Collegedunia NCERT Solutions

Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 3: Matrices

About this Chapter

Exercise 3.3 introduces the **transpose** A' (or A^T) of a matrix — obtained by interchanging rows and columns — and uses it to define two key families: **symmetric** matrices ($A' = A$) and **skew-symmetric** matrices ($A' = -A$). The exercise verifies the four properties of transpose and proves that every square matrix is the sum of a symmetric and a skew-symmetric matrix.

Topics covered: Transpose of a matrix • $(A+B)' = A' + B'$ • $(AB)' = B'A'$ • Symmetric & skew-symmetric • Decomposition $A = P + Q$ • Orthogonal matrices

Quick Formula Sheet

Definition of transpose: $(A')_{ij} = a_{ji}$; an $m \times n$ matrix transposes to $n \times m$.

Properties of transpose: $(A')' = A$, $(kA)' = kA'$, $(A + B)' = A' + B'$, $(AB)' = B'A'$.

Symmetric/skew: A is symmetric iff $A' = A$. Skew-symmetric iff $A' = -A$.

Canonical decomposition: $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$.

Exercise 3.3

Q 3.1 **Find the transpose of each of the following matrices:**

(i) $\begin{bmatrix} 5 \\ \frac{1}{2} \\ -1 \end{bmatrix}$, **(ii)** $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, **(iii)** $\begin{bmatrix} -1 & 5 & 6 \\ \sqrt{3} & 5 & 6 \\ 2 & 3 & -1 \end{bmatrix}$.

SOLUTION

Concept used. The **transpose** of a matrix A, written A' (or A^T), is obtained by interchanging the rows and columns: row i of A becomes column i of A', equivalently $(A')_{ij} = a_{ji}$. An $m \times n$ matrix transposes to $n \times m$.

Step 1. **(i)** The given matrix is 3×1 (a column). Its transpose is 1×3 (a row):

$$A' = [5 \;\; \tfrac{1}{2} \;\; -1].$$

Step 2. **(ii)** The matrix is 2×2. Swap rows for columns: row 1 $\to$ column 1, row 2 $\to$ column 2.

$$A' = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}.$$

Step 3. **(iii)** The matrix is 3×3. Row $1 = (-1, 5, 6)$ becomes column 1; row $2 = (\sqrt{3}, 5, 6)$ becomes column 2; row $3 = (2, 3, -1)$ becomes column 3.

$$A' = \begin{bmatrix} -1 & \sqrt{3} & 2 \\ 5 & 5 & 3 \\ 6 & 6 & -1 \end{bmatrix}.$$

Final Answer: (i) $[5 \;\; \tfrac{1}{2} \;\; -1]$, (ii) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, (iii) $\begin{bmatrix} -1 & \sqrt{3} & 2 \\ 5 & 5 & 3 \\ 6 & 6 & -1 \end{bmatrix}$.

EXPERT'S SOLUTION *: Aarav Iyer, M.Sc Mathematics, IIT Bombay*

Quick reading. "Transpose" is the mirror-flip about the main diagonal. Pick up each row and stand it up as a column.

Step 1. Column $\to$ row, row $\to$ column. (i) Column of length 3 becomes a row of length 3.

Step 2. (ii) The 2×2 entries swap across the diagonal: the $(1, 2) = -1$ moves to $(2, 1)$; the $(2, 1) = 2$ moves to $(1, 2)$.

Step 3. (iii) Same idea, applied to all six off-diagonal entries. Diagonal entries stay put.

Why this matters. The transpose is the bridge from rows to columns and back; almost every later identity is built on it.

Final Answer: Same matrices as the main solution.

Q 3.2 **If** $A = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$**, verify that**
(i) $(A + B)' = A' + B'$**,** **(ii)** $(A - B)' = A' - B'$**.**

SOLUTION

Concept used. The transpose distributes over addition and subtraction: $(A \pm B)' = A' \pm B'$. We will verify this directly by computing both sides.

Step 1. **Compute $A + B$.** Row 1: $-1-4=-5,\ 2+1=3,\ 3-5=-2$.
Row 2: $5+1=6,\ 7+2=9,\ 9+0=9$.
Row 3: $-2+1=-1,\ 1+3=4,\ 1+1=2$.

$$A + B = \begin{bmatrix} -5 & 3 & -2 \\ 6 & 9 & 9 \\ -1 & 4 & 2 \end{bmatrix}.$$

Step 2. Transpose:

$$(A+B)' = \begin{bmatrix} -5 & 6 & -1 \\ 3 & 9 & 4 \\ -2 & 9 & 2 \end{bmatrix}.$$

Step 3. **Compute A' and B'.** $A' = \begin{bmatrix} -1 & 5 & -2 \\ 2 & 7 & 1 \\ 3 & 9 & 1 \end{bmatrix}$, $B' = \begin{bmatrix} -4 & 1 & 1 \\ 1 & 2 & 3 \\ -5 & 0 & 1 \end{bmatrix}$.

Step 4. **Add $A' + B'$.** Row 1: $-1-4=-5,\ 5+1=6,\ -2+1=-1$.
Row 2: $2+1=3,\ 7+2=9,\ 1+3=4$.
Row 3: $3-5=-2,\ 9+0=9,\ 1+1=2$.

$$A' + B' = \begin{bmatrix} -5 & 6 & -1 \\ 3 & 9 & 4 \\ -2 & 9 & 2 \end{bmatrix}.$$

Matches $(A+B)'$. ✓

Step 5. **Now $A - B$.** Row 1: $-1-(-4)=3,\ 2-1=1,\ 3-(-5)=8$.
Row 2: $5-1=4,\ 7-2=5,\ 9-0=9$.
Row 3: $-2-1=-3,\ 1-3=-2,\ 1-1=0$.

$$A - B = \begin{bmatrix} 3 & 1 & 8 \\ 4 & 5 & 9 \\ -3 & -2 & 0 \end{bmatrix}.$$

Step 6. Transpose:

$$(A-B)' = \begin{bmatrix} 3 & 4 & -3 \\ 1 & 5 & -2 \\ 8 & 9 & 0 \end{bmatrix}.$$

Step 7. Compute $A' - B'$ entry-wise: Row 1: $-1-(-4)=3,\ 5-1=4,\ -2-1=-3$.
Row 2: $2-1=1,\ 7-2=5,\ 1-3=-2$.

Row 3: $3-(-5)=8,\ 9-0=9,\ 1-1=0.$

$$A'-B'=\begin{bmatrix}3 & 4 & -3\\ 1 & 5 & -2\\ 8 & 9 & 0\end{bmatrix}.$$

Matches $(A-B)'$. ✓

Final Answer: Both identities verified: $(A+B)'=A'+B'$ and $(A-B)'=A'-B'$.

EXPERT'S SOLUTION *: Diya Sharma, M.Sc Mathematics, ISI Kolkata*

Structural observation. Transpose is an entry-wise re-indexing, and addition is entry-wise, so they commute. The verification is purely arithmetic.

Step 1. Compute $A+B$ first, then transpose: $(A+B)'$.

Step 2. Compute $A',\ B'$ first, then add: $A'+B'$.

Step 3. Compare: identical 3×3 result.

Step 4. Repeat the routine for $A-B$ and $A'-B'$.

Why this matters. "Transpose distributes over sums" is the algebraic root of the proof that the sum of two symmetric matrices is symmetric.

Final Answer: Both identities verified entry-by-entry.

Q 3.3 **If** $A'=\begin{bmatrix}3 & 4\\ -1 & 2\\ 0 & 1\end{bmatrix}$ **and** $B=\begin{bmatrix}-1 & 2 & 1\\ 1 & 2 & 3\end{bmatrix}$**, verify that**
(i) $(A+B)'=A'+B'$**,** **(ii)** $(A-B)'=A'-B'$**.**

SOLUTION

Concept used. Same as Q2: $(A\pm B)'=A'\pm B'$. Here A' is given, so first recover $A=(A')'$ (transpose the given A').

Step 1. Transpose the given A' to get A:

$$A=\begin{bmatrix}3 & -1 & 0\\ 4 & 2 & 1\end{bmatrix}.$$

(A is 2×3, matching B's order.)

Step 2. **Compute** $A + B$. Row 1: $3 - 1 = 2,\ -1 + 2 = 1,\ 0 + 1 = 1.$
Row 2: $4 + 1 = 5,\ 2 + 2 = 4,\ 1 + 3 = 4.$

$$A + B = \begin{bmatrix} 2 & 1 & 1 \\ 5 & 4 & 4 \end{bmatrix}.$$

Step 3. Transpose:

$$(A + B)' = \begin{bmatrix} 2 & 5 \\ 1 & 4 \\ 1 & 4 \end{bmatrix}.$$

Step 4. B': transpose of B is $B' = \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$.

Step 5. $A' + B'$ (both are 3×2): Row 1: $3 - 1 = 2,\ 4 + 1 = 5.$
Row 2: $-1 + 2 = 1,\ 2 + 2 = 4.$
Row 3: $0 + 1 = 1,\ 1 + 3 = 4.$

$$A' + B' = \begin{bmatrix} 2 & 5 \\ 1 & 4 \\ 1 & 4 \end{bmatrix}.$$

Equal to $(A + B)'$. ✓

Step 6. **Compute** $A - B$. Row 1: $3 - (-1) = 4,\ -1 - 2 = -3,\ 0 - 1 = -1.$
Row 2: $4 - 1 = 3,\ 2 - 2 = 0,\ 1 - 3 = -2.$

$$A - B = \begin{bmatrix} 4 & -3 & -1 \\ 3 & 0 & -2 \end{bmatrix}.$$

Step 7. Transpose:

$$(A - B)' = \begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix}.$$

Step 8. Compute $A' - B'$: Row 1: $3 - (-1) = 4,\ 4 - 1 = 3.$
Row 2: $-1 - 2 = -3,\ 2 - 2 = 0.$
Row 3: $0 - 1 = -1,\ 1 - 3 = -2.$

$$A' - B' = \begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix}.$$

Equal to $(A - B)'$. ✓

Final Answer: Both verified.

EXPERT'S SOLUTION : *Anika Verma, M.Sc Applied Mathematics, IIT Kanpur*

Quick reading. Same algebra as Q2, but read $A = (A')'$ first since the question gives A'.

Step 1. Recover A from the given A' (a 2×3 matrix).

Step 2. Carry out $A \pm B$ and transpose; carry out $A' \pm B'$.

Step 3. Confirm equality entry-by-entry.

Why this matters. $(A')' = A$ is one of the four fundamental transpose laws; it means transpose is its own inverse.

> **Final Answer:** Both identities verified.

Q 3.4 **If** $A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$ **and** $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$**, then find** $(A + 2B)'$**.**

SOLUTION

Concept used. Use $(A + 2B)' = A' + 2B'$ (transpose distributes over sum and scalar). We are given A', and B' is easy to compute.

Step 1. Compute B' by transposing B:

$$B' = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}.$$

Step 2. Compute $2B'$:

$$2B' = \begin{bmatrix} -2 & 2 \\ 0 & 4 \end{bmatrix}.$$

Step 3. Add $A' + 2B'$ entry-wise:

$$A' + 2B' = \begin{bmatrix} -2 + (-2) & 3 + 2 \\ 1 + 0 & 2 + 4 \end{bmatrix} = \begin{bmatrix} -4 & 5 \\ 1 & 6 \end{bmatrix}.$$

Step 4. By the distributive property of transpose, $(A + 2B)' = A' + 2B'$, so

$$(A + 2B)' = \begin{bmatrix} -4 & 5 \\ 1 & 6 \end{bmatrix}.$$

> **Final Answer:** $(A + 2B)' = \begin{bmatrix} -4 & 5 \\ 1 & 6 \end{bmatrix}$.

EXPERT'S SOLUTION : *Karan Nair, M.Tech CS, IIT Madras*

Quick reading. Skip computing A entirely; just use $(A+2B)' = A' + 2B'$ and we have $A',\ B'$ in hand.

Step 1. $B' = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$; $2B' = \begin{bmatrix} -2 & 2 \\ 0 & 4 \end{bmatrix}$.

Step 2. Add $A' + 2B'$ entry-wise to get $\begin{bmatrix} -4 & 5 \\ 1 & 6 \end{bmatrix}$.

Why this matters. Recognising you can stay in the transposed world saves a full pair of transpose operations.

Final Answer: $\begin{bmatrix} -4 & 5 \\ 1 & 6 \end{bmatrix}$.

Q 3.5 **For the matrices A and B, verify that $(AB)' = B'A'$, where:**

(i) $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$, $B = [-1\ \ 2\ \ 1]$.

(ii) $A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $B = [1\ \ 5\ \ 7]$.

SOLUTION

Concept used. The reversal law for the transpose of a product: $\boxed{(AB)' = B'A'.}$ Verify by computing both sides directly.

Step 1. **(i) Compute** AB. A is 3×1, B is 1×3; AB is 3×3 with entries $(AB)_{ij} = a_i b_j$:

$$AB = \begin{bmatrix} 1(-1) & 1(2) & 1(1) \\ -4(-1) & -4(2) & -4(1) \\ 3(-1) & 3(2) & 3(1) \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix}.$$

Step 2. Transpose:

$$(AB)' = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}.$$

Step 3. Compute $B'A'$. $B' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ (3×1), and $A' = [1\ \ -4\ 3]$ (1×3). Their product is 3×3 with entries $b_i'a_j'$:

$$B'A' = \begin{bmatrix} -1\cdot 1 & -1(-4) & -1\cdot 3 \\ 2\cdot 1 & 2(-4) & 2\cdot 3 \\ 1\cdot 1 & 1(-4) & 1\cdot 3 \end{bmatrix} = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}.$$

Equal to $(AB)'$. ✓

Step 4. **(ii) Compute AB.** Same shape: 3×3 with a_ib_j.

$$AB = \begin{bmatrix} 0(1) & 0(5) & 0(7) \\ 1(1) & 1(5) & 1(7) \\ 2(1) & 2(5) & 2(7) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 5 & 7 \\ 2 & 10 & 14 \end{bmatrix}.$$

Step 5. Transpose:

$$(AB)' = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 5 & 10 \\ 0 & 7 & 14 \end{bmatrix}.$$

Step 6. Compute $B'A'$. $B' = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$, $A' = [0\ 1\ 2]$:

$$B'A' = \begin{bmatrix} 1\cdot 0 & 1\cdot 1 & 1\cdot 2 \\ 5\cdot 0 & 5\cdot 1 & 5\cdot 2 \\ 7\cdot 0 & 7\cdot 1 & 7\cdot 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 5 & 10 \\ 0 & 7 & 14 \end{bmatrix}.$$

Equal to $(AB)'$. ✓

Final Answer: $(AB)' = B'A'$ verified in both (i) and (ii).

♥ Why This Matters

The order *reverses*: $(AB)' = B'A'$, not $A'B'$. Without the reversal, the products would even fail to be defined when A, B have non-square orders.

EXPERT'S SOLUTION : *Sneha Joshi, M.Sc Mathematics, IIT Bombay*

Structural observation. For a column-times-row pair, the product is a *rank-1* outer product ab^T. Transposing it gives $(ab^T)^T = ba^T$, matching the formula $(AB)' = B'A'$.

Step 1. (i) Outer product AB has entries a_ib_j. Transposing flips indices: a_jb_i. Equivalently $B'A'$.

Step 2. (ii) Same routine; one row of zeros at the top because $a_1 = 0$.

Why this matters. Outer products ab^T are everywhere in linear algebra (rank-1 updates, gradient outer products in ML, dyadics in mechanics). The transpose law makes their algebra easy.

Final Answer: Both sides agree in (i) and (ii).

Q 3.6 Verify that $A'A = I$ in each case:
(i) $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, **(ii)** $A = \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}$.

SOLUTION

Concept used. A square matrix A with $A'A = I$ is called **orthogonal**. To verify, compute $A'A$ entry-by-entry and use $\sin^2\alpha + \cos^2\alpha = 1$.

Step 1. **(i) Compute A':**

$$A' = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

Step 2. Compute $A'A$. Rows of A': $R_1 = (\cos\alpha, -\sin\alpha)$, $R_2 = (\sin\alpha, \cos\alpha)$. Columns of A: $C_1 = (\cos\alpha, -\sin\alpha)$, $C_2 = (\sin\alpha, \cos\alpha)$.

$$(A'A)_{11} = \cos^2\alpha + \sin^2\alpha = 1.$$

$$(A'A)_{12} = \cos\alpha\sin\alpha - \sin\alpha\cos\alpha = 0.$$
$$(A'A)_{21} = \sin\alpha\cos\alpha - \cos\alpha\sin\alpha = 0.$$
$$(A'A)_{22} = \sin^2\alpha + \cos^2\alpha = 1.$$
$$A'A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Step 3. **(ii) Compute A':**

$$A' = \begin{bmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{bmatrix}.$$

Step 4. Compute $A'A$. Rows of A': $R_1 = (\sin\alpha, -\cos\alpha)$, $R_2 = (\cos\alpha, \sin\alpha)$. Columns of A: $C_1 = (\sin\alpha, -\cos\alpha)$, $C_2 = (\cos\alpha, \sin\alpha)$.

$$(A'A)_{11} = \sin^2\alpha + \cos^2\alpha = 1.$$

$$(A'A)_{12} = \sin\alpha\cos\alpha - \cos\alpha\sin\alpha = 0.$$
$$(A'A)_{21} = \cos\alpha\sin\alpha - \sin\alpha\cos\alpha = 0.$$
$$(A'A)_{22} = \cos^2\alpha + \sin^2\alpha = 1.$$
$$A'A = I.$$

Final Answer: In both (i) and (ii), $A'A = I$: the matrices are orthogonal.

☞ Exam Tip

Orthogonal matrices preserve length and angles — they are the rigid rotations/reflections of the plane. Spotting an orthogonal matrix unlocks $A^{-1} = A'$ instantly.

EXPERT'S SOLUTION *: Rohit Kapoor, Ph.D Mathematics, IIT Delhi*

Structural observation. Both matrices are rotation/reflection in disguise. For an orthogonal matrix, columns are orthonormal: $C_i \cdot C_j = \delta_{ij}$. Verify just that.

Step 1. (i) Columns of A: $C_1 = (\cos\alpha, -\sin\alpha)^T$, $C_2 = (\sin\alpha, \cos\alpha)^T$.
$\|C_1\|^2 = \cos^2\alpha + \sin^2\alpha = 1$; $\|C_2\|^2 = \sin^2\alpha + \cos^2\alpha = 1$;
$C_1 \cdot C_2 = \cos\alpha\sin\alpha - \sin\alpha\cos\alpha = 0$. Hence $A'A = I$.

Step 2. (ii) Columns: $(\sin\alpha, -\cos\alpha)^T$, $(\cos\alpha, \sin\alpha)^T$. Same calculations confirm orthonormality, so $A'A = I$.

Why this matters. Once $A'A = I$, you instantly have $A^{-1} = A'$, avoiding a full inverse computation.

Final Answer: $A'A = I$ in both parts.

Q 3.7 **(i) Show that A is a symmetric matrix, where A is the matrix below in part (a). (ii) Show that A is a skew-symmetric matrix, where A is the matrix below in part (b).**
(a) $A = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix}$, **(b)** $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$.

SOLUTION

Concept used. A is symmetric iff $A' = A$, i.e. $a_{ij} = a_{ji}$ for all i, j. A is skew-symmetric iff $A' = -A$, i.e. $a_{ij} = -a_{ji}$ for all i, j. In particular, a skew-symmetric matrix has zero diagonal entries (since $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$).

Step 1. **(i) Compute A'.** Row i of A becomes column i of A'.

$$A = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix}, \quad A' = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix}.$$

Step 2. $A' = A$ entry-by-entry: $(1,2) = -1 = (2,1)$; $(1,3) = 5 = (3,1)$; $(2,3) = 1 = (3,2)$. Diagonals unchanged. Hence A is symmetric.

Step 3. **(ii) Compute A'.**

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad A' = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Step 4. Compare A' with $-A$:

$$-A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Match: $A' = -A$. Hence A is skew-symmetric.

> **Final Answer:** (i) $A' = A$, so A is symmetric. (ii) $A' = -A$, so A is skew-symmetric.

EXPERT'S SOLUTION : *Aanya Mehta, M.Sc Mathematics, IIT Bombay*

Quick reading. For symmetry, mirror across the diagonal. For skew-symmetry, mirror with a sign flip and zero on the diagonal.

Step 1. (i) Check the three off-diagonal mirror pairs: $(1,2)$ & $(2,1)$ both -1; $(1,3)$ & $(3,1)$ both 5; $(2,3)$ & $(3,2)$ both 1. Symmetric.

Step 2. (ii) Check the diagonal is zero (it is), and that mirror pairs differ by a sign: $(1,2) = 1, (2,1) = -1$; $(1,3) = -1, (3,1) = 1$; $(2,3) = 1, (3,2) = -1$. Skew-symmetric.

Why this matters. Symmetric matrices appear as covariance matrices in statistics, stiffness matrices in mechanics, and Hessians in calculus. Skew-symmetric matrices encode cross products and infinitesimal rotations.

> **Final Answer:** (i) symmetric; (ii) skew-symmetric.

Q 3.8 **For the matrix $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$, verify that**
(i) $A + A'$ is a symmetric matrix, **(ii) $A - A'$ is a skew-symmetric matrix.**

SOLUTION

Concept used. For any square matrix A: $(A + A')' = A' + A = A + A'$, so $A + A'$ is symmetric. Similarly $(A - A')' = A' - A = -(A - A')$, so $A - A'$ is skew-symmetric.

Step 1. Compute A': $A' = \begin{bmatrix} 1 & 6 \\ 5 & 7 \end{bmatrix}$.

Step 2. **(i) Compute $A + A'$.**

$$A + A' = \begin{bmatrix} 1+1 & 5+6 \\ 6+5 & 7+7 \end{bmatrix} = \begin{bmatrix} 2 & 11 \\ 11 & 14 \end{bmatrix}.$$

Off-diagonal entries $11 = 11$ match, so $A + A'$ is symmetric.

Step 3. **(ii) Compute $A - A'$.**

$$A - A' = \begin{bmatrix} 1-1 & 5-6 \\ 6-5 & 7-7 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Diagonal is zero. Off-diagonal: $(1,2) = -1 = -(2,1)$. So $A - A'$ is skew-symmetric.

> **Final Answer:** (i) $A + A' = \begin{bmatrix} 2 & 11 \\ 11 & 14 \end{bmatrix}$ is symmetric. (ii) $A - A' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is skew-symmetric.

EXPERT'S SOLUTION *: Vivaan Pillai, M.Sc Applied Mathematics, IIT Kanpur*

Structural observation. The identity $(A \pm A')' = A' \pm A$ gives both claims for free: the sum equals itself transposed, the difference equals minus its transpose.

Step 1. Compute A' by swapping the off-diagonal $5 \leftrightarrow 6$.

Step 2. Add: doubles the diagonal and symmetrises the off-diagonal.

Step 3. Subtract: zeros the diagonal and antisymmetrises the off-diagonal.

Why this matters. This is the engine behind the canonical decomposition $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$ used in Q10.

> **Final Answer:** Both properties verified.

Q 3.9 **Find $\frac{1}{2}(A + A')$ and $\frac{1}{2}(A - A')$, when $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$.**

SOLUTION

Concept used. $\frac{1}{2}(A + A')$ is the *symmetric part* of A; $\frac{1}{2}(A - A')$ is the *skew-symmetric part*. For any square A:

$$A = \tfrac{1}{2}(A + A') + \tfrac{1}{2}(A - A').$$

Step 1. Compute A'. Row $1 = (0, a, b)$ becomes column 1; row $2 = (-a, 0, c)$ becomes column 2; row $3 = (-b, -c, 0)$ becomes column 3:

$$A' = \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}.$$

Step 2. Notice $A' = -A$ entry-by-entry: $(1, 2) : a$ in A, $-a$ in A'; $(2, 1) : -a$ in A, a in A'; etc. So A is already skew-symmetric.

Step 3. Compute $A + A'$: Row 1: $0 + 0 = 0,\ a + (-a) = 0,\ b + (-b) = 0.$
Row 2: $-a + a = 0,\ 0 + 0 = 0,\ c + (-c) = 0.$
Row 3: $-b + b = 0,\ -c + c = 0,\ 0 + 0 = 0.$

$$A + A' = O \Rightarrow \tfrac{1}{2}(A + A') = O.$$

Step 4. Compute $A - A'$: Row 1: $0 - 0 = 0,\ a - (-a) = 2a,\ b - (-b) = 2b.$
Row 2: $-a - a = -2a,\ 0 - 0 = 0,\ c - (-c) = 2c.$
Row 3: $-b - b = -2b,\ -c - c = -2c,\ 0 - 0 = 0.$

$$A - A' = \begin{bmatrix} 0 & 2a & 2b \\ -2a & 0 & 2c \\ -2b & -2c & 0 \end{bmatrix}.$$

$$\tfrac{1}{2}(A - A') = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = A.$$

Final Answer: $\frac{1}{2}(A + A') = O$ and $\frac{1}{2}(A - A') = A$. The given A is purely skew-symmetric.

EXPERT'S SOLUTION : *Ananya Singh, M.Sc Mathematics, ISI Kolkata*

Structural observation. A is already skew-symmetric ($A' = -A$), so its symmetric part is zero and its skew part is A itself.

Step 1. Verify $A' = -A$ entry-by-entry. All match.

Step 2. Then $A + A' = A + (-A) = O \Rightarrow \frac{1}{2}(A + A') = O$.

Step 3. And $A - A' = A - (-A) = 2A \Rightarrow \frac{1}{2}(A - A') = A$.

Why this matters. A matrix is skew-symmetric exactly when its symmetric part vanishes. This is the cleanest non-trivial example of the canonical decomposition.

Final Answer: Symmetric part $= O$; skew-symmetric part $= A$.

Q 3.10 Express the following matrices as the sum of a symmetric and a skew-symmetric matrix:

(i) $\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$, **(ii)** $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$,

(iii) $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$, **(iv)** $\begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}$.

SOLUTION

Concept used. For any square matrix A,

$$A = \underbrace{\tfrac{1}{2}(A + A')}_{\text{symmetric } P} + \underbrace{\tfrac{1}{2}(A - A')}_{\text{skew-symmetric } Q}.$$

$P = \frac{1}{2}(A + A')$ satisfies $P' = P$; $Q = \frac{1}{2}(A - A')$ satisfies $Q' = -Q$.

Step 1. **(i)** $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$. $A' = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$.

$$A + A' = \begin{bmatrix} 6 & 6 \\ 6 & -2 \end{bmatrix}, \quad P = \tfrac{1}{2}(A + A') = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix}.$$

$$A - A' = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}, \quad Q = \tfrac{1}{2}(A - A') = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

Check: $P + Q = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} = A \checkmark$.

Step 2. **(ii)** $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$. A already looks symmetric; verify $A' = A$. Indeed $(1, 2) = -2 = (2, 1)$; $(1, 3) = 2 = (3, 1)$; $(2, 3) = -1 = (3, 2)$. So A is symmetric.

Hence $P = A$ and $Q = O$:

$$P = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}.$$

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Step 3. **(iii)** $A = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$. $A' = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$.

$$A + A' = \begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix},$$

$$P = \tfrac{1}{2}(A + A') = \begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & -2 & -2 \\ -\frac{5}{2} & -2 & 2 \end{bmatrix}.$$

$$A - A' = \begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix},$$

$$Q = \tfrac{1}{2}(A - A') = \begin{bmatrix} 0 & \frac{5}{2} & \frac{3}{2} \\ -\frac{5}{2} & 0 & 3 \\ -\frac{3}{2} & -3 & 0 \end{bmatrix}.$$

Verify $P + Q$: entry $(1, 2)$: $\frac{1}{2} + \frac{5}{2} = 3$ ✓.

Step 4. **(iv)** $A = \begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}$. $A' = \begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix}$.

$$A + A' = \begin{bmatrix} 2 & 4 \\ 4 & 4 \end{bmatrix}, \quad P = \tfrac{1}{2}(A + A') = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}.$$

$$A - A' = \begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix}, \quad Q = \tfrac{1}{2}(A - A') = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}.$$

Check: $P + Q = \begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix} = A$ ✓.

Final Answer: (i)–(iv): decompositions as listed; in each case $A = P + Q$.

♥ Why This Matters

The decomposition $A = P + Q$ is unique. It is the matrix analogue of splitting a function f into its even and odd parts: $f(x) = \frac{1}{2}\big(f(x) + f(-x)\big) + \frac{1}{2}\big(f(x) - f(-x)\big)$.

EXPERT'S SOLUTION *: Tara Reddy, Ph.D Pure Mathematics, IISc Bangalore*

Strategic angle. Two operations only: $A + A'$ to get the symmetric pile, $A - A'$ to get the antisymmetric pile, each halved.

Step 1. Pre-compute A' for each matrix.

Step 2. Compute $P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A')$. Spot already-symmetric matrices (like (ii)) to skip computation.

Step 3. Sanity check by adding $P + Q$ and confirming it equals A.

Why this matters. Decomposing into symmetric and skew parts is used in the stress/strain tensor split in mechanics and in the Hodge decomposition in geometry.

> **Final Answer:** Decompositions as in the main solution.

Q 3.11 **If A and B are symmetric matrices of the same order, then $AB - BA$ is a: (A) skew-symmetric matrix (B) symmetric matrix (C) zero matrix (D) identity matrix.**

SOLUTION

Concept used. Use $(XY)' = Y'X'$ and the assumptions $A' = A$, $B' = B$. Compute $(AB - BA)'$.

Step 1. Apply the transpose to each term:

$$(AB - BA)' = (AB)' - (BA)'.$$

Step 2. By the reversal law: $(AB)' = B'A'$ and $(BA)' = A'B'$.

Step 3. Substitute $A' = A$ and $B' = B$:

$$(AB - BA)' = BA - AB = -(AB - BA).$$

Step 4. Therefore $(AB - BA)' = -(AB - BA)$, the defining property of a skew-symmetric matrix.

> **Final Answer:** Correct answer: (A) skew-symmetric.

EXPERT'S SOLUTION : *Yash Bhat, M.Sc Mathematics, IIT Bombay*

Quick reading. $[A, B] = AB - BA$ (the commutator) has the property $[A, B]^T = [B, A] = -[A, B]$ when A, B are symmetric.

Step 1. Take transpose: $(AB - BA)' = B'A' - A'B' = BA - AB$.

Step 2. Since this equals $-(AB - BA)$, the matrix is skew-symmetric.

Why this matters. "Commutator of two symmetric matrices is skew-symmetric" is foundational in Lie algebras and quantum mechanics.

> **Final Answer:** (A).

Q 3.12 **If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ and $A + A' = I$, then the value of α is:**

(A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{3}$ **(C)** π **(D)** $\frac{3\pi}{2}$.

SOLUTION

Concept used. Compute A', then add to A, and equate to I.

Step 1. Compute A':

$$A' = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}.$$

Step 2. Add $A + A'$:

$$A + A' = \begin{bmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{bmatrix} = 2\cos\alpha\, I.$$

Step 3. Set equal to I: $2\cos\alpha = 1$, so $\cos\alpha = \frac{1}{2}$.

Step 4. The principal-value solution in the standard $[0, 2\pi]$ range is $\alpha = \frac{\pi}{3}$ (and $\alpha = \frac{5\pi}{3}$).

Step 5. Among the four options, only (B) $\frac{\pi}{3}$ matches.

> **Final Answer:** Correct answer: (B) $\alpha = \frac{\pi}{3}$.

EXPERT'S SOLUTION : *Pooja Verma, M.Sc Mathematics, IIT Bombay*

Quick reading. The sum $A + A'$ has off-diagonals zero (sines cancel) and diagonals $2\cos\alpha$. Match to I.

Step 1. Diagonal: $2\cos\alpha = 1 \Rightarrow \cos\alpha = \frac{1}{2}$.

Step 2. Principal solution in $[0, 2\pi]$: $\alpha = \frac{\pi}{3}$.

Step 3. Sanity: $\cos(\frac{\pi}{3}) = \frac{1}{2}$ ✓.

Why this matters. For a rotation matrix $R(\alpha)$, the combination $R + R^T = 2\cos\alpha \cdot I$ is the trace identity that appears whenever angles are extracted from a rotation.

Final Answer: (B) $\dfrac{\pi}{3}$.

Key Takeaways

- Transpose flips an $m \times n$ matrix into $n \times m$ with $(A')_{ij} = a_{ji}$. The four core laws are $(A')' = A$, $(kA)' = kA'$, $(A + B)' = A' + B'$, and $(AB)' = B'A'$.
- Symmetric: $A' = A$. Skew-symmetric: $A' = -A$ (forces zero diagonal).
- Every square matrix A uniquely splits as $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$, a symmetric piece plus a skew-symmetric piece.
- For symmetric A, B, the commutator $AB - BA$ is skew-symmetric.
- A matrix with $A'A = I$ is orthogonal; equivalently $A^{-1} = A'$. Rotation/reflection matrices are the textbook examples.

End of Exercise 3.3