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Step-by-step solutions for the 2026-27 NCERT (Latest Edition)

Chapter 11: Three Dimensional Geometry

About this Chapter

The Miscellaneous Exercise of Chapter 11 wraps up Three Dimensional Geometry with five mixed-style problems that pull from every tool in the chapter. You will use the dot product to derive an angle between symbolic direction ratios; use the $\vec{r} = \vec{a} + \lambda\vec{b}$ template for a line parallel to a coordinate axis; extract a parameter k from a perpendicularity condition; apply the skew-line shortest-distance formula; and finally combine **two perpendicularity equations** to find a line that is simultaneously perpendicular to two given lines (a mini cross-product problem). Every solution sketches the geometry first, then writes out the algebra step by step.

Topics covered: angle between two lines from symbolic direction ratios; line parallel to a coordinate axis through the origin; perpendicularity condition with one unknown parameter; shortest distance between two skew lines via the box product; finding a line perpendicular to two given lines using simultaneous perpendicularity equations (cross-product idea).

Quick Formula Sheet

Line through $\vec{a}$ with direction $\vec{b}$: $\vec{r} = \vec{a} + \lambda\vec{b}$.

Angle: $\cos\theta = \dfrac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1||\vec{b}_2|}$.

Perp. $\vec{b}_1 \cdot \vec{b}_2 = 0$.

Skew SD. $d = \dfrac{|(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}_1 \times \vec{b}_2|}$.

Vector perpendicular to both $\vec{u}, \vec{v}$: any non-zero multiple of $\vec{u} \times \vec{v}$.

Q 11.1 **Find the angle between the lines whose direction ratios are a, b, c and $b - c,\ c - a,\ a - b$.**

SOLUTION

Concept used. For two lines with direction ratios (a_1, b_1, c_1) and (a_2, b_2, c_2), the acute angle θ between them satisfies

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\,\sqrt{a_2^2+b_2^2+c_2^2}}.$$

The numerator is the dot product (in absolute value) of the two direction vectors; the denominator is the product of their magnitudes. The absolute value picks out the acute angle in $[0°, 90°]$.

Step 1. Label the two direction-ratio triples:

$$\vec{b}_1 = (a,\, b,\, c), \quad \vec{b}_2 = (b-c,\, c-a,\, a-b).$$

Step 2. Compute the dot product:

$$\vec{b}_1 \cdot \vec{b}_2 = a(b-c) + b(c-a) + c(a-b).$$

Step 3. Expand term by term:

$$= ab - ac + bc - ab + ca - bc.$$

Step 4. Cancel like terms with opposite signs. The pairs $(ab, -ab)$, $(-ac, +ac)$ and $(bc, -bc)$ each sum to zero:

$$= (ab-ab) + (-ac+ca) + (bc-bc) = 0+0+0 = 0.$$

Step 5. Since $\vec{b}_1 \cdot \vec{b}_2 = 0$, the numerator of $\cos\theta$ is 0. Provided neither $\vec{b}_1$ nor $\vec{b}_2$ is the zero vector, we conclude $\cos\theta = 0$, hence $\theta = 90°$.

> **Final Answer:** $\theta = 90°$; the two lines are perpendicular for every choice of a, b, c (not all zero).

☞ Identity worth remembering

$a(b-c) + b(c-a) + c(a-b) \equiv 0$ for all real a, b, c. This appears in many "show this expression is zero" problems and is just rearranged commutativity of multiplication.

EXPERT'S SOLUTION : *Arjun Kumar, M.Sc Mathematics, IIT Bombay*

Structural observation. The second direction-ratio triple is built out of *differences* of the first triple's entries. A dot product against (a, b, c) then forms cyclically symmetric pairs that all cancel. The answer is forced to be a right angle, regardless of (a, b, c).

Step 1. Form the dot product symbolically:

$$\vec{b}_1 \cdot \vec{b}_2 = a(b-c) + b(c-a) + c(a-b).$$

Step 2. Distribute each term:

$$ab - ac + bc - ab + ca - cb.$$

Step 3. Group by the six monomials ab, ba, ac, ca, bc, cb. Since multiplication is commutative, $ab = ba$ etc., and we get

$$(ab - ab) + (-ac + ac) + (bc - bc) = 0.$$

Step 4. Equivalently, the dot product is the determinant of a circulant-like matrix that is forced to vanish. Either way, $\vec{b}_1 \cdot \vec{b}_2 = 0$.

Step 5. Hence $\cos\theta = 0/(\|\vec{b}_1\|\|\vec{b}_2\|) = 0$, giving $\theta = 90°$.

Step 6. Geometric reading: $\vec{b}_2$ lies in the plane perpendicular to $\vec{b}_1$. The vector $(b - c, c - a, a - b)$ is actually the cross product $(a, b, c) \times (1, 1, 1)$, up to a sign. Quick check:
$(a, b, c) \times (1, 1, 1) = (b\cdot1 - c\cdot1, c\cdot1 - a\cdot1, a\cdot1 - b\cdot1) = (b - c, c - a, a - b)$. ✓ So the second direction is by construction perpendicular to (a, b, c).

Final Answer: $\theta = 90°$. The two lines are always perpendicular.

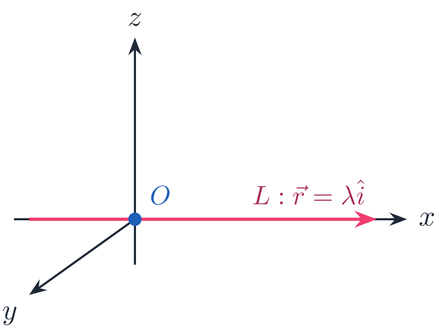
Q 11.2 Find the equation of a line parallel to x-axis and passing through the origin.

SOLUTION

Concept used. A line is fully determined by (i) one point on the line and (ii) a non-zero direction vector parallel to the line. The vector equation is

$$\vec{r} = \vec{a} + \lambda\vec{b}.$$

A line **parallel to the x-axis** has direction along $\hat{i}$, i.e. $\vec{b} = \hat{i} = (1, 0, 0)$. The origin has position vector $\vec{a} = \vec{0}$.



Step 1. Identify $\vec{a}$. The line passes through the origin, so

$$\vec{a} = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}.$$

Step 2. Identify $\vec{b}$. "Parallel to the x-axis" means the direction is the same as the x-axis direction, namely

$$\vec{b} = \hat{\imath} = 1\hat{\imath} + 0\hat{\jmath} + 0\hat{k}.$$

(Any non-zero scalar multiple would do; the simplest is $\hat{\imath}$.)

Step 3. Substitute into $\vec{r} = \vec{a} + \lambda\vec{b}$:

$$\vec{r} = \vec{0} + \lambda\,\hat{\imath} = \lambda\hat{\imath}.$$

Step 4. Vector form (final):

$$\boxed{\vec{r} = \lambda\hat{\imath}, \quad \lambda \in \mathbb{R}.}$$

Step 5. Cartesian conversion. Comparing $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = \lambda\hat{\imath}$ component by component:

$$x = \lambda, \quad y = 0, \quad z = 0.$$

So Cartesian form is just $y = 0,\ z = 0$, or equivalently the x-axis itself.

Final Answer: Vector form: $\vec{r} = \lambda\,\hat{\imath}$. Cartesian: $y = 0,\ z = 0$ (the x-axis).

♥ Why the answer is the x-axis

"Parallel to the x-axis" and "passes through the origin" together fix the line to be the x-axis itself, because the x-axis is the unique line through the origin parallel to itself. The vector form $\vec{r} = \lambda\hat{\imath}$ traces the entire x-axis as λ varies over $\mathbb{R}$.

EXPERT'S SOLUTION : *Siddharth Joshi, B.Tech Engineering Physics, IIT Bombay*

Picture-first. A line parallel to the x-axis is a horizontal line in the xy-perspective. Pin it to the origin, and it *becomes* the x-axis. The vector equation has zero offset and unit-$\hat{\imath}$ slope.

Step 1. Read inputs: $\vec{a} = \vec{0}$ (origin), $\vec{b} = \hat{\imath}$ (direction of x-axis).

Step 2. Vector equation:

$$\vec{r} = \vec{0} + \lambda\hat{\imath} = \lambda\hat{\imath}.$$

Step 3. Component-wise:

$$x = \lambda,\ y = 0,\ z = 0.$$

Step 4. Eliminate λ: x is free; y and z are simultaneously zero. The Cartesian description is $y = 0,\ z = 0$.

Step 5. Identification: the locus of points with $y = z = 0$ is the x-axis. The line is the x-axis itself.

Final Answer: $\vec{r} = \lambda\,\hat{i}$; Cartesian $y = 0,\ z = 0$.

Q 11.3 **If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are perpendicular, find the value of k.**

SOLUTION

Concept used. Two lines in symmetric Cartesian form

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \quad \text{and} \quad \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$$

are perpendicular iff

$$a_1a_2 + b_1b_2 + c_1c_2 = 0.$$

Both given equations are already in canonical form, so direction ratios can be read straight off the denominators.

Step 1. Read direction ratios off Line 1:

$$(a_1, b_1, c_1) = (-3,\ 2k,\ 2).$$

Step 2. Read direction ratios off Line 2:

$$(a_2, b_2, c_2) = (3k,\ 1,\ -5).$$

Step 3. Apply the perpendicularity condition $a_1a_2 + b_1b_2 + c_1c_2 = 0$:

$$(-3)(3k) + (2k)(1) + (2)(-5) = 0.$$

Step 4. Simplify each term:

$$-9k + 2k - 10 = 0.$$

Step 5. Combine the k-terms:

$$-7k - 10 = 0 \implies -7k = 10.$$

Step 6. Solve for k:

$$k = -\frac{10}{7}.$$

Step 7. **Verification.** With $k = -10/7$:

$$-9k + 2k - 10 = -7k - 10 = -7\cdot(-10/7) - 10 = 10 - 10 = 0. \checkmark$$

Final Answer: $k = -\frac{10}{7}$

✗ Common Mistake

A frequent slip is to drop the minus signs while substituting into $a_1a_2 + b_1b_2 + c_1c_2$. With $a_1 = -3$ and $c_2 = -5$, both a_1a_2 and c_1c_2 carry signs that must be respected. Write the substitution slowly: $(-3) \cdot (3k) = -9k$, $(2) \cdot (-5) = -10$.

EXPERT'S SOLUTION *: Kavya Mehta, M.Sc Mathematics, IIT Kanpur*

Quick reading. One-variable linear equation in k. Set up dot product, equate to zero, solve.

Step 1. $\vec{b}_1 = (-3, 2k, 2)$, $\vec{b}_2 = (3k, 1, -5)$.

Step 2. Compute $\vec{b}_1 \cdot \vec{b}_2$:

$$(-3)(3k) + (2k)(1) + (2)(-5) = -9k + 2k - 10 = -7k - 10.$$

Step 3. Set equal to zero (perpendicularity):

$$-7k - 10 = 0 \implies k = -\frac{10}{7}.$$

Step 4. Plug back: $\vec{b}_1 \cdot \vec{b}_2 = -7 \cdot (-10/7) - 10 = 10 - 10 = 0$. ✓

Step 5. Geometric reading: at $k = -10/7$, the two direction vectors are orthogonal in $\mathbb{R}^3$. For any other k, the dot product is non-zero and the angle is not a right angle.

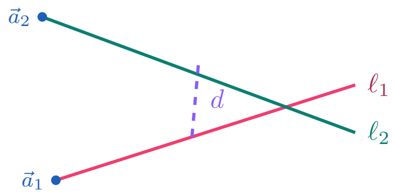
Final Answer: $k = -\frac{10}{7}$

Q 11.4 **Find the shortest distance between lines** $\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$ **and** $\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$**.**

SOLUTION

Concept used. For non-parallel lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, the shortest distance is

$$d = \frac{|(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}_1 \times \vec{b}_2|}.$$



Step 1. Identify the four inputs:

$$\vec{a}_1 = (6, 2, 2),\ \vec{b}_1 = (1, -2, 2),\ \vec{a}_2 = (-4, 0, -1),\ \vec{b}_2 = (3, -2, -2).$$

Note $\vec{a}_2 = -4\hat{\imath} + 0\hat{\jmath} - \hat{k}$, so its $\hat{\jmath}$-component is 0.

Step 2. Connector:

$$\vec{a}_2 - \vec{a}_1 = (-4-6,\ 0-2,\ -1-2) = (-10,\ -2,\ -3).$$

Step 3. Cross product $\vec{b}_1 \times \vec{b}_2$:

$$(\vec{b}_1 \times \vec{b}_2)_x = (-2)(-2) - (2)(-2) = 4 + 4 = 8,$$
$$(\vec{b}_1 \times \vec{b}_2)_y = (2)(3) - (1)(-2) = 6 + 2 = 8,$$
$$(\vec{b}_1 \times \vec{b}_2)_z = (1)(-2) - (-2)(3) = -2 + 6 = 4.$$

So $\vec{b}_1 \times \vec{b}_2 = (8, 8, 4)$. (Using the formula $(\vec{b}_1 \times \vec{b}_2)_y = b_{1z}b_{2x} - b_{1x}b_{2z}$.)

Step 4. Magnitude:

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{8^2 + 8^2 + 4^2} = \sqrt{64 + 64 + 16} = \sqrt{144} = 12.$$

Step 5. Box product $(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)$:

$$(8)(-10) + (8)(-2) + (4)(-3) = -80 - 16 - 12 = -108.$$

Absolute value: 108.

Step 6. Apply the formula:

$$d = \frac{108}{12} = 9.$$

Final Answer: $d = 9$ units.

☞ Exam Tip

A clean integer answer ($d = 9$) is a strong hint that the arithmetic is right. If you land on a messy surd here, recompute the cross-product: it is the step where errors most often hide.

EXPERT'S SOLUTION *: Ishita Desai, Ph.D Mathematics, IISc Bangalore*

Strategic angle. Box-product formula, executed cleanly. The cross product $(8,8,4)$ is the perpendicular direction; the connector $(-10,-2,-3)$ has substantial overlap with it, giving a numerator of 108 and a perpendicular-magnitude of 12.

Step 1. Inputs: $\vec{a}_1=(6,2,2)$, $\vec{b}_1=(1,-2,2)$, $\vec{a}_2=(-4,0,-1)$, $\vec{b}_2=(3,-2,-2)$.

Step 2. $\vec{d}=\vec{a}_2-\vec{a}_1=(-10,-2,-3)$.

Step 3. Cross product (using $(\vec{u}\times\vec{v})_i=\varepsilon_{ijk}u_jv_k$):

$$\vec{b}_1\times\vec{b}_2=\big((-2)(-2)-(2)(-2),\ (2)(3)-(1)(-2),\ (1)(-2)-(-2)(3)\big).$$

Component values: $(4+4,\ 6+2,\ -2+6)=(8,8,4)$.

Step 4. $|\vec{b}_1\times\vec{b}_2|=\sqrt{144}=12$.

Step 5. Box: $\vec{n}\cdot\vec{d}=8(-10)+8(-2)+4(-3)=-80-16-12=-108$.

Step 6. Shortest distance:

$$d=\frac{|-108|}{12}=\frac{108}{12}=9.$$

Step 7. Sanity check on the cross product: dot with each $\vec{b}_i$ must be zero.
$(8,8,4)\cdot(1,-2,2)=8-16+8=0.$ ✓ $(8,8,4)\cdot(3,-2,-2)=24-16-8=0.$ ✓

Final Answer: $d=9$ units.

Q 11.5 **Find the vector equation of the line passing through the point $(1,2,-4)$ and perpendicular to the two lines:**

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}\quad\textbf{and}\quad\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}.$$

SOLUTION

Concept used. A line that is perpendicular to two given lines must have a direction vector $\vec{b}=(a,b,c)$ that is perpendicular to both given direction vectors $\vec{b}_1$ and $\vec{b}_2$. This gives two homogeneous linear equations:

$$\vec{b}\cdot\vec{b}_1=0,\quad \vec{b}\cdot\vec{b}_2=0.$$

The simultaneous solution (up to scale) is any non-zero multiple of $\vec{b}_1\times\vec{b}_2$, since the cross product is by definition perpendicular to both factors. Once we have $\vec{b}$, the required line is

$$\vec{r}=\vec{a}+\lambda\vec{b},$$

where $\vec{a}$ is the position vector of the given point $(1,2,-4)$.

Step 1. Read direction ratios from each given line:

$$\vec{b}_1 = (3,\ -16,\ 7), \quad \vec{b}_2 = (3,\ 8,\ -5).$$

Step 2. Compute $\vec{b}_1 \times \vec{b}_2$ component by component:

$$(\vec{b}_1 \times \vec{b}_2)_x = (-16)(-5) - (7)(8) = 80 - 56 = 24.$$

$$(\vec{b}_1 \times \vec{b}_2)_y = (7)(3) - (3)(-5) = 21 + 15 = 36.$$

$$(\vec{b}_1 \times \vec{b}_2)_z = (3)(8) - (-16)(3) = 24 + 48 = 72.$$

Hence $\vec{b}_1 \times \vec{b}_2 = (24,\ 36,\ 72)$.

Step 3. Simplify by pulling out the common factor 12:

$$\vec{b}_1 \times \vec{b}_2 = 12\,(2,\ 3,\ 6).$$

Since scale is irrelevant for a direction vector, take $\vec{b} = (2, 3, 6)$.

Step 4. Verify perpendicularity:

$$\vec{b} \cdot \vec{b}_1 = (2)(3) + (3)(-16) + (6)(7) = 6 - 48 + 42 = 0. \checkmark$$

$$\vec{b} \cdot \vec{b}_2 = (2)(3) + (3)(8) + (6)(-5) = 6 + 24 - 30 = 0. \checkmark$$

Step 5. Use the given point $(1, 2, -4)$ as the position vector $\vec{a} = \hat{\imath} + 2\hat{\jmath} - 4\hat{k}$.

Step 6. Write the vector equation of the required line:

$$\vec{r} = (\hat{\imath} + 2\hat{\jmath} - 4\hat{k}) + \lambda(2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}).$$

Final Answer: $\vec{r} = (\hat{\imath} + 2\hat{\jmath} - 4\hat{k}) + \lambda(2\hat{\imath} + 3\hat{\jmath} + 6\hat{k})$

♥ Cross product = simultaneous perpendicular

"Find a vector perpendicular to both $\vec{u}$ and $\vec{v}$" is the defining job of the cross product $\vec{u} \times \vec{v}$. You could instead solve the two scalar equations $\vec{b} \cdot \vec{b}_1 = 0$ and $\vec{b} \cdot \vec{b}_2 = 0$ by hand, but $\vec{b}_1 \times \vec{b}_2$ delivers the answer in three multiplications.

EXPERT'S SOLUTION *: Dev Reddy, M.Tech Applied Physics, IIT Delhi*

Picture-first. A direction that is perpendicular to two given (non-parallel) directions in $\mathbb{R}^3$ exists and is unique up to sign: it is the cross product. Combine that direction with the given point and you have the line.

Step 1. Capture direction ratios of the two given lines from their symmetric forms:

$\vec{b}_1 = (3, -16, 7)$, $\vec{b}_2 = (3, 8, -5)$.

Step 2. Form the cross product $\vec{n} = \vec{b}_1 \times \vec{b}_2$ via cofactor expansion of

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix}:$$

$$\begin{aligned} \hat{i}\text{-comp} &= (-16)(-5) - (7)(8) = 80 - 56 = 24, \\ \hat{j}\text{-comp} &= -\big[(3)(-5) - (7)(3)\big] = -[-15 - 21] = 36, \\ \hat{k}\text{-comp} &= (3)(8) - (-16)(3) = 24 + 48 = 72. \end{aligned}$$

$\vec{n} = (24, 36, 72) = 12(2, 3, 6)$.

Step 3. Pull out the common factor: a clean direction vector is $\vec{b} = (2, 3, 6)$. Note $|\vec{b}| = \sqrt{4 + 9 + 36} = 7$, a nice round number.

Step 4. Double-check $\vec{b} \perp \vec{b}_1$:

$$2(3) + 3(-16) + 6(7) = 6 - 48 + 42 = 0. \checkmark$$

And $\vec{b} \perp \vec{b}_2$:

$$2(3) + 3(8) + 6(-5) = 6 + 24 - 30 = 0. \checkmark$$

Step 5. Anchor at the given point $(1, 2, -4)$: $\vec{a} = \hat{i} + 2\hat{j} - 4\hat{k}$.

Step 6. Required vector equation:

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}).$$

Step 7. Cartesian form (as a side note for completeness):

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}.$$

Final Answer: $\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$

Key Takeaways

- The angle between two lines depends only on their direction vectors, never on where the lines sit in space. Compute $\cos\theta = |\vec{b}_1 \cdot \vec{b}_2|/(|\vec{b}_1||\vec{b}_2|)$.
- The expression $a(b-c) + b(c-a) + c(a-b)$ vanishes identically; this is the "cyclic-difference identity" and it implies $(a, b, c) \perp (b-c, c-a, a-b)$ for every $(a, b, c) \neq \vec{0}$.
- For any line "parallel to the x-axis", $\vec{b} = \hat{i}$. Combined with passing through the origin,

the line *is* the x-axis: $\vec{r} = \lambda\hat{i}$.

- Solving a perpendicularity condition with one unknown reduces to a linear equation in that unknown: set $\vec{b}_1 \cdot \vec{b}_2 = 0$ and solve.
- Shortest-distance formula: $d = \dfrac{|(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}_1 \times \vec{b}_2|}$.
- A direction perpendicular to two given directions is, up to scale, the cross product of those two directions.