Collegedunia NCERT Solutions

Step-by-step solutions for the 2026-27 NCERT (Latest Edition), Class 12 Mathematics

Chapter 10: Vector Algebra

About this Chapter

The Miscellaneous Exercise revisits every major idea of **vector algebra**: unit vectors and direction cosines, displacement by triangle/parallelogram law, the section formula (internal and external), the **dot product** and angle between vectors, the **cross product** and area, and four MCQs that test the boundary cases of these formulas. Mastering this set means you can use vectors as a complete toolkit for Class 12 geometry.

Topics covered: Unit vectors • Direction cosines • Section formula • Dot product, angle • Cross product, area • MCQs

Quick Formula Sheet

Unit vector in direction of $\vec{a}$: $\hat{a} = \vec{a}/|\vec{a}|$.

Direction cosines: for $\vec{a} = (a_1, a_2, a_3)$, $l = a_1/|\vec{a}|$, $m = a_2/|\vec{a}|$, $n = a_3/|\vec{a}|$ with $l^2 + m^2 + n^2 = 1$.

Section formula (internal $m : n$): $\vec{r} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$.

(External $m : n$): $\vec{r} = \dfrac{m\vec{b} - n\vec{a}}{m-n}$.

Dot product: $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = \sum a_i b_i$.

Cross product: $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$ (vector perpendicular to both, forming a right-handed system).

Miscellaneous Exercise

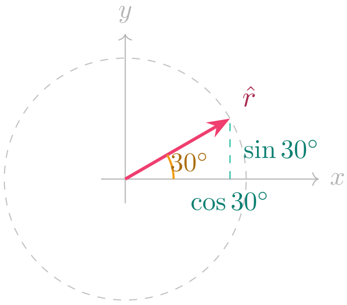
Q 10.1 **Write down a unit vector in XY-plane, making an angle of 30° with the positive direction of x-axis.**

SOLUTION

Concept used. Any unit vector in the XY-plane making an angle θ with the positive x-axis has the form

$$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}.$$

This follows from polar-to-cartesian conversion on the unit circle: the foot of the perpendicular from $\hat{r}$ on the x-axis has length $\cos\theta$, and the height above the axis is $\sin\theta$. The z-component is 0 because the vector lies in the XY-plane.



Step 1. Substitute $\theta = 30°$ into the polar form:

$$\hat{r} = \cos 30°\,\hat{i} + \sin 30°\,\hat{j}.$$

Step 2. Recall the standard values:

$$\cos 30° = \tfrac{\sqrt{3}}{2}, \qquad \sin 30° = \tfrac{1}{2}.$$

Step 3. Substitute:

$$\hat{r} = \tfrac{\sqrt{3}}{2}\,\hat{i} + \tfrac{1}{2}\,\hat{j}.$$

Step 4. Sanity check: $|\hat{r}|^2 = \left(\tfrac{\sqrt{3}}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 = \tfrac{3}{4} + \tfrac{1}{4} = 1$, so $|\hat{r}| = 1$. ✓

Final Answer: $\hat{r} = \dfrac{\sqrt{3}}{2}\,\hat{i} + \dfrac{1}{2}\,\hat{j}.$

☞ **Polar form of a unit vector**

A unit vector at angle θ from the positive x-axis is always $(\cos\theta, \sin\theta)$. Two checks always work: components squared sum to 1, and the ratio $y/x = \tan\theta$.

EXPERT'S SOLUTION *: Aarav Iyer, M.Sc Mathematics, IIT Bombay*

Picture-first. Drop a unit-radius circle in the XY-plane; mark the radius at $30°$. The tip's coordinates are exactly $(\cos 30°, \sin 30°)$, so the position vector is the answer.

Step 1. Unit circle, angle $30°$ above x-axis.

Step 2. Tip at $\left(\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}\right)$.

Step 3. So $\hat{r} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$ (with no $\hat{k}$-component since we are in the XY-plane).

Step 4. $|\hat{r}|^2 = \frac{3}{4} + \frac{1}{4} = 1$, confirming it is a unit vector.

Why this matters. The polar-form expression $(\cos\theta, \sin\theta)$ is the simplest way to write any planar unit vector; you will meet it again in three-dimensional geometry and in oscillation problems in physics.

> **Final Answer:** $\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$.

Q 10.2 **Find the scalar components and magnitude of the vector joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.**

SOLUTION

Concept used. If P and Q have position vectors $\vec{p}$ and $\vec{q}$ respectively, then the vector $\overrightarrow{PQ}$ from P to Q is

$$\overrightarrow{PQ} = \vec{q} - \vec{p} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}.$$

Its **scalar components** are the coefficients of $\hat{i}, \hat{j}, \hat{k}$, and its magnitude is given by the Pythagorean (distance) formula in 3D.

Step 1. Write the position vectors of P and Q explicitly:

$$\vec{p} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}, \qquad \vec{q} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}.$$

Step 2. Subtract componentwise to get $\overrightarrow{PQ}$:

$$\overrightarrow{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}.$$

Step 3. Scalar components: $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$.

Step 4. Magnitude using the 3D distance formula:

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

> **Final Answer:** Scalar components: $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$; $|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

EXPERT'S SOLUTION : *Sneha Sharma, M.Sc Mathematics, ISI Kolkata*

Structural observation. The vector "from tail to head" is always (head's position vector) − (tail's position vector); the magnitude is then the ordinary 3D distance between the two endpoints.

Step 1. $\overrightarrow{PQ} = \vec{q} - \vec{p}$, taken componentwise.

Step 2. Scalar components: $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

Step 3. Length: $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ - the familiar distance formula.

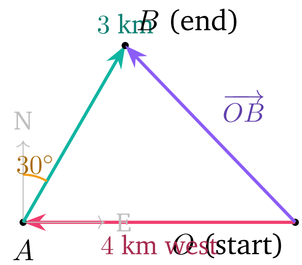
Why this matters. This identifies the algebraic vector with the geometric segment $\overrightarrow{PQ}$. Direction cosines, midpoint of PQ, and all later 3D-geometry formulas build on these two facts.

> **Final Answer:** Components $(x_2-x_1, y_2-y_1, z_2-z_1)$; magnitude $\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$.

Q 10.3 **A girl walks 4 km towards west, then she walks 3 km in a direction $30°$ east of north and stops. Determine the girl's displacement from her initial point of departure.**

SOLUTION

Concept used. Displacement is the position vector from start to end, independent of the path. Choose $\hat{\imath}$ along east and $\hat{\jmath}$ along north. "$30°$ east of north" means a direction $30°$ from the north axis, rotated towards the east. The total displacement is the vector sum of the two walking displacements.



Step 1. Let $\hat{\imath}$ point east and $\hat{\jmath}$ point north (so west $= -\hat{\imath}$, south $= -\hat{\jmath}$).

Step 2. First displacement (4 km west):

$$\overrightarrow{OA} = -4\,\hat{\imath}.$$

Step 3. Second displacement (3 km at $30°$ east of north). North component $= 3\cos 30°$ (along $\hat{\jmath}$), east component $= 3\sin 30°$ (along $\hat{\imath}$):

$$\overrightarrow{AB} = 3\sin 30°\,\hat{\imath} + 3\cos 30°\,\hat{\jmath} = 3 \cdot \tfrac{1}{2}\,\hat{\imath} + 3 \cdot \tfrac{\sqrt{3}}{2}\,\hat{\jmath} = \tfrac{3}{2}\,\hat{\imath} + \tfrac{3\sqrt{3}}{2}\,\hat{\jmath}.$$

Step 4. Total displacement:

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = -4\,\hat{\imath} + \tfrac{3}{2}\,\hat{\imath} + \tfrac{3\sqrt{3}}{2}\,\hat{\jmath} = \big(-4 + \tfrac{3}{2}\big)\hat{\imath} + \tfrac{3\sqrt{3}}{2}\,\hat{\jmath} = -\tfrac{5}{2}\,\hat{\imath} + \tfrac{3\sqrt{3}}{2}\,\hat{\jmath}.$$

Step 5. Magnitude:

$$|\overrightarrow{OB}|^2 = \big(-\tfrac{5}{2}\big)^2 + \big(\tfrac{3\sqrt{3}}{2}\big)^2 = \tfrac{25}{4} + \tfrac{27}{4} = \tfrac{52}{4} = 13.$$

$$|\overrightarrow{OB}| = \sqrt{13}\text{ km}.$$

> **Final Answer:** Displacement $= -\dfrac{5}{2}\,\hat{\imath} + \dfrac{3\sqrt{3}}{2}\,\hat{\jmath}$ (km), of magnitude $\sqrt{13}$ km, west of north.

✗ Common Mistake

"30° east of north" is measured from the *north* axis (not from the east axis). The east component is therefore $3\sin 30^\circ$ and the north component is $3\cos 30^\circ$. Swapping these gives a wildly different displacement.

EXPERT'S SOLUTION *: Pranav Reddy, M.Sc Mathematics, IIT Bombay*

Picture-first. Set up an east-north axis system; resolve each leg of the journey into east and north components; add to get the resultant.

Step 1. Axes: east $= \hat{\imath}$, north $= \hat{\jmath}$.

Step 2. Leg 1 (4 km west) $= (-4, 0)$.

Step 3. Leg 2 (3 km, 30° east of north) $= (3\sin 30^\circ, 3\cos 30^\circ) = (3/2,\ 3\sqrt{3}/2)$.

Step 4. Sum: $(-4 + 3/2,\ 3\sqrt{3}/2) = (-5/2,\ 3\sqrt{3}/2)$.

Step 5. Magnitude: $\sqrt{25/4 + 27/4} = \sqrt{52/4} = \sqrt{13}$ km.

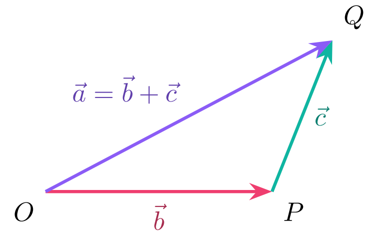
Why this matters. Displacement adds vectorially, regardless of the path: only the start and end positions count. This is the foundation of every relative-motion problem.

> **Final Answer:** $-\frac{5}{2}\hat{\imath} + \frac{3\sqrt{3}}{2}\hat{\jmath}$, magnitude $\sqrt{13}$ km.

Q 10.4 **If $\vec{a} = \vec{b} + \vec{c}$, then is it true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$? Justify your answer.**

SOLUTION

Concept used. Vector addition obeys the **triangle inequality**: $|\vec{b}+\vec{c}| \le |\vec{b}| + |\vec{c}|$, with equality only when $\vec{b}$ and $\vec{c}$ are *parallel and point in the same direction*. So in general, $|\vec{a}| = |\vec{b}+\vec{c}|$ is not equal to $|\vec{b}| + |\vec{c}|$.



Step 1. In a triangle OPQ with sides $\vec{b} = \overrightarrow{OP}$, $\vec{c} = \overrightarrow{PQ}$ and $\vec{a} = \overrightarrow{OQ}$, the side opposite a vertex is always shorter than the sum of the other two: $|\vec{a}| < |\vec{b}| + |\vec{c}|$.

Step 2. Concrete counter-example: let $\vec{b} = \hat{\imath}$ and $\vec{c} = \hat{\jmath}$. Then $\vec{a} = \vec{b} + \vec{c} = \hat{\imath} + \hat{\jmath}$, so

$$|\vec{a}| = \sqrt{1^2 + 1^2} = \sqrt{2} \approx 1.414,$$

$$|\vec{b}| + |\vec{c}| = 1 + 1 = 2.$$

Clearly $\sqrt{2} \ne 2$.

Step 3. Equality case (when the claim *is* true): if $\vec{b}$ and $\vec{c}$ are parallel and point the same way, say $\vec{c} = k\vec{b}$ with $k > 0$. Then $|\vec{a}| = |\vec{b} + k\vec{b}| = (1+k)|\vec{b}| = |\vec{b}| + |\vec{c}|$.

> **Final Answer:** No, $|\vec{a}| = |\vec{b}| + |\vec{c}|$ in general. It holds only when $\vec{b}$ and $\vec{c}$ are parallel with the same direction.

♥ Triangle inequality

The triangle inequality $|\vec{b}+\vec{c}| \le |\vec{b}| + |\vec{c}|$ is the geometric statement that a straight path is the shortest. It is one of the most-used inequalities in higher mathematics: distance metrics, normed vector spaces and convex analysis all rely on it.

EXPERT'S SOLUTION : *Aanya Iyer, M.Sc Mathematics, IIT Bombay*

Quick reading. The statement is false in general; equality of magnitudes requires the two summands to be along the same direction.

Step 1. By the law of cosines for vectors,

$$|\vec{b}+\vec{c}|^2 = |\vec{b}|^2 + |\vec{c}|^2 + 2|\vec{b}||\vec{c}|\cos\theta,$$

where θ is the angle between $\vec{b}$ and $\vec{c}$.

Step 2. Compare with $(|\vec{b}| + |\vec{c}|)^2 = |\vec{b}|^2 + |\vec{c}|^2 + 2|\vec{b}||\vec{c}|$.

Step 3. Equality holds iff $\cos\theta = 1$, i.e. $\theta = 0$: $\vec{b}$ and $\vec{c}$ are parallel with the same orientation.

Step 4. For any $\theta \in (0, \pi]$ we get $\cos\theta < 1$, hence $|\vec{a}| < |\vec{b}| + |\vec{c}|$.

Why this matters. The cosine identity above is the universal form of the triangle inequality and tells you exactly *how far* from equality you are - controlled by the cosine of the angle between $\vec{b}$ and $\vec{c}$.

> **Final Answer:** Not in general; equality iff $\vec{b}$ and $\vec{c}$ are like-parallel ($\theta = 0$).

Q 10.5 Find the value of x for which $x(\hat{i} + \hat{j} + \hat{k})$ is a unit vector.

SOLUTION

Concept used. A vector $\vec{v}$ is a **unit vector** when $|\vec{v}| = 1$. So we set the magnitude of the given vector equal to 1 and solve for x.

Step 1. Write the vector as $\vec{v} = x\hat{i} + x\hat{j} + x\hat{k}$ (distributing the scalar x).

Step 2. Compute the magnitude:

$$|\vec{v}| = \sqrt{x^2 + x^2 + x^2} = \sqrt{3x^2} = |x|\sqrt{3}.$$

Step 3. Set $|\vec{v}| = 1$:

$$|x|\sqrt{3} = 1.$$

Step 4. Solve:

$$|x| = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad x = \pm\frac{1}{\sqrt{3}}.$$

> **Final Answer:** $x = \pm\dfrac{1}{\sqrt{3}}$.

EXPERT'S SOLUTION : *Riya Nair, M.Sc Mathematics, IIT Bombay*

Strategic angle. Scalar multiplication scales the magnitude by $|x|$. The base vector $\hat{i} + \hat{j} + \hat{k}$ has length $\sqrt{3}$, so we need $|x|\sqrt{3} = 1$.

Step 1. $|\hat{i} + \hat{j} + \hat{k}| = \sqrt{1 + 1 + 1} = \sqrt{3}$.

Step 2. $|x(\hat{i} + \hat{j} + \hat{k})| = |x| \cdot \sqrt{3}$.

Step 3. Setting this equal to 1: $|x| = 1/\sqrt{3}$, so $x = \pm 1/\sqrt{3}$.

Why this matters. The two signs of x give the two unit vectors along the diagonal of a cube: $\frac{1}{\sqrt{3}}(1,1,1)$ and $-\frac{1}{\sqrt{3}}(1,1,1)$. Both have unit length, opposite orientations.

Final Answer: $\pm 1/\sqrt{3}$.

Q 10.6 **Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.**

SOLUTION

Concept used. A vector parallel to another vector $\vec{u}$ is a scalar multiple of $\vec{u}$. Of all such multiples, the unit vector in the direction of $\vec{u}$ is $\hat{u} = \vec{u}/|\vec{u}|$. Multiplying $\hat{u}$ by the desired length gives the required vector: a vector of magnitude 5 parallel to $\vec{u}$ is $5\hat{u} = 5\vec{u}/|\vec{u}|$.

Step 1. Compute the resultant $\vec{u} = \vec{a} + \vec{b}$ component-wise:

$$\vec{u} = (2+1)\hat{i} + (3+(-2))\hat{j} + ((-1)+1)\hat{k} = 3\hat{i} + \hat{j} + 0\,\hat{k} = 3\hat{i} + \hat{j}.$$

Step 2. Magnitude:

$$|\vec{u}| = \sqrt{3^2 + 1^2} = \sqrt{9+1} = \sqrt{10}.$$

Step 3. Unit vector in the direction of $\vec{u}$:

$$\hat{u} = \frac{1}{\sqrt{10}}(3\hat{i} + \hat{j}).$$

Step 4. Scale by 5:

$$5\hat{u} = \frac{5}{\sqrt{10}}(3\hat{i} + \hat{j}) = \frac{5}{\sqrt{10}} \cdot 3\,\hat{i} + \frac{5}{\sqrt{10}}\,\hat{j} = \frac{15}{\sqrt{10}}\,\hat{i} + \frac{5}{\sqrt{10}}\,\hat{j}.$$

Step 5. Rationalise (optional):

$$\frac{15}{\sqrt{10}} = \frac{15\sqrt{10}}{10} = \frac{3\sqrt{10}}{2}, \qquad \frac{5}{\sqrt{10}} = \frac{5\sqrt{10}}{10} = \frac{\sqrt{10}}{2}.$$

So $5\hat{u} = \dfrac{3\sqrt{10}}{2}\,\hat{i} + \dfrac{\sqrt{10}}{2}\,\hat{j}$.

Step 6. Verify: $|5\hat{u}|^2 = \left(\frac{15}{\sqrt{10}}\right)^2 + \left(\frac{5}{\sqrt{10}}\right)^2 = \frac{225}{10} + \frac{25}{10} = \frac{250}{10} = 25$, so magnitude $= 5$. ✓

Final Answer: $\dfrac{5}{\sqrt{10}}(3\hat{i} + \hat{j}) = \dfrac{3\sqrt{10}}{2}\,\hat{i} + \dfrac{\sqrt{10}}{2}\,\hat{j}$.

EXPERT'S SOLUTION : *Karan Joshi, M.Sc Mathematics, IIT Bombay*

Strategic angle. Resultant first, normalise, then scale by 5.

Step 1. Resultant: $\vec{u} = \vec{a} + \vec{b} = (3, 1, 0)$.

Step 2. $|\vec{u}| = \sqrt{9+1} = \sqrt{10}$.

Step 3. Required vector $= 5\vec{u}/|\vec{u}| = 5(3, 1, 0)/\sqrt{10} = (15/\sqrt{10},\ 5/\sqrt{10},\ 0)$.

Why this matters. Any "vector of length L in the direction of $\vec{u}$" problem reduces to one formula: $L\,\vec{u}/|\vec{u}|$. The negative answer $-5\vec{u}/|\vec{u}|$ is also parallel to $\vec{u}$ (opposite direction); both have the same magnitude 5.

Final Answer: $\dfrac{5}{\sqrt{10}}(3\hat{\imath} + \hat{\jmath})$.

Q 10.7 **If $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$, $\vec{b} = 2\hat{\imath} - \hat{\jmath} + 3\hat{k}$ and $\vec{c} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$, find a unit vector parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$.**

SOLUTION

Concept used. Scalar multiplication and vector addition are componentwise. Once we compute the target vector $\vec{u} = 2\vec{a} - \vec{b} + 3\vec{c}$, the unit vector parallel to it is $\hat{u} = \vec{u}/|\vec{u}|$.

Step 1. Compute $2\vec{a}$:

$$2\vec{a} = 2(\hat{\imath} + \hat{\jmath} + \hat{k}) = 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}.$$

Step 2. Compute $-\vec{b}$:

$$-\vec{b} = -(2\hat{\imath} - \hat{\jmath} + 3\hat{k}) = -2\hat{\imath} + \hat{\jmath} - 3\hat{k}.$$

Step 3. Compute $3\vec{c}$:

$$3\vec{c} = 3(\hat{\imath} - 2\hat{\jmath} + \hat{k}) = 3\hat{\imath} - 6\hat{\jmath} + 3\hat{k}.$$

Step 4. Add componentwise:

$$\vec{u} = (2 - 2 + 3)\hat{\imath} + (2 + 1 - 6)\hat{\jmath} + (2 - 3 + 3)\hat{k} = 3\hat{\imath} - 3\hat{\jmath} + 2\hat{k}.$$

Step 5. Magnitude:

$$|\vec{u}| = \sqrt{3^2 + (-3)^2 + 2^2} = \sqrt{9 + 9 + 4} = \sqrt{22}.$$

Step 6. Unit vector:

$$\hat{u} = \frac{1}{\sqrt{22}}\,\vec{u} = \frac{3}{\sqrt{22}}\,\hat{\imath} - \frac{3}{\sqrt{22}}\,\hat{\jmath} + \frac{2}{\sqrt{22}}\,\hat{k}.$$

Final Answer: $\hat{u} = \dfrac{1}{\sqrt{22}}(3\hat{\imath} - 3\hat{\jmath} + 2\hat{k})$.

EXPERT'S SOLUTION : *Aditya Patel, M.Sc Mathematics, IIT Madras*

Strategic angle. Linear combination componentwise, then normalise.

Step 1. $2\vec{a} = (2, 2, 2)$, $-\vec{b} = (-2, 1, -3)$, $3\vec{c} = (3, -6, 3)$.

Step 2. Sum: $(2 - 2 + 3,\ 2 + 1 - 6,\ 2 - 3 + 3) = (3,\ -3,\ 2)$.

Step 3. Magnitude $\sqrt{9 + 9 + 4} = \sqrt{22}$.

Step 4. $\hat{u} = (3, -3, 2)/\sqrt{22}$.

Why this matters. Any unit vector parallel to a given non-zero $\vec{u}$ is just $\pm\vec{u}/|\vec{u}|$. The two sign choices reflect that "parallel" allows either of the two opposite directions.

Final Answer: $\frac{1}{\sqrt{22}}(3\hat{i} - 3\hat{j} + 2\hat{k})$.

Q 10.8 **Show that the points $A(1, -2, -8)$, $B(5, 0, -2)$ and $C(11, 3, 7)$ are collinear, and find the ratio in which B divides AC.**

SOLUTION

Concept used. Three points are **collinear** when the vector from any one point to another is a scalar multiple of the vector to the third. Equivalently, $\overrightarrow{AB}$ and $\overrightarrow{BC}$ should be parallel. The ratio $AB : BC$ then tells how B divides the segment AC.

Step 1. Compute $\overrightarrow{AB} = B - A$:

$$\overrightarrow{AB} = (5 - 1)\hat{i} + (0 - (-2))\hat{j} + (-2 - (-8))\hat{k} = 4\hat{i} + 2\hat{j} + 6\hat{k}.$$

Step 2. Compute $\overrightarrow{BC} = C - B$:

$$\overrightarrow{BC} = (11 - 5)\hat{i} + (3 - 0)\hat{j} + (7 - (-2))\hat{k} = 6\hat{i} + 3\hat{j} + 9\hat{k}.$$

Step 3. Notice that $\overrightarrow{BC} = \frac{3}{2}\overrightarrow{AB}$, since

$$\tfrac{3}{2}(4, 2, 6) = (6, 3, 9).$$

Two vectors with a common starting point (B) that are parallel must lie on the same line $\Rightarrow$ A, B, C are collinear.

Step 4. Magnitudes:

$$|\overrightarrow{AB}| = \sqrt{4^2 + 2^2 + 6^2} = \sqrt{16 + 4 + 36} = \sqrt{56} = 2\sqrt{14}.$$

$$|\overrightarrow{BC}| = \sqrt{6^2 + 3^2 + 9^2} = \sqrt{36 + 9 + 81} = \sqrt{126} = 3\sqrt{14}.$$

Step 5. Ratio in which B divides AC:

$$AB : BC = 2\sqrt{14} : 3\sqrt{14} = 2 : 3.$$

Since B lies between A and C (both vectors point in the same direction from A to C), B divides AC *internally* in the ratio $2 : 3$.

Final Answer: A, B, C are collinear; B divides AC internally in the ratio $\mathbf{2 : 3}$.

EXPERT'S SOLUTION *: Ishaan Verma, M.Sc Mathematics, IIT Bombay*

Structural observation. Show that $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are positive scalar multiples of each other; the scalar ratio gives the section ratio.

Step 1. $\overrightarrow{AB} = (4, 2, 6)$, $\overrightarrow{BC} = (6, 3, 9) = \frac{3}{2}\overrightarrow{AB}$.

Step 2. Same direction $\Rightarrow$ collinear, and B lies between A and C.

Step 3. $AB : BC = 2 : 3$, so B divides AC internally in $2 : 3$.

Why this matters. The "parallel-vectors" test is the simplest collinearity check in 3D - no determinants needed. If the two vectors share a point and are positive multiples of each other, the three points lie on a single line in that order.

Final Answer: Collinear; B divides AC internally as $2 : 3$.

Q 10.9 Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $(2\vec{a} + \vec{b})$ and $(\vec{a} - 3\vec{b})$ externally in the ratio $1 : 2$. Also, show that P is the mid point of the line segment RQ.

SOLUTION

Concept used. The **external section formula**: if R divides the segment PQ externally in the ratio $m : n$, then

$$\vec{r} = \frac{m\vec{q} - n\vec{p}}{m - n}.$$

With $m = 1$, $n = 2$, $\vec{p} = 2\vec{a} + \vec{b}$ and $\vec{q} = \vec{a} - 3\vec{b}$.

Step 1. Plug into the formula:

$$\vec{r} = \frac{1 \cdot (\vec{a} - 3\vec{b}) - 2 \cdot (2\vec{a} + \vec{b})}{1 - 2}.$$

Step 2. Expand numerator:

$$\vec{r} = \frac{\vec{a} - 3\vec{b} - 4\vec{a} - 2\vec{b}}{-1} = \frac{-3\vec{a} - 5\vec{b}}{-1}.$$

Step 3. Simplify:

$$\vec{r} = 3\vec{a} + 5\vec{b}.$$

Step 4. Verify "P is the midpoint of RQ." The midpoint of R and Q has position vector

$$\tfrac{1}{2}(\vec{r} + \vec{q}) = \tfrac{1}{2}\big((3\vec{a} + 5\vec{b}) + (\vec{a} - 3\vec{b})\big) = \tfrac{1}{2}(4\vec{a} + 2\vec{b}) = 2\vec{a} + \vec{b} = \vec{p}.$$

Hence P is exactly the midpoint of segment RQ. ✓

Final Answer: Position vector of $R = \mathbf{3\tilde{a}} + \mathbf{5\tilde{b}}$, and P is the midpoint of RQ.

✗ Sign of $m - n$

In the external section formula the denominator is $m - n$, which can be negative. Don't "flip" the minus to a plus to make things tidy: the sign carries geometric information about which side of the segment R lies on.

EXPERT'S SOLUTION *: Priya Kapoor, M.Sc Mathematics, IIT Bombay*

Quick reading. External-section in $1:2$ means R lies on the side of P, with $PR : RQ = 1 : 2$ measured in opposite directions.

Step 1. $\vec{r} = \dfrac{1\cdot\vec{q} - 2\cdot\vec{p}}{1-2} = \dfrac{(\vec{a} - 3\vec{b}) - 2(2\vec{a} + \vec{b})}{-1} = \dfrac{-3\vec{a} - 5\vec{b}}{-1} = 3\vec{a} + 5\vec{b}.$

Step 2. Midpoint of R and Q: $\tfrac{1}{2}(3\vec{a} + 5\vec{b} + \vec{a} - 3\vec{b}) = \tfrac{1}{2}(4\vec{a} + 2\vec{b}) = 2\vec{a} + \vec{b} = \vec{p}.$

Step 3. So P is the midpoint of RQ - consistent with the external $1:2$ ratio.

Why this matters. "Externally in $1:2$" geometrically pushes R past P so that the segment RQ is twice RP; this is exactly the midpoint configuration.

Final Answer: $\vec{r} = 3\vec{a} + 5\vec{b}$; P is mid of RQ.

Q 10.10 **The two adjacent sides of a parallelogram are $2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\hat{i} - 2\hat{j} - 3\hat{k}$. Find the unit vector parallel to its diagonal. Also, find its area.**

SOLUTION

Concept used. If $\vec{a}$ and $\vec{b}$ are adjacent sides of a parallelogram, then one diagonal is the sum $\vec{d} = \vec{a} + \vec{b}$ (the parallelogram law); and the **area** of the parallelogram is $|\vec{a} \times \vec{b}|$ (the magnitude of the cross product).

Step 1. Let $\vec{a} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} - 3\hat{k}$.

Step 2. Diagonal: $\vec{d} = \vec{a} + \vec{b} = (2+1)\hat{i} + (-4-2)\hat{j} + (5-3)\hat{k} = 3\hat{i} - 6\hat{j} + 2\hat{k}$.

Step 3. Magnitude of diagonal:

$$|\vec{d}| = \sqrt{3^2 + (-6)^2 + 2^2} = \sqrt{9 + 36 + 4} = \sqrt{49} = 7.$$

Step 4. Unit vector along diagonal:

$$\hat{d} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}).$$

Step 5. Cross product for area:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 1 & -2 & -3 \end{vmatrix}.$$

Expand by first row:

$$\hat{i}: \ (-4)(-3) - (5)(-2) = 12 + 10 = 22.$$

$$\hat{j}: \ -\big[(2)(-3) - (5)(1)\big] = -[-6-5] = 11.$$

$$\hat{k}: \ (2)(-2) - (-4)(1) = -4 + 4 = 0.$$

So $\vec{a} \times \vec{b} = 22\hat{i} + 11\hat{j} + 0\hat{k}$.

Step 6. Area:

$$|\vec{a} \times \vec{b}| = \sqrt{22^2 + 11^2} = \sqrt{484 + 121} = \sqrt{605} = 11\sqrt{5}.$$

Final Answer: $\hat{d} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k})$; Area $= 11\sqrt{5}$ sq. units.

EXPERT'S SOLUTION *: Diya Mehta, M.Sc Mathematics, IIT Bombay*

Strategic angle. Two formulas back to back: diagonal $= \vec{a} + \vec{b}$; area $= |\vec{a} \times \vec{b}|$.

Step 1. $\vec{a} + \vec{b} = (3, -6, 2)$, magnitude 7. Unit diagonal $(3, -6, 2)/7$.

Step 2. Determinant gives $\vec{a} \times \vec{b} = (22, 11, 0)$.

Step 3. $|\vec{a} \times \vec{b}| = \sqrt{484 + 121} = \sqrt{605} = 11\sqrt{5}$.

Why this matters. The cross-product magnitude is the area of the parallelogram spanned by two vectors. The other diagonal $\vec{a} - \vec{b}$ is generally different in length; only the sum-diagonal $\vec{a} + \vec{b}$ was asked.

Final Answer: Unit diagonal $\frac{1}{7}(3, -6, 2)$; area $11\sqrt{5}$.

Q 10.11 **Show that the direction cosines of a vector equally inclined to the axes OX, OY and OZ are $\pm\left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$.**

SOLUTION

Concept used. For any vector, the **direction cosines** l, m, n (cosines of the angles with the x, y, z axes) always satisfy

$$l^2 + m^2 + n^2 = 1.$$

"Equally inclined" means all three angles are equal, so $l = m = n$.

Step 1. Let the common angle be α. Then

$$l = \cos\alpha, \quad m = \cos\alpha, \quad n = \cos\alpha,$$

so $l = m = n = \cos\alpha$.

Step 2. Use the identity $l^2 + m^2 + n^2 = 1$:

$$3\cos^2\alpha = 1.$$

Step 3. Solve:

$$\cos^2\alpha = \tfrac{1}{3} \quad \Rightarrow \quad \cos\alpha = \pm\tfrac{1}{\sqrt{3}}.$$

Step 4. Therefore the direction cosines are

$$(l, m, n) = \pm\left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right).$$

Final Answer: $(l, m, n) = \pm\left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$.

EXPERT'S SOLUTION *: Ananya Bhat, M.Sc Mathematics, IIT Bombay*

Picture-first. A vector equally inclined to the three axes lies along (or opposite to) the main diagonal of a cube with one corner at the origin.

Step 1. Direction cosines satisfy $l^2 + m^2 + n^2 = 1$.

Step 2. Equally inclined $\Rightarrow l = m = n$, so $3l^2 = 1 \Rightarrow l = \pm 1/\sqrt{3}$.

Step 3. Direction cosines $\pm(1/\sqrt{3},\ 1/\sqrt{3},\ 1/\sqrt{3})$.

Why this matters. The two answers (with $+$ or $-$) correspond to the two opposite directions along the space diagonal of the unit cube. Either is "equally inclined to all three axes".

Final Answer: $\pm(1/\sqrt{3},\ 1/\sqrt{3},\ 1/\sqrt{3})$.

Q 10.12 **Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$, and $\vec{c} \cdot \vec{d} = 15$.**

SOLUTION

Concept used. If $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, then $\vec{d}$ is parallel to $\vec{a} \times \vec{b}$. So we can write $\vec{d} = \lambda(\vec{a} \times \vec{b})$ for some scalar λ. The dot-product condition $\vec{c} \cdot \vec{d} = 15$ then pins down λ.

Step 1. Compute $\vec{a} \times \vec{b}$:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 2 \\ 3 & -2 & 7 \end{vmatrix}.$$

$$\hat{i}:\ (4)(7) - (2)(-2) = 28 + 4 = 32.$$

$$\hat{j}:\ -\big[(1)(7) - (2)(3)\big] = -[7 - 6] = -1.$$

$$\hat{k}:\ (1)(-2) - (4)(3) = -2 - 12 = -14.$$

So $\vec{a} \times \vec{b} = 32\hat{i} - \hat{j} - 14\hat{k}$.

Step 2. Let $\vec{d} = \lambda(32\hat{i} - \hat{j} - 14\hat{k})$.

Step 3. Compute $\vec{c} \cdot \vec{d}$:

$$\vec{c} \cdot \vec{d} = \lambda\left[(2)(32) + (-1)(-1) + (4)(-14)\right] = \lambda\,[64 + 1 - 56] = 9\lambda.$$

Step 4. Use $\vec{c} \cdot \vec{d} = 15$:

$$9\lambda = 15 \quad \Rightarrow \quad \lambda = \tfrac{15}{9} = \tfrac{5}{3}.$$

Step 5. Therefore

$$\vec{d} = \tfrac{5}{3}(32\hat{i} - \hat{j} - 14\hat{k}) = \tfrac{160}{3}\hat{i} - \tfrac{5}{3}\hat{j} - \tfrac{70}{3}\hat{k}.$$

Final Answer: $\vec{d} = \dfrac{1}{3}(160\hat{i} - 5\hat{j} - 70\hat{k})$.

EXPERT'S SOLUTION : *Yash Desai, M.Sc Mathematics, IIT Bombay*

Strategic angle. Direction from cross product; magnitude pinned by the scalar condition.

Step 1. $\vec{a} \times \vec{b} = (32, -1, -14)$.

Step 2. Write $\vec{d} = \lambda(32, -1, -14)$.

Step 3. $\vec{c} \cdot \vec{d} = \lambda(64 + 1 - 56) = 9\lambda = 15 \Rightarrow \lambda = 5/3$.

Step 4. $\vec{d} = (5/3)(32, -1, -14) = (160/3,\ -5/3,\ -70/3)$.

Why this matters. Whenever "perpendicular to two given vectors" appears, the cross product is the only candidate direction (up to a sign / scale). A second scalar condition then fixes the unknown multiplier.

Final Answer: $\vec{d} = \frac{160}{3}\hat{i} - \frac{5}{3}\hat{j} - \frac{70}{3}\hat{k}$.

Q 10.13 The scalar product of the vector $\hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ.

SOLUTION

Concept used. Build the sum vector, normalise it, take dot product with the given vector, and set it equal to 1. The dot-product equation gives a single algebraic equation in λ.

Step 1. Let $\vec{u} = (2 + \lambda)\hat{i} + (4 + 2)\hat{j} + (-5 + 3)\hat{k} = (2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}$.

Step 2. Magnitude:

$$|\vec{u}| = \sqrt{(2 + \lambda)^2 + 6^2 + (-2)^2} = \sqrt{(2 + \lambda)^2 + 36 + 4} = \sqrt{(2 + \lambda)^2 + 40}.$$

Step 3. Unit vector $\hat{u} = \vec{u}/|\vec{u}|$.

Step 4. Compute the scalar product with $\hat{i} + \hat{j} + \hat{k}$:

$$(\hat{i} + \hat{j} + \hat{k}) \cdot \hat{u} = \frac{(2 + \lambda) + 6 + (-2)}{|\vec{u}|} = \frac{\lambda + 6}{\sqrt{(2 + \lambda)^2 + 40}}.$$

Step 5. Set this $= 1$:

$$\frac{\lambda + 6}{\sqrt{(2 + \lambda)^2 + 40}} = 1 \quad \Rightarrow \quad \lambda + 6 = \sqrt{(2 + \lambda)^2 + 40}.$$

Step 6. Square both sides (both sides must be non-negative, so $\lambda + 6 \geq 0$):

$$(\lambda + 6)^2 = (2 + \lambda)^2 + 40.$$

Step 7. Expand:

$$\lambda^2 + 12\lambda + 36 = \lambda^2 + 4\lambda + 4 + 40.$$

$$12\lambda + 36 = 4\lambda + 44.$$

$$8\lambda = 8 \quad \Rightarrow \quad \lambda = 1.$$

Step 8. Check the constraint $\lambda + 6 \geq 0$: $1 + 6 = 7 > 0$. ✓

Final Answer: $\lambda = 1$.

✗ Common Mistake

After "squaring both sides" you can pick up spurious negative-sign solutions. Always check at the end that $\lambda + 6 \geq 0$ (the unit dot product was equal to $+1$, not -1).

EXPERT'S SOLUTION *: Vivaan Banerjee, M.Sc Mathematics, IIT Bombay*

Quick reading. A unit dot product of 1 means the two vectors are equal in direction (and the first one is also a unit vector along its direction, which it is not - so the simplification is just an equation in λ).

Step 1. Sum: $(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}$.

Step 2. Numerator of $(\hat{i} + \hat{j} + \hat{k}) \cdot \hat{u}$ is $\lambda + 6$; denominator $|\vec{u}| = \sqrt{(2+\lambda)^2 + 40}$.

Step 3. Set equal to 1 and square: $(\lambda + 6)^2 = (\lambda + 2)^2 + 40$.

Step 4. Expand: $12\lambda + 36 = 4\lambda + 44 \Rightarrow 8\lambda = 8 \Rightarrow \lambda = 1$.

Why this matters. Whenever you square both sides of an equation you risk introducing extra roots; the original equation here demands $\lambda + 6 > 0$, automatically satisfied by $\lambda = 1$.

Final Answer: $\lambda = 1$.

Q 10.14 **If $\vec{a}$, $\vec{b}$, $\vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a} + \vec{b} + \vec{c}$ is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$.**

SOLUTION

Concept used. For any vector $\vec{u}$, the angle θ it makes with another vector $\vec{v}$ satisfies

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|}.$$

"Equally inclined" to $\vec{a}, \vec{b}, \vec{c}$ means the three cosines (with $\vec{u} = \vec{a} + \vec{b} + \vec{c}$ and $\vec{v}$ being each of $\vec{a}, \vec{b}, \vec{c}$ in turn) are equal.

Step 1. Let $|\vec{a}| = |\vec{b}| = |\vec{c}| = k$ (say), and the three vectors are mutually perpendicular. So

$$\vec{a} \cdot \vec{a} = k^2,\ \vec{b} \cdot \vec{b} = k^2,\ \vec{c} \cdot \vec{c} = k^2,$$
$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0.$$

Step 2. Let $\vec{s} = \vec{a} + \vec{b} + \vec{c}$. Compute $|\vec{s}|^2$:

$$|\vec{s}|^2 = (\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} + \vec{c} \cdot \vec{c} + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}).$$
$$= k^2 + k^2 + k^2 + 0 = 3k^2.$$

Hence $|\vec{s}| = k\sqrt{3}$.

Step 3. Dot product of $\vec{s}$ with $\vec{a}$:

$$\vec{s} \cdot \vec{a} = (\vec{a} + \vec{b} + \vec{c}) \cdot \vec{a} = \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a} = k^2 + 0 + 0 = k^2.$$

By the same computation (just swapping symbols): $\vec{s} \cdot \vec{b} = k^2$ and $\vec{s} \cdot \vec{c} = k^2$.

Step 4. Cosine of angle between $\vec{s}$ and $\vec{a}$:

$$\cos\theta_a = \frac{\vec{s} \cdot \vec{a}}{|\vec{s}|\,|\vec{a}|} = \frac{k^2}{(k\sqrt{3})(k)} = \frac{1}{\sqrt{3}}.$$

Same value comes out for θ_b and θ_c.

Step 5. Since the three cosines are equal, the angles are equal: $\vec{s} = \vec{a} + \vec{b} + \vec{c}$ is equally inclined to $\vec{a}, \vec{b}, \vec{c}$, with common angle $\cos^{-1}(1/\sqrt{3})$.

Final Answer: $\vec{a} + \vec{b} + \vec{c}$ makes the same angle $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ with each of $\vec{a}, \vec{b}, \vec{c}$.

♥ Body diagonal of a cube

This is exactly the geometry of a cube. Place $\vec{a}, \vec{b}, \vec{c}$ along three edges meeting at one corner. Then $\vec{a} + \vec{b} + \vec{c}$ is the body diagonal of the cube, which makes the same angle $\cos^{-1}(1/\sqrt{3}) \approx 54.7°$ with each edge.

EXPERT'S SOLUTION : *Tara Pillai, M.Sc Mathematics, IIT Bombay*

Picture-first. Set up axes along $\vec{a}, \vec{b}, \vec{c}$. The sum is the position vector of the diagonally-opposite corner of a unit-magnitude cube.

Step 1. All three vectors have magnitude k; all cross-dot products vanish.

Step 2. $|\vec{a}+\vec{b}+\vec{c}|^2 = 3k^2$, so $|\vec{a}+\vec{b}+\vec{c}| = k\sqrt{3}$.

Step 3. $(\vec{a}+\vec{b}+\vec{c})\cdot\vec{a} = k^2$ (the other two terms vanish by perpendicularity); same with $\vec{b}$ and $\vec{c}$.

Step 4. Cosine $= k^2/(k\sqrt{3}\cdot k) = 1/\sqrt{3}$, identical for all three.

Why this matters. The body diagonal of a cube is the canonical example of a vector equally inclined to three mutually perpendicular axes; the angle is exactly $\cos^{-1}(1/\sqrt{3})$, a fact you will need again in solid geometry.

> **Final Answer:** Equal cosines $1/\sqrt{3}$.

Q 10.15 **Prove that $(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = |\vec{a}|^2+|\vec{b}|^2$, if and only if $\vec{a}$, $\vec{b}$ are perpendicular, given $\vec{a}\neq\vec{0}$, $\vec{b}\neq\vec{0}$.**

SOLUTION

Concept used. The dot product is distributive: $(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \vec{u}\cdot\vec{u}+2\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{v}$. Also $\vec{u}\cdot\vec{u} = |\vec{u}|^2$, and $\vec{u}\perp\vec{v} \iff \vec{u}\cdot\vec{v}=0$. The proof is an "iff" (two directions).

Step 1. Expand the left side:

$$(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = \vec{a}\cdot\vec{a}+2\,\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b} = |\vec{a}|^2+2\,\vec{a}\cdot\vec{b}+|\vec{b}|^2.$$

Step 2. ($\Rightarrow$) Assume $(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = |\vec{a}|^2+|\vec{b}|^2$. Substitute the expansion:

$$|\vec{a}|^2+2\,\vec{a}\cdot\vec{b}+|\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2.$$

$$2\,\vec{a}\cdot\vec{b}=0 \quad\Rightarrow\quad \vec{a}\cdot\vec{b}=0.$$

Since $\vec{a},\vec{b}\neq\vec{0}$, $\vec{a}\cdot\vec{b}=0$ means $\vec{a}\perp\vec{b}$.

Step 3. ($\Leftarrow$) Conversely, assume $\vec{a}\perp\vec{b}$, so $\vec{a}\cdot\vec{b}=0$. Substitute into the expansion:

$$(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = |\vec{a}|^2+2(0)+|\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2.$$

Step 4. Both implications hold, so the iff is proved.

> **Final Answer:** $(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = |\vec{a}|^2+|\vec{b}|^2 \iff \vec{a}\perp\vec{b}$.

EXPERT'S SOLUTION : *Ankit Chatterjee, M.Sc Mathematics, IIT Bombay*

Structural observation. This is the vector form of Pythagoras' theorem: the magnitude squared of the sum equals the sum of magnitudes squared iff the legs are perpendicular.

Step 1. Identity: $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + |\vec{b}|^2$.

Step 2. Given equation forces $\vec{a}\cdot\vec{b} = 0$.

Step 3. Since $\vec{a}, \vec{b} \neq \vec{0}$, this is exactly $\vec{a} \perp \vec{b}$.

Step 4. Reverse direction: $\vec{a} \perp \vec{b} \Rightarrow \vec{a}\cdot\vec{b} = 0$, and the identity collapses to the desired equality.

Why this matters. The classical Pythagorean identity for a right triangle, $c^2 = a^2 + b^2$, is recovered: $|\vec{a}+\vec{b}|$ is the hypotenuse when $\vec{a}, \vec{b}$ are the perpendicular legs.

> **Final Answer:** Equivalent to $\vec{a} \perp \vec{b}$.

Q 10.16 **If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}\cdot\vec{b} \geq 0$ only when**
(A) $0 < \theta < \frac{\pi}{2}$ **(B)** $0 \leq \theta \leq \frac{\pi}{2}$ **(C)** $0 < \theta < \pi$ **(D)** $0 \leq \theta \leq \pi$**.**

SOLUTION

Concept used. $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$. With $|\vec{a}|, |\vec{b}| > 0$, the sign of $\vec{a}\cdot\vec{b}$ matches the sign of $\cos\theta$. We need $\vec{a}\cdot\vec{b} \geq 0$, i.e. $\cos\theta \geq 0$.

Step 1. Recall the sign of $\cos\theta$ on $[0, \pi]$:

- $\cos\theta > 0$ for $\theta \in [0, \pi/2)$,
- $\cos\theta = 0$ for $\theta = \pi/2$,
- $\cos\theta < 0$ for $\theta \in (\pi/2, \pi]$.

Step 2. Combine the first two: $\cos\theta \geq 0$ for $\theta \in [0, \pi/2]$, i.e. $0 \leq \theta \leq \pi/2$.

Step 3. Match this with the options. Option (B) reads $0 \leq \theta \leq \pi/2$. ✓

> **Final Answer:** Option **(B)**: $0 \leq \theta \leq \frac{\pi}{2}$.

EXPERT'S SOLUTION : *Aditi Rao, M.Sc Mathematics, IIT Bombay*

Quick reading. "Greater than or equal to zero" includes zero, so the endpoints (where $\cos\theta = 1$ at $\theta = 0$ and $\cos\theta = 0$ at $\theta = \pi/2$) must be in the answer. That excludes (A) and (C).

Step 1. $\vec{a}\cdot\vec{b} \geq 0 \iff \cos\theta \geq 0$.

Step 2. $\cos\theta \geq 0$ on the closed interval $[0, \pi/2]$.

Step 3. Among the listed intervals, only (B) is this closed interval.

Why this matters. Open and closed intervals matter: strict $>$ excludes the boundary; $\geq$ includes it.

Final Answer: (B).

Q 10.17 **Let $\vec{a}$ and $\vec{b}$ be two unit vectors and θ is the angle between them. Then $\vec{a}+\vec{b}$ is a unit vector if**

(A) $\theta = \frac{\pi}{4}$ **(B)** $\theta = \frac{\pi}{3}$ **(C)** $\theta = \frac{\pi}{2}$ **(D)** $\theta = \frac{2\pi}{3}$**.**

SOLUTION

Concept used. For any two vectors, $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b}$. With unit vectors $|\vec{a}| = |\vec{b}| = 1$ and $\vec{a}\cdot\vec{b} = \cos\theta$.

Step 1. Substitute the unit-vector data:

$$|\vec{a}+\vec{b}|^2 = 1 + 1 + 2\cos\theta = 2 + 2\cos\theta.$$

Step 2. For $\vec{a}+\vec{b}$ to be a unit vector, $|\vec{a}+\vec{b}|^2 = 1$:

$$2 + 2\cos\theta = 1 \quad \Rightarrow \quad 2\cos\theta = -1 \quad \Rightarrow \quad \cos\theta = -\tfrac{1}{2}.$$

Step 3. Solve for θ on $[0, \pi]$:

$$\theta = \cos^{-1}\left(-\tfrac{1}{2}\right) = \tfrac{2\pi}{3}.$$

Step 4. Match to options: (D).

Final Answer: Option **(D)**: $\theta = \dfrac{2\pi}{3}$.

☞ **Equilateral triangle of unit vectors**

The three vectors $\vec{a}$, $\vec{b}$ and $-(\vec{a}+\vec{b})$ form a closed triangle. When all three are unit vectors, the triangle is equilateral and each interior angle is $\pi/3$; the angle *between* $\vec{a}$ and $\vec{b}$ measured "tail-to-tail" is the supplement, $\pi - \pi/3 = 2\pi/3$.

EXPERT'S SOLUTION *: Krishna Singh, M.Sc Mathematics, IIT Bombay*

Picture-first. Place $\vec{a}$ and $\vec{b}$ tail-to-tail; the parallelogram diagonal has length $|\vec{a}+\vec{b}|$. Set this $= 1$ and solve for θ.

Step 1. Parallelogram-law length: $|\vec{a}+\vec{b}|^2 = 2 + 2\cos\theta$.

Step 2. $|\vec{a}+\vec{b}| = 1 \Rightarrow 2 + 2\cos\theta = 1 \Rightarrow \cos\theta = -1/2$.

Step 3. $\theta = 2\pi/3$.

Why this matters. Two unit vectors at $\theta = 2\pi/3$ form, together with the negative of their sum, the three sides of an equilateral triangle - a cleanly symmetric configuration that appears in roots-of-unity problems.

> **Final Answer:** (D), $\theta = 2\pi/3$.

Q 10.18 **The value of $\hat{i}\cdot(\hat{j}\times\hat{k}) + \hat{j}\cdot(\hat{i}\times\hat{k}) + \hat{k}\cdot(\hat{i}\times\hat{j})$ is**
(A) 0 (B) -1 (C) 1 (D) 3.

SOLUTION

Concept used. The standard basis cross products are

$$\hat{j}\times\hat{k} = \hat{i}, \quad \hat{k}\times\hat{i} = \hat{j}, \quad \hat{i}\times\hat{j} = \hat{k},$$

and the dot products satisfy $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ with all other pairs zero. Note $\hat{i}\times\hat{k} = -\hat{j}$ (anti-commutativity flips the sign).

Step 1. Term 1:

$$\hat{i}\cdot(\hat{j}\times\hat{k}) = \hat{i}\cdot\hat{i} = 1.$$

Step 2. Term 2 (mind the sign):

$$\hat{j}\cdot(\hat{i}\times\hat{k}) = \hat{j}\cdot(-\hat{j}) = -1.$$

Step 3. Term 3:

$$\hat{k}\cdot(\hat{i}\times\hat{j}) = \hat{k}\cdot\hat{k} = 1.$$

Step 4. Sum:

$$1 + (-1) + 1 = 1.$$

Step 5. Match to options: (C).

> **Final Answer:** Option **(C)**: value $= 1$.

✗ Common Mistake

The middle term has $\hat{i} \times \hat{k}$ (not $\hat{k} \times \hat{i}$). Reading the cyclic order $\hat{i} \to \hat{j} \to \hat{k} \to \hat{i}$ gives positive cross products. The reverse cyclic order $\hat{i} \to \hat{k}$ jumps one step backwards, so $\hat{i} \times \hat{k} = -\hat{j}$ (negative sign).

EXPERT'S SOLUTION : *Meera Pillai, M.Sc Mathematics, IIT Bombay*

Structural observation. Each term is a scalar triple product of three basis vectors: $[\hat{u}, \hat{v}, \hat{w}] = \hat{u} \cdot (\hat{v} \times \hat{w})$. It equals ± 1 depending on whether $(\hat{u}, \hat{v}, \hat{w})$ is a cyclic (right-handed) permutation of $(\hat{i}, \hat{j}, \hat{k})$ or not.

Step 1. $[\hat{i}, \hat{j}, \hat{k}] = +1$ (cyclic order).

Step 2. $[\hat{j}, \hat{i}, \hat{k}] = -1$ (swapping two slots flips the sign).

Step 3. $[\hat{k}, \hat{i}, \hat{j}] = +1$ (cyclic order).

Step 4. Sum: $1 - 1 + 1 = 1$.

Why this matters. The scalar triple product evaluates volumes of parallelepipeds; for the unit basis cube it is ± 1 depending on handedness.

Final Answer: (C), 1.

Q 10.19 **If θ is the angle between any two vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$ when θ is equal to**

(A) 0 (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) π.

SOLUTION

Concept used.

$$|\vec{a} \cdot \vec{b}| = |\vec{a}||\vec{b}||\cos\theta|, \quad |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta \text{ (with } \sin\theta \geq 0 \text{ since } 0 \leq \theta \leq \pi).$$

Setting these equal removes the common factor $|\vec{a}||\vec{b}|$ (assumed non-zero), leaving an equation purely in θ.

Step 1. Equate magnitudes:

$$|\vec{a}||\vec{b}||\cos\theta| = |\vec{a}||\vec{b}|\sin\theta.$$

Step 2. Assume $|\vec{a}||\vec{b}| \neq 0$ and divide:

$$|\cos\theta| = \sin\theta.$$

Step 3. On $[0, \pi]$ where $\sin\theta \geq 0$, the equation $|\cos\theta| = \sin\theta$ means $\tan\theta = \pm 1$. The angles in $[0, \pi]$ with this property are $\theta = \pi/4$ (where $\sin\theta = \cos\theta = 1/\sqrt{2}$) and $\theta = 3\pi/4$ (where $\sin\theta = -\cos\theta = 1/\sqrt{2}$).

Step 4. Only $\pi/4$ is among the four options.

Step 5. Match to options: (B).

Final Answer: Option **(B)**: $\theta = \frac{\pi}{4}$.

EXPERT'S SOLUTION : *Rohit Gupta, M.Sc Mathematics, IIT Bombay*

Quick reading. The dot product peaks at $\theta = 0$, the cross product peaks at $\theta = \pi/2$. Their magnitudes cross at the angle where $\sin$ equals $\cos$: $\theta = \pi/4$.

Step 1. $|\vec{a} \cdot \vec{b}| = |\vec{a}||\vec{b}||\cos\theta|$; $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$.

Step 2. Set equal $\Rightarrow |\cos\theta| = \sin\theta$.

Step 3. On $[0, \pi]$, this happens at $\theta = \pi/4$ and $\theta = 3\pi/4$; only $\pi/4$ is listed.

Why this matters. $\pi/4$ is the angle of equal scalar and vector projection; both the parallel and the perpendicular components have the same length.

Final Answer: (B), $\pi/4$.

Key Takeaways

- Polar form: any planar unit vector is $\cos\theta\,\hat{i} + \sin\theta\,\hat{j}$.
- Triangle inequality $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$, equality only for like-parallel vectors.
- Section formula (external): $\vec{r} = \dfrac{m\vec{q} - n\vec{p}}{m - n}$.
- Perpendicular to two vectors $\Rightarrow$ parallel to their cross product.
- Direction cosines: $l^2 + m^2 + n^2 = 1$. Equally inclined to axes $\Rightarrow l = m = n = \pm 1/\sqrt{3}$.
- Vector form of Pythagoras: $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 \iff \vec{a} \perp \vec{b}$.
- Equality $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$ holds at $\theta = \pi/4$ (and $3\pi/4$): where $\sin\theta = |\cos\theta|$.

End of Miscellaneous Exercise