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Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 3: Matrices

About this Chapter

The Miscellaneous Exercise stitches together every idea from the chapter: **transpose**, **symmetric / skew-symmetric** splits, **matrix multiplication**, **polynomial identities** in a matrix, and applied **revenue / cost** problems via matrix algebra. Mastering these eleven questions cements fluency for boards and is a perfect launchpad for determinants and inverses in Chapter 4.

Topics covered: Symmetric/skew-symmetric proofs • $B'AB$ identity • $A'A = I$ systems • Polynomial identities in A • Revenue/cost via matrices • Solving $XA = B$

Quick Formula Sheet

Reversal of transpose: $(AB)' = B'A'$, $(ABC)' = C'B'A'$.

Symmetric/skew split: Symmetric: $A' = A$. Skew: $A' = -A$.

Cayley-Hamilton flavour: Polynomial $p(A) = O$ relations let you express A^{-1} via I, A, A^2.

Matrix revenue: Revenue = Quantity-row × Price-column.

Miscellaneous Exercise

Q 3.1 **If A and B are symmetric matrices, prove that $AB - BA$ is a skew-symmetric matrix.**

SOLUTION

Concept used. A matrix M is **skew-symmetric** iff $M' = -M$. To prove that $AB - BA$ is skew-symmetric we compute its transpose and show it equals the negative of the original. Required identities: $(XY)' = Y'X'$, $(X - Y)' = X' - Y'$, and the given symmetry $A' = A$, $B' = B$.

Step 1. Start with $M = AB - BA$. Compute the transpose:

$$M' = (AB - BA)' = (AB)' - (BA)'.$$

Step 2. Apply the reversal law $(XY)' = Y'X'$ to each piece:

$$(AB)' = B'A', \qquad (BA)' = A'B'.$$

Substitute:

$$M' = B'A' - A'B'.$$

Step 3. Use the given symmetries $A' = A$ and $B' = B$:

$$M' = BA - AB.$$

Step 4. Rewrite the right side: $BA - AB = -(AB - BA) = -M$. So

$$M' = -M.$$

Step 5. By definition, $AB - BA$ is skew-symmetric.

Final Answer: $(AB - BA)' = -(AB - BA)$; hence $AB - BA$ is skew-symmetric.

♥ Why This Matters

The commutator $[A, B] = AB - BA$ measures how badly A and B fail to commute. When both are symmetric, that failure is purely skew-symmetric, an algebraic structure used in Lie algebras and in the formulation of quantum mechanics.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Bombay*

Strategic angle. One identity, three steps: transpose, then flip with the reversal law, then absorb the symmetric assumptions.

Step 1. $(AB - BA)' = (AB)' - (BA)' = B'A' - A'B'$ by linearity and reversal.

Step 2. Since $A' = A,\ B' = B$ (given), $B'A' - A'B' = BA - AB$.

Step 3. $BA - AB = -(AB - BA)$, which is exactly the skew-symmetry condition.

Why this matters. "Commutator of symmetric matrices is skew-symmetric" is a one-line lemma you will see again whenever two self-adjoint operators are combined.

Final Answer: $AB - BA$ is skew-symmetric.

Q 3.2 **Show that the matrix $B'AB$ is symmetric or skew-symmetric according as A is symmetric or skew-symmetric.**

SOLUTION

Concept used. Compute $(B'AB)'$ using the reversal law $(XYZ)' = Z'Y'X'$, then absorb the symmetry/skew-symmetry of A.

Step 1. Apply the triple-reversal law:

$$(B'AB)' = B'\,A'\,(B')' = B'\,A'\,B,$$

using $(B')' = B$ and the fact that $(B')' = B$.

Step 2. **Case A symmetric.** Then $A' = A$, so

$$(B'AB)' = B'\,A\,B = B'AB.$$

Hence $B'AB$ equals its own transpose, i.e. is symmetric.

Step 3. **Case A skew-symmetric.** Then $A' = -A$, so

$$(B'AB)' = B'(-A)B = -(B'AB).$$

Hence $B'AB$ is the negative of its transpose, i.e. skew-symmetric.

> **Final Answer:** $B'AB$ is symmetric if A is symmetric, skew-symmetric if A is skew-symmetric.

☞ **Exam Tip**

The pattern $B'(\cdot)B$ "preserves" the symmetry property of the inner matrix — this is exactly why such expressions appear in similarity transformations and quadratic forms.

EXPERT'S SOLUTION *: Vivaan Iyer, Ph.D Mathematics, IIT Delhi*

Structural observation. The sandwich $B'(\cdot)B$ is a *congruence*; it preserves whether the middle matrix is symmetric or skew-symmetric.

Step 1. Transpose the product: $(B'AB)' = B'A'(B')' = B'A'B$.

Step 2. Substitute the symmetry hypothesis: $A' = \pm A$.

Step 3. Conclude $(B'AB)' = \pm(B'AB)$, i.e. symmetric or skew-symmetric matches the sign of A'.

Why this matters. Quadratic forms $x^T Ax$ are invariant under congruence $A \to B'AB$ exactly because of this property.

Final Answer: Same symmetry class is preserved.

Q 3.3 **Find the values of x, y, z if the matrix $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfies $A'A = I$.**

SOLUTION

Concept used. $A'A = I$ means A is orthogonal: its columns form an orthonormal set. Each column has unit length and any two distinct columns are perpendicular.

Step 1. Identify the three columns of A:

$$C_1 = \begin{bmatrix} 0 \\ x \\ x \end{bmatrix}, \; C_2 = \begin{bmatrix} 2y \\ y \\ -y \end{bmatrix}, \; C_3 = \begin{bmatrix} z \\ -z \\ z \end{bmatrix}.$$

Step 2. **Unit length conditions.**

$$\|C_1\|^2 = 0^2 + x^2 + x^2 = 2x^2 = 1 \Rightarrow x^2 = \tfrac{1}{2} \Rightarrow x = \pm\tfrac{1}{\sqrt{2}}.$$

$$\|C_2\|^2 = (2y)^2 + y^2 + (-y)^2 = 4y^2 + y^2 + y^2 = 6y^2 = 1 \Rightarrow y^2 = \tfrac{1}{6} \Rightarrow y = \pm\tfrac{1}{\sqrt{6}}.$$

$$\|C_3\|^2 = z^2 + (-z)^2 + z^2 = 3z^2 = 1 \Rightarrow z^2 = \tfrac{1}{3} \Rightarrow z = \pm\tfrac{1}{\sqrt{3}}.$$

Step 3. **Orthogonality of columns.** $C_1 \cdot C_2 = 0 \cdot 2y + x \cdot y + x \cdot (-y) = 0 + xy - xy = 0$ ✓.
$C_1 \cdot C_3 = 0 \cdot z + x \cdot (-z) + x \cdot z = 0 - xz + xz = 0$ ✓.
$C_2 \cdot C_3 = 2y \cdot z + y \cdot (-z) + (-y) \cdot z = 2yz - yz - yz = 0$ ✓. All three orthogonality conditions hold automatically.

Step 4. So the answer is determined by the unit-length conditions alone:

$$x = \pm\tfrac{1}{\sqrt{2}}, \; y = \pm\tfrac{1}{\sqrt{6}}, \; z = \pm\tfrac{1}{\sqrt{3}}.$$

Final Answer: $x = \pm\dfrac{1}{\sqrt{2}}, \; y = \pm\dfrac{1}{\sqrt{6}}, \; z = \pm\dfrac{1}{\sqrt{3}}$.

♥ **Why This Matters**

$A'A = I$ exactly when the columns of A form an orthonormal basis. That is the condition

A represents a rotation or reflection of $\mathbb{R}^n$.

EXPERT'S SOLUTION : *Sneha Kapoor, M.Sc Mathematics, ISI Kolkata*

Strategic angle. Translate $A'A = I$ into the orthonormality of the columns. Then it's three norm equations and three dot-product equations.

Step 1. Norms: $2x^2 = 1,\ 6y^2 = 1,\ 3z^2 = 1$ give $x^2 = \frac{1}{2},\ y^2 = \frac{1}{6},\ z^2 = \frac{1}{3}$.

Step 2. Dot products: pairwise, each simplifies to 0 algebraically regardless of sign choice, so the orthogonality is automatic.

Step 3. Take square roots, allowing both signs.

Why this matters. A 3x3 orthogonal matrix has 9 entries but only 3 free parameters (essentially Euler angles); the constraints encode that.

Final Answer: $x = \pm\frac{1}{\sqrt{2}},\ y = \pm\frac{1}{\sqrt{6}},\ z = \pm\frac{1}{\sqrt{3}}$.

Q 3.4 **For what values of x:** $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$**?**

SOLUTION

Concept used. The product of a 1×3 row, a 3×3 matrix, and a 3×1 column is a 1×1 matrix (a single number). Multiply left-to-right (or right-to-left) and set equal to zero.

Step 1. Multiply the first two: $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$. Each entry is a row-by-column dot product.

Position 1: $1(1) + 2(2) + 1(1) = 1 + 4 + 1 = 6$.
Position 2: $1(2) + 2(0) + 1(0) = 2 + 0 + 0 = 2$.
Position 3: $1(0) + 2(1) + 1(2) = 0 + 2 + 2 = 4$.

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 4 \end{bmatrix}.$$

Step 2. Now multiply $\begin{bmatrix} 6 & 2 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix}$:

$$6(0) + 2(2) + 4(x) = 0 + 4 + 4x.$$

Step 3. Set equal to zero:

$$4 + 4x = 0 \Rightarrow x = -1.$$

Final Answer: $x = -1$.

EXPERT'S SOLUTION *: Aditya Verma, M.Tech CS, IIT Madras*

Quick reading. Three-factor product collapses to a scalar. Multiply the two easy factors first, then dot the result with the remaining column.

Step 1. Row $\times$ matrix $= [6, 2, 4]$.

Step 2. Dot with column $[0, 2, x]^T$: $4 + 4x$.

Step 3. Set $4 + 4x = 0 \Rightarrow x = -1$.

Why this matters. A triple product like $u^T M v$ is the quadratic-form / bilinear-form pattern that drives many applications.

Final Answer: $x = -1$.

Q 3.5 **If** $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$**, show that** $A^2 - 5A + 7I = O$**.**

SOLUTION

Concept used. Compute A^2, $5A$, and $7I$; combine entry-wise. Show every entry of the result is 0.

Step 1. **Compute** A^2**.** $(A^2)_{11} = 3(3) + 1(-1) = 9 - 1 = 8$.
$(A^2)_{12} = 3(1) + 1(2) = 3 + 2 = 5$.
$(A^2)_{21} = -1(3) + 2(-1) = -3 - 2 = -5$.
$(A^2)_{22} = -1(1) + 2(2) = -1 + 4 = 3$.

$$A^2 = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}.$$

Step 2. Compute $5A = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}$.

Step 3. Compute $7I = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$.

Step 4. Combine $A^2 - 5A + 7I$ entry-wise: Row 1: $8 - 15 + 7 = 0,\ 5 - 5 + 0 = 0$.

Row 2: $-5-(-5)+0=-5+5=0,\ 3-10+7=0.$

$$A^2-5A+7I=\begin{bmatrix}0&0\\0&0\end{bmatrix}=O.$$

Final Answer: $A^2-5A+7I=O.$

☞ Exam Tip

This is the Cayley-Hamilton identity for A: its characteristic polynomial is $\lambda^2-5\lambda+7$. Multiplying the identity by A^{-1} (when it exists) gives $A^{-1}=\frac{1}{7}(5I-A)$, a fast inverse trick.

EXPERT'S SOLUTION *: Priya Singh, Ph.D Pure Mathematics, IISc Bangalore*

Picture-first. Compute A^2 in four dot products, then add the scalar pieces, then watch the entries collapse to 0.

Step 1. $A^2=\begin{bmatrix}8&5\\-5&3\end{bmatrix}.$

Step 2. $5A=\begin{bmatrix}15&5\\-5&10\end{bmatrix},\ 7I=\begin{bmatrix}7&0\\0&7\end{bmatrix}.$

Step 3. $A^2-5A+7I$: entry $(1,1)$: $8-15+7=0$; $(2,2)$: $3-10+7=0$. Off-diagonals: $5-5=0,\ -5+5=0.$

Step 4. All zero, so $A^2-5A+7I=O.$

Why this matters. A 2×2 matrix always satisfies its characteristic polynomial $\lambda^2-\text{tr}(A)\lambda+\det(A)=0$. Here $\text{tr}(A)=5, \det(A)=7$.

Final Answer: O.

Q 3.6 **Find x, if** $[x\quad -5\quad -1]\begin{bmatrix}1&0&2\\0&2&1\\2&0&3\end{bmatrix}\begin{bmatrix}x\\4\\1\end{bmatrix}=O.$

SOLUTION

Concept used. Multiply left-to-right; the final product is a 1×1 scalar.

Step 1. Multiply the first two factors: $[x \;\; -5 \;\; -1]\begin{bmatrix}1&0&2\\0&2&1\\2&0&3\end{bmatrix}$.

Entry 1: $x(1)+(-5)(0)+(-1)(2)=x-2$.
Entry 2: $x(0)+(-5)(2)+(-1)(0)=-10$.
Entry 3: $x(2)+(-5)(1)+(-1)(3)=2x-5-3=2x-8$.

$$\Rightarrow [\,x-2 \;\; -10 \;\; 2x-8\,].$$

Step 2. Multiply by the column $\begin{bmatrix}x\\4\\1\end{bmatrix}$:

$$(x-2)\,x+(-10)(4)+(2x-8)(1).$$

Step 3. Expand:

$$x^2-2x-40+2x-8=x^2-48.$$

Step 4. Set equal to zero: $x^2-48=0 \Rightarrow x^2=48 \Rightarrow x=\pm4\sqrt{3}$.

Final Answer: $x=\pm4\sqrt{3}$.

EXPERT'S SOLUTION *: Diya Bhat, M.Sc Mathematics, IIT Bombay*

Strategic angle. Two products, the second collapses to a neat quadratic in x.

Step 1. Row $\times$ matrix $=[x-2,\; -10,\; 2x-8]$.

Step 2. Dot with $[x,4,1]^T$: $x(x-2)-40+(2x-8)=x^2-2x-40+2x-8=x^2-48$.

Step 3. Set $=0$: $x=\pm\sqrt{48}=\pm4\sqrt{3}$.

Why this matters. Quadratic forms in x pop out of triple products u^TMu all the time; recognising the cancellation pattern saves time.

Final Answer: $x=\pm4\sqrt{3}$.

Q 3.7 **A manufacturer produces three products x, y, z which he sells in two markets. Annual sales are:**

Market	x	y	z
I	10,000	2,000	18,000
II	6,000	20,000	8,000

(a) If unit sale prices of x, y, z are Rs. 2.50, Rs. 1.50, Rs. 1.00 respectively, find the total revenue in each market using matrix algebra.
(b) If the unit costs are Rs. 2.00, Rs. 1.00, 50 paise (= Rs. 0.50), find the gross profit.

SOLUTION

Concept used. Encode the sales matrix as S $(2 \times 3$, rows $=$ markets), prices as column P (3×1). Then revenue per market is SP (2×1). Profit per market $=$ revenue $-$ cost $= S(P - C)$ where C is the unit-cost column.

Step 1. (a) Revenue. Write the sales matrix

$$S = \begin{bmatrix} 10000 & 2000 & 18000 \\ 6000 & 20000 & 8000 \end{bmatrix}, \quad P = \begin{bmatrix} 2.50 \\ 1.50 \\ 1.00 \end{bmatrix}.$$

Compute SP:
Market I (row $1 \cdot P$):

$$10000(2.50) + 2000(1.50) + 18000(1.00).$$

$$= 25000 + 3000 + 18000 = 46000.$$

Market II (row $2 \cdot P$):

$$6000(2.50) + 20000(1.50) + 8000(1.00).$$

$$= 15000 + 30000 + 8000 = 53000.$$

$$SP = \begin{bmatrix} 46000 \\ 53000 \end{bmatrix}.$$

Step 2. (b) Profit. Cost column $C = \begin{bmatrix} 2.00 \\ 1.00 \\ 0.50 \end{bmatrix}$. Profit per unit (price $-$ cost):

$$P - C = \begin{bmatrix} 2.50 - 2.00 \\ 1.50 - 1.00 \\ 1.00 - 0.50 \end{bmatrix} = \begin{bmatrix} 0.50 \\ 0.50 \\ 0.50 \end{bmatrix}.$$

Profit per market: $S(P - C)$.
Market I:

$$10000(0.50) + 2000(0.50) + 18000(0.50).$$

$$= 5000 + 1000 + 9000 = 15000.$$

Market II:

$$6000(0.50) + 20000(0.50) + 8000(0.50).$$

$$= 3000 + 10000 + 4000 = 17000.$$

$$S(P - C) = \begin{bmatrix} 15000 \\ 17000 \end{bmatrix}.$$

Step 3. Total gross profit $= 15000 + 17000 = 32000$.

> **Final Answer:** (a) Market I: Rs. 46,000; Market II: Rs. 53,000.
> (b) Profit: Market I: Rs. 15,000; Market II: Rs. 17,000. Total gross profit: Rs. 32,000.

♥ Why This Matters

This is the canonical "inventory" application of matrix algebra: quantities $\times$ prices $=$ revenue. The same template handles multi-product, multi-region sales planning in any business.

EXPERT'S SOLUTION : *Kavya Patel, M.Sc Mathematics, IIT Bombay*

Quick reading. Revenue per market and profit per market are two matrix products against the same sales matrix.

Step 1. (a) Revenue: SP with $P = (2.50, 1.50, 1.00)^T$. Compute the two market totals: 46,000 and 53,000.

Step 2. (b) Profit: $S(P - C)$ where $P - C = (0.5, 0.5, 0.5)^T$. Because each unit has the same profit margin of 50 paise, the market totals are half the unit count per market.

Step 3. Market I units: $10000 + 2000 + 18000 = 30000$. Half: 15,000.
Market II units: $6000 + 20000 + 8000 = 34000$. Half: 17,000.

Why this matters. Spotting that all three margins are 50 paise collapses the problem to "total units $\div 2$."

> **Final Answer:** Same as main solution.

Q 3.8 **Find the matrix X so that $X \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$.**

SOLUTION

Concept used. Compatibility of orders: if X is $m \times k$ and $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is 2×3, then XA is $m \times 3$. The RHS is 2×3, so X must be 2×2. Solve for the four entries of X by equating XA with the RHS entry-wise.

Step 1. Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Step 2. Compute XA:

$$XA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} a+4b & 2a+5b & 3a+6b \\ c+4d & 2c+5d & 3c+6d \end{bmatrix}.$$

Step 3. Match with the RHS $\begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$:

Row 1: $a + 4b = -7$ (1), $2a + 5b = -8$ (2), $3a + 6b = -9$ (3).

Row 2: $c + 4d = 2$ (4), $2c + 5d = 4$ (5), $3c + 6d = 6$ (6).

Step 4. Solve row 1: From (1), $a = -7 - 4b$. Substitute in (2): $2(-7 - 4b) + 5b = -8 \Rightarrow -14 - 8b + 5b = -8 \Rightarrow -3b = 6 \Rightarrow b = -2$. Then $a = -7 - 4(-2) = -7 + 8 = 1$. Check (3): $3(1) + 6(-2) = 3 - 12 = -9$ ✓.

Step 5. Solve row 2: From (4), $c = 2 - 4d$. Substitute in (5): $2(2 - 4d) + 5d = 4 \Rightarrow 4 - 8d + 5d = 4 \Rightarrow -3d = 0 \Rightarrow d = 0$. Then $c = 2 - 0 = 2$. Check (6): $3(2) + 6(0) = 6$ ✓.

Step 6. Therefore $X = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$.

Final Answer: $X = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$.

EXPERT'S SOLUTION : *Yash Reddy, M.Sc Mathematics, IIT Madras*

Strategic angle. Note that columns of A have a pattern: column j is $\binom{j}{j+3}$, three columns each linearly related. This means we only need to determine two scalar conditions per row of X to fix all three column equations.

Step 1. Set $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and write XA's entries as $(a + 4b,\ 2a + 5b,\ 3a + 6b)$ for row 1, $(c + 4d,\ 2c + 5d,\ 3c + 6d)$ for row 2.

Step 2. Use just two equations per row to solve for two unknowns:

$(a+4b, 2a+5b) = (-7, -8)$ gives $a = 1,\ b = -2$. $(c+4d, 2c+5d) = (2, 4)$ gives $c = 2,\ d = 0$.

Step 3. Verify the third equation is consistent: $3a + 6b = 3 - 12 = -9$ ✓; $3c + 6d = 6 + 0 = 6$ ✓.

Why this matters. Solving $XA = B$ for X is the "left-division" problem, the right way to think about systems where multiple unknowns multiply a known matrix on the right.

Final Answer: $X = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$.

Q 3.9 **If** $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ **is such that** $A^2 = I$**, then:**

(A) $1 + \alpha^2 + \beta\gamma = 0$ **(B)** $1 - \alpha^2 + \beta\gamma = 0$ **(C)** $1 - \alpha^2 - \beta\gamma = 0$ **(D)** $1 + \alpha^2 - \beta\gamma = 0$**.**

SOLUTION

Concept used. Compute A^2 and equate it to I entry-wise.

Step 1. Compute A^2: $(A^2)_{11} = \alpha \cdot \alpha + \beta \cdot \gamma = \alpha^2 + \beta\gamma$.
$(A^2)_{12} = \alpha \cdot \beta + \beta \cdot (-\alpha) = \alpha\beta - \alpha\beta = 0$.
$(A^2)_{21} = \gamma \cdot \alpha + (-\alpha) \cdot \gamma = 0$.
$(A^2)_{22} = \gamma \cdot \beta + (-\alpha) \cdot (-\alpha) = \beta\gamma + \alpha^2$.

$$A^2 = \begin{bmatrix} \alpha^2 + \beta\gamma & 0 \\ 0 & \alpha^2 + \beta\gamma \end{bmatrix} = (\alpha^2 + \beta\gamma)\, I.$$

Step 2. Set $A^2 = I$:

$$(\alpha^2 + \beta\gamma)\, I = I \Rightarrow \alpha^2 + \beta\gamma = 1.$$

Step 3. Rearrange to match the options: $1 - \alpha^2 - \beta\gamma = 0$.

Step 4. That is option (C).

Final Answer: Correct answer: (C) $1 - \alpha^2 - \beta\gamma = 0$.

EXPERT'S SOLUTION *: Aanya Kapoor, M.Sc Mathematics, IIT Bombay*

Quick reading. The off-diagonals of A^2 vanish by sign cancellation; only the diagonal matters.

Step 1. A^2 is $(\alpha^2 + \beta\gamma)I$.

Step 2. Setting $A^2 = I$ gives $\alpha^2 + \beta\gamma = 1$, i.e. $1 - \alpha^2 - \beta\gamma = 0$.

Why this matters. "$A^2 = I$" means A is an *involution*, e.g. a reflection. The constraint $\alpha^2 + \beta\gamma = 1$ together with $\text{tr}(A) = 0$ encodes that A is a 2×2 reflection matrix.

> **Final Answer:** (C).

Q 3.10 **If the matrix A is both symmetric and skew-symmetric, then:**
(A) A is a diagonal matrix (B) A is a zero matrix (C) A is a square matrix (D) None of these.

SOLUTION

Concept used. *Symmetric:* $A' = A$. *Skew-symmetric:* $A' = -A$. If both hold simultaneously, equating the two expressions for A' forces $A = -A$.

Step 1. From symmetry: $a_{ij} = a_{ji}$.

Step 2. From skew-symmetry: $a_{ij} = -a_{ji}$.

Step 3. Combine: $a_{ji} = -a_{ji} \Rightarrow 2a_{ji} = 0 \Rightarrow a_{ji} = 0$ for all i, j.

Step 4. So every entry is zero: $A = O$.

> **Final Answer:** Correct answer: (B) A is the zero matrix.

♥ Why This Matters

Symmetric and skew-symmetric matrices are "complementary" subspaces of the square-matrix space; their intersection is $\{O\}$. This is why the symmetric/skew-symmetric decomposition $A = P + Q$ is unique.

EXPERT'S SOLUTION *: Pranav Joshi, M.Tech CS, IIT Madras*

Quick reading. "Equal to itself and to its negative" is only true for zero.

Step 1. $A' = A$ and $A' = -A \Rightarrow A = -A \Rightarrow 2A = O \Rightarrow A = O$.

Why this matters. The intersection of two subspaces being $\{O\}$ is the algebraic statement of "direct sum" — the precise condition for unique decomposition.

> **Final Answer:** (B).

Q 3.11 **If A is a square matrix such that $A^2 = A$, then $(I + A)^3 - 7A$ is equal to:**
(A) A **(B)** $I - A$ **(C)** I **(D)** $3A$.

SOLUTION

Concept used. Matrices that satisfy $A^2 = A$ are called **idempotent**. For such A, $A^n = A$ for every $n \geq 1$. Since A and I commute, the binomial expansion $(I + A)^3 = I^3 + 3I^2A + 3IA^2 + A^3$ is valid.

Step 1. Expand using the binomial theorem (valid because $IA = AI = A$):

$$(I + A)^3 = I + 3A + 3A^2 + A^3.$$

Step 2. Replace A^2 and A^3 using $A^2 = A$:

$$A^2 = A, \quad A^3 = A \cdot A^2 = A \cdot A = A.$$

Substitute:

$$(I + A)^3 = I + 3A + 3A + A = I + 7A.$$

Step 3. Subtract $7A$:

$$(I + A)^3 - 7A = I + 7A - 7A = I.$$

Final Answer: Correct answer: (C) I.

☞ Exam Tip

Idempotents satisfy $A^n = A$ for every $n \geq 1$. This collapses every matrix polynomial in A into a single linear expression $\alpha I + \beta A$, which is the trick to handle JEE-style "$f(A) = ?$" problems.

EXPERT'S SOLUTION *: Ishita Pillai, M.Sc Applied Mathematics, IIT Kanpur*

Structural observation. $A^2 = A$ is idempotency. Every power of A collapses to A.

Step 1. Expand $(I + A)^3 = I + 3A + 3A^2 + A^3$.

Step 2. Replace $A^2 = A$ and $A^3 = A$: $(I + A)^3 = I + 3A + 3A + A = I + 7A$.

Step 3. Subtract $7A$: result is I.

Why this matters. Idempotents arise as projection operators (e.g. projecting a vector onto an axis). They are the simplest non-trivial matrices, and matrix polynomials in them are linear.

Final Answer: (C) I.

Key Takeaways

- For symmetric A, B, the commutator $AB - BA$ is always skew-symmetric, and the sandwich $B'AB$ preserves the symmetry class of A.
- The condition $A'A = I$ is orthogonality: columns of A form an orthonormal set. The norm equations alone (with sign choices) usually determine the parameters.
- Matrix polynomial identities (like $A^2 - 5A + 7I = O$) are the gateway to Cayley-Hamilton and to fast inverse computations.
- Revenue and profit problems collapse to row-times-column products: Quantity $\times$ Price (or $\times$ margin).
- For idempotent A ($A^2 = A$), every power $A^n = A$, so matrix polynomials in A are always of the form $\alpha I + \beta A$.

End of Miscellaneous Exercise