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Chapter 10: Vector Algebra

About this Chapter

Exercise 10.3 focuses on the **scalar (dot) product** $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta = a_1b_1 + a_2b_2 + a_3b_3$. Master the angle formula, projections, perpendicularity tests, and applications to triangles.

Topics covered: Dot product • Angle between vectors • Projection • Perpendicular vectors • Geometric applications

Quick Formula Sheet

Definition:
$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$, where θ is the angle between them.

Component form:
If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

Angle:
$\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$.

Projection:
Projection of $\vec{a}$ on $\vec{b}$ is $\dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$.

Perpendicular: $\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0$.

Exercise 10.3

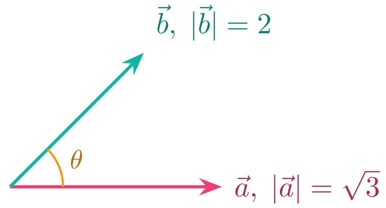
Q 10.1 **Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2, respectively having $\vec{a} \cdot \vec{b} = \sqrt{6}$.**

SOLUTION

Concept used. The angle θ between two non-zero vectors satisfies

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}.$$

With $\theta \in [0, \pi]$ we then take $\theta = \cos^{-1}(\cdot)$.



Step 1. Substitute the given data:

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{\sqrt{6}}{\sqrt{3}\cdot 2}.$$

Step 2. Simplify the right-hand side:

$$\frac{\sqrt{6}}{2\sqrt{3}} = \frac{\sqrt{6}}{2\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{18}}{6} = \frac{3\sqrt{2}}{6} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}.$$

Step 3. Hence $\cos\theta = \dfrac{1}{\sqrt{2}}$, so $\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \dfrac{\pi}{4}$.

Final Answer: $\theta = \dfrac{\pi}{4}$ (i.e. $45°$).

EXPERT'S SOLUTION : *Aarav Singh, M.Sc Mathematics, IIT Bombay*

Quick reading. One formula, one substitution, one inverse cosine.

Step 1. $\cos\theta = \dfrac{\sqrt{6}}{\sqrt{3}\cdot 2} = \dfrac{1}{\sqrt{2}}$.

Step 2. $\theta = \pi/4$.

Why this matters. The dot product packages magnitude information into a single scalar; given any two of $|\vec{a}|, |\vec{b}|, \theta, \vec{a}\cdot\vec{b}$ you can solve for the others.

Final Answer: $\theta = \pi/4$.

Q 10.2 Find the angle between the vectors $\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$ and $3\hat{\imath} - 2\hat{\jmath} + \hat{k}$.

SOLUTION

Concept used. Use $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$, where the dot product is the sum of products of corresponding components.

Step 1. Let $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$.

Step 2. Compute $\vec{a} \cdot \vec{b}$ component-wise:

$$\vec{a} \cdot \vec{b} = (1)(3) + (-2)(-2) + (3)(1) = 3 + 4 + 3 = 10.$$

Step 3. Compute magnitudes:

$$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}.$$

$$|\vec{b}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9 + 4 + 1} = \sqrt{14}.$$

Step 4. Apply the formula:

$$\cos\theta = \frac{10}{\sqrt{14} \cdot \sqrt{14}} = \frac{10}{14} = \frac{5}{7}.$$

Step 5. Hence $\theta = \cos^{-1}\left(\frac{5}{7}\right)$.

> **Final Answer:** $\theta = \cos^{-1}\left(\frac{5}{7}\right)$.

EXPERT'S SOLUTION : *Sneha Gupta, M.Sc Mathematics, ISI Kolkata*

Strategic angle. Numerator from dot product, denominator from two magnitudes; both are symmetric here.

Step 1. $\vec{a} \cdot \vec{b} = 3 + 4 + 3 = 10$.

Step 2. $|\vec{a}|^2 = |\vec{b}|^2 = 14$, so $|\vec{a}||\vec{b}| = 14$.

Step 3. $\cos\theta = 10/14 = 5/7$.

Why this matters. When both vectors have equal magnitudes the denominator becomes that common magnitude squared - a slight time-saver in computation.

> **Final Answer:** $\cos^{-1}(5/7)$.

Q 10.3 **Find the projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$.**

SOLUTION

Concept used. The (scalar) projection of $\vec{a}$ on $\vec{b}$ is

$$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}.$$

It measures the signed length of the shadow of $\vec{a}$ along the direction of $\vec{b}$.

Step 1. Let $\vec{a} = \hat{i} - \hat{j}$, $\vec{b} = \hat{i} + \hat{j}$.

Step 2. Compute $\vec{a} \cdot \vec{b}$:

$$\vec{a} \cdot \vec{b} = (1)(1) + (-1)(1) + 0 = 1 - 1 = 0.$$

Step 3. Magnitude of $\vec{b}$:

$$|\vec{b}| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

Step 4. Projection:

$$\text{proj}_{\vec{b}}\vec{a} = \frac{0}{\sqrt{2}} = 0.$$

A zero projection means $\vec{a} \perp \vec{b}$: $\hat{i} - \hat{j}$ is perpendicular to $\hat{i} + \hat{j}$ in the plane.

Final Answer: Projection $= 0$.

EXPERT'S SOLUTION *: Aanya Patel, M.Sc Mathematics, IIT Bombay*

Picture-first. $\hat{i} + \hat{j}$ points along the line $y = x$; $\hat{i} - \hat{j}$ points along $y = -x$. The two lines are perpendicular, so the shadow of one along the other is zero.

Step 1. $\vec{a} \cdot \vec{b} = 1 - 1 = 0$.

Step 2. Projection $= 0/\sqrt{2} = 0$.

Why this matters. Whenever the dot product is zero, the two vectors are perpendicular and either vector has zero projection onto the other.

Final Answer: 0.

Q 10.4 **Find the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $7\hat{i} - \hat{j} + 8\hat{k}$.**

SOLUTION

Concept used. $\text{proj}_{\vec{b}}\vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$.

Step 1. Let $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ and $\vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}$.

Step 2. Compute dot product:

$$\vec{a} \cdot \vec{b} = (1)(7) + (3)(-1) + (7)(8) = 7 - 3 + 56 = 60.$$

Step 3. Compute $|\vec{b}|$:

$$|\vec{b}| = \sqrt{7^2 + (-1)^2 + 8^2} = \sqrt{49 + 1 + 64} = \sqrt{114}.$$

Step 4. Projection:

$$\text{proj}_{\vec{b}}\vec{a} = \frac{60}{\sqrt{114}}.$$

Final Answer: Projection $= \dfrac{60}{\sqrt{114}}$.

EXPERT'S SOLUTION *: Vivaan Mehta, M.Sc Mathematics, IIT Bombay*

Quick reading. Dot product over length of the second vector.

Step 1. $\vec{a} \cdot \vec{b} = 7 - 3 + 56 = 60.$

Step 2. $|\vec{b}| = \sqrt{114}.$

Step 3. Projection $= 60/\sqrt{114}.$

Why this matters. Projection is the building block for the component-decomposition of one vector along another and is the algebraic version of "what part of $\vec{a}$ acts along $\vec{b}$?".

Final Answer: $60/\sqrt{114}$.

Q 10.5 Show that each of the given three vectors is a unit vector:

$$\tfrac{1}{7}(2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}), \quad \tfrac{1}{7}(3\hat{\imath} - 6\hat{\jmath} + 2\hat{k}), \quad \tfrac{1}{7}(6\hat{\imath} + 2\hat{\jmath} - 3\hat{k}).$$

Also, show that they are mutually perpendicular to each other.

SOLUTION

Concept used. A vector $\vec{u}$ is a **unit vector** iff $|\vec{u}| = 1$, i.e. $\vec{u} \cdot \vec{u} = 1$. Two vectors are **mutually perpendicular** iff their dot product is zero.

Step 1. Write the vectors:

$$\vec{u}_1 = \tfrac{1}{7}(2, 3, 6), \quad \vec{u}_2 = \tfrac{1}{7}(3, -6, 2), \quad \vec{u}_3 = \tfrac{1}{7}(6, 2, -3).$$

Step 2. Unit-vector check (each).

$$|\vec{u}_1|^2 = \tfrac{1}{49}(2^2 + 3^2 + 6^2) = \tfrac{4+9+36}{49} = \tfrac{49}{49} = 1.$$

$$|\vec{u}_2|^2 = \tfrac{1}{49}(3^2 + (-6)^2 + 2^2) = \tfrac{9+36+4}{49} = \tfrac{49}{49} = 1.$$

$$|\vec{u}_3|^2 = \frac{1}{49}(6^2 + 2^2 + (-3)^2) = \frac{36+4+9}{49} = \frac{49}{49} = 1.$$

So each has unit magnitude.

Step 3. **Perpendicularity check (three dot products).**

$$\vec{u}_1 \cdot \vec{u}_2 = \frac{1}{49}\big[(2)(3) + (3)(-6) + (6)(2)\big] = \frac{6-18+12}{49} = \frac{0}{49} = 0.$$

$$\vec{u}_2 \cdot \vec{u}_3 = \frac{1}{49}\big[(3)(6) + (-6)(2) + (2)(-3)\big] = \frac{18-12-6}{49} = 0.$$

$$\vec{u}_3 \cdot \vec{u}_1 = \frac{1}{49}\big[(6)(2) + (2)(3) + (-3)(6)\big] = \frac{12+6-18}{49} = 0.$$

Step 4. All three dot products vanish, so the vectors are mutually perpendicular.

> **Final Answer:** Each $|\vec{u}_i| = 1$ and $\vec{u}_i \cdot \vec{u}_j = 0$ for $i \neq j$, so the three vectors are mutually perpendicular unit vectors.

♥ Orthonormal basis

A set of three mutually perpendicular unit vectors in 3D is an *orthonormal basis*. Any vector $\vec{v}$ in 3D can be uniquely written as $\vec{v} = (\vec{v}\cdot\vec{u}_1)\vec{u}_1 + (\vec{v}\cdot\vec{u}_2)\vec{u}_2 + (\vec{v}\cdot\vec{u}_3)\vec{u}_3$. The standard basis $\hat{i}, \hat{j}, \hat{k}$ is just one example.

EXPERT'S SOLUTION *: Aditi Bhat, M.Sc Mathematics, IIT Bombay*

Strategic angle. Six checks: three unit-vector tests, three perpendicularity tests.

Step 1. Each vector's component-squares sum to 49; with the $\frac{1}{7}$ outside, the magnitude is 1.

Step 2. Pairwise dot products: $(2, 3, 6) \cdot (3, -6, 2) = 6 - 18 + 12 = 0$. By symmetry the other two pairs also give 0.

Why this matters. The three vectors form an orthonormal basis - a "rotated" version of $\hat{i}, \hat{j}, \hat{k}$.

> **Final Answer:** Three orthonormal vectors.

Q 10.6 **Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$.**

SOLUTION

Concept used. For any two vectors,

$$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} - \vec{b}\cdot\vec{b} = |\vec{a}|^2 - |\vec{b}|^2,$$

because dot product is commutative ($\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$) and $\vec{a}\cdot\vec{a} = |\vec{a}|^2$.

Step 1. Use the identity:

$$|\vec{a}|^2 - |\vec{b}|^2 = 8.$$

Step 2. Substitute $|\vec{a}| = 8|\vec{b}|$:

$$(8|\vec{b}|)^2 - |\vec{b}|^2 = 8 \implies 64|\vec{b}|^2 - |\vec{b}|^2 = 8 \implies 63|\vec{b}|^2 = 8.$$

Step 3. Solve for the magnitude of $\vec{b}$:

$$|\vec{b}|^2 = \frac{8}{63}.$$

Taking square roots, $|\vec{b}| = \dfrac{2\sqrt{2}}{3\sqrt{7}}$ (after simplifying $\sqrt{8/63}$).

Step 4. Then $|\vec{a}| = 8|\vec{b}| = \dfrac{16\sqrt{2}}{3\sqrt{7}}$.

Final Answer: $|\vec{a}| = \dfrac{16\sqrt{2}}{3\sqrt{7}}, \quad |\vec{b}| = \dfrac{2\sqrt{2}}{3\sqrt{7}}.$

EXPERT'S SOLUTION *: Rohit Verma, M.Sc Mathematics, ISI Kolkata*

Strategic angle. The identity $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$ collapses the data to a single equation in $|\vec{b}|$.

Step 1. $|\vec{a}|^2 - |\vec{b}|^2 = 8$.

Step 2. With $|\vec{a}| = 8|\vec{b}|$: $63|\vec{b}|^2 = 8$.

Step 3. $|\vec{b}|^2 = 8/63 \Rightarrow |\vec{b}| = 2\sqrt{2}/(3\sqrt{7})$, $|\vec{a}| = 16\sqrt{2}/(3\sqrt{7})$.

Why this matters. Vector identities mimic algebraic identities; recognising $(\vec{a}+\vec{b})(\vec{a}-\vec{b}) \leftrightarrow a^2 - b^2$ saves component-by-component computation.

Final Answer: $|\vec{a}| = 16\sqrt{2}/(3\sqrt{7}),\ |\vec{b}| = 2\sqrt{2}/(3\sqrt{7})$.

Q 10.7 **Evaluate the product** $(3\vec{a} - 5\vec{b})\cdot(2\vec{a} + 7\vec{b})$**.**

SOLUTION

Concept used. The dot product is bilinear (distributive over both arguments and pulls out scalars). Treat the expression like the product of two binomials in algebra, replacing $a \cdot a$ by $|\vec{a}|^2$, $b \cdot b$ by $|\vec{b}|^2$, and using $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.

Step 1. Expand:

$$(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b}) = 3\vec{a} \cdot 2\vec{a} + 3\vec{a} \cdot 7\vec{b} - 5\vec{b} \cdot 2\vec{a} - 5\vec{b} \cdot 7\vec{b}.$$

Step 2. Pull out scalar factors and use commutativity of dot product:

$$= 6\,(\vec{a} \cdot \vec{a}) + 21\,(\vec{a} \cdot \vec{b}) - 10\,(\vec{b} \cdot \vec{a}) - 35\,(\vec{b} \cdot \vec{b}).$$

Step 3. Combine like terms ($\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$):

$$= 6|\vec{a}|^2 + (21 - 10)\,(\vec{a} \cdot \vec{b}) - 35|\vec{b}|^2 = 6|\vec{a}|^2 + 11\,(\vec{a} \cdot \vec{b}) - 35|\vec{b}|^2.$$

Final Answer: $(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b}) = 6|\vec{a}|^2 + 11\,(\vec{a} \cdot \vec{b}) - 35|\vec{b}|^2$.

EXPERT'S SOLUTION : *Tara Reddy, M.Sc Mathematics, IIT Bombay*

Quick reading. Treat the dot product like multiplication, but $a^2 \to |\vec{a}|^2$.

Step 1. Expand FOIL-style: $6a \cdot a + 21a \cdot b - 10b \cdot a - 35b \cdot b$.

Step 2. Combine: $6|\vec{a}|^2 + 11(\vec{a} \cdot \vec{b}) - 35|\vec{b}|^2$.

Why this matters. Identities like $(\vec{a} \pm \vec{b}) \cdot (\vec{a} \pm \vec{b})$ and $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b})$ reduce many problems to algebraic manipulation on magnitudes and one dot product.

Final Answer: $6|\vec{a}|^2 + 11(\vec{a} \cdot \vec{b}) - 35|\vec{b}|^2$.

Q 10.8 **Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is $\frac{1}{2}$.**

SOLUTION

Concept used. $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$.

Step 1. Let $|\vec{a}| = |\vec{b}| = r$ (same magnitude).

Step 2. Substitute into the dot-product formula with $\theta = 60^\circ$ ($\cos 60^\circ = 1/2$):

$$\vec{a} \cdot \vec{b} = r \cdot r \cdot \tfrac{1}{2} = \tfrac{r^2}{2}.$$

Step 3. Given $\vec{a}\cdot\vec{b}=\frac{1}{2}$:

$$\frac{r^2}{2}=\frac{1}{2} \implies r^2=1 \implies r=1.$$

Final Answer: $|\vec{a}|=|\vec{b}|=1$.

EXPERT'S SOLUTION *: Karan Iyer, M.Sc Mathematics, IIT Bombay*

Strategic angle. Single unknown r, one equation: $r^2\cos 60^\circ=$ given.

Step 1. $r^2\cdot\frac{1}{2}=\frac{1}{2}$.

Step 2. $r=1$.

Why this matters. When magnitudes are equal, $\vec{a}\cdot\vec{b}=r^2\cos\theta$, so the dot product directly tells you r^2 once θ is known.

Final Answer: 1 (each).

Q 10.9 **Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=12$.**

SOLUTION

Concept used. Expand $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})$ using $(\vec{u}-\vec{v})\cdot(\vec{u}+\vec{v})=|\vec{u}|^2-|\vec{v}|^2$.

Step 1. Apply the identity:

$$(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=|\vec{x}|^2-|\vec{a}|^2.$$

Step 2. Since $\vec{a}$ is a unit vector, $|\vec{a}|=1$:

$$|\vec{x}|^2-1=12 \implies |\vec{x}|^2=13.$$

Step 3. Hence $|\vec{x}|=\sqrt{13}$ (magnitude is non-negative).

Final Answer: $|\vec{x}|=\sqrt{13}$.

EXPERT'S SOLUTION *: Aanya Sharma, M.Sc Mathematics, IIT Bombay*

Quick reading. Difference-of-squares identity.

Step 1. $|\vec{x}|^2-1=12 \Rightarrow |\vec{x}|=\sqrt{13}$.

Why this matters. The $|\vec{u}|^2-|\vec{v}|^2$ identity is a workhorse - any problem of the form

$(\vec{u}-\vec{v})\cdot(\vec{u}+\vec{v})$ collapses to a one-line equation.

Final Answer: $\sqrt{13}$.

Q 10.10 **If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.**

SOLUTION

Concept used. $\vec{u} \perp \vec{v} \iff \vec{u}\cdot\vec{v} = 0$. Compute $\vec{a}+\lambda\vec{b}$ in component form and set its dot product with $\vec{c}$ to zero.

Step 1. Form $\vec{a}+\lambda\vec{b}$ component-wise:

$$\vec{a}+\lambda\vec{b} = (2-\lambda)\hat{i} + (2+2\lambda)\hat{j} + (3+\lambda)\hat{k}.$$

Step 2. Compute $(\vec{a}+\lambda\vec{b})\cdot\vec{c}$ with $\vec{c} = 3\hat{i} + \hat{j} + 0\,\hat{k}$:

$$(\vec{a}+\lambda\vec{b})\cdot\vec{c} = (2-\lambda)(3) + (2+2\lambda)(1) + (3+\lambda)(0).$$

Step 3. Expand:

$$= 6 - 3\lambda + 2 + 2\lambda + 0 = 8 - \lambda.$$

Step 4. Set to zero:

$$8 - \lambda = 0 \implies \lambda = 8.$$

Final Answer: $\lambda = 8$.

EXPERT'S SOLUTION *: Pranav Joshi, M.Sc Mathematics, IIT Bombay*

Strategic angle. One linear equation in λ from one perpendicularity condition.

Step 1. Dot $(\vec{a}+\lambda\vec{b})$ with $\vec{c}$: $(2-\lambda)(3) + (2+2\lambda)(1) = 8 - \lambda$.

Step 2. Set $= 0 \Rightarrow \lambda = 8$.

Why this matters. Perpendicularity is one scalar equation; one unknown is determined by one equation, so the problem is well-posed.

Final Answer: $\lambda = 8$.

Q 10.11 **Show that $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$, for any two non-zero vectors $\vec{a}$ and $\vec{b}$.**

SOLUTION

Concept used. Two vectors are perpendicular iff their dot product is zero. Use the identity $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = |\vec{u}|^2 - |\vec{v}|^2$ with $\vec{u} = |\vec{a}|\vec{b}$ and $\vec{v} = |\vec{b}|\vec{a}$.

Step 1. Let $\vec{u} = |\vec{a}|\,\vec{b}$ and $\vec{v} = |\vec{b}|\,\vec{a}$. Then the given expression is $(\vec{u}+\vec{v})$ and $(\vec{u}-\vec{v})$.

Step 2. Apply the identity:

$$(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = |\vec{u}|^2 - |\vec{v}|^2.$$

Step 3. Compute $|\vec{u}|^2$:

$$|\vec{u}|^2 = \left(|\vec{a}|\right)^2 |\vec{b}|^2 = |\vec{a}|^2\,|\vec{b}|^2.$$

Similarly,

$$|\vec{v}|^2 = |\vec{b}|^2\,|\vec{a}|^2 = |\vec{a}|^2\,|\vec{b}|^2.$$

Step 4. So $|\vec{u}|^2 - |\vec{v}|^2 = 0$.

The dot product of the two given vectors is zero, hence they are perpendicular.

> **Final Answer:** $\left(|\vec{a}|\vec{b} + |\vec{b}|\vec{a}\right)\cdot\left(|\vec{a}|\vec{b} - |\vec{b}|\vec{a}\right) = 0$, so the two vectors are perpendicular.

EXPERT'S SOLUTION *: Ishaan Desai, M.Sc Mathematics, IIT Bombay*

Structural observation. Set $\vec{u} = |\vec{a}|\vec{b}$, $\vec{v} = |\vec{b}|\vec{a}$; note both have the same magnitude $|\vec{a}||\vec{b}|$.

Step 1. $|\vec{u}| = |\vec{a}|\cdot|\vec{b}| = |\vec{v}|$.

Step 2. $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = |\vec{u}|^2 - |\vec{v}|^2 = 0$.

Why this matters. Sum and difference of two equal-magnitude vectors are always perpendicular - the same fact powers the diagonals of a rhombus.

> **Final Answer:** Dot product $= 0$, hence perpendicular.

Q 10.12 **If $\vec{a}\cdot\vec{a} = 0$ and $\vec{a}\cdot\vec{b} = 0$, then what can be concluded about the vector $\vec{b}$?**

SOLUTION

Concept used. $\vec{a}\cdot\vec{a} = |\vec{a}|^2$, and $|\vec{a}|^2 = 0 \iff \vec{a} = \vec{0}$. If $\vec{a}$ is the zero vector, then $\vec{a}\cdot\vec{b} = 0$ holds for *every* $\vec{b}$.

Step 1. From $\vec{a}\cdot\vec{a}=0$: $|\vec{a}|^2=0$, so $|\vec{a}|=0$, i.e. $\vec{a}=\vec{0}$ (the zero vector).

Step 2. Substitute into the second condition: $\vec{0}\cdot\vec{b}=0$. This is trivially true regardless of $\vec{b}$.

Step 3. Therefore $\vec{b}$ can be *any* vector.

Final Answer: $\vec{a}=\vec{0}$, so $\vec{b}$ may be any vector.

EXPERT'S SOLUTION : *Diya Reddy, M.Sc Mathematics, ISI Kolkata*

Quick reading. "$\vec{a}\cdot\vec{a}=0$" is the only condition that survives; it forces $\vec{a}$ to be zero.

Step 1. $|\vec{a}|^2=0\Rightarrow\vec{a}=\vec{0}$.

Step 2. $\vec{0}\cdot\vec{b}=0$ for any $\vec{b}$.

Why this matters. The dot product can be zero either because one vector is zero or because the two are perpendicular - here the first condition picks the first case.

Final Answer: $\vec{b}$ is arbitrary.

Q 10.13 **If $\vec{a},\vec{b},\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, find the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$.**

SOLUTION

Concept used. Take the dot product of $(\vec{a}+\vec{b}+\vec{c})$ with itself; the result equals $|\vec{a}+\vec{b}+\vec{c}|^2$, but the given hypothesis says this is $\vec{0}\cdot\vec{0}=0$.

Step 1. Expand $(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$:

$$|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2\,(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0.$$

Step 2. Substitute the unit-vector data $|\vec{a}|=|\vec{b}|=|\vec{c}|=1$:

$$1+1+1+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0.$$

Step 3. Simplify:

$$3+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0\implies\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=-\tfrac{3}{2}.$$

Final Answer: $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=-\dfrac{3}{2}$.

♥ Equilateral triangle of unit vectors

Three unit vectors summing to zero form a closed triangle on the unit sphere with $120°$ between any pair. Then each pairwise dot product is $\cos 120° = -\frac{1}{2}$, and their sum is $-\frac{3}{2}$.

EXPERT'S SOLUTION *: Aditi Mehta, M.Sc Mathematics, IIT Bombay*

Strategic angle. Square the given equation (in the vector sense), use the unit-vector data, isolate the symmetric sum.

Step 1. $|\vec{a}+\vec{b}+\vec{c}|^2 = 0$.

Step 2. Expansion gives $3 + 2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = 0$.

Step 3. $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a} = -3/2$.

Why this matters. The same "square the sum" trick works for any number of vectors with constraints on the sum.

Final Answer: $-3/2$.

Q 10.14 If either vector $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}\cdot\vec{b}=0$. But the converse need not be true. Justify your answer with an example.

SOLUTION

Concept used. The dot product $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ vanishes when (i) $|\vec{a}|=0$, or (ii) $|\vec{b}|=0$, or (iii) $\cos\theta = 0$ (i.e. the vectors are perpendicular). The first two cases give the implication in the statement; the third is the source of counter-examples to the converse.

Step 1. **Forward direction.** If $\vec{a}=\vec{0}$, then for any $\vec{b}$, $\vec{0}\cdot\vec{b}=0$. Similarly if $\vec{b}=\vec{0}$. So $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$ implies $\vec{a}\cdot\vec{b}=0$.

Step 2. **Counter-example to the converse.** Take $\vec{a}=\hat{i}$ (non-zero) and $\vec{b}=\hat{j}$ (non-zero):

$$\vec{a}\cdot\vec{b} = (1)(0)+(0)(1)+(0)(0) = 0.$$

Here both $\vec{a}$ and $\vec{b}$ are non-zero, but $\vec{a}\cdot\vec{b}=0$.

Step 3. Hence the converse "$\vec{a}\cdot\vec{b}=0 \Rightarrow \vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$" is false.

Final Answer: Counter-example: $\vec{a}=\hat{i}$, $\vec{b}=\hat{j}$. Both are non-zero, but $\vec{a}\cdot\vec{b}=0$ (they are perpendicular).

EXPERT'S SOLUTION : *Sneha Rao, M.Sc Mathematics, IIT Bombay*

Strategic angle. Recall the three ways the dot product can be zero; the perpendicular case kills the converse.

Step 1. $\hat{i} \cdot \hat{j} = 0$, yet $|\hat{i}| = |\hat{j}| = 1 \neq 0$.

Why this matters. A zero dot product carries one of two pieces of information - either a vector is zero, or the vectors are perpendicular; you need additional data to choose between them.

> **Final Answer:** $\hat{i} \cdot \hat{j} = 0$ disproves the converse.

Q 10.15 **If the vertices A, B, C of a triangle ABC are $(1, 2, 3), (-1, 0, 0), (0, 1, 2)$ respectively, then find $\angle ABC$. [$\angle ABC$ is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.]**

SOLUTION

Concept used. The angle at vertex B in triangle ABC is the angle between $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Compute these as $A - B$ and $C - B$, then apply $\cos\theta = \dfrac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|}$.

Step 1. Compute $\overrightarrow{BA}$ and $\overrightarrow{BC}$:

$$\overrightarrow{BA} = A - B = (1 - (-1))\hat{i} + (2 - 0)\hat{j} + (3 - 0)\hat{k} = 2\hat{i} + 2\hat{j} + 3\hat{k}.$$

$$\overrightarrow{BC} = C - B = (0 - (-1))\hat{i} + (1 - 0)\hat{j} + (2 - 0)\hat{k} = \hat{i} + \hat{j} + 2\hat{k}.$$

Step 2. Dot product:

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = (2)(1) + (2)(1) + (3)(2) = 2 + 2 + 6 = 10.$$

Step 3. Magnitudes:

$$|\overrightarrow{BA}| = \sqrt{2^2 + 2^2 + 3^2} = \sqrt{4 + 4 + 9} = \sqrt{17}.$$
$$|\overrightarrow{BC}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{1 + 1 + 4} = \sqrt{6}.$$

Step 4. Apply the cosine formula:

$$\cos(\angle ABC) = \frac{10}{\sqrt{17} \cdot \sqrt{6}} = \frac{10}{\sqrt{102}}.$$

Step 5. Hence $\angle ABC = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$.

> **Final Answer:** $\angle ABC = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$.

EXPERT'S SOLUTION : *Yash Iyer, M.Sc Mathematics, IIT Bombay*

Strategic angle. At any vertex V of a triangle the angle is between the two outgoing edge-vectors from V.

Step 1. $\overrightarrow{BA} = (2, 2, 3),\ \overrightarrow{BC} = (1, 1, 2)$.

Step 2. Dot $= 10$; magnitudes $\sqrt{17}, \sqrt{6}$.

Step 3. $\angle ABC = \cos^{-1}(10/\sqrt{102})$.

Why this matters. The cosine formula bypasses the law of cosines (which needs three side-lengths); two vectors plus a dot product give the angle directly.

Final Answer: $\cos^{-1}(10/\sqrt{102})$.

Q 10.16 **Show that the points $A(1, 2, 7), B(2, 6, 3)$ and $C(3, 10, -1)$ are collinear.**

SOLUTION

Concept used. Three points A, B, C are collinear iff $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel (i.e. $\overrightarrow{BC} = \lambda\,\overrightarrow{AB}$ for some scalar λ), equivalently iff $|\overrightarrow{AC}| = |\overrightarrow{AB}| + |\overrightarrow{BC}|$ when B lies between A and C.

Step 1. Compute side vectors:

$$\overrightarrow{AB} = (2-1, 6-2, 3-7) = (1, 4, -4).$$

$$\overrightarrow{BC} = (3-2, 10-6, -1-3) = (1, 4, -4).$$

$$\overrightarrow{AC} = (3-1, 10-2, -1-7) = (2, 8, -8).$$

Step 2. Observe that $\overrightarrow{BC} = \overrightarrow{AB}$ (component-by-component) and $\overrightarrow{AC} = 2\,\overrightarrow{AB}$.

Step 3. Since $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ are scalar multiples of one another, all three points lie on a single straight line.

Step 4. Verify magnitudes:

$$|\overrightarrow{AB}| = \sqrt{1+16+16} = \sqrt{33}.$$

$$|\overrightarrow{BC}| = \sqrt{1+16+16} = \sqrt{33}.$$

$$|\overrightarrow{AC}| = \sqrt{4+64+64} = \sqrt{132} = 2\sqrt{33}.$$

Hence $|\overrightarrow{AB}| + |\overrightarrow{BC}| = 2\sqrt{33} = |\overrightarrow{AC}|$. ✓

Final Answer: $\overrightarrow{AB} \parallel \overrightarrow{BC}$ and $|\overrightarrow{AC}| = |\overrightarrow{AB}| + |\overrightarrow{BC}|$, so A, B, C are collinear.

EXPERT'S SOLUTION : *Pooja Gupta, M.Sc Mathematics, IIT Bombay*

Strategic angle. Compute $\overrightarrow{AB}$ and $\overrightarrow{BC}$; if they are identical (or proportional), the three points are collinear.

Step 1. $\overrightarrow{AB} = (1, 4, -4) = \overrightarrow{BC}$.

Step 2. Same direction, same magnitude: B is the midpoint of AC; the three points lie on a common line.

Why this matters. The cleanest collinearity test in 3D is "$\overrightarrow{AB}$ parallel to $\overrightarrow{AC}$". If they happen to be equal, B is also the midpoint.

Final Answer: $\overrightarrow{AB} = \overrightarrow{BC}$, so collinear.

Q 10.17 **Show that the vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ form the vertices of a right-angled triangle.**

SOLUTION

Concept used. Treat the three given vectors as the position vectors of A, B, C. Compute the three side-vectors $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CA}$, find their squared lengths, and apply the converse of Pythagoras: if one squared length equals the sum of the other two, the triangle is right-angled.

Step 1. Let $\vec{A} = 2\hat{i} - \hat{j} + \hat{k},\ \vec{B} = \hat{i} - 3\hat{j} - 5\hat{k},\ \vec{C} = 3\hat{i} - 4\hat{j} - 4\hat{k}$.

Step 2. Compute the side vectors:

$$\overrightarrow{AB} = \vec{B} - \vec{A} = (-1, -2, -6).$$

$$\overrightarrow{BC} = \vec{C} - \vec{B} = (2, -1, 1).$$

$$\overrightarrow{CA} = \vec{A} - \vec{C} = (-1, 3, 5).$$

Step 3. Squared lengths:

$$|\overrightarrow{AB}|^2 = 1 + 4 + 36 = 41.$$

$$|\overrightarrow{BC}|^2 = 4 + 1 + 1 = 6.$$

$$|\overrightarrow{CA}|^2 = 1 + 9 + 25 = 35.$$

Step 4. Test: $|\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2 = 6 + 35 = 41 = |\overrightarrow{AB}|^2$.

Step 5. Hence the triangle is right-angled, with the right angle at the vertex C (opposite to the longest side AB).

Final Answer: $|\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2 = |\overrightarrow{AB}|^2$, so it is a right-angled triangle (right angle at C).

EXPERT'S SOLUTION *: Krishna Verma, M.Sc Mathematics, IIT Bombay*

Strategic angle. Squared side-lengths, then Pythagoras.

Step 1. $|AB|^2 = 41,\ |BC|^2 = 6,\ |CA|^2 = 35.$

Step 2. $6 + 35 = 41 = |AB|^2$, confirms right angle at C.

Why this matters. Working with squared lengths sidesteps surds and is the practical way to check Pythagoras in coordinates.

Final Answer: Right-angled at C.

Q 10.18 **If $\vec{a}$ is a non-zero vector of magnitude 'a' and λ a non-zero scalar, then $\lambda\vec{a}$ is unit vector if**

(A) $\lambda = 1$ **(B)** $\lambda = -1$ **(C)** $a = |\lambda|$ **(D)** $a = 1/|\lambda|$.

SOLUTION

Concept used. The magnitude of $\lambda\vec{a}$ is $|\lambda||\vec{a}| = |\lambda|\, a$. The vector $\lambda\vec{a}$ is a unit vector iff this magnitude equals 1.

Step 1. Compute $|\lambda\vec{a}|$:

$$|\lambda\vec{a}| = |\lambda| \cdot |\vec{a}| = |\lambda| \cdot a.$$

Step 2. Set equal to 1:

$$|\lambda| \cdot a = 1 \iff a = \frac{1}{|\lambda|}.$$

Step 3. Compare with the options - this matches **(D)**.

Final Answer: Option **(D)** $a = 1/|\lambda|$.

EXPERT'S SOLUTION *: Ananya Patel, M.Sc Mathematics, IIT Bombay*

Quick reading. Multiply magnitudes; require the product equal 1.

Step 1. $|\lambda\vec{a}| = |\lambda|\, a.$

Step 2. Unit iff $|\lambda|\, a = 1 \Leftrightarrow a = 1/|\lambda|$.

Why this matters. Scaling a vector by λ stretches the magnitude by $|\lambda|$; to land exactly at length 1, the original length must reciprocate $|\lambda|$.

Final Answer: (D).

Key Takeaways

- $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta = a_1b_1 + a_2b_2 + a_3b_3$.
- Angle: $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$. Projection of $\vec{a}$ on $\vec{b}$: $\dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$.
- Perpendicularity test: $\vec{a} \cdot \vec{b} = 0$ (when both vectors are non-zero).
- Identity $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = |\vec{u}|^2 - |\vec{v}|^2$ is the workhorse for magnitude problems.
- Three unit vectors summing to zero $\Rightarrow$ pairwise angles $120°$, pairwise dot products $-1/2$.

End of Exercise 10.3