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Step-by-step solutions for the 2026-27 NCERT (Latest Edition), Class 12 Mathematics

Chapter 10: Vector Algebra

About this Chapter

Exercise 10.4 develops the **vector (cross) product** $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$, evaluated as a 3×3 determinant. You will use it to find perpendicular unit vectors, compute areas of triangles and parallelograms, and verify distributivity-type identities.

Topics covered: Cross product • Determinant form • Area of triangle/parallelogram • Perpendicular vectors

Quick Formula Sheet

Definition:
$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$, where $\hat{n}$ is the unit normal forming a right-handed system with $\vec{a}, \vec{b}$.

Determinant form:
$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$.

Area of triangle: $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$.
Area of parallelogram: $|\vec{a} \times \vec{b}|$ for adjacent sides $\vec{a}, \vec{b}$.

$\vec{a} \parallel \vec{b}$: $\vec{a} \times \vec{b} = \vec{0}$.

Exercise 10.4

Q 10.1 **Find $|\vec{a} \times \vec{b}|$, if $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$.**

SOLUTION

Concept used. The cross product of $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Expanding by the first row gives a vector whose magnitude is then computed by the usual Pythagoras formula.

Step 1. Set up the determinant:

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix}.$$

Step 2. Expand along the first row:

$$\vec{a}\times\vec{b} = \hat{i}\big[(-7)(2)-(7)(-2)\big] - \hat{j}\big[(1)(2)-(7)(3)\big] + \hat{k}\big[(1)(-2)-(-7)(3)\big].$$

Compute each minor:

$$\hat{i}\text{-minor} : -14+14 = 0.$$

$$\hat{j}\text{-minor} : 2-21 = -19,\ \text{prefixed by } - \Rightarrow +19.$$

$$\hat{k}\text{-minor} : -2+21 = 19.$$

Step 3. Combine:

$$\vec{a}\times\vec{b} = 0\,\hat{i} + 19\,\hat{j} + 19\,\hat{k} = 19\hat{j} + 19\hat{k}.$$

Step 4. Magnitude:

$$|\vec{a}\times\vec{b}| = \sqrt{0^2+19^2+19^2} = \sqrt{2\cdot 19^2} = 19\sqrt{2}.$$

> **Final Answer:** $|\vec{a}\times\vec{b}| = 19\sqrt{2}$.

✗ Common Mistake

A sign slip in the $\hat{j}$-term is the commonest error in cross-product calculations. The cofactor expansion has alternating signs $+,-,+$ on $\hat{i},\hat{j},\hat{k}$; do not forget the minus on the middle term.

EXPERT'S SOLUTION *: Aarav Mehta, M.Sc Mathematics, IIT Bombay*

Strategic angle. Determinant expansion + magnitude.

Step 1. Cofactors: $\hat{i} : (-14)-(-14) = 0$; $\hat{j} : -(2-21) = 19$; $\hat{k} : (-2)-(-21) = 19$.

Step 2. $\vec{a}\times\vec{b} = (0, 19, 19)$.

Step 3. Magnitude $= \sqrt{19^2+19^2} = 19\sqrt{2}$.

Why this matters. The vector $\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a},\vec{b}$ and its length is the area of the parallelogram they span; here the parallelogram has area $19\sqrt{2}$.

> **Final Answer:** $19\sqrt{2}$.

Q 10.2 **Find a unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$.**

SOLUTION

Concept used. The vector $(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})$ is perpendicular to both $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ (cross product is always orthogonal to its operands). Dividing by its magnitude gives the desired unit perpendicular.

Step 1. Compute $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$:

$$\vec{a}+\vec{b}=(3+1)\hat{i}+(2+2)\hat{j}+(2-2)\hat{k}=4\hat{i}+4\hat{j}+0\,\hat{k}.$$

$$\vec{a}-\vec{b}=(3-1)\hat{i}+(2-2)\hat{j}+(2-(-2))\hat{k}=2\hat{i}+0\,\hat{j}+4\hat{k}.$$

Step 2. Cross product:

$$(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 4 & 0\\ 2 & 0 & 4\end{vmatrix}.$$

Expand by the first row:

$$\hat{i}\text{-minor}:(4)(4)-(0)(0)=16.$$

$$\hat{j}\text{-minor}:(4)(4)-(0)(2)=16,\ \text{with sign }-\Rightarrow -16.$$
$$\hat{k}\text{-minor}:(4)(0)-(4)(2)=-8.$$

So

$$(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})=16\hat{i}-16\hat{j}-8\hat{k}.$$

Step 3. Magnitude:

$$|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})|=\sqrt{16^2+(-16)^2+(-8)^2}=\sqrt{256+256+64}=\sqrt{576}=24.$$

Step 4. Unit vector:

$$\hat{n}=\frac{16\hat{i}-16\hat{j}-8\hat{k}}{24}=\frac{2}{3}\hat{i}-\frac{2}{3}\hat{j}-\frac{1}{3}\hat{k}.$$

Final Answer: $\hat{n}=\frac{2}{3}\hat{i}-\frac{2}{3}\hat{j}-\frac{1}{3}\hat{k}$ (or its negative).

EXPERT'S SOLUTION *: Sneha Banerjee, M.Sc Mathematics, IIT Bombay*

Strategic angle. Cross product of the two given combinations is orthogonal to both; normalise.

Step 1. $\vec{a}+\vec{b}=(4,4,0)$, $\vec{a}-\vec{b}=(2,0,4)$.

Step 2. Cross product $= (16, -16, -8)$.

Step 3. Magnitude $= 24$. Unit vector $= (16, -16, -8)/24 = (2/3, -2/3, -1/3)$.

Why this matters. There are exactly two unit vectors perpendicular to a plane (positive and negative). Reversing the order of the cross product flips the sign.

> **Final Answer:** $\pm\left(\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}\right)$.

Q 10.3 **If a unit vector $\vec{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence the components of $\vec{a}$.**

SOLUTION

Concept used. If $\vec{a}$ is a unit vector making angles α, β, γ with the x, y, z axes, then its direction cosines $\cos\alpha, \cos\beta, \cos\gamma$ are the components of $\vec{a}$, and they satisfy

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Step 1. Given $\alpha = \pi/3$, $\beta = \pi/4$, $\gamma = \theta$ (acute).

Step 2. Substitute:

$$\cos^2\left(\frac{\pi}{3}\right) + \cos^2\left(\frac{\pi}{4}\right) + \cos^2\theta = 1.$$
$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\theta = 1.$$
$$\frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1.$$
$$\cos^2\theta = 1 - \frac{3}{4} = \frac{1}{4}.$$

Step 3. Take the positive root (acute angle $\Rightarrow \cos\theta > 0$):

$$\cos\theta = \frac{1}{2}, \quad \theta = \frac{\pi}{3}.$$

Step 4. Components of $\vec{a}$:

$$\vec{a} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\theta\,\hat{k} = \frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}.$$

> **Final Answer:** $\theta = \dfrac{\pi}{3}$; $\quad \vec{a} = \frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}$.

EXPERT'S SOLUTION : *Aanya Iyer, M.Sc Mathematics, IIT Bombay*

Strategic angle. The identity $l^2 + m^2 + n^2 = 1$ uniquely determines the third direction cosine (up to sign); the acute-angle constraint picks the sign.

Step 1. Two cosines: $\cos(\pi/3) = 1/2$, $\cos(\pi/4) = 1/\sqrt{2}$.

Step 2. Third: $\cos^2\theta = 1 - 1/4 - 1/2 = 1/4 \Rightarrow \cos\theta = 1/2 \Rightarrow \theta = \pi/3$.

Step 3. $\vec{a} = (1/2, 1/\sqrt{2}, 1/2)$.

Why this matters. Specifying any two direction cosines determines the third only up to a sign; the "acute angle" condition fixes that sign.

> **Final Answer:** $\theta = \pi/3$; $\vec{a} = \frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}$.

Q 10.4 **Show that** $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$.

SOLUTION

Concept used. Cross product is **distributive over addition**: $(\vec{u} \pm \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} \pm \vec{v} \times \vec{w}$. Also $\vec{a} \times \vec{a} = \vec{0}$ (parallel vectors) and $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$ (anticommutative).

Step 1. Distribute the left side:

$$(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = (\vec{a} - \vec{b}) \times \vec{a} + (\vec{a} - \vec{b}) \times \vec{b}.$$

Step 2. Distribute again:

$$= \vec{a} \times \vec{a} - \vec{b} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{b}.$$

Step 3. Use $\vec{a} \times \vec{a} = \vec{0}$ and $\vec{b} \times \vec{b} = \vec{0}$:

$$= \vec{0} - \vec{b} \times \vec{a} + \vec{a} \times \vec{b} - \vec{0}.$$

Step 4. Use $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$:

$$= -(-\vec{a} \times \vec{b}) + \vec{a} \times \vec{b} = \vec{a} \times \vec{b} + \vec{a} \times \vec{b} = 2(\vec{a} \times \vec{b}).$$

> **Final Answer:** $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$.

EXPERT'S SOLUTION : *Pranav Reddy, M.Sc Mathematics, IIT Bombay*

Strategic angle. Expand like algebra; use $\vec{x} \times \vec{x} = \vec{0}$ and anticommutativity.

Step 1. $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = \vec{a} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{a} - \vec{b} \times \vec{b}$.

Step 2. Self-cross terms vanish; $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$.

Step 3. Result: $\vec{a} \times \vec{b} + \vec{a} \times \vec{b} = 2(\vec{a} \times \vec{b})$.

Why this matters. The cross product is bilinear and anticommutative; both rules together produce these factor-of-2 identities just as $(a - b)(a + b) = a^2 - b^2$ does for ordinary multiplication.

> **Final Answer:** $2(\vec{a} \times \vec{b})$.

Q 10.5 **Find λ and μ if $(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}$.**

SOLUTION

Concept used. $\vec{u} \times \vec{v} = \vec{0}$ iff $\vec{u} \parallel \vec{v}$, i.e. corresponding components are proportional: $\frac{u_1}{v_1} = \frac{u_2}{v_2} = \frac{u_3}{v_3}$ (when no $v_i = 0$).

Step 1. Cross product is zero $\Rightarrow$ vectors are parallel. Components of the two vectors:

$$(2, 6, 27) \quad \text{and} \quad (1, \lambda, \mu).$$

Step 2. Equate ratios:

$$\frac{2}{1} = \frac{6}{\lambda} = \frac{27}{\mu}.$$

Step 3. From $\frac{2}{1} = \frac{6}{\lambda}$: $2\lambda = 6 \Rightarrow \lambda = 3$.

Step 4. From $\frac{2}{1} = \frac{27}{\mu}$: $2\mu = 27 \Rightarrow \mu = \frac{27}{2}$.

> **Final Answer:** $\lambda = 3,\ \mu = \frac{27}{2}$.

EXPERT'S SOLUTION : *Riya Singh, M.Sc Mathematics, IIT Bombay*

Quick reading. Zero cross product $\Rightarrow$ parallel $\Rightarrow$ proportional components.

Step 1. Ratios: $2/1 = 6/\lambda = 27/\mu = 2$.

Step 2. $\lambda = 3$, $\mu = 27/2$.

Why this matters. Two parallel non-zero vectors carry exactly one degree of freedom (the common ratio); fixing one ratio determines the rest.

Final Answer: $\lambda = 3,\ \mu = 27/2$.

Q 10.6 **Given that $\vec{a}\cdot\vec{b} = 0$ and $\vec{a}\times\vec{b} = \vec{0}$. What can you conclude about the vectors $\vec{a}$ and $\vec{b}$?**

SOLUTION

Concept used. For two vectors, the dot product is zero when either vector is zero *or* they are perpendicular; the cross product is zero when either vector is zero *or* they are parallel. Two non-zero vectors cannot be simultaneously perpendicular and parallel, so the only way both equations hold is if at least one of $\vec{a}, \vec{b}$ is the zero vector.

Step 1. Suppose both $\vec{a}, \vec{b}$ are non-zero with $|\vec{a}|, |\vec{b}| > 0$.

Step 2. $\vec{a}\cdot\vec{b} = 0 \Rightarrow$ they are perpendicular ($\theta = \pi/2$).

Step 3. $\vec{a}\times\vec{b} = \vec{0} \Rightarrow |\vec{a}||\vec{b}|\sin\theta = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$ or π.

Step 4. Conflict: θ cannot be both $\pi/2$ and 0 or π. Hence the assumption fails: at least one of $\vec{a}, \vec{b}$ is the zero vector.

Final Answer: At least one of $\vec{a}, \vec{b}$ must be the zero vector ($\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$).

EXPERT'S SOLUTION : *Aditi Verma, M.Sc Mathematics, IIT Bombay*

Strategic angle. Dot zero $\Leftrightarrow$ perpendicular (or a vector is zero); cross zero $\Leftrightarrow$ parallel (or a vector is zero). Both at once forces a zero vector.

Step 1. Assume both non-zero $\Rightarrow \theta = \pi/2$ from dot, but $\sin\theta = 0$ from cross.

Step 2. Contradiction $\Rightarrow$ at least one is the zero vector.

Why this matters. Dot and cross products give complementary information; combining their constraints often pins down the configuration uniquely (here, "one of them is zero").

Final Answer: $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$.

Q 10.7 **Let the vectors $\vec{a}, \vec{b}, \vec{c}$ be given as $a_1\hat{i} + a_2\hat{j} + a_3\hat{k},\ b_1\hat{i} + b_2\hat{j} + b_3\hat{k},\ c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$.**

Then show that $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$**.**

SOLUTION

Concept used. Cross product is **distributive over vector addition** on the right. We prove this in component form by computing both sides as 3×3 determinant expansions and observing they agree component-by-component.

Step 1. Compute $\vec{b}+\vec{c}$:

$$\vec{b}+\vec{c}=(b_1+c_1)\hat{i}+(b_2+c_2)\hat{j}+(b_3+c_3)\hat{k}.$$

Step 2. Compute $\vec{a}\times(\vec{b}+\vec{c})$ as a determinant:

$$\vec{a}\times(\vec{b}+\vec{c})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1+c_1 & b_2+c_2 & b_3+c_3\end{vmatrix}.$$

Expand:

$$\hat{i}\text{-comp}: a_2(b_3+c_3)-a_3(b_2+c_2)=(a_2b_3-a_3b_2)+(a_2c_3-a_3c_2).$$

$$\hat{j}\text{-comp}: -[a_1(b_3+c_3)-a_3(b_1+c_1)]=-(a_1b_3-a_3b_1)-(a_1c_3-a_3c_1).$$

$$\hat{k}\text{-comp}: a_1(b_2+c_2)-a_2(b_1+c_1)=(a_1b_2-a_2b_1)+(a_1c_2-a_2c_1).$$

Step 3. Each component is a sum of two terms. Group them:

$$\vec{a}\times(\vec{b}+\vec{c})=\underbrace{(a_2b_3-a_3b_2)\hat{i}-(a_1b_3-a_3b_1)\hat{j}+(a_1b_2-a_2b_1)\hat{k}}_{\vec{a}\times\vec{b}}$$

$$+\underbrace{(a_2c_3-a_3c_2)\hat{i}-(a_1c_3-a_3c_1)\hat{j}+(a_1c_2-a_2c_1)\hat{k}}_{\vec{a}\times\vec{c}}.$$

Step 4. Therefore $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$, as required.

> **Final Answer:** Proved: $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$.

EXPERT'S SOLUTION *: Rohit Nair, M.Sc Mathematics, IIT Bombay*

Strategic angle. Both sides are determinants of 3×3 matrices; the splitting follows from the row-linearity of determinants.

Step 1. Row-linearity: $\det[\hat{i},\hat{j},\hat{k};\vec{a};\vec{b}+\vec{c}]=\det[\hat{i},\hat{j},\hat{k};\vec{a};\vec{b}]+\det[\hat{i},\hat{j},\hat{k};\vec{a};\vec{c}]$.

Step 2. The two determinants on the right are $\vec{a}\times\vec{b}$ and $\vec{a}\times\vec{c}$ respectively.

Why this matters. Once you know determinants are multilinear in their rows/columns, every distributivity property of the cross product follows immediately.

Final Answer: Distributivity verified.

Q 10.8 If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a} \times \vec{b} = \vec{0}$. Is the converse true? Justify your answer with an example.

SOLUTION

Concept used. $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$. This vanishes when (i) $|\vec{a}| = 0$, (ii) $|\vec{b}| = 0$, or (iii) $\sin\theta = 0$ (i.e. $\vec{a} \parallel \vec{b}$). The first two cases give the forward implication; case (iii) breaks the converse.

Step 1. Forward. If $\vec{a} = \vec{0}$, then $\vec{a} \times \vec{b} = \vec{0} \times \vec{b} = \vec{0}$ for any $\vec{b}$. Similarly if $\vec{b} = \vec{0}$.

Step 2. Converse - counter-example. Take $\vec{a} = \hat{i}$ and $\vec{b} = 2\hat{i}$. Both are non-zero.

$$\vec{a} \times \vec{b} = \hat{i} \times (2\hat{i}) = 2(\hat{i} \times \hat{i}) = 2\vec{0} = \vec{0}.$$

So $\vec{a} \times \vec{b} = \vec{0}$ even though neither is zero. Thus the converse is **not true**.

Final Answer: Converse is false. Counter-example: $\vec{a} = \hat{i}$, $\vec{b} = 2\hat{i}$. Both non-zero, but $\vec{a} \times \vec{b} = \vec{0}$ (they are parallel).

EXPERT'S SOLUTION *: Tara Patel, M.Sc Mathematics, ISI Kolkata*

Strategic angle. Cross product zero ⇔ "parallel" (which includes "one is zero" as a degenerate case); pick parallel non-zero vectors.

Step 1. $\hat{i} \times (2\hat{i}) = 2(\hat{i} \times \hat{i}) = \vec{0}$.

Why this matters. A zero cross product carries one of two pieces of information; parallel-and-non-zero is the more interesting one in practice.

Final Answer: Converse false; e.g. $\hat{i} \times 2\hat{i} = \vec{0}$.

Q 10.9 Find the area of the triangle with vertices $A(1, 1, 2)$, $B(2, 3, 5)$ and $C(1, 5, 5)$.

SOLUTION

Concept used. Area of $\triangle ABC$ in 3D is

$$\text{Area} = \tfrac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|.$$

Step 1. Compute the two edge vectors from A:

$$\overrightarrow{AB} = B - A = (1, 2, 3), \quad \overrightarrow{AC} = C - A = (0, 4, 3).$$

Step 2. Cross product:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix}.$$

Expand:

$$\hat{\imath} : (2)(3) - (3)(4) = 6 - 12 = -6.$$

$$\hat{\jmath} : -[(1)(3) - (3)(0)] = -(3 - 0) = -3.$$

$$\hat{k} : (1)(4) - (2)(0) = 4 - 0 = 4.$$

So $\overrightarrow{AB} \times \overrightarrow{AC} = -6\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$.

Step 3. Magnitude:

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-6)^2 + (-3)^2 + 4^2} = \sqrt{36 + 9 + 16} = \sqrt{61}.$$

Step 4. Area:

$$\text{Area} = \tfrac{1}{2}\sqrt{61}.$$

Final Answer: Area $= \dfrac{\sqrt{61}}{2}$ square units.

EXPERT'S SOLUTION : *Diya Joshi, M.Sc Mathematics, IIT Bombay*

Strategic angle. Two edges from one vertex; half the magnitude of their cross product.

Step 1. $\overrightarrow{AB} = (1, 2, 3)$, $\overrightarrow{AC} = (0, 4, 3)$.

Step 2. Cross product $= (-6, -3, 4)$.

Step 3. Magnitude $\sqrt{61}$. Half $\Rightarrow \sqrt{61}/2$.

Why this matters. This is the cleanest area formula in 3D: no need to find side-lengths or angles individually.

Final Answer: $\sqrt{61}/2$.

Q 10.10 **Find the area of the parallelogram whose adjacent sides are determined by the vectors $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.**

SOLUTION

Concept used. Area of a parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$ is $|\vec{a} \times \vec{b}|$.

Step 1. Cross product:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix}.$$

Expand:

$$\hat{i} : (-1)(1) - (3)(-7) = -1 + 21 = 20.$$

$$\hat{j} : -[(1)(1) - (3)(2)] = -(1 - 6) = 5.$$

$$\hat{k} : (1)(-7) - (-1)(2) = -7 + 2 = -5.$$

So $\vec{a} \times \vec{b} = 20\hat{i} + 5\hat{j} - 5\hat{k}$.

Step 2. Magnitude:

$$|\vec{a} \times \vec{b}| = \sqrt{20^2 + 5^2 + (-5)^2} = \sqrt{400 + 25 + 25} = \sqrt{450} = 15\sqrt{2}.$$

Final Answer: Area $= 15\sqrt{2}$ square units.

EXPERT'S SOLUTION *: Karan Bhat, M.Sc Mathematics, IIT Bombay*

Quick reading. Determinant, magnitude.

Step 1. $\vec{a} \times \vec{b} = (20, 5, -5)$.

Step 2. Magnitude $= \sqrt{450} = 15\sqrt{2}$.

Why this matters. The parallelogram area $|\vec{a} \times \vec{b}|$ doubles the triangle area; same machinery, halved or full.

Final Answer: $15\sqrt{2}$.

Q 10.11 **Let the vectors $\vec{a}, \vec{b}$ be such that $|\vec{a}| = 3$ and $|\vec{b}| = \frac{\sqrt{2}}{3}$, then $\vec{a} \times \vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is**

(A) $\pi/6$ **(B)** $\pi/4$ **(C)** $\pi/3$ **(D)** $\pi/2$.

SOLUTION

Concept used. $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$. Unit vector means this equals 1.

Step 1. Set up: $|\vec{a}||\vec{b}|\sin\theta = 1$.

Step 2. Substitute:

$$3 \cdot \frac{\sqrt{2}}{3} \cdot \sin\theta = 1 \implies \sqrt{2}\sin\theta = 1 \implies \sin\theta = \frac{1}{\sqrt{2}}.$$

Step 3. Hence $\theta = \pi/4$.

> **Final Answer:** Option **(B)** $\theta = \pi/4$.

EXPERT'S SOLUTION *: Neha Sharma, M.Sc Mathematics, IIT Bombay*

Quick reading. Solve $|\vec{a}||\vec{b}|\sin\theta = 1$ for θ.

Step 1. $3 \cdot (\sqrt{2}/3) \cdot \sin\theta = 1 \Rightarrow \sin\theta = 1/\sqrt{2}$.

Step 2. $\theta = \pi/4$.

Why this matters. The cross product's magnitude packages magnitude-of-each times sine of angle - one of the cleanest closed-form identities in vector algebra.

> **Final Answer:** (B).

Q 10.12 **Area of a rectangle having vertices A, B, C and D with position vectors $-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$ and $-\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$, respectively, is**
(A) $1/2$ **(B)** 1 **(C)** 2 **(D)** 4.

SOLUTION

Concept used. Area of the rectangle $= |\overrightarrow{AB}| \cdot |\overrightarrow{AD}|$ where $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are adjacent sides. (Or, equivalently, $|\overrightarrow{AB} \times \overrightarrow{AD}|$.)

Step 1. Compute adjacent side vectors. With $\vec{A} = (-1, \frac{1}{2}, 4)$, $\vec{B} = (1, \frac{1}{2}, 4)$, $\vec{D} = (-1, -\frac{1}{2}, 4)$:

$$\overrightarrow{AB} = \vec{B} - \vec{A} = (1 - (-1), \tfrac{1}{2} - \tfrac{1}{2}, 4 - 4) = (2, 0, 0).$$

$$\overrightarrow{AD} = \vec{D} - \vec{A} = (-1 - (-1), -\tfrac{1}{2} - \tfrac{1}{2}, 4 - 4) = (0, -1, 0).$$

Step 2. Magnitudes (sides):

$$|\overrightarrow{AB}| = \sqrt{2^2} = 2, \quad |\overrightarrow{AD}| = \sqrt{(-1)^2} = 1.$$

Step 3. Sanity: $\overrightarrow{AB} \cdot \overrightarrow{AD} = (2)(0) + (0)(-1) + (0)(0) = 0$, so the sides are perpendicular (so it is indeed a rectangle).

Step 4. Area:

$$\text{Area} = |\overrightarrow{AB}| \cdot |\overrightarrow{AD}| = 2 \cdot 1 = 2.$$

Final Answer: Option **(C)** area $= 2$.

EXPERT'S SOLUTION : *Pooja Kapoor, M.Sc Mathematics, IIT Bombay*

Strategic angle. Compute two adjacent sides; product of their lengths is the rectangle's area.

Step 1. $\overrightarrow{AB} = (2, 0, 0)$, $\overrightarrow{AD} = (0, -1, 0)$.

Step 2. Perpendicular: $\overrightarrow{AB} \cdot \overrightarrow{AD} = 0$.

Step 3. Side lengths 2 and $1 \Rightarrow$ area $= 2$.

Why this matters. For a rectangle the dot product of adjacent sides is zero, so $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin 90° = |\vec{a}||\vec{b}|$; either formula gives the same answer.

Final Answer: (C).

Key Takeaways

- $\vec{a} \times \vec{b}$ via 3×3 determinant; expand by the first row.
- Cofactor signs $+, -, +$ on $\hat{i}, \hat{j}, \hat{k}$ - never forget the minus on the middle term.
- Area of $\triangle ABC$ in 3D: $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$. Area of parallelogram: $|\vec{a} \times \vec{b}|$.
- Cross product $\vec{0} \iff$ vectors are parallel (or one is zero).
- Three direction cosines satisfy $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.

End of Exercise 10.4