in each part.

Q 3.15 **Find $A^2 - 5A + 6I$, if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$.**

SOLUTION

Concept used. For a square matrix A of order n, $A^2 = A \cdot A$. The identity matrix I_n satisfies kI_n being diagonal with k on the diagonal. The polynomial $A^2 - 5A + 6I$ is evaluated entry-wise after computing each term.

Step 1. **Compute** $A^2 = A \cdot A$. Rows of A: $R_1 = (2, 0, 1), R_2 = (2, 1, 3), R_3 = (1, -1, 0)$. Columns of A: $C_1 = (2, 2, 1), C_2 = (0, 1, -1), C_3 = (1, 3, 0)$.

$(A^2)_{11} = R_1 \cdot C_1 = 2(2) + 0(2) + 1(1) = 4 + 0 + 1 = 5.$
$(A^2)_{12} = R_1 \cdot C_2 = 2(0) + 0(1) + 1(-1) = 0 + 0 - 1 = -1.$
$(A^2)_{13} = R_1 \cdot C_3 = 2(1) + 0(3) + 1(0) = 2 + 0 + 0 = 2.$
$(A^2)_{21} = R_2 \cdot C_1 = 2(2) + 1(2) + 3(1) = 4 + 2 + 3 = 9.$
$(A^2)_{22} = R_2 \cdot C_2 = 2(0) + 1(1) + 3(-1) = 0 + 1 - 3 = -2.$
$(A^2)_{23} = R_2 \cdot C_3 = 2(1) + 1(3) + 3(0) = 2 + 3 + 0 = 5.$
$(A^2)_{31} = R_3 \cdot C_1 = 1(2) + (-1)(2) + 0(1) = 2 - 2 + 0 = 0.$
$(A^2)_{32} = R_3 \cdot C_2 = 1(0) + (-1)(1) + 0(-1) = -1.$
$(A^2)_{33} = R_3 \cdot C_3 = 1(1) + (-1)(3) + 0(0) = -2.$

$$A^2 = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}.$$

Step 2. Compute $5A = \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix}$.

Step 3. Compute $6I = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

Step 4. Combine $A^2 - 5A + 6I$ entry-wise. Row 1:

$5-10+6=1,\ -1-0+0=-1,\ 2-5+0=-3.$
Row 2: $9-10+0=-1,\ -2-5+6=-1,\ 5-15+0=-10.$
Row 3: $0-5+0=-5,\ -1-(-5)+0=4,\ -2-0+6=4.$

$$A^2-5A+6I=\begin{bmatrix}1 & -1 & -3\\ -1 & -1 & -10\\ -5 & 4 & 4\end{bmatrix}.$$

Final Answer: $A^2-5A+6I=\begin{bmatrix}1 & -1 & -3\\ -1 & -1 & -10\\ -5 & 4 & 4\end{bmatrix}.$

EXPERT'S SOLUTION *: Sanya Reddy, M.Sc Mathematics, IIT Bombay*

Strategic angle. Three computations in sequence: A^2, then $5A$, then add $6I$ along the diagonal only, finally entry-wise combine.

Step 1. Calculate A^2 by listing rows of A and columns of A, then computing 9 dot products. Result as above.

Step 2. $5A$ scales every entry of A by 5.

Step 3. $6I$ adds 6 only to the three diagonal entries.

Step 4. Combine: $A^2-5A+6I$. Diagonal entries get the extra $+6$. Off-diagonals just take A^2-5A.

Why this matters. Evaluating polynomials in A is the gateway to the Cayley-Hamilton theorem and to matrix inverses.

Final Answer: $\begin{bmatrix}1 & -1 & -3\\ -1 & -1 & -10\\ -5 & 4 & 4\end{bmatrix}.$

Q 3.16 **If** $A=\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}$**, prove that** $A^3-6A^2+7A+2I=0$**.**

SOLUTION

Concept used. Compute A^2 and $A^3 = A \cdot A^2$ directly, then form the linear combination and verify each entry is 0.

Step 1. Compute A^2. Rows of A: $R_1 = (1, 0, 2), R_2 = (0, 2, 1), R_3 = (2, 0, 3)$.
Columns of A: $C_1 = (1, 0, 2), C_2 = (0, 2, 0), C_3 = (2, 1, 3)$.
$(A^2)_{11} = 1(1) + 0(0) + 2(2) = 1 + 0 + 4 = 5.$
$(A^2)_{12} = 1(0) + 0(2) + 2(0) = 0.$
$(A^2)_{13} = 1(2) + 0(1) + 2(3) = 2 + 0 + 6 = 8.$
$(A^2)_{21} = 0(1) + 2(0) + 1(2) = 2.$
$(A^2)_{22} = 0(0) + 2(2) + 1(0) = 4.$
$(A^2)_{23} = 0(2) + 2(1) + 1(3) = 0 + 2 + 3 = 5.$
$(A^2)_{31} = 2(1) + 0(0) + 3(2) = 2 + 0 + 6 = 8.$
$(A^2)_{32} = 2(0) + 0(2) + 3(0) = 0.$
$(A^2)_{33} = 2(2) + 0(1) + 3(3) = 4 + 0 + 9 = 13.$

$$A^2 = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}.$$

Step 2. Compute $A^3 = A \cdot A^2$. Use rows of A and columns of A^2.
Columns of A^2: $C_1^2 = (5, 2, 8), C_2^2 = (0, 4, 0), C_3^2 = (8, 5, 13)$.
$(A^3)_{11} = 1(5) + 0(2) + 2(8) = 5 + 0 + 16 = 21.$
$(A^3)_{12} = 1(0) + 0(4) + 2(0) = 0.$
$(A^3)_{13} = 1(8) + 0(5) + 2(13) = 8 + 0 + 26 = 34.$
$(A^3)_{21} = 0(5) + 2(2) + 1(8) = 0 + 4 + 8 = 12.$
$(A^3)_{22} = 0(0) + 2(4) + 1(0) = 8.$
$(A^3)_{23} = 0(8) + 2(5) + 1(13) = 0 + 10 + 13 = 23.$
$(A^3)_{31} = 2(5) + 0(2) + 3(8) = 10 + 0 + 24 = 34.$
$(A^3)_{32} = 2(0) + 0(4) + 3(0) = 0.$
$(A^3)_{33} = 2(8) + 0(5) + 3(13) = 16 + 0 + 39 = 55.$

$$A^3 = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}.$$

Step 3. Compute $6A^2$.

$$6A^2 = \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix}.$$

Step 4. Compute $7A$.

$$7A = \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix}.$$

Step 5. $2I = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Step 6. Form $A^3 - 6A^2 + 7A + 2I$ entry-wise.

Row 1: $21 - 30 + 7 + 2 = 0$, $0 - 0 + 0 + 0 = 0$, $34 - 48 + 14 + 0 = 0$.

Row 2: $12 - 12 + 0 + 0 = 0$, $8 - 24 + 14 + 2 = 0$, $23 - 30 + 7 + 0 = 0$.

Row 3: $34 - 48 + 14 + 0 = 0$, $0 - 0 + 0 + 0 = 0$, $55 - 78 + 21 + 2 = 0$.

Every entry is 0, so $A^3 - 6A^2 + 7A + 2I = O$.

Final Answer: $A^3 - 6A^2 + 7A + 2I = O$, the 3×3 zero matrix.

☞ **Exam Tip**

The relation $A^3 - 6A^2 + 7A + 2I = O$ is the characteristic equation of A (Cayley-Hamilton). You can use it to express A^{-1} in terms of lower powers of A, a trick repeatedly seen in JEE problems.

EXPERT'S SOLUTION : *Aditya Bhat, Ph.D Mathematics, IIT Delhi*

Picture-first. Two products to compute (A^2, A^3), then a plain entry-wise combination. The structure tells us A satisfies its own characteristic polynomial $\lambda^3 - 6\lambda^2 + 7\lambda + 2$.

Step 1. A^2 via 9 dot products (above).

Step 2. $A^3 = A \cdot A^2$ via 9 more dot products (above).

Step 3. Form $A^3 - 6A^2 + 7A + 2I$ entry-wise. The diagonal entries each sum to 0 because A obeys its characteristic equation.

Step 4. Verify off-diagonal $(1, 3)$ entry: $34 - 48 + 14 + 0 = 0$ ✓.

Why this matters. A matrix satisfies its own characteristic polynomial (Cayley-Hamilton). This lets you compute powers of A as linear combinations of I, A, A^2, and is the standard route to A^{-1} when known to exist.

Final Answer: O.

Q 3.17 **If** $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ **and** $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, **find k so that** $A^2 = kA - 2I$.

SOLUTION

Concept used. Compute A^2, match it against $kA - 2I$ entry-wise, and solve for k.

Step 1. Compute A^2: $(A^2)_{11} = 3(3) + (-2)(4) = 9 - 8 = 1$.
$(A^2)_{12} = 3(-2) + (-2)(-2) = -6 + 4 = -2$.
$(A^2)_{21} = 4(3) + (-2)(4) = 12 - 8 = 4$.
$(A^2)_{22} = 4(-2) + (-2)(-2) = -8 + 4 = -4$.

$$A^2 = \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix}.$$

Step 2. Compute $kA - 2I = \begin{bmatrix} 3k-2 & -2k \\ 4k & -2k-2 \end{bmatrix}$.

Step 3. Equate entries:

$$(1,1): \ 3k - 2 = 1 \Rightarrow k = 1.$$

$$(1,2): \ -2k = -2 \Rightarrow k = 1.$$

$$(2,1): \ 4k = 4 \Rightarrow k = 1.$$

$$(2,2): \ -2k - 2 = -4 \Rightarrow -2k = -2 \Rightarrow k = 1.$$

Step 4. All four entries give the same value $k = 1$.

> **Final Answer:** $k = 1$.

EXPERT'S SOLUTION : *Ishaan Kapoor, M.Sc Mathematics, ISI Kolkata*

Quick reading. Compute A^2, then "divide" the equation $A^2 + 2I = kA$ entry-wise to read off k.

Step 1. $A^2 = \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix}$.

Step 2. $A^2 + 2I = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} = A$.

Step 3. So $kA = A$, giving $k = 1$.

Why this matters. The relation $A^2 = A - 2I + 2I = \ldots$ is Hamilton's identity for A in disguise, the same idea as Q16.

> **Final Answer:** $k = 1$.

Q 3.18 **If** $A = \begin{bmatrix} 0 & -\tan(\alpha/2) \\ \tan(\alpha/2) & 0 \end{bmatrix}$ **and** I **is the identity matrix of order** 2**, show that** $I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

SOLUTION

Concept used. Use the half-angle identities: $\cos\alpha = \dfrac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)}$ and $\sin\alpha = \dfrac{2\tan(\alpha/2)}{1+\tan^2(\alpha/2)}$.

Step 1. Set $t = \tan(\alpha/2)$ for brevity. Then $A = \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix}$, $I + A = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}$,

$$I - A = \begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}.$$

Step 2. Compute the RHS product $(I-A)R$ where $R = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$:

$$(1,1):\ 1\cos\alpha + t\sin\alpha.$$

$$(1,2):\ 1(-\sin\alpha) + t\cos\alpha = t\cos\alpha - \sin\alpha.$$

$$(2,1):\ -t\cos\alpha + 1\sin\alpha = \sin\alpha - t\cos\alpha.$$

$$(2,2):\ -t(-\sin\alpha) + 1\cos\alpha = t\sin\alpha + \cos\alpha.$$

Step 3. Substitute the half-angle formulas. With $s = 1 + t^2$: $\cos\alpha = \dfrac{1-t^2}{s}$, $\sin\alpha = \dfrac{2t}{s}$.

Step 4. Entry $(1,1)$: $\cos\alpha + t\sin\alpha = \dfrac{1-t^2}{s} + t\cdot\dfrac{2t}{s} = \dfrac{1-t^2+2t^2}{s} = \dfrac{1+t^2}{s} = 1.$

Step 5. Entry $(1,2)$:

$$t\cos\alpha - \sin\alpha = \frac{t(1-t^2)}{s} - \frac{2t}{s} = \frac{t-t^3-2t}{s} = \frac{-t-t^3}{s} = \frac{-t(1+t^2)}{s} = -t.$$

Step 6. Entry $(2,1)$: $\sin\alpha - t\cos\alpha = \dfrac{2t}{s} - \dfrac{t(1-t^2)}{s} = \dfrac{2t-t+t^3}{s} = \dfrac{t+t^3}{s} = t.$

Step 7. Entry $(2,2)$: $t\sin\alpha + \cos\alpha = \dfrac{2t^2}{s} + \dfrac{1-t^2}{s} = \dfrac{t^2+1}{s} = 1.$

Step 8. Assemble: $(I-A)R = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix} = I + A$. Done.

Final Answer: $I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, proved using $\cos\alpha = \frac{1-t^2}{1+t^2}$, $\sin\alpha = \frac{2t}{1+t^2}$ with $t = \tan(\alpha/2)$.

♥ Why This Matters

The matrix $\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$ is the standard plane rotation by α. The identity links half-angle algebra to rotations and is closely related to the Cayley transform.

EXPERT'S SOLUTION *: Rahul Verma, Ph.D Pure Mathematics, IISc Bangalore*

Structural observation. Set $t = \tan(\alpha/2)$; the algebra reduces to verifying two identities, $(1+t^2)\cdot$ stuff.

Step 1. $I + A,\ I - A$ are $\begin{bmatrix}1 & -t\\ t & 1\end{bmatrix}, \begin{bmatrix}1 & t\\ -t & 1\end{bmatrix}$.

Step 2. Multiply $(I - A)R$ keeping t symbolic. The four entries become $\frac{1+t^2}{1+t^2} = 1$ (diagonal) and $\pm\frac{t(1+t^2)}{1+t^2} = \pm t$ (off-diagonal), matching $I + A$.

Step 3. No need to expand fully; the $1 + t^2$ factor cancels cleanly.

Why this matters. "Half-angle" identities are a coordinate choice that simplify rotation matrices. The same parameter t appears in stereographic projection and Cayley transform contexts.

Final Answer: Identity verified entry-by-entry.

Q 3.19 **A trust fund has Rs. 30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide Rs. 30,000 among the two types of bonds, if the trust fund must obtain an annual total interest of (a) Rs. 1800, (b) Rs. 2000.**

SOLUTION

Concept used. Let x be the amount invested in the 5% bond and $30000 - x$ in the 7% bond. The total interest is the matrix product

$$\begin{bmatrix}x & 30000 - x\end{bmatrix}\begin{bmatrix}5/100\\ 7/100\end{bmatrix} = \text{(total interest)}.$$

Step 1. **(a) Target interest Rs. 1800.** The matrix equation is

$$\begin{bmatrix}x & 30000 - x\end{bmatrix}\begin{bmatrix}0.05\\ 0.07\end{bmatrix} = 1800.$$

Multiply out:

$$0.05x + 0.07(30000 - x) = 1800.$$

Expand:

$$0.05x + 2100 - 0.07x = 1800.$$

Combine: $-0.02x = 1800 - 2100 = -300.$

$$x = \frac{-300}{-0.02} = 15000.$$

Then $30000 - x = 15000.$

Allocation: Rs. 15,000 in 5% bond and Rs. 15,000 in 7% bond.

Step 2. **(b) Target interest Rs. 2000.** Same setup:

$$0.05x + 0.07(30000 - x) = 2000.$$

$$0.05x + 2100 - 0.07x = 2000.$$

$$-0.02x = -100 \Rightarrow x = \frac{-100}{-0.02} = 5000.$$

Then $30000 - x = 25000.$

Allocation: Rs. 5,000 in 5% bond and Rs. 25,000 in 7% bond.

Step 3. Verify (a): $0.05(15000) + 0.07(15000) = 750 + 1050 = 1800$ ✓.
Verify (b): $0.05(5000) + 0.07(25000) = 250 + 1750 = 2000$ ✓.

Final Answer: (a) Rs. 15,000 + Rs. 15,000.
(b) Rs. 5,000 in 5% bond, Rs. 25,000 in 7% bond.

EXPERT'S SOLUTION *: Nikhil Sharma, M.Tech CS, IIT Madras*

Strategic angle. The matrix equation $[x,\ 30000 - x] \cdot [0.05,\ 0.07]^T = I$ reduces to a single linear equation in x. Solve, then read off $30000 - x$.

Step 1. Set x = amount in the 5% bond. The interest equation is $0.05x + 0.07(30000 - x) = I$ where I is the target.

Step 2. Simplify: $2100 - 0.02x = I$, so $x = \dfrac{2100 - I}{0.02}$.

Step 3. (a) $I = 1800$: $x = \dfrac{300}{0.02} = 15000$. (b) $I = 2000$: $x = \dfrac{100}{0.02} = 5000$.

Why this matters. Word problems on portfolio allocation, inventory totals, and revenue all reduce to linear equations of this type. Matrix notation lets you stack many such equations into $AX = B$.

Final Answer: (a) 15,000/15,000. (b) 5,000/25,000.

Q 3.20 **The bookshop of a particular school has 10 dozen chemistry books, 8 dozen physics books and 10 dozen economics books. Their selling prices are Rs. 80, Rs. 60 and Rs. 40 each respectively. Find the total amount the bookshop will receive from selling all the books using matrix algebra.**

SOLUTION

Concept used. Encode quantities (in copies, not dozens) as a 1×3 row matrix and prices as a 3×1 column matrix. The 1×1 product is the total revenue.

Step 1. Convert dozens to copies (one dozen $= 12$ copies):

$$10 \text{ dozen} = 120,\ 8 \text{ dozen} = 96,\ 10 \text{ dozen} = 120.$$

Step 2. Quantities row matrix $Q = [120\ \ 96\ \ 120]$ and price column $P = \begin{bmatrix} 80 \\ 60 \\ 40 \end{bmatrix}$.

Step 3. Compute the revenue QP:

$$QP = 120(80) + 96(60) + 120(40).$$

Step 4. Term by term: $120 \times 80 = 9600$;
$96 \times 60 = 5760$;
$120 \times 40 = 4800$.

Step 5. Total: $9600 + 5760 + 4800 = 20160$.

Final Answer: Total revenue $=$ Rs. $20{,}160$.

EXPERT'S SOLUTION *: Dev Iyer, M.Sc Mathematics, IIT Bombay*

Quick reading. Quantities $\times$ prices is the textbook example of a matrix product modeling a real situation.

Step 1. Quantities (copies): $[120, 96, 120]$.

Step 2. Prices: $[80, 60, 40]^T$.

Step 3. Product: $120 \cdot 80 + 96 \cdot 60 + 120 \cdot 40 = 9600 + 5760 + 4800 = 20160$.

Why this matters. The matrix way of writing $\sum_i q_i p_i$ generalises to many books in many shops, where the quantity becomes a matrix and the price becomes a column.

Final Answer: Rs. 20,160.

Q 3.21 **Assume X, Y, Z, W, P are matrices of order $2 \times n,\ 3 \times k,\ 2 \times p,\ n \times 3,\ p \times k$, respectively. The restriction on n, k and p so that $PY + WY$ will be defined are:**
(A) $k = 3,\ p = n$ **(B)** k **is arbitrary,** $p = 2$ **(C)** p **is arbitrary,** $k = 3$ **(D)** $k = 2,\ p = 3$**.**

SOLUTION

Concept used. For AB to be defined, the number of columns of A must equal the number of rows of B. For $A + B$ to be defined, A and B must have the same order.

Step 1. P is $p \times k$, Y is $3 \times k$. For PY to be defined, columns of P $(= k)$ must equal rows of Y $(= 3)$, so $k = 3$. Then PY has order $p \times k = p \times 3$.

Step 2. W is $n \times 3$, Y is $3 \times k$. For WY, columns of W $(= 3)$ match rows of Y $(= 3)$ automatically. Then WY has order $n \times k = n \times 3$ (since $k = 3$).

Step 3. For $PY + WY$ to be defined, the orders must match: $p \times 3 = n \times 3$, so $p = n$.

Step 4. Combine: $k = 3$ and $p = n$, which is option (A).

Final Answer: Correct answer: (A) $k = 3,\ p = n$.

EXPERT'S SOLUTION *: Pooja Patel, B.Tech Electrical Engineering, IIT Bombay*

Quick reading. Two compatibility conditions: multiplication (PY defined) gives $k = 3$; addition ($PY + WY$ defined) gives $p = n$.

Step 1. Multiplication: $P_{p\times k}Y_{3\times k}$ requires $k = 3$.

Step 2. Resulting orders: PY is $p \times 3$, WY is $n \times 3$.

Step 3. Addition: $p \times 3 = n \times 3$ forces $p = n$.

Step 4. Match with options: (A) is exactly this.

Why this matters. Quick order-checks save tons of arithmetic in exam settings.

Final Answer: (A).

Q 3.22 **Assume X, Y, Z, W, P are as in Q21. If $n = p$, then the order of the matrix**

$7X - 5Z$ **is:**

(A) $p \times 2$ **(B)** $2 \times n$ **(C)** $n \times 3$ **(D)** $p \times n$.

SOLUTION

Concept used. kA has the same order as A. $A - B$ requires A and B to have the same order, and the result has that same order.

Step 1. X has order $2 \times n$; $7X$ has order $2 \times n$.

Step 2. Z has order $2 \times p$; $5Z$ has order $2 \times p$.

Step 3. For $7X - 5Z$ to be defined, $2 \times n = 2 \times p$, so $n = p$ (given).

Step 4. Common order: $2 \times n$. Among the choices, that is (B).

Final Answer: Correct answer: (B) $2 \times n$.

EXPERT'S SOLUTION *: Aditya Singh, M.Tech CS, IIT Madras*

Quick reading. Scalars preserve order; subtraction needs matching orders; result keeps that order.

Step 1. $7X$ is $2 \times n$, $5Z$ is $2 \times p = 2 \times n$ (since $n = p$).

Step 2. Their difference is $2 \times n$.

Step 3. Option (B).

Why this matters. Tracking "order" is the cheapest sanity check; nail it before any arithmetic.

Final Answer: (B) $2 \times n$.

Key Takeaways

- Matrix addition and subtraction are entry-wise and require equal order. Scalar multiplication scales every entry.
- For $A_{m\times n}B_{n\times p}$, the entry $(AB)_{ij} = \sum_k a_{ik}b_{kj}$ is the dot product of row i of A with column j of B. Inner dimensions must match.
- Matrix multiplication is non-commutative: $AB \neq BA$ in general.
- For matrix equations $aX + bY = C$, treat X, Y like scalar unknowns: eliminate one by linear combination of the given equations.
- Real-world quantities $\times$ price columns model revenue as a matrix product; quick to set up and to evaluate.

End of Exercise 3.2