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Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 12: Linear Programming

About this Chapter

A **linear programming problem (LPP)** asks us to find the maximum or minimum value of a **linear objective function** $Z = ax + by$, subject to a system of linear inequalities called **constraints**. The set of all points satisfying every constraint is the **feasible region**. By the **corner-point theorem**, when the optimum exists it occurs at a vertex (corner) of the feasible region. In this exercise we solve ten LPPs graphically: draw each constraint line, shade the feasible region, list its corner points, evaluate Z at each corner, and pick the optimum.

Topics covered: Objective function • Linear constraints • Feasible region • Corner-point method • Bounded vs unbounded regions

Quick Formula Sheet

Objective function:
$Z = ax + by$ (to be maximised or minimised)

Constraints (general form):
$a_i x + b_i y \le c_i$ or $\ge c_i$, with $x, y \ge 0$

Corner-point theorem:
If the optimum of Z on a bounded feasible region exists, it occurs at a corner point of the region.

Unbounded region rule:
Max exists iff the open half-plane $ax + by > M$ has no common point with the feasible region (where M is the largest corner value); similarly for the minimum.

Exercise 12.1

Q 12.1 **Maximise $Z = 3x + 4y$ subject to the constraints: $x + y \le 4$, $x \ge 0$, $y \ge 0$.**

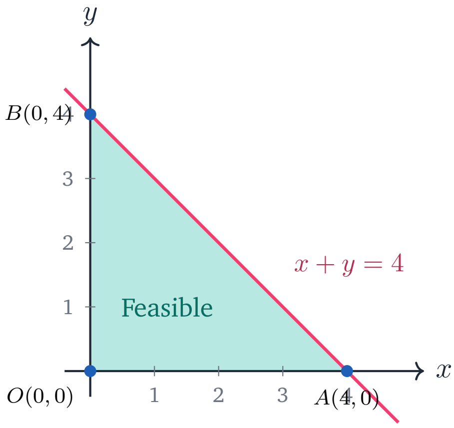
SOLUTION

Concept used. A **linear programming problem** consists of three parts: (i) the *objective function* $Z = ax + by$ to be optimised, (ii) the *linear constraints* (a set of linear inequalities in x and y), and (iii) the *non-negativity restrictions* $x \geq 0$, $y \geq 0$. The set of all (x, y) satisfying every constraint is the **feasible region**. By the **corner-point theorem**, if the feasible region is bounded then Z attains both its maximum and its minimum value at some corner (vertex) of the region. So our recipe is: plot constraints, find corner points, evaluate Z, pick the optimum.

Step 1. **Identify the constraint lines.** The boundary line of $x + y \leq 4$ is $x + y = 4$. Its x-intercept is $(4, 0)$ (set $y = 0$) and its y-intercept is $(0, 4)$ (set $x = 0$). The lines $x = 0$ and $y = 0$ are the coordinate axes.

Step 2. **Test the half-plane.** Substitute the origin $(0, 0)$ into $x + y \leq 4$: $0 + 0 = 0 \leq 4$, true. So the feasible side of $x + y = 4$ is the side containing the origin.

Step 3. **Plot and shade the feasible region.** Combined with $x \geq 0$ and $y \geq 0$, the feasible region is the closed triangle with vertices $O(0, 0)$, $A(4, 0)$, $B(0, 4)$.



Step 4. **List the corner points.** The vertices of the feasible region are $O(0, 0)$, $A(4, 0)$, $B(0, 4)$.

Step 5. **Evaluate $Z = 3x + 4y$ at each corner.**

$$\begin{aligned} \text{At } O(0,0) &: \ Z = 3(0) + 4(0) = 0. \\ \text{At } A(4,0) &: \ Z = 3(4) + 4(0) = 12. \\ \text{At } B(0,4) &: \ Z = 3(0) + 4(4) = 16. \end{aligned}$$

Step 6. **Pick the optimum.** The largest value is 16, achieved at $B(0, 4)$. Since the feasible region is bounded, this is the global maximum.

Final Answer: Maximum value of Z is **16**, attained at $(x, y) = (0, 4)$.

☞ **Exam Tip**

For a maximisation LPP, after building the table of corner values, the answer is just the row with the largest Z. Always write the corner explicitly along with the value, e.g. "$Z_{\text{max}} = 16$ at $(0, 4)$"; that is the form CBSE expects.

EXPERT'S SOLUTION *: Pranav Sharma, M.Sc Mathematics, IIT Bombay*

Strategic angle. The objective $Z = 3x + 4y$ has a larger coefficient on y than on x $(4 > 3)$, so among corner points lying on the boundary line $x + y = 4$ we expect the one with the bigger y to win. That is a useful gut-check before any arithmetic.

Step 1. **Recognise the structure.** The constraint $x + y \leq 4$ together with $x, y \geq 0$ traps the feasible region inside the right triangle with legs on the axes and hypotenuse $x + y = 4$. This is the simplest non-trivial bounded LP region.

Step 2. **Use the corner-point theorem.** Because the feasible set is bounded and closed, and Z is linear (hence continuous), Z attains its maximum somewhere on the set. A standard theorem says that maximum lies at a vertex. So we only need to inspect three points: $O(0, 0)$, $A(4, 0)$, $B(0, 4)$.

Step 3. **Walk along the boundary.** On the segment OA $(y = 0)$, $Z = 3x$ grows from 0 to 12. On segment OB $(x = 0)$, $Z = 4y$ grows from 0 to 16. On hypotenuse AB, $y = 4 - x$, so

$$Z = 3x + 4(4 - x) = 16 - x.$$

On AB, Z is largest when x is smallest, i.e. at $x = 0$, giving $Z = 16$ at $B(0, 4)$.

Step 4. **Compare.** Maximum on OA: 12. Maximum on OB: 16. Maximum on AB: 16 at B. The global maximum on the triangle is **16** at $B(0, 4)$.

Why this matters. The coefficient-comparison shortcut works for all bounded triangular LP regions with the hypotenuse $x + y = k$: between the two non-origin corners, Z is larger at the corner sitting on the axis matching the larger coefficient.

Final Answer: $Z_{\text{max}} =$ **16** at $(0, 4)$.

Q 12.2 **Minimise** $Z = -3x + 4y$ **subject to** $x + 2y \leq 8$, $3x + 2y \leq 12$, $x \geq 0$, $y \geq 0$.

SOLUTION

Concept used. For a minimisation LPP on a bounded feasible region, the same **corner-point theorem** applies: the minimum of $Z = ax + by$ is attained at one of the vertices of the feasible region. Negative coefficients in Z are fine: a negative coefficient simply means that growing the corresponding variable *decreases* Z.

Step 1. Constraint lines and intercepts.

$$x + 2y = 8 \quad \Rightarrow \text{ intercepts } (8,0) \text{ and } (0,4),$$
$$3x + 2y = 12 \quad \Rightarrow \text{ intercepts } (4,0) \text{ and } (0,6).$$

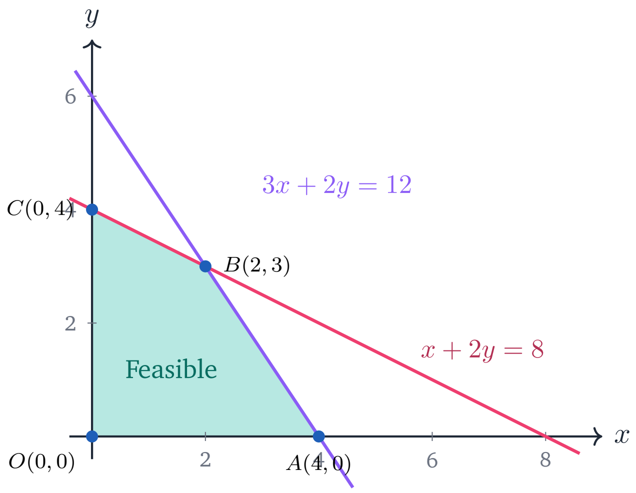
Step 2. Test the half-planes. Origin in $x + 2y \le 8$: $0 \le 8$ ✓. Origin in $3x + 2y \le 12$: $0 \le 12$ ✓. So the feasible side of each line is the origin side.

Step 3. Find the intersection of the two slanted lines. Solve simultaneously:

$$x + 2y = 8,$$
$$3x + 2y = 12.$$

Subtract: $2x = 4 \Rightarrow x = 2$. Substitute: $2 + 2y = 8 \Rightarrow y = 3$. Intersection $(2,3)$.

Step 4. Plot the feasible region. The binding boundary on the x-axis side comes from $3x + 2y = 12$ at $(4,0)$ (since $(4,0)$ satisfies $x + 2y = 4 \le 8$ but $(8,0)$ violates $3x + 2y = 24 \le 12$). On the y-axis side it comes from $x + 2y = 8$ at $(0,4)$. Hence the vertices are $O(0,0)$, $A(4,0)$, $B(2,3)$, $C(0,4)$.



Step 5. Evaluate $Z = -3x + 4y$ at each corner.

At $O(0,0)$: $Z = -3(0) + 4(0) = 0.$

At $A(4,0)$: $Z = -3(4) + 4(0) = -12.$

At $B(2,3)$: $Z = -3(2) + 4(3) = -6 + 12 = 6.$

At $C(0,4)$: $Z = -3(0) + 4(4) = 16.$

Step 6. Pick the minimum. The smallest of $\{0, -12, 6, 16\}$ is -12, attained at $A(4,0)$.

Final Answer: Minimum value of Z is $\mathbf{-12}$, attained at $(x,y) = (4,0)$.

✗ Common Mistake

A common slip is mis-identifying the boundary on the x-axis: students draw $(8, 0)$ as a corner because $x + 2y = 8$ meets the axis there. But $(8, 0)$ violates $3x + 2y \leq 12$. Always test each candidate vertex against *every* constraint before adding it to the corner list.

EXPERT'S SOLUTION *: Aanya Iyer, M.Tech CS, IIT Madras*

Strategic angle. Since $Z = -3x + 4y$, growing x *decreases* Z while growing y increases it. So the minimum should sit as far right and as low as possible; i.e. on the x-axis at the largest feasible x. That predicts $(4, 0)$ before any arithmetic.

Step 1. **Find the rightmost feasible x.** On $y = 0$, the binding constraint is $3x + 2y \leq 12 \Rightarrow 3x \leq 12 \Rightarrow x \leq 4$. So the rightmost feasible point on the x-axis is $(4, 0)$.

Step 2. **Verify via the corner table.** The four vertices are obtained by pairing constraints two at a time and keeping only those satisfying the others:

- $y = 0,\ x = 0 \Rightarrow O(0, 0)$.
- $y = 0,\ 3x + 2y = 12 \Rightarrow A(4, 0)$. Check $x + 2y = 4 \leq 8$ ✓.
- $x + 2y = 8,\ 3x + 2y = 12 \Rightarrow B(2, 3)$. Both ≥ 0 ✓.
- $x = 0,\ x + 2y = 8 \Rightarrow C(0, 4)$. Check $3x + 2y = 8 \leq 12$ ✓.

Step 3. **Evaluate.** $Z(O) = 0$, $Z(A) = -12$, $Z(B) = 6$, $Z(C) = 16$. Minimum $= -12$ at $A(4, 0)$.

Step 4. **Geometric reading.** The level set $-3x + 4y = k$ is a family of parallel lines with slope $3/4$. As we slide k down (more negative), the line slides to the lower-right. The last vertex it touches before leaving the feasible region is $A(4, 0)$, confirming the minimum.

Why this matters. Reading off the sign and size of the coefficients in Z tells you, before computing anything, which corner is the prime suspect. This corner-prediction shortcut catches plenty of arithmetic slips.

Final Answer: $Z_{\text{min}} = \mathbf{-12}$ at $(4, 0)$.

Q 12.3 **Maximise** $Z = 5x + 3y$ **subject to** $3x + 5y \leq 15$, $5x + 2y \leq 10$, $x \geq 0$, $y \geq 0$.

SOLUTION

Concept used. Same recipe: locate the feasible polygon's corner points, compute Z at each, pick the largest. Two non-axis constraints generally produce four candidate corners (O, one on each axis, and the slanted-line intersection).

Step 1. Constraint lines and intercepts.

$$3x + 5y = 15 \quad \Rightarrow \text{ intercepts } (5,0) \text{ and } (0,3),$$
$$5x + 2y = 10 \quad \Rightarrow \text{ intercepts } (2,0) \text{ and } (0,5).$$

Origin satisfies both ($0 \le 15$, $0 \le 10$).

Step 2. Intersection of the two slanted lines.

$$3x + 5y = 15, \quad \text{(A)}$$
$$5x + 2y = 10. \quad \text{(B)}$$

Multiply (A) by 2 and (B) by 5:

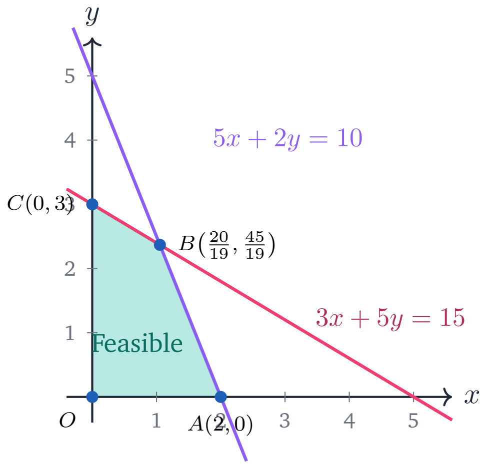
$$6x + 10y = 30,$$
$$25x + 10y = 50.$$

Subtract: $19x = 20 \Rightarrow x = \frac{20}{19}$. Substitute into (B):

$$5 \cdot \frac{20}{19} + 2y = 10 \Rightarrow \frac{100}{19} + 2y = 10 \Rightarrow 2y = 10 - \frac{100}{19} = \frac{190-100}{19} = \frac{90}{19} \Rightarrow y = \frac{45}{19}.$$

Intersection: $\left(\frac{20}{19}, \frac{45}{19}\right)$.

Step 3. Identify the corners. On the x-axis, the tighter bound is $5x + 2y \le 10 \Rightarrow x \le 2$, giving corner $A(2,0)$. On the y-axis, the tighter bound is $3x + 5y \le 15 \Rightarrow y \le 3$, giving corner $C(0,3)$. Together with $O(0,0)$ and $B\left(\frac{20}{19}, \frac{45}{19}\right)$, the polygon has four vertices.



Step 4. Evaluate $Z = 5x + 3y$ at each corner.

$$\text{At } O(0,0): \; Z = 5(0) + 3(0) = 0.$$
$$\text{At } A(2,0): \; Z = 5(2) + 3(0) = 10.$$
$$\text{At } B\left(\tfrac{20}{19}, \tfrac{45}{19}\right): \; Z = 5 \cdot \frac{20}{19} + 3 \cdot \frac{45}{19} = \frac{100+135}{19} = \frac{235}{19}.$$
$$\text{At } C(0,3): \; Z = 5(0) + 3(3) = 9.$$

Now $\frac{235}{19} = 12.368\ldots$

Step 5. Compare. $\{0, 10, \frac{235}{19}, 9\}$. Since $\frac{235}{19} \approx 12.37 > 10$, the maximum is $\frac{235}{19}$ at B.

Final Answer: Maximum value of Z is $\dfrac{235}{19} \approx 12.37$, attained at $\left(\frac{20}{19}, \frac{45}{19}\right)$.

♥ Why fractional corners are fine

LP corner points need not have integer coordinates. The corner-point theorem is purely geometric: vertices of a polygon defined by linear inequalities can have any rational coordinates. If the problem additionally required integer values (an *integer* LP), we would need different techniques. Plain LP keeps fractions.

EXPERT'S SOLUTION *: Karan Mehta, Ph.D Mathematics, IIT Delhi*

Strategic angle. Both slanted lines have positive x- and y-intercepts, so the feasible region is a quadrilateral with one vertex at the origin. The interior vertex (where the slanted lines meet) is usually the optimum candidate when both constraints are "binding". Let us check directly.

Step 1. Solve the system as a 2×2. Write the active constraints in matrix form at the intersection vertex:

$$\begin{pmatrix} 3 & 5 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ 10 \end{pmatrix}.$$

Determinant: $3(2) - 5(5) = 6 - 25 = -19$. By Cramer's rule:

$$x = \frac{1}{-19}\det\begin{pmatrix} 15 & 5 \\ 10 & 2 \end{pmatrix} = \frac{30-50}{-19} = \frac{-20}{-19} = \frac{20}{19},$$

$$y = \frac{1}{-19}\det\begin{pmatrix} 3 & 15 \\ 5 & 10 \end{pmatrix} = \frac{30-75}{-19} = \frac{-45}{-19} = \frac{45}{19}.$$

Step 2. Read off Z at this vertex. $Z = 5 \cdot \frac{20}{19} + 3 \cdot \frac{45}{19} = \frac{100+135}{19} = \frac{235}{19}$.

Step 3. Compare with the axial vertices. $Z(2,0) = 10$, $Z(0,3) = 9$. Both smaller than $\frac{235}{19} \approx 12.37$. $Z(0,0) = 0$.

Step 4. Decision. The maximum is $\frac{235}{19}$ at $\left(\frac{20}{19}, \frac{45}{19}\right)$.

Why this matters. When both slanted constraints are binding at the optimum, both inequalities turn into equalities, and the vertex is the unique solution of a 2×2 linear system. Cramer's rule (or elimination) gives it in one shot.

Final Answer: $Z_{\text{max}} = \dfrac{235}{19}$ at $\left(\frac{20}{19}, \frac{45}{19}\right)$.

Q 12.4 **Minimise $Z = 3x + 5y$ such that $x + 3y \geq 3$, $x + y \geq 2$, $x, y \geq 0$.**

SOLUTION

Concept used. When constraints are of the "$\geq$" type the feasible region is typically **unbounded**. For an unbounded region the corner-point method needs an extra check: if M is the smallest value of Z at the corner points, then M is the actual minimum *only if* the open half-plane $ax + by < M$ has no point in common with the feasible region. Otherwise Z has no minimum.

Step 1. Constraint lines.

$$x + 3y = 3 \quad \Rightarrow \text{ intercepts } (3, 0) \text{ and } (0, 1),$$
$$x + y = 2 \quad \Rightarrow \text{ intercepts } (2, 0) \text{ and } (0, 2).$$

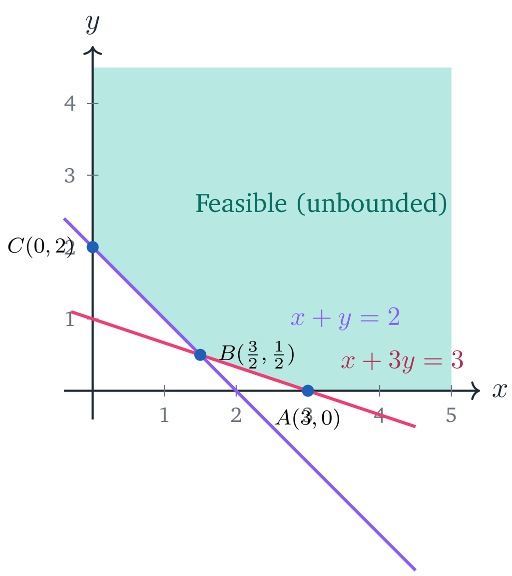
Origin in $x + 3y \geq 3$: $0 \geq 3$ false. Origin in $x + y \geq 2$: $0 \geq 2$ false. So the feasible side is the side *away* from the origin for both.

Step 2. Intersection of the two lines.

$$x + 3y = 3,$$
$$x + y = 2.$$

Subtract: $2y = 1 \Rightarrow y = \frac{1}{2}$. Substitute: $x + \frac{1}{2} = 2 \Rightarrow x = \frac{3}{2}$. Intersection $\left(\frac{3}{2}, \frac{1}{2}\right)$.

Step 3. Locate the corners. Along the x-axis, the binding constraint is $x + y \geq 2 \Rightarrow x \geq 2$, so $A(2, 0)$ is a corner (note $x + 3y = 2 < 3$ would fail; check at $(2, 0)$: $x + 3y = 2 < 3$, so $(2, 0)$ is *not* in the feasible region!). Re-check: at $(2, 0)$, $x + 3y = 2$, which is < 3, so $(2, 0)$ violates the first constraint. The binding x-axis corner is on $x + 3y = 3$ at $(3, 0)$. Check $(3, 0)$ in $x + y \geq 2$: $3 \geq 2$ ✓. So $A(3, 0)$. On the y-axis, the binding constraint is $x + y \geq 2 \Rightarrow y \geq 2$, giving $C(0, 2)$. Check C in $x + 3y \geq 3$: $0 + 6 = 6 \geq 3$ ✓. Corners: $A(3, 0)$, $B\left(\frac{3}{2}, \frac{1}{2}\right)$, $C(0, 2)$.



Step 4. **Evaluate $Z = 3x + 5y$ at each corner.**

$$\begin{aligned} \text{At } A(3,0): &\; Z = 3(3) + 5(0) = 9. \\ \text{At } B\left(\tfrac{3}{2}, \tfrac{1}{2}\right): &\; Z = 3 \cdot \tfrac{3}{2} + 5 \cdot \tfrac{1}{2} = \tfrac{9}{2} + \tfrac{5}{2} = \tfrac{14}{2} = 7. \\ \text{At } C(0,2): &\; Z = 3(0) + 5(2) = 10. \end{aligned}$$

Smallest corner value: $M = 7$ at B.

Step 5. **Unbounded-region check.** We must verify $3x + 5y < 7$ has no point in the feasible region. Equivalently, ask: does the open half-plane $3x + 5y < 7$ intersect the feasible region? Plot the line $3x + 5y = 7$. The origin gives $0 < 7$, true, so the half-plane $3x + 5y < 7$ contains the origin. But the origin is *not* feasible. We need to check whether the half-plane $3x + 5y < 7$ intersects the (unbounded) feasible region.

Test some feasible points. $(3,0)$: $3(3) + 5(0) = 9 \geq 7$. $(0,2)$: $0 + 10 = 10 \geq 7$. $(1.5, 0.5)$: $4.5 + 2.5 = 7$ (on the line). For any point in the feasible region with $y \geq \frac{1}{2}$ and $x \geq \frac{3}{2}$, we have $3x + 5y \geq 3 \cdot \frac{3}{2} + 5 \cdot \frac{1}{2} = 7$. More carefully: every feasible point lies on or above both lines $x + 3y = 3$ and $x + y = 2$, which together with $x, y \geq 0$ means $3x + 5y \geq 7$ throughout. So no feasible point makes $3x + 5y < 7$.

Therefore $Z = 7$ is the genuine minimum.

Final Answer: Minimum value of Z is **7**, attained at $\left(\frac{3}{2}, \frac{1}{2}\right)$.

☞ Exam Tip

For an unbounded region with $\geq$ constraints, the open half-plane $Z < M$ check is mandatory in CBSE. Write it explicitly: "We check whether $3x + 5y < 7$ has any common point with the feasible region. It does not. So $Z_{\min} = 7$." Skipping this line costs a mark.

EXPERT'S SOLUTION *: Aditi Reddy, Ph.D Pure Mathematics, IISc Bangalore*

Strategic angle. The constraints $x + 3y \geq 3$ and $x + y \geq 2$ both push the feasible region away from the origin into the first quadrant. Linear combinations of these inequalities directly bound $Z = 3x + 5y$ from below.

Step 1. **Find a clever combination.** Multiply $x + 3y \geq 3$ by 1 and $x + y \geq 2$ by 2:

$$x + 3y \geq 3, \qquad 2x + 2y \geq 4.$$

Add: $3x + 5y \geq 7$. So $Z \geq 7$ for every feasible point; this is a hard lower bound by direct inequality manipulation.

Step 2. **Find a point achieving the bound.** The bound is tight when both original inequalities are equalities, i.e. at the intersection of $x + 3y = 3$ and $x + y = 2$. Solving: $y = \frac{1}{2}$, $x = \frac{3}{2}$. Check non-negativity: ✓. So $\left(\frac{3}{2}, \frac{1}{2}\right)$ is feasible and gives $Z = 7$.

Step 3. Conclude. The lower bound 7 is attained, hence it is the minimum.

Why this matters. The "multiply-and-add" trick (formally called **linear programming duality**) often produces a clean lower (or upper) bound on Z that matches a vertex value. When it does, you have proved optimality without invoking the corner-point theorem.

> **Final Answer:** $Z_{\text{min}} = \mathbf{7}$ at $\left(\frac{3}{2}, \frac{1}{2}\right)$.

Q 12.5 **Maximise** $Z = 3x + 2y$ **subject to** $x + 2y \leq 10$, $3x + y \leq 15$, $x, y \geq 0$.

SOLUTION

Concept used. Standard bounded LPP: vertices, evaluate, pick the maximum.

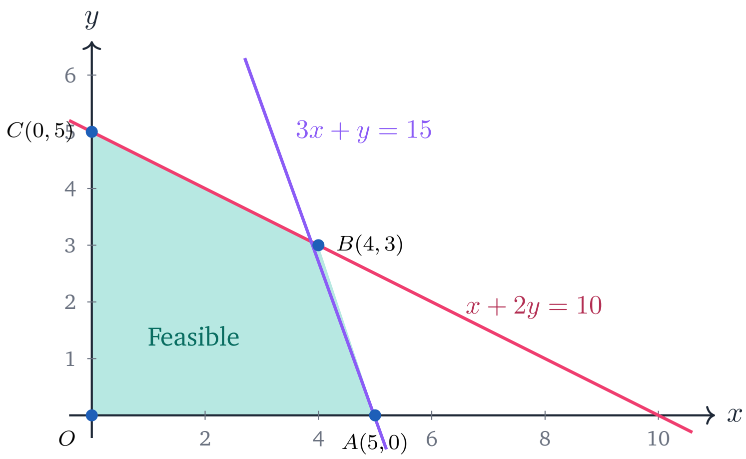
Step 1. Constraint lines.

$$\begin{aligned} x + 2y &= 10 \quad && \Rightarrow (10, 0) \text{ and } (0, 5), \\ 3x + y &= 15 \quad && \Rightarrow (5, 0) \text{ and } (0, 15). \end{aligned}$$

Origin satisfies both ($0 \leq 10$, $0 \leq 15$).

Step 2. Intersection. Solve $x + 2y = 10$ and $3x + y = 15$. Multiply the second by 2: $6x + 2y = 30$. Subtract the first: $5x = 20 \Rightarrow x = 4$. Substitute into $3x + y = 15$: $12 + y = 15 \Rightarrow y = 3$. So $(4, 3)$.

Step 3. Corner list. On the x-axis, tighter is $3x + y \leq 15 \Rightarrow x \leq 5$, giving $A(5, 0)$. On the y-axis, tighter is $x + 2y \leq 10 \Rightarrow y \leq 5$, giving $C(0, 5)$. Vertices: $O(0, 0)$, $A(5, 0)$, $B(4, 3)$, $C(0, 5)$.



Step 4. Evaluate $Z = 3x + 2y$.

$$\begin{aligned}
&\text{At } O(0,0): \ Z = 0. \\
&\text{At } A(5,0): \ Z = 3(5) + 2(0) = 15. \\
&\text{At } B(4,3): \ Z = 3(4) + 2(3) = 12 + 6 = 18. \\
&\text{At } C(0,5): \ Z = 3(0) + 2(5) = 10.
\end{aligned}$$

Step 5. Decision. Maximum is 18 at $B(4,3)$.

> **Final Answer:** Maximum value of Z is **18**, attained at $(4,3)$.

EXPERT'S SOLUTION : *Vivaan Kapoor, M.Sc Mathematics, ISI Kolkata*

Strategic angle. For a maximisation problem with both constraints binding at the optimum, the optimum vertex is the intersection of the two slanted lines. Here, Z's slope $-3/2$ lies between the slopes of the constraint lines $-1/2$ and -3, so geometrically the level-set line will rest on the intersection vertex when slid as far north-east as possible.

Step 1. Slopes of the constraint lines. $x + 2y = 10$ has slope $-\frac{1}{2}$. $3x + y = 15$ has slope -3. The level set $3x + 2y = k$ has slope $-\frac{3}{2}$. Since $-3 < -\frac{3}{2} < -\frac{1}{2}$, the level set's slope lies strictly between the slopes of the two binding constraints. This is precisely the condition under which the optimum sits at the intersection vertex.

Step 2. Solve for the intersection.

$$\begin{aligned}
x + 2y &= 10, \\
3x + y &= 15.
\end{aligned}$$

From the second, $y = 15 - 3x$. Substitute:
$x + 2(15 - 3x) = 10 \Rightarrow x + 30 - 6x = 10 \Rightarrow -5x = -20 \Rightarrow x = 4$. Then $y = 15 - 12 = 3$.

Step 3. Compute Z. $Z = 3(4) + 2(3) = 12 + 6 = 18$.

Step 4. Sanity check. At the axial corners, $Z(5,0) = 15 < 18$ and $Z(0,5) = 10 < 18$. So $(4,3)$ wins.

Why this matters. Comparing the slope of the objective's level set with the slopes of the binding constraints tells you at a glance which vertex is the optimum, without computing Z at every corner.

> **Final Answer:** $Z_{\max} = \mathbf{18}$ at $(4,3)$.

Q 12.6 **Minimise $Z = x + 2y$ subject to $2x + y \geq 3$, $x + 2y \geq 6$, $x, y \geq 0$. Show that the minimum of Z occurs at more than two points.**

SOLUTION

Concept used. When the level-set line $ax + by = k$ is **parallel** to one of the binding boundary lines, the minimum (or maximum) is attained *along the entire edge*, not just at a vertex. So the optimum is attained at infinitely many points.

Step 1. Constraint lines.

$$2x + y = 3 \quad \Rightarrow \quad (1.5, 0) \text{ and } (0, 3),$$
$$x + 2y = 6 \quad \Rightarrow \quad (6, 0) \text{ and } (0, 3).$$

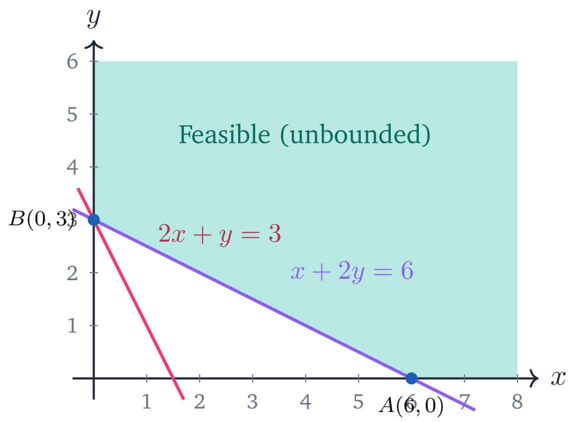
Both pass through $(0, 3)$. Origin gives $0 \geq 3$, false, for both. Feasible side is the side away from the origin.

Step 2. Find their intersection. From $2x + y = 3$: $y = 3 - 2x$. Substitute into $x + 2y = 6$: $x + 2(3 - 2x) = 6 \Rightarrow x + 6 - 4x = 6 \Rightarrow -3x = 0 \Rightarrow x = 0$, $y = 3$. So they intersect on the y-axis at $(0, 3)$.

Step 3. Corners of the feasible region.

- On the x-axis, the binding constraint is $x + 2y \geq 6 \Rightarrow x \geq 6$ (since $2x + y \geq 3$ on the axis gives only $x \geq 1.5$). So $(6, 0)$ is a corner. Check $2x + y = 12 \geq 3$ ✓.
- The intersection $(0, 3)$. Check $x \geq 0$, $y \geq 0$ ✓.
- Above $(0, 3)$ the constraint $x + 2y \geq 6$ is satisfied automatically (since $y \geq 3$). The region extends upwards unboundedly.

Corners: $A(6, 0)$ and $B(0, 3)$.



Step 4. Evaluate $Z = x + 2y$.

$$\text{At } A(6, 0): \quad Z = 6 + 0 = 6.$$
$$\text{At } B(0, 3): \quad Z = 0 + 6 = 6.$$

Same value at both corners!

Step 5. **Inspect the edge AB.** The edge from $A(6,0)$ to $B(0,3)$ lies on the line $x+2y=6$. The objective is also $Z=x+2y$. So on this entire edge, $Z=6$. That is, the minimum value 6 is attained at every point of the segment AB, not just at the two corners. This proves the minimum occurs at more than two points.

Step 6. **Unbounded-region check.** Is there any feasible point with $x+2y<6$? Every feasible point must satisfy $x+2y\geq 6$ by assumption. So no. Hence $Z_{\min}=6$.

> **Final Answer:** Minimum value of Z is **6**, attained at every point on the segment joining $(6,0)$ and $(0,3)$ (infinitely many points).

♥ Parallel level sets give edge optima

The phenomenon "minimum at more than two points" is not an exotic edge case: it happens precisely when the level-set line $ax+by=k$ is parallel to one of the binding constraint boundaries. Recognising this from the coefficients ($1:2$ here matches $1:2$ on the constraint $x+2y=6$) is faster than computing every corner.

EXPERT'S SOLUTION : *Tara Bhat, M.Sc Applied Mathematics, IIT Kanpur*

Strategic angle. The objective $Z=x+2y$ has exactly the same coefficient ratio as one of the constraints, $x+2y\geq 6$. So the level set $Z=k$ is parallel to that constraint's boundary line. We should expect a degenerate optimum sitting along that whole edge.

Step 1. **Parallelism observation.** The objective level set $x+2y=k$ has slope $-\frac{1}{2}$. The constraint boundary $x+2y=6$ also has slope $-\frac{1}{2}$. They are parallel.

Step 2. **Lower bound from the constraint.** Since every feasible point satisfies $x+2y\geq 6$, we immediately get $Z=x+2y\geq 6$. The bound is 6.

Step 3. **Where is the bound achieved?** It is achieved precisely on the edge $x+2y=6$, intersected with the feasible region. The intersection is the segment from $A(6,0)$ to $B(0,3)$ (check $2x+y$ at A: $12\geq 3$ ✓; at B: $3\geq 3$ ✓; in between, $2x+y$ varies linearly from 12 down to 3, always ≥ 3 ✓).

Step 4. **Conclude.** Every point on segment AB is feasible and gives $Z=6$. So the minimum value 6 occurs at infinitely many points, including but not limited to A and B.

Why this matters. A linear program has a non-unique optimum iff the objective's level set is parallel to a binding constraint at the optimum. Spotting this from coefficients alone saves time.

> **Final Answer:** $Z_{\min}=\mathbf{6}$ along the entire segment from $(6,0)$ to $(0,3)$.

Q 12.7 **Minimise and Maximise** $Z = 5x + 10y$ **subject to** $x + 2y \leq 120$, $x + y \geq 60$, $x - 2y \geq 0$, $x, y \geq 0$.

SOLUTION

Concept used. A mixed system with $\leq$ and $\geq$ constraints can yield a bounded region. We find every vertex by pairwise intersection of the boundary lines, keep only those satisfying *all* constraints, then evaluate Z at each.

Step 1. **Constraint lines.**

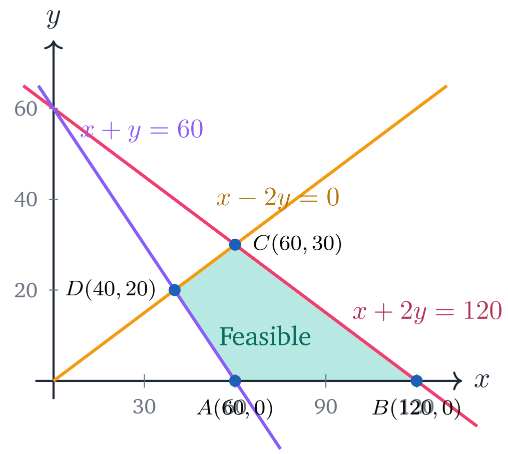
$$\begin{aligned} x + 2y = 120 &\quad \Rightarrow \quad (120, 0) \text{ and } (0, 60), \\ x + y = 60 &\quad \Rightarrow \quad (60, 0) \text{ and } (0, 60), \\ x - 2y = 0 &\quad \Rightarrow \quad \text{line through origin, slope } \tfrac{1}{2}. \end{aligned}$$

Step 2. **Half-plane sides.** $x + 2y \leq 120$: side of origin. $x + y \geq 60$: side away from origin. $x - 2y \geq 0$, i.e. $x \geq 2y$: side below the line $y = x/2$ (the right-down side).

Step 3. **Find candidate vertices.**

- $(x + y = 60) \cap (y = 0)$: $x = 60 \Rightarrow (60, 0)$. Check: $x + 2y = 60 \leq 120$ ✓; $x - 2y = 60 \geq 0$ ✓.
- $(x + 2y = 120) \cap (y = 0)$: $x = 120 \Rightarrow (120, 0)$. Check: $x + y = 120 \geq 60$ ✓; $x - 2y = 120 \geq 0$ ✓.
- $(x + 2y = 120) \cap (x - 2y = 0)$: add: $2x = 120 \Rightarrow x = 60$, $y = 30 \Rightarrow (60, 30)$. Check: $x + y = 90 \geq 60$ ✓.
- $(x + y = 60) \cap (x - 2y = 0)$: $x = 2y$, so $3y = 60$, $y = 20$, $x = 40 \Rightarrow (40, 20)$. Check: $x + 2y = 80 \leq 120$ ✓.

So the feasible polygon has corners $A(60, 0)$, $B(120, 0)$, $C(60, 30)$, $D(40, 20)$.



Step 4. **Evaluate $Z = 5x + 10y$.**

$$\begin{aligned} \text{At } A(60,0):\ & Z = 5(60) + 10(0) = 300. \\ \text{At } B(120,0):\ & Z = 5(120) + 10(0) = 600. \\ \text{At } C(60,30):\ & Z = 5(60) + 10(30) = 300 + 300 = 600. \\ \text{At } D(40,20):\ & Z = 5(40) + 10(20) = 200 + 200 = 400. \end{aligned}$$

Step 5. **Read off optima.**

- Minimum: 300 at $A(60,0)$.
- Maximum: 600, attained at both $B(120,0)$ and $C(60,30)$. Since $Z(B) = Z(C)$, $Z = 5x + 10y$ takes the value 600 along the entire edge BC, i.e. the segment on $x + 2y = 120$ between $(120,0)$ and $(60,30)$.

Final Answer: $Z_{\text{min}} =$ **300** at $(60,0)$; $Z_{\text{max}} =$ **600**, attained at every point on the segment joining $(120,0)$ and $(60,30)$.

✗ Common Mistake

The maximum here is *not* attained at a single vertex. Many students stop after seeing $Z(B) = 600$ and miss the fact that $Z(C)$ also equals 600, so the maximum is attained along the whole edge BC. Always evaluate Z at *every* corner before declaring the optimum.

EXPERT'S SOLUTION : *Riya Joshi, B.Tech CSE, IIT Roorkee*

Strategic angle. The objective $Z = 5x + 10y$ factors as $5(x + 2y)$. So Z is literally 5 times the LHS of the constraint $x + 2y \leq 120$. That instantly gives $Z \leq 600$, and equality holds along the binding edge.

Step 1. **Factor the objective.** $Z = 5x + 10y = 5(x + 2y)$.

Step 2. **Upper bound from a constraint.** Since $x + 2y \leq 120$ in the feasible region, $Z = 5(x + 2y) \leq 5(120) = 600$. So $Z_{\text{max}} = 600$, with equality precisely on the binding edge $x + 2y = 120$.

Step 3. **Where on that edge?** Intersect $x + 2y = 120$ with the feasible region: from the constraint $x - 2y \geq 0$ and $y \geq 0$ we get the segment from $(60,30)$ (where $x - 2y = 0$) to $(120,0)$ (where $y = 0$). All points on this segment are feasible (verify $x + y \geq 60$: on the segment $x + y = 120 - y \geq 120 - 30 = 90 \geq 60$ ✓). So the maximum is attained along the entire segment.

Step 4. **For the minimum.** The smallest corner value is 300 at $(60,0)$. Since the region is bounded, this is the global minimum.

Why this matters. When the objective is proportional to a constraint's LHS, the optimum bound and its attainment locus pop out of the constraint immediately. Look for proportionality before grinding through the corner table.

Final Answer: $Z_{\min} = 300$ at $(60, 0)$; $Z_{\max} = 600$ along segment from $(120, 0)$ to $(60, 30)$.

Q 12.8 Minimise and Maximise $Z = x + 2y$ subject to $x + 2y \geq 100$, $2x - y \leq 0$, $2x + y \leq 200$, $x, y \geq 0$.

SOLUTION

Concept used. A bounded LP with three slanted constraints can still produce a quadrilateral region. Once again: find every pairwise intersection, retain only those satisfying all constraints, evaluate Z at each.

Step 1. Constraint lines.

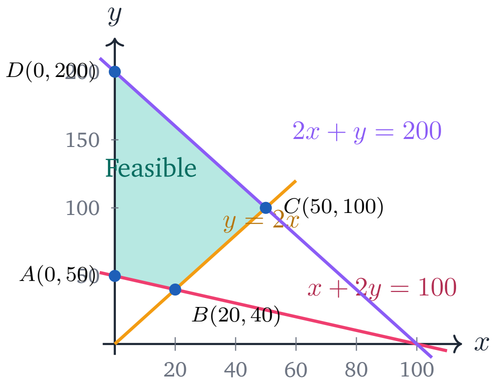
$$\begin{aligned} x + 2y = 100 &\quad\Rightarrow\quad (100, 0) \text{ and } (0, 50), \\ 2x - y = 0 &\quad\Rightarrow\quad \text{line through origin, slope } 2, \\ 2x + y = 200 &\quad\Rightarrow\quad (100, 0) \text{ and } (0, 200). \end{aligned}$$

Half-planes: $x + 2y \geq 100$ (away from origin), $2x - y \leq 0$, i.e. $y \geq 2x$ (above the line $y = 2x$), $2x + y \leq 200$ (toward origin).

Step 2. Find candidate vertices.

- $(x + 2y = 100) \cap (x = 0)$: $(0, 50)$. Check: $2x - y = -50 \leq 0$ ✓; $2x + y = 50 \leq 200$ ✓.
- $(x + 2y = 100) \cap (2x - y = 0)$: $y = 2x$, so $x + 4x = 100$, $x = 20$, $y = 40$ $\Rightarrow (20, 40)$. Check: $2x + y = 40 + 40 = 80 \leq 200$ ✓.
- $(2x - y = 0) \cap (2x + y = 200)$: add: $4x = 200$, $x = 50$, $y = 100 \Rightarrow (50, 100)$. Check: $x + 2y = 50 + 200 = 250 \geq 100$ ✓.
- $(2x + y = 200) \cap (x = 0)$: $(0, 200)$. Check: $x + 2y = 400 \geq 100$ ✓; $2x - y = -200 \leq 0$ ✓.

Corners: $A(0, 50)$, $B(20, 40)$, $C(50, 100)$, $D(0, 200)$.



Step 3. Evaluate $Z = x + 2y$.

$$\begin{aligned} \text{At } A(0,50): &\quad Z = 0 + 100 = 100. \\ \text{At } B(20,40): &\quad Z = 20 + 80 = 100. \\ \text{At } C(50,100): &\quad Z = 50 + 200 = 250. \\ \text{At } D(0,200): &\quad Z = 0 + 400 = 400. \end{aligned}$$

Step 4. Read off optima.

- Minimum: 100, attained at both $A(0, 50)$ and $B(20, 40)$. Since the level set $x + 2y = 100$ is parallel to (in fact coincides with) the binding constraint boundary, $Z = 100$ along the whole edge AB.
- Maximum: 400 at $D(0, 200)$.

Final Answer: $Z_{\text{min}} = \mathbf{100}$, attained at every point on the segment joining $(0, 50)$ and $(20, 40)$; $Z_{\text{max}} = \mathbf{400}$ at $(0, 200)$.

EXPERT'S SOLUTION *: Krishna Nair, Ph.D Mathematics, IIT Delhi*

Strategic angle. The objective $Z = x + 2y$ matches the constraint $x + 2y \geq 100$ in coefficients. So we expect a degenerate minimum sitting along the edge $x + 2y = 100$.

Step 1. Lower bound. For any feasible point, $x + 2y \geq 100$, so $Z = x + 2y \geq 100$. Equality holds on the edge $x + 2y = 100$.

Step 2. The edge segment. Intersect $x + 2y = 100$ with the rest of the feasible region. The other constraints become $y \geq 2x$ (so $x + 2(2x) = 5x \leq 100$? no, we want points with $x + 2y = 100$ AND $y \geq 2x$, which means $x + 2y = 100$, $y \geq 2x$, i.e. $100 - x \geq 4x$, i.e. $x \leq 20$) and $2x + y \leq 200$ (since $y = 50 - x/2$, $2x + 50 - x/2 = 3x/2 + 50 \leq 200$, always true for $x \leq 100$). And $x \geq 0$. So the segment runs from $x = 0$ ($\Rightarrow (0, 50)$) to $x = 20$ ($\Rightarrow (20, 40)$).

Step 3. Upper bound. Z grows with y (coefficient $2 > 0$) and with x (coefficient

$1 > 0$). Maximum vertex is the one farthest up-right. Among corners, $D(0, 200)$ wins with $Z = 400$.

Step 4. Final answer. $Z_{\text{min}} = 100$ on segment A–B; $Z_{\text{max}} = 400$ at $D(0, 200)$.

Why this matters. As in Q6, a degenerate optimum is the signal of coefficient-parallelism between objective and one constraint. The minimum here lives on an edge, not just a vertex.

Final Answer: $Z_{\text{min}} = 100$ on segment $(0, 50)$ to $(20, 40)$; $Z_{\text{max}} = 400$ at $(0, 200)$.

Q 12.9 **Maximise $Z = -x + 2y$, subject to the constraints: $x \geq 3$, $x + y \geq 5$, $x + 2y \geq 6$, $y \geq 0$.**

SOLUTION

Concept used. Same recipe, with the unbounded-region check. For maximisation on an unbounded region: if M is the largest value of Z at the corners, M is the actual maximum *only if* the open half-plane $ax + by > M$ has no point in common with the feasible region. If it does, the maximum does not exist.

Step 1. Constraint lines.

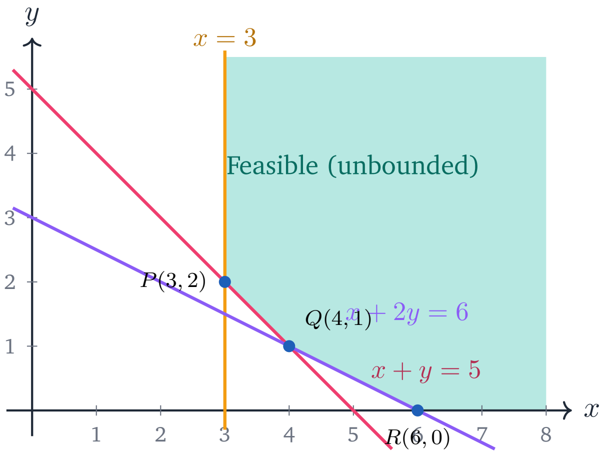
$$\begin{aligned} x &= 3 && \text{vertical line,} \\ x + y &= 5 && \Rightarrow (5, 0), (0, 5), \\ x + 2y &= 6 && \Rightarrow (6, 0), (0, 3). \end{aligned}$$

Step 2. Half-plane sides. $x \geq 3$: right of $x = 3$. $x + y \geq 5$: away from origin. $x + 2y \geq 6$: away from origin. $y \geq 0$: above x-axis.

Step 3. Candidate vertices.

- $x = 3$ and $x + y = 5$: $y = 2 \Rightarrow (3, 2)$. Check $x + 2y = 7 \geq 6$ ✓.
- $x = 3$ and $x + 2y = 6$: $y = \frac{3}{2} \Rightarrow (3, \frac{3}{2})$. Check $x + y = 4.5 \geq 5$? *No*, $4.5 < 5$. Reject.
- $x + y = 5$ and $x + 2y = 6$: subtract, $y = 1$, $x = 4 \Rightarrow (4, 1)$. Check $x \geq 3$ ✓.
- $x + y = 5$ and $y = 0$: $(5, 0)$. Check $x + 2y = 5 \geq 6$? No, reject.
- $x + 2y = 6$ and $y = 0$: $(6, 0)$. Check $x \geq 3$ ✓; $x + y = 6 \geq 5$ ✓.

Corners: $P(3, 2)$, $Q(4, 1)$, $R(6, 0)$. The region extends upward unboundedly along $x = 3$.



Step 4. **Evaluate $Z = -x + 2y$ at the corners.**

$$\text{At } P(3,2): \; Z = -3 + 4 = 1.$$
$$\text{At } Q(4,1): \; Z = -4 + 2 = -2.$$
$$\text{At } R(6,0): \; Z = -6 + 0 = -6.$$

Largest corner value: $M = 1$ at P.

Step 5. **Unbounded-region check for maximum.** Does the open half-plane $-x + 2y > 1$ contain any feasible point? Consider the feasible ray $\{(3, y) : y \geq 2\}$ along the line $x = 3$. At such a point, $Z = -3 + 2y$. For y arbitrarily large, Z is arbitrarily large. For instance, at $(3, 100)$, $Z = -3 + 200 = 197 \gg 1$.

Hence the half-plane $-x + 2y > 1$ has many feasible points. Therefore $Z = 1$ is *not* the maximum: Z can grow without bound on the feasible region.

Step 6. **Conclusion.** Z has *no maximum* on this feasible region.

Final Answer: The maximum value of Z does *not* exist (i.e., Z is unbounded above on the feasible region).

☞ Exam Tip

"Maximum does not exist" is a legitimate, full-mark CBSE answer when the unbounded-region check fails. State it explicitly with a sentence like "Since the open half-plane $-x + 2y > 1$ contains feasible points (e.g. $(3, 100)$), the maximum is unbounded."

EXPERT'S SOLUTION : *Ananya Verma, M.Sc Mathematics, IIT Bombay*

Strategic angle. $Z = -x + 2y$ rewards increasing y and penalises increasing x. The feasible region permits y to grow arbitrarily large (along $x = 3$ in particular), with x *not* forced to grow with it. That recipe immediately predicts "no maximum".

Step 1. **Look along the boundary $x = 3$.** For any $y \geq 2$, the point $(3, y)$ satisfies $x \geq 3$ ✓; $x + y = 3 + y \geq 5$ ✓when $y \geq 2$; $x + 2y = 3 + 2y \geq 6$ ✓when $y \geq 1.5$; $y \geq 0$ ✓. So the entire ray $\{(3, y) : y \geq 2\}$ is feasible.

Step 2. **Z on this ray.** $Z = -3 + 2y \to +\infty$ as $y \to +\infty$.

Step 3. **Conclude.** No finite maximum exists.

Step 4. **Sanity check.** Note that the corner-point method would have given the misleading answer $Z = 1$ at $(3, 2)$. The unbounded-region check is the safeguard that rejects this and reports "no max".

Why this matters. Without the unbounded-region check, an LP solver naively reading off the largest corner value would give a wrong answer. CBSE marking schemes explicitly require this check.

> **Final Answer:** Z has no maximum (unbounded above).

Q 12.10 **Maximise $Z = x + y$, subject to $x - y \leq -1$, $-x + y \leq 0$, $x, y \geq 0$.**

SOLUTION

Concept used. When constraints are mutually contradictory, the feasible region is **empty** and the LPP has *no feasible solution* (and hence no optimum). To check, we look for any single (x, y) that satisfies every constraint.

Step 1. **Rewrite the inequalities.**

$$\begin{aligned} x - y \leq -1 &\quad\Leftrightarrow\quad y \geq x + 1, \\ -x + y \leq 0 &\quad\Leftrightarrow\quad y \leq x. \end{aligned}$$

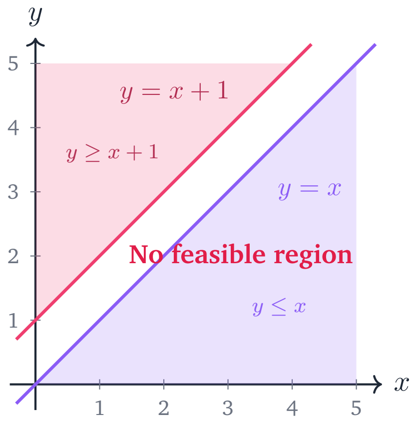
Plus $x \geq 0$ and $y \geq 0$.

Step 2. **Combine.** The first says $y \geq x + 1$; the second says $y \leq x$. Together:

$$x + 1 \leq y \leq x.$$

Subtracting, $x + 1 \leq x$, i.e. $1 \leq 0$, which is impossible.

Step 3. **Visualise.** Plot the boundary lines $y = x + 1$ and $y = x$. The line $y = x + 1$ lies *above* $y = x$ at every x. The constraint $y \geq x + 1$ asks us to be above $y = x + 1$ (the higher line); the constraint $y \leq x$ asks us to be below $y = x$ (the lower line). There is no region between them: the higher line is entirely above the lower line, so "between" is empty.



Step 4. Conclusion. The two shaded half-planes do not overlap. The feasible region is empty. Hence the LPP has no solution: there is no (x, y) at which Z can even be evaluated under the given constraints.

Final Answer: There is no feasible solution: the constraints are inconsistent, so Z has *no maximum* (and no minimum).

✗ Common Mistake

"No feasible solution" is different from "no maximum on an unbounded region". Here the feasible *set* is empty; in Q9 the set was non-empty but Z grew without bound. Both deserve full marks if stated correctly, but they are not the same conclusion.

EXPERT'S SOLUTION : *Ishaan Desai, M.Tech CS, IIT Madras*

Strategic angle. The two constraints $y \geq x + 1$ and $y \leq x$ are direct opposites: one demands y exceed x by at least 1, the other demands y not exceed x at all. No real number can do both.

Step 1. Algebraic infeasibility. Add the two original inequalities:
$(x - y) + (-x + y) \leq -1 + 0$, i.e. $0 \leq -1$, which is false. Since adding feasible inequalities should yield a true statement, the system is inconsistent.

Step 2. Geometric infeasibility. Sketch $y = x$ and $y = x + 1$. Both have slope 1; they are parallel lines with $y = x + 1$ a vertical unit above $y = x$. The strip between them is open; demanding "above $y = x + 1$ AND below $y = x$" demands a point in a region that lies on both sides of the strip, which is empty.

Step 3. Implication for Z. Since there is no feasible point, the objective function $Z = x + y$ is not defined anywhere on the feasible set. Hence neither a maximum nor a minimum exists.

Why this matters. Whenever two inequalities $Ax + By \leq c$ and $A'x + B'y \leq c'$ can be added to produce $0 \leq$ (negative), the system is inconsistent and the LPP has no solution. Spot the contradiction first; it saves an entire graphical solve.

Final Answer: No feasible solution; Z has no maximum.

Key Takeaways

- A linear programming problem (LPP) optimises a linear objective $Z = ax + by$ subject to linear inequality constraints and $x, y \geq 0$.
- The feasible region is the intersection of all constraint half-planes; if bounded, Z attains both its max and its min at a corner point.
- For an unbounded feasible region, the corner-point method gives a candidate optimum, but you must check whether the open half-plane "better than the candidate" has any feasible point. If yes, that optimum does not exist.
- If the level set of Z is parallel to a binding constraint, the optimum is attained along an entire edge, not just at a vertex (multiple optima).
- Contradictory constraints leave an empty feasible region; the LPP then has no solution at all.

End of Exercise 12.1