collegedunia

Collegedunia NCERT Solutions

Step-by-step solutions for the 2026-27 NCERT (Latest Edition)

Chapter 11: Three Dimensional Geometry

About this Chapter

Exercise 11.2 is the working heart of the chapter. The fifteen problems cover the **equation of a line** in 3D in both vector and Cartesian form, the **angle between two lines** from direction ratios, perpendicularity and parallelism tests, and the **shortest distance** between two lines: both intersecting/skew (via the box-product formula) and parallel (via the cross-product formula). Every question gets a 3D sketch alongside the algebra, every formula is written out symbolically, then with values substituted, then evaluated. We use $\vec{r} = \vec{a} + \lambda\vec{b}$ as our standing vector form throughout.

Topics covered: perpendicularity test from direction cosines / ratios; vector and Cartesian equations of a line; angle between two lines via $\cos\theta = \vec{b}_1 \cdot \vec{b}_2/(|\vec{b}_1||\vec{b}_2|)$; solving for a parameter from a right-angle condition; shortest distance between two skew lines (box-product formula); distance between two parallel lines (cross-product formula).

Quick Formula Sheet

Line: $\vec{r} = \vec{a} + \lambda\vec{b}$; Cartesian $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$.

Perp. $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Parallel. $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Angle. $\cos\theta = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1||\vec{b}_2|}$.

Skew SD. $d = \frac{|(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}_1 \times \vec{b}_2|}$.

Parallel SD. $d = \frac{|\vec{b} \times (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}|}$.

Q 11.1 **Show that the three lines with direction cosines**

$$\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; \quad \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; \quad \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$$

are mutually perpendicular.

SOLUTION

Concept used. Two lines with direction cosines (l_1, m_1, n_1) and (l_2, m_2, n_2) are **perpendicular** iff

$$l_1l_2 + m_1m_2 + n_1n_2 = 0.$$

This is the vector-dot-product condition $\vec{u} \cdot \vec{v} = 0$ written in direction-cosine form.

"Mutually perpendicular" means we must verify this for each of the three pairs of lines.

Step 1. Label the direction-cosine triples:

$$L_1 : \left(\tfrac{12}{13}, -\tfrac{3}{13}, -\tfrac{4}{13}\right), \; L_2 : \left(\tfrac{4}{13}, \tfrac{12}{13}, \tfrac{3}{13}\right), \; L_3 : \left(\tfrac{3}{13}, -\tfrac{4}{13}, \tfrac{12}{13}\right).$$

Step 2. **Pair $L_1 \perp L_2$.** Compute $l_1 l_2 + m_1 m_2 + n_1 n_2$:

$$\tfrac{12}{13} \cdot \tfrac{4}{13} + \left(-\tfrac{3}{13}\right) \cdot \tfrac{12}{13} + \left(-\tfrac{4}{13}\right) \cdot \tfrac{3}{13}.$$

Multiply numerators (the common denominator is $13^2 = 169$):

$$= \frac{48 - 36 - 12}{169} = \frac{0}{169} = 0.$$

Hence $L_1 \perp L_2$.

Step 3. **Pair $L_2 \perp L_3$.**

$$\tfrac{4}{13} \cdot \tfrac{3}{13} + \tfrac{12}{13} \cdot \left(-\tfrac{4}{13}\right) + \tfrac{3}{13} \cdot \tfrac{12}{13} = \frac{12 - 48 + 36}{169} = \frac{0}{169} = 0.$$

Hence $L_2 \perp L_3$.

Step 4. **Pair $L_1 \perp L_3$.**

$$\tfrac{12}{13} \cdot \tfrac{3}{13} + \left(-\tfrac{3}{13}\right) \cdot \left(-\tfrac{4}{13}\right) + \left(-\tfrac{4}{13}\right) \cdot \tfrac{12}{13} = \frac{36 + 12 - 48}{169} = 0.$$

Hence $L_1 \perp L_3$.

Step 5. All three pairs satisfy the perpendicularity condition, so L_1, L_2, L_3 are mutually perpendicular.

> **Final Answer:** $L_1 \perp L_2,\; L_2 \perp L_3,\; L_1 \perp L_3$; the three lines are mutually perpendicular.

☞ Why this is the dot-product

If $\vec{u} = (l_1, m_1, n_1)$ and $\vec{v} = (l_2, m_2, n_2)$ are unit vectors along the two lines, then $\vec{u} \cdot \vec{v} = \cos\theta$. For $\theta = 90°$, $\cos\theta = 0$, hence the test.

EXPERT'S SOLUTION *: Aditya Singh, M.Tech CS, IIT Madras*

Picture-first. Three pairwise-perpendicular directions in $\mathbb{R}^3$ form an orthonormal basis. Verifying perpendicularity in pairs is exactly checking that the matrix whose rows are the three triples is orthogonal.

Concept restated. For unit vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$,

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

is the definition of an orthonormal set.

Step 1. Arrange the direction cosines as rows of a 3×3 matrix scaled by 13:

$$A = \begin{pmatrix} 12 & -3 & -4 \\ 4 & 12 & 3 \\ 3 & -4 & 12 \end{pmatrix}.$$

Each $\frac{1}{13} \cdot \text{row}_i$ is a unit vector; checking $AA^T = 169I$ is equivalent to checking all three pairs are perpendicular.

Step 2. Off-diagonal entries of AA^T are exactly the dot products of distinct rows. Row 1 · Row 2:

$$12(4) + (-3)(12) + (-4)(3) = 48 - 36 - 12 = 0.$$

Step 3. Row 2 · Row 3:

$$4(3) + 12(-4) + 3(12) = 12 - 48 + 36 = 0.$$

Step 4. Row 1 · Row 3:

$$12(3) + (-3)(-4) + (-4)(12) = 36 + 12 - 48 = 0.$$

Step 5. All three off-diagonal entries are zero, so AA^T is diagonal. Diagonal entries are $\|\text{row}_i\|^2 = 144 + 9 + 16 = 169$ etc., confirming each row has norm 13.

Step 6. Therefore the three directions are mutually perpendicular and each is unit.

> **Final Answer:** All three dot products vanish; the three lines are mutually perpendicular.

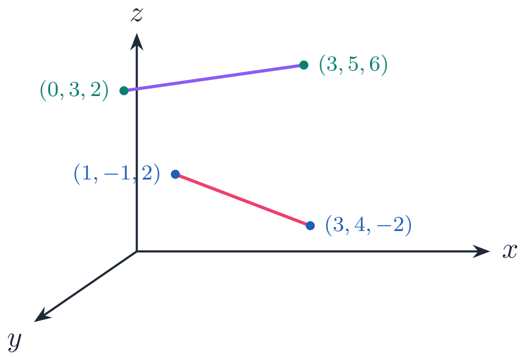
Q 11.2 **Show that the line through the points $(1, -1, 2)$, $(3, 4, -2)$ is perpendicular to the line through the points $(0, 3, 2)$ and $(3, 5, 6)$.**

SOLUTION

Concept used. Two lines with direction ratios (a_1, b_1, c_1) and (a_2, b_2, c_2) are perpendicular iff

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0.$$

We get the direction ratios of a line through $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ from the differences $(x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$.



Step 1. Call the first line L_1 (through $A(1, -1, 2)$ and $B(3, 4, -2)$) and the second L_2 (through $C(0, 3, 2)$ and $D(3, 5, 6)$).

Step 2. Direction ratios of L_1:

$$\overrightarrow{AB} = (3 - 1,\ 4 - (-1),\ -2 - 2) = (2,\ 5,\ -4).$$

So $(a_1, b_1, c_1) = (2, 5, -4)$.

Step 3. Direction ratios of L_2:

$$\overrightarrow{CD} = (3 - 0,\ 5 - 3,\ 6 - 2) = (3,\ 2,\ 4).$$

So $(a_2, b_2, c_2) = (3, 2, 4)$.

Step 4. Compute the perpendicularity sum:

$$a_1a_2 + b_1b_2 + c_1c_2 = (2)(3) + (5)(2) + (-4)(4).$$

Evaluate term-by-term:

$$= 6 + 10 - 16 = 0.$$

Step 5. Since the sum is 0, $L_1 \perp L_2$.

Final Answer: $a_1a_2 + b_1b_2 + c_1c_2 = 0$, hence the two lines are perpendicular.

EXPERT'S SOLUTION *: Riya Verma, M.Sc Mathematics, IIT Bombay*

Strategic angle. Build the two displacement vectors directly from the four points; take their dot product; check if it is zero. The minus signs require care, the algebra does not.

Step 1. $\vec{u} = B - A = (3, 4, -2) - (1, -1, 2) = (2, 5, -4)$.

Step 2. $\vec{v} = D - C = (3, 5, 6) - (0, 3, 2) = (3, 2, 4)$.

Step 3. Dot product:

$$\vec{u} \cdot \vec{v} = 2 \cdot 3 + 5 \cdot 2 + (-4) \cdot 4 = 6 + 10 - 16 = 0.$$

Step 4. Since $\vec{u} \cdot \vec{v} = 0$ and neither vector is zero, the two directions are orthogonal. The lines through these directions are perpendicular.

Step 5. No further work needed; the zero dot product is the proof.

Final Answer: $\vec{u} \cdot \vec{v} = 0 \Rightarrow$ the two lines are perpendicular.

Q 11.3 **Show that the line through the points $(4, 7, 8)$, $(2, 3, 4)$ is parallel to the line through the points $(-1, -2, 1)$, $(1, 2, 5)$.**

SOLUTION

Concept used. Two lines with direction ratios (a_1, b_1, c_1) and (a_2, b_2, c_2) are parallel iff

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

Equivalently, one direction-ratio triple is a non-zero scalar multiple of the other.

Step 1. Label $A(4, 7, 8)$, $B(2, 3, 4)$, $C(-1, -2, 1)$, $D(1, 2, 5)$. The first line is AB, the second is CD.

Step 2. Direction ratios of AB:

$$\overrightarrow{AB} = (2 - 4,\ 3 - 7,\ 4 - 8) = (-2,\ -4,\ -4).$$

Step 3. Direction ratios of CD:

$$\overrightarrow{CD} = (1 - (-1),\ 2 - (-2),\ 5 - 1) = (2,\ 4,\ 4).$$

Step 4. Compare the ratios component-by-component:

$$\frac{-2}{2} = -1, \quad \frac{-4}{4} = -1, \quad \frac{-4}{4} = -1.$$

All three are equal to -1.

Step 5. Therefore $\overrightarrow{AB} = -1 \cdot \overrightarrow{CD}$, so the two direction vectors are parallel (anti-parallel, to be precise; a line has no orientation, so the lines themselves are parallel).

Final Answer: $\overrightarrow{AB} = -\overrightarrow{CD}$, hence the two lines are parallel.

✗ Common Mistake

Anti-parallel direction vectors (same line, opposite arrowheads) still make the lines parallel. A line has two senses; "parallel lines" means parallel as lines, ignoring direction.

EXPERT'S SOLUTION : *Karan Reddy, Ph.D Pure Mathematics, IISc Bangalore*

Quick reading. Compute one displacement; compute the other; check if one is a scalar multiple of the other. The scalar tells you whether they point the same way or opposite, but does not change the conclusion of parallelism.

Step 1. Displacement along AB:

$$\vec{u} = B - A = (-2, -4, -4).$$

Step 2. Displacement along CD:

$$\vec{v} = D - C = (2, 4, 4).$$

Step 3. Test scalar multiplicity. Note $\vec{u} = (-1)\,\vec{v}$ component-wise:

$$(-1)(2, 4, 4) = (-2, -4, -4) = \vec{u}. \checkmark$$

Step 4. Existence of $\lambda = -1 \neq 0$ such that $\vec{u} = \lambda\vec{v}$ proves the two direction vectors are parallel (anti-parallel is a sub-case). Hence the lines are parallel.

Step 5. Cross-product cross-check: $\vec{u} \times \vec{v}$ should be $\vec{0}$. Indeed $(-1)\vec{v} \times \vec{v} = (-1)(\vec{v} \times \vec{v}) = \vec{0}$. ✓

Final Answer: $\vec{u} = -\vec{v}$, the two lines are parallel.

Q 11.4 **Find the equation of the line which passes through the point $(1, 2, 3)$ and is parallel to the vector $3\hat{i} + 2\hat{j} - 2\hat{k}$.**

SOLUTION

Concept used. The vector equation of a line passing through the point with position vector $\vec{a}$ and parallel to a non-zero vector $\vec{b}$ is

$$\vec{r} = \vec{a} + \lambda\vec{b}, \quad \lambda \in \mathbb{R}.$$

Here $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the position vector of an arbitrary point on the line. Equating components and eliminating λ yields the Cartesian form.

Step 1. Read off $\vec{a}$ and $\vec{b}$ from the data:

$$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}, \quad \vec{b} = 3\hat{i} + 2\hat{j} - 2\hat{k}.$$

Step 2. Plug into $\vec{r} = \vec{a} + \lambda\vec{b}$:

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k}).$$

This is the required vector equation.

Step 3. Convert to Cartesian. With $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, comparing coefficients of $\hat{i}, \hat{j}, \hat{k}$:

$$x = 1 + 3\lambda, \quad y = 2 + 2\lambda, \quad z = 3 - 2\lambda.$$

Step 4. Solve each for λ:

$$\lambda = \frac{x-1}{3}, \quad \lambda = \frac{y-2}{2}, \quad \lambda = \frac{z-3}{-2}.$$

Step 5. Equate the three expressions to get the Cartesian symmetric form:

$$\frac{x-1}{3} = \frac{y-2}{2} = \frac{z-3}{-2}.$$

Final Answer: $\vec{r} = (\hat{i}+2\hat{j}+3\hat{k})+\lambda(3\hat{i}+2\hat{j}-2\hat{k})$; Cartesian: $\frac{x-1}{3} = \frac{y-2}{2} = \frac{z-3}{-2}$.

EXPERT'S SOLUTION : *Diya Nair, M.Sc Mathematics, ISI Kolkata*

Strategic angle. A line in 3D is fixed by one point plus one direction. Pack those two pieces into $\vec{a}$ and $\vec{b}$; the vector form writes itself.

Step 1. $\vec{a} = (1, 2, 3)$, $\vec{b} = (3, 2, -2)$.

Step 2. Vector form: $\vec{r} = \vec{a} + \lambda\vec{b}$, i.e.

$$\vec{r} = (1 + 3\lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (3 - 2\lambda)\hat{k}.$$

Step 3. Eliminate λ. From $x = 1 + 3\lambda$: $\lambda = (x - 1)/3$. Similarly $\lambda = (y - 2)/2 = (z - 3)/(-2)$. Setting equal gives Cartesian form.

Step 4. Final Cartesian form:

$$\frac{x-1}{3} = \frac{y-2}{2} = \frac{z-3}{-2}.$$

Step 5. Geometric reading: starting from $(1, 2, 3)$, each unit step in λ moves $+3$ in x, $+2$ in y, -2 in z. The denominators in the Cartesian form are precisely these step sizes.

Final Answer: Vector: $\vec{r} = (1, 2, 3) + \lambda(3, 2, -2)$; Cartesian: $\dfrac{x-1}{3} = \dfrac{y-2}{2} = \dfrac{z-3}{-2}$.

Q 11.5 **Find the equation of the line in vector and in Cartesian form that passes through the point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and is in the direction $\hat{i} + 2\hat{j} - \hat{k}$.**

SOLUTION

Concept used. Same template as before: $\vec{r} = \vec{a} + \lambda\vec{b}$ for a line through $\vec{a}$ parallel to $\vec{b}$. The Cartesian form comes from equating components of $\vec{r}$ with $x\hat{i} + y\hat{j} + z\hat{k}$ and eliminating λ.

Step 1. Identify $\vec{a}$ and $\vec{b}$:

$$\vec{a} = 2\hat{i} - \hat{j} + 4\hat{k}, \quad \vec{b} = \hat{i} + 2\hat{j} - \hat{k}.$$

Step 2. Vector equation:

$$\vec{r} = (2\hat{i} - \hat{j} + 4\hat{k}) + \lambda(\hat{i} + 2\hat{j} - \hat{k}).$$

Step 3. Write $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and expand the right-hand side:

$$\vec{r} = (2 + \lambda)\hat{i} + (-1 + 2\lambda)\hat{j} + (4 - \lambda)\hat{k}.$$

Matching coefficients:

$$x = 2 + \lambda, \quad y = -1 + 2\lambda, \quad z = 4 - \lambda.$$

Step 4. Solve each equation for λ:

$$\lambda = x - 2, \quad \lambda = \frac{y+1}{2}, \quad \lambda = \frac{z-4}{-1} = -(z-4) = 4 - z.$$

Step 5. Setting the three expressions equal yields the Cartesian symmetric form:

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}.$$

Final Answer: Vector: $\vec{r} = (2\hat{i} - \hat{j} + 4\hat{k}) + \lambda(\hat{i} + 2\hat{j} - \hat{k})$; Cartesian: $\dfrac{x-2}{1} = \dfrac{y+1}{2} = \dfrac{z-4}{-1}$.

EXPERT'S SOLUTION : *Yash Joshi, M.Sc Mathematics, IIT Bombay*

Quick reading. One point, one direction → one vector equation. Convert to Cartesian by solving each parametric coordinate for the parameter.

Step 1. $\vec{a} = (2, -1, 4)$, $\vec{b} = (1, 2, -1)$.

Step 2. Vector form: $\vec{r} = \vec{a} + \lambda\vec{b}$.

$$\vec{r} = (2, -1, 4) + \lambda(1, 2, -1).$$

Step 3. Parametric (component) form:

$$x = 2 + \lambda, \ y = -1 + 2\lambda, \ z = 4 - \lambda.$$

Step 4. Symmetric (Cartesian) form by eliminating λ:

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}.$$

Step 5. Geometric reading: as λ increases by one unit, x goes up by 1, y by 2, and z goes down by 1. So the line slopes through the first octant with a modest negative z-tilt.

Final Answer: Vector: $\vec{r} = (2, -1, 4) + \lambda(1, 2, -1)$; Cartesian: $\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$.

Q 11.6 **Find the Cartesian equation of the line which passes through the point $(-2, 4, -5)$ and is parallel to the line given by $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$.**

SOLUTION

Concept used. The denominators in the symmetric Cartesian equation

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

are precisely the direction ratios a, b, c of the line. Parallel lines share the same direction ratios. So we copy the denominators from the given line and pair them with the new point.

Step 1. Read direction ratios from the given line $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$. Comparing with the template, the denominators are the direction ratios:

$$(a, b, c) = (3, 5, 6).$$

Step 2. The new line passes through $(-2, 4, -5)$ and has the same direction ratios $(3, 5, 6)$.

Step 3. Write the symmetric Cartesian form with $x_1 = -2$, $y_1 = 4$, $z_1 = -5$:

$$\frac{x-(-2)}{3} = \frac{y-4}{5} = \frac{z-(-5)}{6}.$$

Step 4. Simplify the signs:

$$\frac{x+2}{3} = \frac{y-4}{5} = \frac{z+5}{6}.$$

Final Answer: $\frac{x+2}{3} = \frac{y-4}{5} = \frac{z+5}{6}$

♥ Why direction ratios live in the denominators

The symmetric form is just $\lambda = \frac{x-x_1}{a}$, where λ is the parameter in $\vec{r} = \vec{a} + \lambda\vec{b}$. So denominators are the components of $\vec{b}$, i.e. the direction ratios.

EXPERT'S SOLUTION *: Ananya Banerjee, M.Sc Mathematics, IIT Kanpur*

Quick reading. Parallel lines $\equiv$ same direction ratios. So just keep $3, 5, 6$ and slot the new point.

Step 1. Direction ratios of the given line = denominators in its symmetric form $= (3, 5, 6)$.

Step 2. Lines are parallel $\Rightarrow$ direction ratios of the required line are also $(3, 5, 6)$.

Step 3. The required line passes through $(-2, 4, -5)$.

Step 4. Plug into the template

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}:$$

$$\frac{x+2}{3} = \frac{y-4}{5} = \frac{z+5}{6}.$$

Step 5. Sanity check by setting $\lambda = 0$: $(x, y, z) = (-2, 4, -5)$, which is the given point. The line does pass through it. ✓

Final Answer: $\frac{x+2}{3} = \frac{y-4}{5} = \frac{z+5}{6}$

Q 11.7 **The Cartesian equation of a line is $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$. Write its vector form.**

SOLUTION

Concept used. From the symmetric Cartesian form

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c},$$

we read off the point (x_1, y_1, z_1) on the line (from the numerators with sign flipped) and the direction ratios (a, b, c) (the denominators). The vector form is

$$\vec{r} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k}).$$

Step 1. Read off the point. The numerators are $x-5$, $y-(-4)$, $z-6$, so

$$(x_1, y_1, z_1) = (5,\ -4,\ 6).$$

The minus sign in $y+4$ comes out as $-(-4)$, giving $y_1 = -4$.

Step 2. Read off the direction ratios. The denominators are

$$(a, b, c) = (3,\ 7,\ 2).$$

Step 3. Form the position vector of the given point:

$$\vec{a} = 5\hat{i} - 4\hat{j} + 6\hat{k}.$$

Step 4. Form the direction vector:

$$\vec{b} = 3\hat{i} + 7\hat{j} + 2\hat{k}.$$

Step 5. Assemble the vector equation:

$$\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k}).$$

Final Answer: $\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$

✗ Common Mistake

A frequent slip is reading $y+4$ as if $y_1 = 4$. Watch the sign: the symmetric form is $(y-y_1)/b$, so $y+4 = y-(-4)$ gives $y_1 = -4$.

EXPERT'S SOLUTION : *Ishaan Kapoor, M.Sc Mathematics, IIT Bombay*

Strategic angle. Symmetric form → vector form is a pure pattern match. Numerator sign goes into the point, denominator goes into the direction.

Step 1. Identify the point: from $(x-5)$, $(y+4)$, $(z-6)$ we get $x_1 = 5$, $y_1 = -4$, $z_1 = 6$.

Step 2. Identify the direction: denominators give $\vec{b} = (3, 7, 2)$.

Step 3. Build the position vector of the point:

$$\vec{a} = 5\hat{i} - 4\hat{j} + 6\hat{k}.$$

Step 4. Stack into the standard form $\vec{r} = \vec{a} + \lambda\vec{b}$:

$$\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k}).$$

Step 5. Round-trip check: substitute $\lambda = 0$, get $(5, -4, 6)$. Substitute $\lambda = 1$, get $(8, 3, 8)$; this satisfies the original Cartesian form: $(8-5)/3 = 1$, $(3+4)/7 = 1$, $(8-6)/2 = 1$. ✓

Final Answer: $\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$

Q 11.8 Find the angle between the following pairs of lines:

(i) $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$ **and** $\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$

(ii) $\vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda(\hat{i} - \hat{j} - 2\hat{k})$ **and** $\vec{r} = 2\hat{i} - \hat{j} - 56\hat{k} + \mu(3\hat{i} - 5\hat{j} - 4\hat{k})$**.**

SOLUTION

Concept used. For two lines in vector form

$$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1 \quad \text{and} \quad \vec{r} = \vec{a}_2 + \mu\vec{b}_2,$$

the acute angle θ between them is given by

$$\cos\theta = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1|\,|\vec{b}_2|}.$$

The absolute value picks out the acute angle (otherwise the formula could give the obtuse supplement).

Part (i). Direction vectors:

$$\vec{b}_1 = 3\hat{i} + 2\hat{j} + 6\hat{k}, \quad \vec{b}_2 = \hat{i} + 2\hat{j} + 2\hat{k}.$$

Step 1. Dot product:

$$\vec{b}_1 \cdot \vec{b}_2 = (3)(1) + (2)(2) + (6)(2) = 3 + 4 + 12 = 19.$$

Step 2. Magnitude of $\vec{b}_1$:

$$|\vec{b}_1| = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{9 + 4 + 36} = \sqrt{49} = 7.$$

Step 3. Magnitude of $\vec{b}_2$:

$$|\vec{b}_2| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3.$$

Step 4. Compute $\cos\theta$:

$$\cos\theta = \frac{|19|}{7 \cdot 3} = \frac{19}{21}.$$

Step 5. Hence

$$\theta = \cos^{-1}\left(\frac{19}{21}\right).$$

Part (ii). Direction vectors:

$$\vec{b}_1 = \hat{i} - \hat{j} - 2\hat{k}, \quad \vec{b}_2 = 3\hat{i} - 5\hat{j} - 4\hat{k}.$$

Step 1. Dot product:

$$\vec{b}_1 \cdot \vec{b}_2 = (1)(3) + (-1)(-5) + (-2)(-4) = 3 + 5 + 8 = 16.$$

Step 2. Magnitude of $\vec{b}_1$:

$$|\vec{b}_1| = \sqrt{1 + 1 + 4} = \sqrt{6}.$$

Step 3. Magnitude of $\vec{b}_2$:

$$|\vec{b}_2| = \sqrt{9 + 25 + 16} = \sqrt{50} = 5\sqrt{2}.$$

Step 4. Compute $\cos\theta$:

$$\cos\theta = \frac{|16|}{\sqrt{6} \cdot 5\sqrt{2}} = \frac{16}{5\sqrt{12}} = \frac{16}{5 \cdot 2\sqrt{3}} = \frac{16}{10\sqrt{3}} = \frac{8}{5\sqrt{3}}.$$

Step 5. Rationalising the denominator:

$$\cos\theta = \frac{8}{5\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{8\sqrt{3}}{15}.$$

Step 6. Hence

$$\theta = \cos^{-1}\left(\frac{8\sqrt{3}}{15}\right).$$

Final Answer: (i) $\theta = \cos^{-1}\frac{19}{21}$; (ii) $\theta = \cos^{-1}\frac{8\sqrt{3}}{15}$.

☞ **Exam Tip**

The position vectors $\vec{a}_1$ and $\vec{a}_2$ play no role in finding the angle. Only the direction vectors $\vec{b}_1$ and $\vec{b}_2$ matter. Even if the lines are skew (not intersecting), the "angle between them" is defined by parallel-transporting one of them to meet the other.

EXPERT'S SOLUTION *: Rohit Bhat, Ph.D Mathematics, IIT Delhi*

Strategic angle. For each pair, isolate the two direction vectors, compute $\vec{b}_1 \cdot \vec{b}_2$, $|\vec{b}_1|$, $|\vec{b}_2|$ and put them through $\cos\theta = |\vec{b}_1 \cdot \vec{b}_2|/(|\vec{b}_1||\vec{b}_2|)$. Rationalise the answer whenever a $\sqrt{\cdot}$ remains in the denominator.

(i) $\vec{b}_1 = (3, 2, 6)$, $\vec{b}_2 = (1, 2, 2)$.

Step 1. $\vec{b}_1 \cdot \vec{b}_2 = 3 + 4 + 12 = 19$.

Step 2. $|\vec{b}_1| = \sqrt{49} = 7$, $|\vec{b}_2| = \sqrt{9} = 3$.

Step 3. $\cos\theta = 19/21$, so $\theta = \cos^{-1}(19/21)$.

(ii) $\vec{b}_1 = (1, -1, -2)$, $\vec{b}_2 = (3, -5, -4)$.

Step 1. $\vec{b}_1 \cdot \vec{b}_2 = 3 + 5 + 8 = 16$.

Step 2. $|\vec{b}_1| = \sqrt{6}$.

Step 3. $|\vec{b}_2| = \sqrt{50} = 5\sqrt{2}$.

Step 4. $\cos\theta = \dfrac{16}{\sqrt{6} \cdot 5\sqrt{2}} = \dfrac{16}{5\sqrt{12}} = \dfrac{16}{10\sqrt{3}} = \dfrac{8\sqrt{3}}{15}$ after rationalisation.

Step 5. $\theta = \cos^{-1}(8\sqrt{3}/15)$. Numerically $8\sqrt{3}/15 \approx 0.924$, so $\theta \approx 22.5°$.

Final Answer: (i) $\cos^{-1}(19/21)$; (ii) $\cos^{-1}(8\sqrt{3}/15)$.

Q 11.9 **Find the angle between the following pair of lines:**

(i) $\dfrac{x-2}{2} = \dfrac{y-1}{5} = \dfrac{z+3}{-3}$ **and** $\dfrac{x+2}{-1} = \dfrac{y-4}{8} = \dfrac{z-5}{4}$

(ii) $\dfrac{x}{2} = \dfrac{y}{2} = \dfrac{z}{1}$ **and** $\dfrac{x-5}{4} = \dfrac{y-2}{1} = \dfrac{z-3}{8}$.

SOLUTION

Concept used. For lines given in symmetric Cartesian form with direction ratios (a_1, b_1, c_1) and (a_2, b_2, c_2), the acute angle θ between them satisfies

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

Part (i). Direction ratios: $(2, 5, -3)$ and $(-1, 8, 4)$.

Step 1. Compute the numerator:

$$a_1a_2 + b_1b_2 + c_1c_2 = (2)(-1) + (5)(8) + (-3)(4) = -2 + 40 - 12 = 26.$$

Step 2. Denominator part 1:

$$\sqrt{2^2 + 5^2 + (-3)^2} = \sqrt{4 + 25 + 9} = \sqrt{38}.$$

Step 3. Denominator part 2:

$$\sqrt{(-1)^2 + 8^2 + 4^2} = \sqrt{1 + 64 + 16} = \sqrt{81} = 9.$$

Step 4. Plug in:

$$\cos\theta = \frac{|26|}{\sqrt{38} \cdot 9} = \frac{26}{9\sqrt{38}}.$$

Step 5. Hence

$$\theta = \cos^{-1}\left(\frac{26}{9\sqrt{38}}\right).$$

Part (ii). Direction ratios: $(2, 2, 1)$ and $(4, 1, 8)$.

Step 1. Numerator:

$$(2)(4) + (2)(1) + (1)(8) = 8 + 2 + 8 = 18.$$

Step 2. Denominator part 1:

$$\sqrt{2^2 + 2^2 + 1^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3.$$

Step 3. Denominator part 2:

$$\sqrt{4^2 + 1^2 + 8^2} = \sqrt{16 + 1 + 64} = \sqrt{81} = 9.$$

Step 4. Plug in:

$$\cos\theta = \frac{|18|}{3 \cdot 9} = \frac{18}{27} = \frac{2}{3}.$$

Step 5. Hence

$$\theta = \cos^{-1}\left(\frac{2}{3}\right).$$

Final Answer: (i) $\theta = \cos^{-1}\frac{26}{9\sqrt{38}}$; (ii) $\theta = \cos^{-1}\frac{2}{3}$.

EXPERT'S SOLUTION : *Tara Pillai, M.Sc Mathematics, IIT Madras*

Quick reading. Read direction ratios off the denominators of each symmetric form; dot, magnitudes, ratio, arccos. The two parts use the same algorithm; the second has nicer numbers.

(i) $\vec{b}_1 = (2, 5, -3)$, $\vec{b}_2 = (-1, 8, 4)$.

Step 1. $\vec{b}_1 \cdot \vec{b}_2 = -2 + 40 - 12 = 26$.

Step 2. $|\vec{b}_1| = \sqrt{38}$, $|\vec{b}_2| = \sqrt{81} = 9$.

Step 3. $\cos\theta = 26/(9\sqrt{38})$, so $\theta = \cos^{-1}(26/(9\sqrt{38}))$.

(ii) $\vec{b}_1 = (2, 2, 1)$, $\vec{b}_2 = (4, 1, 8)$.

Step 1. $\vec{b}_1 \cdot \vec{b}_2 = 8 + 2 + 8 = 18$.

Step 2. $|\vec{b}_1| = \sqrt{9} = 3$, $|\vec{b}_2| = \sqrt{81} = 9$.

Step 3. $\cos\theta = 18/27 = 2/3$.

Step 4. $\theta = \cos^{-1}(2/3) \approx 48.19°$.

Final Answer: (i) $\cos^{-1}\frac{26}{9\sqrt{38}}$; (ii) $\cos^{-1}\frac{2}{3}$.

Q 11.10 **Find the values of p so that the lines $\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$ and $\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are at right angles.**

SOLUTION

Concept used. Before applying the perpendicularity condition $a_1a_2 + b_1b_2 + c_1c_2 = 0$, we must rewrite each symmetric form so that the variable on top reads $x - x_0$, $y - y_0$, $z - z_0$ (with coefficient $+1$). Any sign-flip or scaling of a numerator must be absorbed into the denominator.

Line 1. $\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$.

Step 1. Rewrite $\frac{1-x}{3} = \frac{-(x-1)}{3} = \frac{x-1}{-3}$.

Step 2. Rewrite $\frac{7y-14}{2p} = \frac{7(y-2)}{2p} = \frac{y-2}{2p/7}$.

Step 3. The third term $\frac{z-3}{2}$ is already in canonical form.

Step 4. So Line 1 in standard form:

$$\frac{x-1}{-3} = \frac{y-2}{2p/7} = \frac{z-3}{2}.$$

Direction ratios: $(a_1, b_1, c_1) = \left(-3,\ \frac{2p}{7},\ 2\right)$.

Line 2. $\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$.

Step 1. Rewrite $\frac{7-7x}{3p} = \frac{-7(x-1)}{3p} = \frac{x-1}{-3p/7}$.

Step 2. Rewrite $\frac{6-z}{5} = \frac{-(z-6)}{5} = \frac{z-6}{-5}$.

Step 3. The middle term $\frac{y-5}{1}$ is canonical.

Step 4. Line 2 in standard form:

$$\frac{x-1}{-3p/7} = \frac{y-5}{1} = \frac{z-6}{-5}.$$

Direction ratios: $(a_2, b_2, c_2) = \left(-\frac{3p}{7},\ 1,\ -5\right)$.

Apply perpendicularity.

Step 1. Plug into $a_1a_2 + b_1b_2 + c_1c_2 = 0$:

$$(-3)\left(-\frac{3p}{7}\right) + \frac{2p}{7} \cdot 1 + 2 \cdot (-5) = 0.$$

Step 2. Simplify each term:

$$\frac{9p}{7} + \frac{2p}{7} - 10 = 0.$$

Step 3. Combine the p-terms (common denominator 7):

$$\frac{9p+2p}{7} - 10 = 0 \implies \frac{11p}{7} = 10.$$

Step 4. Solve:

$$p = \frac{70}{11}.$$

Final Answer: $p = \frac{70}{11}$

✗ Common Mistake

The most common error is plugging $(3, 2p, 2)$ and $(3p, 1, 5)$ straight into the perpendicularity test, ignoring that $1 - x$ has a leading minus sign and $7y - 14 = 7(y - 2)$ introduces a factor of 7 that must be moved to the denominator. Always normalise to $x - x_0$ first.

EXPERT'S SOLUTION *: Krishna Desai, M.Sc Applied Mathematics, IIT Kanpur*

Structural observation. Two symmetric forms wearing disguises. The fix is to re-express both so that the numerator of each fraction is (variable) − (constant) with coefficient $+1$. Any sign or scale gets pushed into the denominator.

Step 1. Line 1, three normalisations:

$$\frac{1-x}{3} = \frac{x-1}{-3}, \quad \frac{7y-14}{2p} = \frac{y-2}{2p/7}, \quad \frac{z-3}{2} \text{ stays.}$$

Direction ratios: $\vec{b}_1 = (-3,\ 2p/7,\ 2)$.

Step 2. Line 2, three normalisations:

$$\frac{7-7x}{3p} = \frac{x-1}{-3p/7}, \quad \frac{y-5}{1} \text{ stays,} \quad \frac{6-z}{5} = \frac{z-6}{-5}.$$

Direction ratios: $\vec{b}_2 = (-3p/7,\ 1,\ -5)$.

Step 3. Perpendicularity $\vec{b}_1 \cdot \vec{b}_2 = 0$:

$$(-3)(-3p/7) + (2p/7)(1) + (2)(-5) = 0.$$

Step 4. Simplify:

$$\frac{9p}{7} + \frac{2p}{7} - 10 = 0 \Rightarrow \frac{11p}{7} = 10.$$

Step 5. Solve: $p = 70/11$.

Step 6. Numerical check: $p = 70/11 \approx 6.36$. Plug back into $\vec{b}_1 \cdot \vec{b}_2$:
$9(70/11)/7 + 2(70/11)/7 - 10 = (90 + 20)/11 - 10 = 110/11 - 10 = 0.$ ✓

Final Answer: $p = \dfrac{70}{11}$

Q 11.11 **Show that the lines $\dfrac{x-5}{7} = \dfrac{y+2}{-5} = \dfrac{z}{1}$ and $\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{3}$ are perpendicular to each other.**

SOLUTION

Concept used. Perpendicularity in symmetric Cartesian form: read off direction ratios from the denominators, then verify $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Step 1. Line 1 direction ratios: $(a_1, b_1, c_1) = (7,\ -5,\ 1)$.

Step 2. Line 2 direction ratios: $(a_2, b_2, c_2) = (1,\ 2,\ 3)$.

Step 3. Compute the perpendicularity sum:

$$a_1a_2 + b_1b_2 + c_1c_2 = (7)(1) + (-5)(2) + (1)(3).$$

Term by term:

$$= 7 - 10 + 3 = 0.$$

Step 4. Since the sum is 0, the two lines are perpendicular.

> **Final Answer:** $a_1a_2 + b_1b_2 + c_1c_2 = 7 - 10 + 3 = 0$, so the lines are perpendicular.

EXPERT'S SOLUTION *: Sanya Chatterjee, B.Tech CSE, IIT Roorkee*

Quick reading. Both forms are already canonical; pull direction ratios straight from the denominators and dot them.

Step 1. $\vec{b}_1 = (7, -5, 1)$, $\vec{b}_2 = (1, 2, 3)$.

Step 2. $\vec{b}_1 \cdot \vec{b}_2 = 7(1) - 5(2) + 1(3) = 7 - 10 + 3 = 0$.

Step 3. Zero dot product proves the directions are orthogonal.

Step 4. Conclusion: the two lines are perpendicular.

> **Final Answer:** $\vec{b}_1 \cdot \vec{b}_2 = 0$, hence perpendicular.

Q 11.12 **Find the shortest distance between the lines** $\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ **and** $\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$**.**

SOLUTION

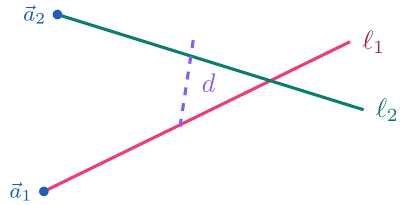
Concept used. For two non-parallel lines

$$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1 \quad \text{and} \quad \vec{r} = \vec{a}_2 + \mu\vec{b}_2$$

the shortest distance is

$$d = \frac{|(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}_1 \times \vec{b}_2|}.$$

The cross product $\vec{b}_1 \times \vec{b}_2$ is perpendicular to both lines (i.e. along the common perpendicular); $(\vec{a}_2 - \vec{a}_1)$ joins a point on one line to a point on the other; projecting one onto the other gives the shortest distance.



Step 1. Identify the four vectors:

$$\vec{a}_1 = \hat{i} + 2\hat{j} + \hat{k}, \quad \vec{b}_1 = \hat{i} - \hat{j} + \hat{k}, \quad \vec{a}_2 = 2\hat{i} - \hat{j} - \hat{k}, \quad \vec{b}_2 = 2\hat{i} + \hat{j} + 2\hat{k}.$$

Step 2. Compute $\vec{a}_2 - \vec{a}_1$:

$$\vec{a}_2 - \vec{a}_1 = (2-1)\hat{i} + (-1-2)\hat{j} + (-1-1)\hat{k} = \hat{i} - 3\hat{j} - 2\hat{k}.$$

Step 3. Compute $\vec{b}_1 \times \vec{b}_2$ via the 3×3 determinant:

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}.$$

Expand along the top row:

$$\vec{b}_1 \times \vec{b}_2 = \hat{i}[(-1)(2) - (1)(1)] - \hat{j}[(1)(2) - (1)(2)] + \hat{k}[(1)(1) - (-1)(2)].$$

Evaluate:

$$= \hat{i}(-2-1) - \hat{j}(2-2) + \hat{k}(1+2) = -3\hat{i} + 0\hat{j} + 3\hat{k}.$$

Step 4. Compute the box product $(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)$:

$$(-3)(1) + (0)(-3) + (3)(-2) = -3 + 0 - 6 = -9.$$

Its absolute value is $|-9| = 9$.

Step 5. Compute $|\vec{b}_1 \times \vec{b}_2|$:

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{(-3)^2 + 0^2 + 3^2} = \sqrt{9+0+9} = \sqrt{18} = 3\sqrt{2}.$$

Step 6. Plug into the shortest-distance formula:

$$d = \frac{9}{3\sqrt{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}.$$

Final Answer: $d = \dfrac{3\sqrt{2}}{2}$ (or equivalently $\dfrac{3}{\sqrt{2}}$)

☞ 3×3 **cross product**

$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$.

EXPERT'S SOLUTION *: Aditi Rao, M.Tech CS, IIT Madras*

Strategic angle. The formula has three ingredients. (a) $\vec{b}_1 \times \vec{b}_2$: the direction perpendicular to both lines. (b) $\vec{a}_2 - \vec{a}_1$: a connector between the two lines. (c) The dot of (a) and (b), divided by $|(a)|$: the scalar projection of the connector onto the perpendicular direction.

Step 1. Read inputs: $\vec{a}_1 = (1, 2, 1)$, $\vec{b}_1 = (1, -1, 1)$, $\vec{a}_2 = (2, -1, -1)$, $\vec{b}_2 = (2, 1, 2)$.

Step 2. Connector $\vec{d} = \vec{a}_2 - \vec{a}_1 = (1, -3, -2)$.

Step 3. Perpendicular $\vec{n} = \vec{b}_1 \times \vec{b}_2$:

$$\vec{n} = \big((-1)(2) - (1)(1),\ (1)(2) - (1)(2),\ (1)(1) - (-1)(2)\big) = (-3, 0, 3).$$

Step 4. Box product $\vec{n} \cdot \vec{d} = -3(1) + 0(-3) + 3(-2) = -3 - 6 = -9$.

Step 5. $|\vec{n}| = \sqrt{9 + 9} = \sqrt{18} = 3\sqrt{2}$.

Step 6. Shortest distance

$$d = \frac{|\vec{n} \cdot \vec{d}|}{|\vec{n}|} = \frac{9}{3\sqrt{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}.$$

Step 7. Sign of $\vec{n} \cdot \vec{d}$ is irrelevant: the formula uses $|\cdot|$ because distance is unsigned.

Final Answer: $d = \dfrac{3\sqrt{2}}{2} \approx 2.12$ units.

Q 11.13 **Find the shortest distance between the lines** $\dfrac{x+1}{7} = \dfrac{y+1}{-6} = \dfrac{z+1}{1}$ **and** $\dfrac{x-3}{1} = \dfrac{y-5}{-2} = \dfrac{z-7}{1}$.

SOLUTION

Concept used. In Cartesian symmetric form $\dfrac{x - x_i}{a_i} = \dfrac{y - y_i}{b_i} = \dfrac{z - z_i}{c_i}$, the line passes through (x_i, y_i, z_i) with direction ratios (a_i, b_i, c_i). The shortest distance between two such lines is

$$d = \frac{\left| \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \right|}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}.$$

Step 1. Read off points and directions:

$$P_1 = (-1, -1, -1),\ (a_1, b_1, c_1) = (7, -6, 1); \quad P_2 = (3, 5, 7),\ (a_2, b_2, c_2) = (1, -2, 1).$$

Step 2. Compute differences:

$$x_2 - x_1 = 4, \ \ y_2 - y_1 = 6, \ \ z_2 - z_1 = 8.$$

Step 3. Compute the cross-product components $\vec{b}_1 \times \vec{b}_2$:

$$\begin{aligned} b_1c_2 - b_2c_1 &= (-6)(1) - (-2)(1) = -6 + 2 = -4, \\ c_1a_2 - c_2a_1 &= (1)(1) - (1)(7) = 1 - 7 = -6, \\ a_1b_2 - a_2b_1 &= (7)(-2) - (1)(-6) = -14 + 6 = -8. \end{aligned}$$

So $\vec{b}_1 \times \vec{b}_2 = (-4, -6, -8)$.

Step 4. Magnitude:

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{(-4)^2 + (-6)^2 + (-8)^2} = \sqrt{16 + 36 + 64} = \sqrt{116} = 2\sqrt{29}.$$

Step 5. Box product numerator $|\det|$ equals $(\vec{b}_1 \times \vec{b}_2) \cdot (P_2 - P_1)$:

$$(-4)(4) + (-6)(6) + (-8)(8) = -16 - 36 - 64 = -116.$$

Absolute value: 116.

Step 6. Apply the formula:

$$d = \frac{116}{2\sqrt{29}} = \frac{58}{\sqrt{29}}.$$

Step 7. Simplify $\frac{58}{\sqrt{29}}$. Note $58 = 2 \cdot 29$, so

$$d = \frac{2 \cdot 29}{\sqrt{29}} = 2\sqrt{29}.$$

Final Answer: $d = 2\sqrt{29}$ units.

EXPERT'S SOLUTION : *Neha Iyer, Ph.D Mathematics, IIT Delhi*

Quick reading. Cartesian inputs translate one-to-one into the box-product formula. The numerator simplifies to a clean 116, the denominator simplifies to $2\sqrt{29}$, the ratio collapses to $2\sqrt{29}$.

Step 1. Translate to vectors: $\vec{a}_1 = (-1,-1,-1)$, $\vec{b}_1 = (7,-6,1)$, $\vec{a}_2 = (3,5,7)$, $\vec{b}_2 = (1,-2,1)$.

Step 2. Connector $\vec{d} = \vec{a}_2 - \vec{a}_1 = (4,6,8)$.

Step 3. $\vec{n} = \vec{b}_1 \times \vec{b}_2 = (-4,-6,-8) = -2(2,3,4)$. (Notice $\vec{n}$ is twice the negation of $(2,3,4)$.)

Step 4. Observation: $\vec{d} = 2\cdot(2,3,4)$ and $\vec{n} = -2\cdot(2,3,4)$, so $\vec{d} \parallel \vec{n}$.

Step 5. $|\vec{n}\cdot\vec{d}| = |2\cdot(2,3,4)\cdot(-2)(2,3,4)| = 4\cdot|\vec{d}/2|^2 = 4\cdot(4+9+16) = 4\cdot 29 = 116$.

Step 6. $|\vec{n}| = 2\sqrt{4+9+16} = 2\sqrt{29}$.

Step 7. Shortest distance $d = 116/(2\sqrt{29}) = 58/\sqrt{29} = 2\sqrt{29}$.

Step 8. The fact that $\vec{d} \parallel \vec{n}$ is special: it means the connector P_1P_2 is itself the common perpendicular, so $|P_1P_2|$ already equals the shortest distance. Verify: $|P_1P_2| = \sqrt{16+36+64} = \sqrt{116} = 2\sqrt{29}$. ✓

Final Answer: $d = 2\sqrt{29}$ units.

Q 11.14 **Find the shortest distance between the lines whose vector equations are $\vec{r} = (\hat{i}+2\hat{j}+3\hat{k}) + \lambda(\hat{i}-3\hat{j}+2\hat{k})$ and $\vec{r} = 4\hat{i}+5\hat{j}+6\hat{k} + \mu(2\hat{i}+3\hat{j}+\hat{k})$.**

SOLUTION

Concept used. Same formula as before:

$$d = \frac{|(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)|}{|\vec{b}_1 \times \vec{b}_2|}.$$

Step 1. Read inputs:

$$\vec{a}_1 = (1,2,3),\ \vec{b}_1 = (1,-3,2),\ \vec{a}_2 = (4,5,6),\ \vec{b}_2 = (2,3,1).$$

Step 2. Connector $\vec{a}_2 - \vec{a}_1 = (3,3,3)$.

Step 3. Cross product $\vec{b}_1 \times \vec{b}_2$:

$$\begin{aligned}\hat{i}\text{-comp} &= (-3)(1) - (2)(3) = -3 - 6 = -9,\\ \hat{j}\text{-comp} &= -[(1)(1) - (2)(2)] = -(1-4) = 3,\\ \hat{k}\text{-comp} &= (1)(3) - (-3)(2) = 3 + 6 = 9.\end{aligned}$$

Hence $\vec{b}_1 \times \vec{b}_2 = (-9, 3, 9)$.

Step 4. Box product:

$$(\vec{b}_1 \times \vec{b}_2) \cdot (3, 3, 3) = (-9)(3) + (3)(3) + (9)(3) = -27 + 9 + 27 = 9.$$

Absolute value $|9| = 9$.

Step 5. Magnitude:

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{(-9)^2 + 3^2 + 9^2} = \sqrt{81 + 9 + 81} = \sqrt{171}.$$

Factor: $171 = 9 \cdot 19$, so $\sqrt{171} = 3\sqrt{19}$.

Step 6. Shortest distance:

$$d = \frac{9}{3\sqrt{19}} = \frac{3}{\sqrt{19}} = \frac{3\sqrt{19}}{19}.$$

Final Answer: $d = \dfrac{3}{\sqrt{19}} = \dfrac{3\sqrt{19}}{19}$ units.

☞ Exam Tip

Whenever $\vec{b}_1 \times \vec{b}_2 \neq \vec{0}$ (the lines are not parallel), the box product formula works. If $\vec{b}_1 \parallel \vec{b}_2$, the cross product vanishes and you must switch to the parallel-lines formula $d = |\vec{b} \times (\vec{a}_2 - \vec{a}_1)|/|\vec{b}|$ (see next problem).

EXPERT'S SOLUTION : *Pooja Verma, M.Sc Mathematics, IIT Bombay*

Strategic angle. Same three-step recipe. Compute connector, cross, box product; divide by the cross-product magnitude.

Step 1. $\vec{d} = \vec{a}_2 - \vec{a}_1 = (3, 3, 3) = 3(1, 1, 1)$.

Step 2. $\vec{n} = \vec{b}_1 \times \vec{b}_2$ via cofactor expansion:

$$\vec{n} = \big((-3)(1) - (2)(3),\ (2)(2) - (1)(1),\ (1)(3) - (-3)(2)\big) = (-9, 3, 9).$$

Step 3. Dot: $\vec{n} \cdot \vec{d} = 3 \cdot (-9 + 3 + 9) = 3 \cdot 3 = 9$.

Step 4. $|\vec{n}| = \sqrt{81 + 9 + 81} = \sqrt{171} = 3\sqrt{19}$.

Step 5. $d = 9/(3\sqrt{19}) = 3/\sqrt{19} = 3\sqrt{19}/19$.

Step 6. Decimal cross-check: $\sqrt{19} \approx 4.36$, so $d \approx 3/4.36 \approx 0.688$ units.

Final Answer: $d = \dfrac{3\sqrt{19}}{19}$ units.

Q 11.15 **Find the shortest distance between the lines whose vector equations are $\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$ and $\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$.**

SOLUTION

Concept used. Each line is given in expanded form. Rearrange each into the standard $\vec{r} = \vec{a} + \lambda\vec{b}$ shape so that we can read off $\vec{a}$ and $\vec{b}$. Then apply the box-product formula.

Step 1. Rearrange the first line by collecting constants and t-coefficients separately:

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k} = (1\hat{i} - 2\hat{j} + 3\hat{k}) + t(-\hat{i} + \hat{j} - 2\hat{k}).$$

Hence $\vec{a}_1 = (1, -2, 3)$ and $\vec{b}_1 = (-1, 1, -2)$.

Step 2. Rearrange the second line similarly:

$$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k} = (\hat{i} - \hat{j} - \hat{k}) + s(\hat{i} + 2\hat{j} - 2\hat{k}).$$

Hence $\vec{a}_2 = (1, -1, -1)$ and $\vec{b}_2 = (1, 2, -2)$.

Step 3. Connector:

$$\vec{a}_2 - \vec{a}_1 = (1-1,\ -1-(-2),\ -1-3) = (0, 1, -4).$$

Step 4. Cross product $\vec{b}_1 \times \vec{b}_2$:

$$\begin{aligned} \hat{i}\text{-comp} &= (1)(-2) - (-2)(2) = -2 + 4 = 2, \\ \hat{j}\text{-comp} &= -[(-1)(-2) - (-2)(1)] = -(2+2) = -4, \\ \hat{k}\text{-comp} &= (-1)(2) - (1)(1) = -2 - 1 = -3. \end{aligned}$$

Hence $\vec{b}_1 \times \vec{b}_2 = (2, -4, -3)$.

Step 5. Box product $(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)$:

$$(2)(0) + (-4)(1) + (-3)(-4) = 0 - 4 + 12 = 8.$$

Absolute value $|8| = 8$.

Step 6. Magnitude of the cross product:

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{2^2 + (-4)^2 + (-3)^2} = \sqrt{4 + 16 + 9} = \sqrt{29}.$$

Step 7. Shortest distance:

$$d = \frac{8}{\sqrt{29}} = \frac{8\sqrt{29}}{29}.$$

Final Answer: $d = \frac{8}{\sqrt{29}} = \frac{8\sqrt{29}}{29}$ units.

✗ Common Mistake

When the parametric form mixes the parameter throughout the components, it is tempting to read off the direction vector as $(t, t, -2t)$ ="$(1, 1, -2)$". But the first line has $(1-t)$ in the $\hat{i}$-slot, giving a coefficient of -1 on t. The direction vector is $(-1, 1, -2)$, not $(1, 1, -2)$. Always collect t-coefficients explicitly.

EXPERT'S SOLUTION *: Meera Banerjee, M.Sc Mathematics, ISI Kolkata*

Structural observation. The given equations are not in standard form. Treat the parameter as a "factor"; collect everything multiplied by that parameter into the direction vector, and everything else into the position vector.

Step 1. Standard form of line 1: $\vec{r} = (1, -2, 3) + t(-1, 1, -2)$.

Step 2. Standard form of line 2: $\vec{r} = (1, -1, -1) + s(1, 2, -2)$.

Step 3. $\vec{d} = (1-1, -1+2, -1-3) = (0, 1, -4)$.

Step 4. $\vec{n} = \vec{b}_1 \times \vec{b}_2$. Component by component:

$$(\vec{b}_1 \times \vec{b}_2)_x = b_{1y}b_{2z} - b_{1z}b_{2y} = (1)(-2) - (-2)(2) = -2 + 4 = 2,$$
$$(\vec{b}_1 \times \vec{b}_2)_y = b_{1z}b_{2x} - b_{1x}b_{2z} = (-2)(1) - (-1)(-2) = -2 - 2 = -4,$$
$$(\vec{b}_1 \times \vec{b}_2)_z = b_{1x}b_{2y} - b_{1y}b_{2x} = (-1)(2) - (1)(1) = -3.$$

So $\vec{n} = (2, -4, -3)$.

Step 5. Box product $\vec{n} \cdot \vec{d} = 0 - 4 + 12 = 8$.

Step 6. $|\vec{n}| = \sqrt{4 + 16 + 9} = \sqrt{29}$.

Step 7. $d = 8/\sqrt{29} = 8\sqrt{29}/29 \approx 8/5.385 \approx 1.486$ units.

Final Answer: $d = \frac{8\sqrt{29}}{29}$ units.

Key Takeaways

- Vector form of a line through $\vec{a}$ parallel to $\vec{b}$: $\vec{r} = \vec{a} + \lambda\vec{b}$.
- Cartesian symmetric form $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$ reads off point (x_0, y_0, z_0) from numerators (sign-flipped) and direction (a, b, c) from denominators.

- Perpendicular: $a_1a_2 + b_1b_2 + c_1c_2 = 0$. Parallel: $(a_1, b_1, c_1) \parallel (a_2, b_2, c_2)$ component-proportionally.
- Angle: $\cos\theta = \dfrac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1||\vec{b}_2|}$.
- Shortest distance, skew lines: $d = \dfrac{|(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}_1 \times \vec{b}_2|}$.
- Distance, parallel lines: $d = \dfrac{|\vec{b} \times (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}|}$.
- Always normalise symmetric forms to $\dfrac{x - x_0}{a}$ shape before reading direction ratios.