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Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 13: Probability

About this Chapter

Chapter 13 builds the formal machinery of **conditional probability**, the **multiplication theorem**, **independence**, the **total probability theorem**, **Bayes' theorem** and the **distribution and mean of a random variable**. Exercise 13.1 drills the very first skill the chapter teaches: computing $P(E \mid F) = \dfrac{P(E \cap F)}{P(F)}$ from sample spaces, set intersections and given probabilities. Every later result in the chapter rests on the definition trained here.

Topics covered: Conditional probability • Sample-space reduction • $P(E \cap F)$ from counting • Properties of $P(\cdot \mid F)$

Quick Formula Sheet

Conditional probability:
$P(E \mid F) = \dfrac{P(E \cap F)}{P(F)}$, $P(F) \neq 0$

Counting form:
$P(E \mid F) = \dfrac{n(E \cap F)}{n(F)}$ for equally-likely outcomes

Multiplication form:
$P(E \cap F) = P(F)\,P(E \mid F) = P(E)\,P(F \mid E)$

Properties:
$P(S \mid F) = P(F \mid F) = 1$; $P(E' \mid F) = 1 - P(E \mid F)$; $P((A \cup B) \mid F) = P(A \mid F) + P(B \mid F) - P((A \cap B) \mid F)$

Exercise 13.1

Q 13.1 **Given that E and F are events such that $P(E) = 0.6$, $P(F) = 0.3$ and $P(E \cap F) = 0.2$, find $P(E \mid F)$ and $P(F \mid E)$.**

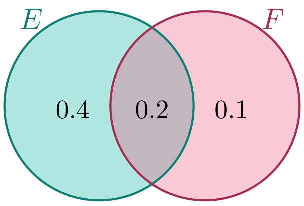
SOLUTION

Concept used. The **conditional probability** of event E given that event F has already occurred is defined by

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}, \quad P(F) \neq 0.$$

The intersection $E \cap F$ is the set of outcomes that lie in both E and F, so $P(E \cap F)$ counts the chance that both events happen simultaneously. Swapping the roles of E and F gives the companion formula $P(F \mid E) = \dfrac{P(F \cap E)}{P(E)}$; since $E \cap F = F \cap E$, the numerator is the same number $P(E \cap F)$.

> ☞ **Why divide by $P(F)$?**
> Conditioning on F shrinks the sample space from S to F. Within this new "universe" of probability mass $P(F)$, we ask what fraction lies in E as well, which is $P(E \cap F)/P(F)$.



$P(E) = 0.6,\ P(F) = 0.3,\ P(E \cap F) = 0.2$

Step 1. Apply the definition to compute $P(E \mid F)$. Substitute $P(E \cap F) = 0.2$ and $P(F) = 0.3$:

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{0.2}{0.3} = \frac{2}{3}.$$

Step 2. Apply the same definition with E and F swapped to compute $P(F \mid E)$. Substitute $P(E \cap F) = 0.2$ and $P(E) = 0.6$:

$$P(F \mid E) = \frac{P(E \cap F)}{P(E)} = \frac{0.2}{0.6} = \frac{1}{3}.$$

Step 3. Sanity check. Both answers lie in $[0, 1]$, as every probability must. Also $P(E \mid F) > P(E) = 0.6$ would be a red flag, but here $P(E \mid F) = 2/3 \approx 0.667 > 0.6$, which simply says knowing F has happened makes E slightly more likely.

Final Answer: $P(E \mid F) = \dfrac{2}{3}$ and $P(F \mid E) = \dfrac{1}{3}$.

✗ Common Mistake

Students sometimes write $P(E \mid F) = P(E \cap F) \cdot P(F)$ by confusing the multiplication theorem with the definition. The conditional probability *divides* by $P(F)$; the multiplication theorem *multiplies* by $P(F)$ and gives $P(E \cap F)$, not $P(E \mid F)$.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Bombay*

Strategic angle. Read off the three given numbers, decide which formula they slot into, then evaluate. With $P(E)$, $P(F)$ and $P(E \cap F)$ all in hand, both conditional probabilities are one substitution away.

Step 1. Identify the formulas. We need

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}, \qquad P(F \mid E) = \frac{P(E \cap F)}{P(E)}.$$

Each requires only the numerator $P(E \cap F)$ and the appropriate denominator.

Step 2. Plug into the first formula:

$$P(E \mid F) = \frac{0.2}{0.3} = \frac{2/10}{3/10} = \frac{2}{3}.$$

Step 3. Plug into the second formula:

$$P(F \mid E) = \frac{0.2}{0.6} = \frac{2/10}{6/10} = \frac{2}{6} = \frac{1}{3}.$$

Step 4. Cross-check via the multiplication theorem. We should have
$P(E \cap F) = P(E)\,P(F \mid E) = 0.6 \times \frac{1}{3} = 0.2$ ✓ and
$P(F)\,P(E \mid F) = 0.3 \times \frac{2}{3} = 0.2$ ✓.

Why this matters. The two conditional probabilities are different numbers even though they share the same numerator. The denominator decides which way you are conditioning.

Final Answer: $P(E \mid F) = \frac{2}{3}$, $P(F \mid E) = \frac{1}{3}$.

Q 13.2 **Compute $P(A \mid B)$, if $P(B) = 0.5$ and $P(A \cap B) = 0.32$.**

SOLUTION

Concept used. By definition, the conditional probability of A given B is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0.$$

This is the only formula required. We are given the numerator and denominator directly.

Step 1. Confirm that $P(B) \neq 0$. Here $P(B) = 0.5 > 0$, so the formula is valid.

Step 2. Substitute the given values:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.32}{0.5}.$$

Step 3. Carry out the division:

$$\frac{0.32}{0.5} = \frac{0.32 \times 2}{0.5 \times 2} = \frac{0.64}{1} = 0.64.$$

Final Answer: $P(A \mid B) = 0.64$.

EXPERT'S SOLUTION *: Sneha Iyer, M.Sc Mathematics, ISI Kolkata*

Quick reading. A direct plug-in question; the only trap is the decimal arithmetic.

Step 1. Use $P(A \mid B) = P(A \cap B)/P(B)$.

Step 2. Substitute: $P(A \mid B) = 0.32/0.5$.

Step 3. Clear the half-decimal denominator: multiply numerator and denominator by 2, getting $0.64/1 = 0.64$.

Why this matters. Always multiply through to remove fractional denominators before doing decimal division; it eliminates a common arithmetic slip.

Final Answer: $P(A \mid B) = 0.64$.

Q 13.3 **If** $P(A) = 0.8$, $P(B) = 0.5$ **and** $P(B \mid A) = 0.4$, **find**
(i) $P(A \cap B)$ **(ii)** $P(A \mid B)$ **(iii)** $P(A \cup B)$.

SOLUTION

Concept used. Three results from §13.2–13.3 are needed:

- **Multiplication theorem:** $P(A \cap B) = P(A)\, P(B \mid A) = P(B)\, P(A \mid B)$.

- **Conditional probability:** $P(A \mid B) = P(A \cap B)/P(B)$, $P(B) \neq 0$.
- **Addition theorem:** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

We use the first to find $P(A \cap B)$, the second to find $P(A \mid B)$, and the third to find $P(A \cup B)$.

Step 1. **(i)** Apply the multiplication theorem with the known pair $P(A)$ and $P(B \mid A)$:

$$P(A \cap B) = P(A)\,P(B \mid A) = 0.8 \times 0.4 = 0.32.$$

Step 2. **(ii)** Apply the definition of conditional probability, swapping roles:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.32}{0.5} = \frac{0.32 \times 2}{0.5 \times 2} = \frac{0.64}{1} = 0.64.$$

Step 3. **(iii)** Apply the addition theorem with the value of $P(A \cap B)$ just computed:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.8 + 0.5 - 0.32.$$

Add first: $0.8 + 0.5 = 1.3$. Then subtract: $1.3 - 0.32 = 0.98$.

$$P(A \cup B) = 0.98.$$

Final Answer: $P(A \cap B) = 0.32,\ P(A \mid B) = 0.64,\ P(A \cup B) = 0.98.$

♥ Multiplication versus addition

Multiplication gives the chance that two events happen *together*; addition gives the chance that at least one of them happens. Both rest on conditional probability under the hood.

EXPERT'S SOLUTION *: Vivaan Gupta, M.Tech CS, IIT Madras*

Strategic angle. Three sub-parts, each a one-formula computation. Order them so each part feeds the next: $P(A \cap B)$ unlocks $P(A \mid B)$ and $P(A \cup B)$.

Step 1. **Compute $P(A \cap B)$.** Multiplication theorem with A as the conditioned event:

$$P(A \cap B) = P(A)\,P(B \mid A) = 0.8 \times 0.4 = 0.32.$$

Step 2. **Compute $P(A \mid B)$.** Conditional probability formula with the just-found numerator:

$$P(A \mid B) = \frac{0.32}{0.5} = 0.64.$$

Step 3. **Compute $P(A \cup B)$.** Addition theorem:

$$P(A \cup B) = 0.8 + 0.5 - 0.32 = 1.30 - 0.32 = 0.98.$$

Step 4. Cross-check: $P(A \cup B) \le 1$ ✓; $P(A \cap B) \le \min(P(A), P(B)) = 0.5$ ✓; $P(A \mid B) \ge P(A \cap B) = 0.32$ ✓.

Why this matters. The three quantities are linked: once any two of $\{P(A \cap B), P(A \mid B), P(B \mid A)\}$ are known together with $P(A)$ and $P(B)$, everything else follows mechanically.

Final Answer: $P(A \cap B) = 0.32,\ P(A \mid B) = 0.64,\ P(A \cup B) = 0.98.$

Q 13.4 **Evaluate $P(A \cup B)$, if $2P(A) = P(B) = \frac{5}{13}$ and $P(A \mid B) = \frac{2}{5}$.**

SOLUTION

Concept used. Two results are required:

- Multiplication theorem: $P(A \cap B) = P(B)\,P(A \mid B)$.
- Addition theorem: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

We first read off $P(A)$ and $P(B)$ from the chained equality, then compute $P(A \cap B)$, then plug into the addition theorem.

Step 1. Read off $P(A)$ and $P(B)$. From $2P(A) = P(B) = \frac{5}{13}$:

$$P(B) = \frac{5}{13}, \qquad 2P(A) = \frac{5}{13} \Rightarrow P(A) = \frac{5}{26}.$$

Step 2. Compute $P(A \cap B)$ from the multiplication theorem:

$$P(A \cap B) = P(B)\,P(A \mid B) = \frac{5}{13} \times \frac{2}{5} = \frac{2}{13}.$$

Step 3. Apply the addition theorem:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{5}{26} + \frac{5}{13} - \frac{2}{13}.$$

Take LCM 26:

$$= \frac{5}{26} + \frac{10}{26} - \frac{4}{26} = \frac{5 + 10 - 4}{26} = \frac{11}{26}.$$

Final Answer: $P(A \cup B) = \frac{11}{26}$.

EXPERT'S SOLUTION : *Pranav Mehta, M.Sc Applied Mathematics, IIT Kanpur*

Structural observation. The chained equality $2P(A) = P(B) = \frac{5}{13}$ packages two facts in one line. Unpack it before doing anything else.

Step 1. Unpack: $P(B) = \frac{5}{13}$ and $P(A) = \frac{1}{2} \cdot \frac{5}{13} = \frac{5}{26}$.

Step 2. Use $P(A \cap B) = P(B)P(A \mid B) = \frac{5}{13} \cdot \frac{2}{5} = \frac{2}{13}$.

Step 3. Use $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{5}{26} + \frac{10}{26} - \frac{4}{26} = \frac{11}{26}$.

Step 4. Check: $\frac{11}{26} < 1$ ✓ and $\frac{11}{26} \geq P(B) = \frac{10}{26}$ ✓.

Why this matters. A chained equality is just a shorthand. Always split it into individual equations first; this prevents algebraic mistakes downstream.

Final Answer: $P(A \cup B) = \dfrac{11}{26}$.

Q 13.5 **If** $P(A) = \dfrac{6}{11}$, $P(B) = \dfrac{5}{11}$ **and** $P(A \cup B) = \dfrac{7}{11}$, **find**
(i) $P(A \cap B)$ **(ii)** $P(A \mid B)$ **(iii)** $P(B \mid A)$.

SOLUTION

Concept used. Rearranging the addition theorem,

$$P(A \cap B) = P(A) + P(B) - P(A \cup B).$$

Then the conditional-probability definitions give

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad P(B \mid A) = \frac{P(A \cap B)}{P(A)}.$$

Step 1. **(i)** Use the rearranged addition theorem:

$$P(A \cap B) = \frac{6}{11} + \frac{5}{11} - \frac{7}{11} = \frac{6+5-7}{11} = \frac{4}{11}.$$

Step 2. **(ii)** Use the conditional-probability formula:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{4/11}{5/11} = \frac{4}{11} \times \frac{11}{5} = \frac{4}{5}.$$

Step 3. **(iii)** Similarly:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{4/11}{6/11} = \frac{4}{11} \times \frac{11}{6} = \frac{4}{6} = \frac{2}{3}.$$

Final Answer: $P(A \cap B) = \dfrac{4}{11}$, $P(A \mid B) = \dfrac{4}{5}$, $P(B \mid A) = \dfrac{2}{3}$.

EXPERT'S SOLUTION : *Aanya Singh, Ph.D Mathematics, IIT Delhi*

Strategic angle. Solve part (i) first; (ii) and (iii) are then two direct substitutions.

Step 1. From $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, isolate $P(A \cap B)$:

$$P(A \cap B) = \tfrac{6}{11} + \tfrac{5}{11} - \tfrac{7}{11} = \tfrac{4}{11}.$$

Step 2. Compute $P(A \mid B) = \dfrac{4/11}{5/11} = \dfrac{4}{5}$.

Step 3. Compute $P(B \mid A) = \dfrac{4/11}{6/11} = \dfrac{2}{3}$.

Step 4. Validate: $P(A \mid B)\,P(B) = \frac{4}{5} \cdot \frac{5}{11} = \frac{4}{11} = P(A \cap B)$ ✓.

Why this matters. The same intersection probability $\frac{4}{11}$ feeds both conditional probabilities; the difference in answers comes purely from the denominators.

Final Answer: $\dfrac{4}{11}, \dfrac{4}{5}, \dfrac{2}{3}$.

Q 13.6 **Determine $P(E \mid F)$. A coin is tossed three times, where**
(i) E: head on third toss, F: heads on first two tosses
(ii) E: at least two heads, F: at most two heads
(iii) E: at most two tails, F: at least one tail.

SOLUTION

Concept used. For equally-likely outcomes,

$$P(E \mid F) = \frac{n(E \cap F)}{n(F)}.$$

The sample space for three coin tosses is

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\},\ n(S) = 8.$$

For each part we list E, F, $E \cap F$, count the outcomes, and apply the formula.

Step 1. **(i)** *Head on third toss, heads on first two tosses.*
$E = \{HHH, HTH, THH, TTH\}$, so $n(E) = 4$.
$F = \{HHH, HHT\}$, so $n(F) = 2$.
$E \cap F = \{HHH\}$, so $n(E \cap F) = 1$.
Therefore

$$P(E \mid F) = \frac{n(E \cap F)}{n(F)} = \frac{1}{2}.$$

Step 2. **(ii)** *At least two heads, at most two heads.*

At least two heads: $E = \{HHH, HHT, HTH, THH\}$, $n(E) = 4$.
At most two heads (i.e. ≤ 2 heads, equivalently not HHH):
$F = \{HHT, HTH, THH, HTT, THT, TTH, TTT\}$, $n(F) = 7$.
$E \cap F = \{HHT, HTH, THH\}$, $n(E \cap F) = 3$.

$$P(E \mid F) = \frac{3}{7}.$$

Step 3. **(iii)** *At most two tails, at least one tail.*
At most two tails (≤ 2 tails, equivalently not TTT):
$E = \{HHH, HHT, HTH, THH, HTT, THT, TTH\}$, $n(E) = 7$.
At least one tail (≥ 1 tail, equivalently not HHH):
$F = \{HHT, HTH, THH, HTT, THT, TTH, TTT\}$, $n(F) = 7$.
$E \cap F = \{HHT, HTH, THH, HTT, THT, TTH\}$, $n(E \cap F) = 6$.

$$P(E \mid F) = \frac{6}{7}.$$

Final Answer: (i) $\frac{1}{2}$ (ii) $\frac{3}{7}$ (iii) $\frac{6}{7}$.

✗ Common Mistake

"At least two heads" means ≥ 2 heads, i.e. exactly 2 or exactly 3. "At most two heads" means ≤ 2 heads, i.e. 0, 1 or 2 heads. Mixing up the two flips the answer.

EXPERT'S SOLUTION : *Arjun Reddy, M.Sc Mathematics, IIT Bombay*

Picture-first. For each part, draw the tree of 8 outcomes once, then highlight the cells that belong to E, F, and $E \cap F$.

$S =$	TTT	TTT	TTT	TTT	HTT	HTT	HTT	HTT

Step 1. **(i)** $F = \{HHH, HHT\}$, of which only HHH has a head on toss 3.
$\Rightarrow P(E \mid F) = 1/2$.

Step 2. **(ii)** F excludes only HHH, so $n(F) = 7$. Of these, E keeps the three two-head outcomes HHT, HTH, THH. $\Rightarrow 3/7$.

Step 3. **(iii)** F excludes only HHH, $n(F) = 7$. Of these, E (at most two tails) keeps everything except TTT, leaving 6 outcomes. $\Rightarrow 6/7$.

Why this matters. For small sample spaces, listing outcomes is faster and less error-prone than juggling formulas.

Final Answer: $\frac{1}{2}, \frac{3}{7}, \frac{6}{7}$.

Q 13.7 **Determine $P(E \mid F)$. Two coins are tossed once, where**
(i) E: tail appears on one coin, F: one coin shows head
(ii) E: no tail appears, F: no head appears.

SOLUTION

Concept used. Sample space for two tosses: $S = \{HH, HT, TH, TT\}$, $n(S) = 4$, equally likely. Use

$$P(E \mid F) = \frac{n(E \cap F)}{n(F)}.$$

Step 1. **(i)** *Tail on one coin / one coin shows head.*
Both phrases mean exactly one tail and exactly one head, namely the outcomes HT and TH.
$E = \{HT, TH\}$, $F = \{HT, TH\}$, so $E \cap F = \{HT, TH\}$.
$n(E \cap F) = 2$, $n(F) = 2$.

$$P(E \mid F) = \frac{2}{2} = 1.$$

Step 2. **(ii)** *No tail / no head.*
E = "no tail" = $\{HH\}$, so $n(E) = 1$.
F = "no head" = $\{TT\}$, so $n(F) = 1$.
$E \cap F = \varnothing$, $n(E \cap F) = 0$.

$$P(E \mid F) = \frac{0}{1} = 0.$$

Final Answer: (i) 1 (ii) 0.

EXPERT'S SOLUTION : *Diya Kapoor, M.Sc Mathematics, IIT Bombay*

Quick reading.

Step 1. In (i), "tail on one coin" and "one coin shows head" both pick out the same two outcomes $\{HT, TH\}$, so $E = F$ and $P(E \mid F) = 1$.

Step 2. In (ii), the events $\{HH\}$ and $\{TT\}$ are disjoint, so $P(E \mid F) = 0$.

Why this matters. $P(E \mid F) = 1$ means E is certain inside F; $P(E \mid F) = 0$ means E is impossible inside F. Both extremes appear naturally when E and F overlap fully or not at all.

> **Final Answer:** 1 and 0.

Q 13.8 **A die is thrown three times. E: 4 appears on the third toss, F: 6 and 5 appears respectively on first two tosses. Find $P(E \mid F)$.**

SOLUTION

Concept used. For three dice tosses, the sample space has $n(S) = 6^3 = 216$ equally-likely ordered triples. Apply $P(E \mid F) = n(E \cap F)/n(F)$.

Step 1. Describe F: the first two tosses are fixed as 6 then 5, the third toss is free. Number of such triples $= 1 \times 1 \times 6 = 6$. So $n(F) = 6$.

Step 2. Describe $E \cap F$: first toss 6, second toss 5, third toss 4. That is the single triple $(6, 5, 4)$, so $n(E \cap F) = 1$.

Step 3. Apply the formula:

$$P(E \mid F) = \frac{n(E \cap F)}{n(F)} = \frac{1}{6}.$$

> **Final Answer:** $P(E \mid F) = \dfrac{1}{6}$.

☞ Exam Tip

For independent successive tosses, the conditional probability of a specific outcome on the third toss given any restriction on the first two equals the unconditional probability of that outcome, $1/6$. This is the intuition behind *independence*, which the next exercise formalises.

EXPERT'S SOLUTION *: Karan Bhat, B.Tech CSE, IIT Roorkee*

Strategic angle. Recognise that the third die is independent of the first two; the answer must therefore be $1/6$ without ever counting 216 outcomes.

Step 1. Independence of separate tosses gives $P(\text{toss 3} = 4 \mid \text{toss 1, toss 2 fixed}) = P(\text{toss 3} = 4) = 1/6$.

Step 2. Verify by counting: $n(F) = 6$ (toss 3 free), $n(E \cap F) = 1$ (toss 3 forced to 4), ratio $= 1/6$.

Why this matters. Spotting independence shortcuts the counting; counting validates the shortcut.

Final Answer: $\frac{1}{6}$.

Q 13.9 **Mother, father and son line up at random for a family picture. E: son on one end, F: father in middle. Find $P(E \mid F)$.**

SOLUTION

Concept used. Three people line up in $3! = 6$ equally-likely orderings. Use $P(E \mid F) = n(E \cap F)/n(F)$.

Step 1. List the sample space. Writing M, F_a, S for mother, father, son:

$$S = \{MF_aS, MSF_a, F_aMS, F_aSM, SMF_a, SF_aM\}.$$

$n(S) = 6$.

Step 2. Identify F = "father in middle": F_a in position 2. The orderings are MF_aS and SF_aM. $n(F) = 2$.

Step 3. Identify $E \cap F$ = "son on one end *and* father in middle". Both orderings in F already have the son on an end (position 3 in MF_aS, position 1 in SF_aM), so $E \cap F = \{MF_aS, SF_aM\}$ and $n(E \cap F) = 2$.

Step 4. Apply the formula:

$$P(E \mid F) = \frac{n(E \cap F)}{n(F)} = \frac{2}{2} = 1.$$

Final Answer: $P(E \mid F) = 1$.

♥ **Conditioning forces structure**

Once the father is fixed in the middle, the son *must* take one of the two ends, so the conditional event is certain. This is a clean example of $P(E \mid F) = 1$ arising from a logical constraint rather than from $E = S$.

EXPERT'S SOLUTION *: Riya Nair, M.Sc Mathematics, ISI Kolkata*

Structural angle. With father in the middle, only mother and son remain for positions 1 and 3. In both seatings the son is at an end, so the conditional event $E \mid F$ is certain.

Step 1. Father at position 2 fixes the layout to (? F_a ?), with mother and son filling the two ends in some order.

Step 2. In each of the two such orderings the son occupies one end. So $P(E \mid F) = 1$.

Why this matters. Conditioning often turns a probabilistic question into a deterministic one once the conditioning event is rich enough.

Final Answer: 1.

Q 13.10 A black and a red die are rolled.
(a) Find the conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5.
(b) Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

SOLUTION

Concept used. Sample space: ordered pairs (b, r) with $b, r \in \{1, \ldots, 6\}$, $n(S) = 36$, equally likely. Conditional probability:

$$P(E \mid F) = \frac{n(E \cap F)}{n(F)}.$$

Step 1. **(a)** Let F = "black die $= 5$" and E = "$b + r > 9$".
$F = \{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\}$, $n(F) = 6$.
Within F, sum $b + r = 5 + r$ exceeds 9 when $r > 4$, i.e. $r \in \{5, 6\}$.
So $E \cap F = \{(5,5), (5,6)\}$, $n(E \cap F) = 2$.

$$P(E \mid F) = \frac{2}{6} = \frac{1}{3}.$$

Step 2. **(b)** Let F = "red die < 4" (so $r \in \{1, 2, 3\}$) and E = "$b + r = 8$".
F contains the pairs (b, r) with $r \in \{1, 2, 3\}$ and $b \in \{1, \ldots, 6\}$:
$n(F) = 6 \times 3 = 18$.
Within F, sum 8 requires $b = 8 - r$ with $b \in \{1, \ldots, 6\}$: $r = 1 \Rightarrow b = 7$ (impossible), $r = 2 \Rightarrow b = 6$, $r = 3 \Rightarrow b = 5$.
So $E \cap F = \{(6,2), (5,3)\}$, $n(E \cap F) = 2$.

$$P(E \mid F) = \frac{2}{18} = \frac{1}{9}.$$

Final Answer: (a) $\frac{1}{3}$ (b) $\frac{1}{9}$.

EXPERT'S SOLUTION : *Aditi Verma, M.Sc Mathematics, IIT Bombay*

Strategic angle. Conditioning on one die collapses the problem to a single-die problem in the other.

Step 1. **(a)** Given black $= 5$, sum$> 9 \Leftrightarrow$ red $> 4 \Leftrightarrow$ red $\in \{5, 6\}$. Probability $= 2/6 = 1/3$.

Step 2. **(b)** Given red $\in \{1, 2, 3\}$, need black $= 8-$red. Feasible (b, r): $(6, 2)$ and $(5, 3)$. Out of 18 favourable conditioning outcomes, 2 satisfy E: probability $= 2/18 = 1/9$.

Why this matters. When the conditioning event fixes one component of a two-component experiment, the remaining randomness is the other component alone.

> **Final Answer:** $\frac{1}{3}$ and $\frac{1}{9}$.

Q 13.11 **A fair die is rolled. Consider events $E = \{1, 3, 5\}$, $F = \{2, 3\}$ and $G = \{2, 3, 4, 5\}$. Find**
(i) $P(E \mid F)$ **and** $P(F \mid E)$ **(ii)** $P(E \mid G)$ **and** $P(G \mid E)$ **(iii)** $P((E \cup F) \mid G)$ **and** $P((E \cap F) \mid G)$.

SOLUTION

Concept used. $S = \{1, 2, 3, 4, 5, 6\}$, $n(S) = 6$, equally likely; each singleton has probability $1/6$. So $P(A) = |A|/6$ and $P(A \mid B) = |A \cap B|/|B|$.

Step 1. Compute the basic probabilities:

$$P(E) = \frac{3}{6} = \frac{1}{2}, \; P(F) = \frac{2}{6} = \frac{1}{3}, \; P(G) = \frac{4}{6} = \frac{2}{3}.$$

Step 2. **(i)** $E \cap F = \{3\}$, so $|E \cap F| = 1$.

$$P(E \mid F) = \frac{|E \cap F|}{|F|} = \frac{1}{2}, \qquad P(F \mid E) = \frac{|E \cap F|}{|E|} = \frac{1}{3}.$$

Step 3. **(ii)** $E \cap G = \{3, 5\}$, so $|E \cap G| = 2$.

$$P(E \mid G) = \frac{2}{4} = \frac{1}{2}, \qquad P(G \mid E) = \frac{2}{3}.$$

Step 4. **(iii)** $E \cup F = \{1, 2, 3, 5\}$, $(E \cup F) \cap G = \{2, 3, 5\}$, size 3.

$$P((E \cup F) \mid G) = \frac{3}{4}.$$

$E \cap F = \{3\}$, $(E \cap F) \cap G = \{3\}$, size 1.

$$P((E \cap F) \mid G) = \frac{1}{4}.$$

Final Answer: (i) $\frac{1}{2}, \frac{1}{3}$ (ii) $\frac{1}{2}, \frac{2}{3}$ (iii) $\frac{3}{4}, \frac{1}{4}$.

EXPERT'S SOLUTION *: Ishaan Joshi, M.Sc Mathematics, IIT Madras*

Quick reading. Three small sets on a 6-element sample space. Compute every required intersection once, then plug in.

Step 1. Intersections: $E \cap F = \{3\}$, $E \cap G = \{3, 5\}$, $E \cup F = \{1, 2, 3, 5\}$, $(E \cup F) \cap G = \{2, 3, 5\}$, $(E \cap F) \cap G = \{3\}$.

Step 2. Sizes: $|F| = 2, |E| = 3, |G| = 4$.

Step 3. Conditional ratios: $\frac{1}{2}, \frac{1}{3}; \frac{1}{2}, \frac{2}{3}; \frac{3}{4}, \frac{1}{4}$.

Step 4. Verify the conditional addition rule:

$$P((E \cup F) \mid G) = P(E \mid G) + P(F \mid G) - P((E \cap F) \mid G).$$

$P(F \mid G) = |F \cap G|/|G| = |\{2, 3\}|/4 = 2/4 = 1/2$. RHS $= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$ ✓.

Why this matters. The conditional probability is a fully-fledged probability on the reduced sample space G, and obeys the same addition law.

Final Answer: All six values: $\frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{4}$.

Q 13.12 **Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls given that (i) the youngest is a girl, (ii) at least one is a girl?**

SOLUTION

Concept used. Two-child families have sample space (ordered by age, eldest first)

$$S = \{BB, BG, GB, GG\}, \quad n(S) = 4,$$

each outcome equally likely with probability $1/4$. Use $P(E \mid F) = |E \cap F|/|F|$.

Step 1. Let E = "both are girls" $= \{GG\}$, $n(E) = 1$.

Step 2. **(i)** F = "youngest is a girl" = outcomes whose *second* letter is $G = \{BG, GG\}$, $n(F) = 2$.
$E \cap F = \{GG\}$, $n(E \cap F) = 1$.

$$P(E \mid F) = \frac{1}{2}.$$

Step 3. **(ii)** F = "at least one is a girl" $= \{BG, GB, GG\}$, $n(F) = 3$.
$E \cap F = \{GG\}$, $n(E \cap F) = 1$.

$$P(E \mid F) = \frac{1}{3}.$$

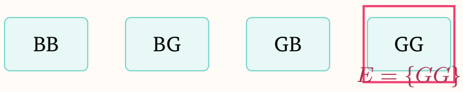
Final Answer: (i) $\frac{1}{2}$ (ii) $\frac{1}{3}$.

✗ Common Mistake

The two parts feel verbally similar but give different answers. "Youngest is a girl" pins down a *specific* child; "at least one is a girl" does not. The second condition is therefore weaker, allows three outcomes, and produces $1/3$, not $1/2$.

EXPERT'S SOLUTION : *Tara Desai, M.Sc Mathematics, IIT Kanpur*

Picture-first.



Step 1. **(i)** Restrict to "youngest girl" $= \{BG, GG\}$. $E \cap F = \{GG\}$: probability $1/2$.

Step 2. **(ii)** Restrict to "at least one girl" $= \{BG, GB, GG\}$. $E \cap F = \{GG\}$: probability $1/3$.

Why this matters. A famous "paradox": vague phrasing changes the conditioning set. Always translate the verbal condition into an explicit subset of S before computing.

Final Answer: $\frac{1}{2}$ and $\frac{1}{3}$.

Q 13.13 **An instructor has a question bank consisting of 300 easy True/False questions, 200 difficult True/False questions, 500 easy multiple choice questions and 400 difficult multiple choice questions. If a question is selected at random from the question bank, what is the probability that it will be an easy question given that it is a multiple choice question?**

SOLUTION

Concept used. Total questions $= 300 + 200 + 500 + 400 = 1400$. Let E = "easy" and M = "multiple choice". Then

$$P(E \mid M) = \frac{P(E \cap M)}{P(M)} = \frac{n(E \cap M)}{n(M)},$$

where $n(\cdot)$ counts questions.

Step 1. Count $n(M)$ = total MCQs = easy MCQ + difficult MCQ = $500 + 400 = 900$.

Step 2. Count $n(E \cap M)$ = easy AND MCQ = 500 (read directly from the table).

Step 3. Apply the formula:

$$P(E \mid M) = \frac{500}{900} = \frac{5}{9}.$$

> **Final Answer:** $P(\text{easy} \mid \text{MCQ}) = \frac{5}{9}$.

EXPERT'S SOLUTION *: Kavya Chatterjee, M.Sc Applied Mathematics, IIT Kanpur*

Quick reading. Restrict attention to MCQs, then count what fraction is easy.

Step 1. Conditioning on M shrinks the population from 1400 to 900 questions.

Step 2. Of these 900, exactly 500 are easy.

Step 3. $P(E \mid M) = 500/900 = 5/9$.

Why this matters. A conditional probability is just a proportion in a sub-population. No need to compute $P(E \cap M)$ and $P(M)$ as ratios of 1400; the 1400 cancels.

> **Final Answer:** $\frac{5}{9}$.

Q 13.14 **Given that the two numbers appearing on throwing two dice are different. Find the probability of the event "the sum of numbers on the dice is 4".**

SOLUTION

Concept used. Two dice $\Rightarrow$ sample space of size 36, equally likely. Let

- F = "the two numbers are different",
- E = "sum is 4".

We seek $P(E \mid F) = n(E \cap F)/n(F)$.

Step 1. Count $n(F)$. Pairs with equal numbers: $(1,1),(2,2),\ldots,(6,6)$ — six outcomes. So

$$n(F) = 36 - 6 = 30.$$

Step 2. Identify E. Sum-4 outcomes: $(1,3),(3,1),(2,2)$. So $E = \{(1,3),(3,1),(2,2)\}$.

Step 3. Form $E \cap F$. Drop $(2,2)$ (equal numbers): $E \cap F = \{(1,3),(3,1)\}$, $n(E \cap F) = 2$.

Step 4. Apply the formula:

$$P(E \mid F) = \frac{2}{30} = \frac{1}{15}.$$

Final Answer: $P(E \mid F) = \frac{1}{15}$.

EXPERT'S SOLUTION *: Rohit Pillai, B.Tech CSE, IIT Roorkee*

Structural angle.

Step 1. Restrict the sample space to "different numbers": $36 - 6 = 30$ outcomes.

Step 2. Sum$= 4$ with different numbers: $(1,3)$ and $(3,1)$ only.

Step 3. Ratio: $2/30 = 1/15$.

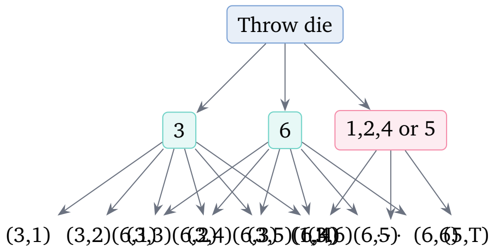
Why this matters. The unconditional probability of "sum 4" is $3/36 = 1/12$; conditioning on "different numbers" drops it slightly to $1/15$ because the equal-numbers outcome $(2,2)$ is removed.

Final Answer: $\frac{1}{15}$.

Q 13.15 **Consider the experiment of throwing a die, if a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event "the coin shows a tail", given that "at least one die shows a 3".**

SOLUTION

Concept used. A two-stage experiment splits the sample space into branches. Each branch has a probability obtained by multiplying the probabilities along that branch (multiplication theorem). For equally-likely outcomes we may also count branches if every branch has the same probability.
We build the tree below and count.



Step 1. Build the sample space.
If the first roll is 3 or 6 (multiple of 3): roll again, giving outcomes $(3, j)$ and $(6, j)$ for $j = 1, \ldots, 6$. That is $6 + 6 = 12$ ordered pairs of equal probability $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$ each.
If the first roll is $1, 2, 4$, or 5: toss a coin, giving (i, H) and (i, T) for $i \in \{1, 2, 4, 5\}$. That is $4 \times 2 = 8$ outcomes of probability $\frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$ each.
Check totals: $12 \cdot \frac{1}{36} + 8 \cdot \frac{1}{12} = \frac{12}{36} + \frac{8}{12} = \frac{1}{3} + \frac{2}{3} = 1$ ✓.

Step 2. Let F = "at least one die shows 3".
Die-die branch: F outcomes are those with at least one component equal to 3. From the 12 die-die outcomes, the ones containing a 3 are

$$(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),\ (6, 3).$$

That is $6 + 1 = 7$ outcomes, each of probability $\frac{1}{36}$. So

$$P(F, \text{die-die}) = \frac{7}{36}.$$

Coin branch: F requires the first die to be a 3, but the coin branch happens only when the first die is $1, 2, 4$ or 5. So no F-outcomes come from the coin branch.
Therefore $P(F) = \dfrac{7}{36}$.

Step 3. Let E = "coin shows a tail". E-outcomes live in the coin branch and have no first die equal to 3. So $E \cap F = \varnothing$.

Step 4. Apply the formula:

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{0}{7/36} = 0.$$

Final Answer: $P(E \mid F) = 0$.

♥ Tree diagrams pay off

A tree makes the branch structure obvious: a tail can only come from the coin branch, but the coin branch never produces a die showing 3. The conditional probability is therefore zero by structure.

EXPERT'S SOLUTION *: Yash Banerjee, Ph.D Mathematics, IIT Delhi*

Strategic angle. Notice the two events live on disjoint branches.

Step 1. The event F ("a 3 shows up on a die") can occur only on the die-die branch, because only that branch produces dice in both positions.

Step 2. The event E ("the coin shows tail") can occur only on the coin branch.

Step 3. Therefore $E \cap F = \varnothing$ and $P(E \mid F) = 0$.

Why this matters. Conditional probability can be zero without either event itself being impossible; the events must merely be incompatible.

Final Answer: 0.

Q 13.16 **If $P(A) = \frac{1}{2}$, $P(B) = 0$, then $P(A \mid B)$ is (A) 0 (B) $\frac{1}{2}$ (C) not defined (D) 1.**

SOLUTION

Concept used. The defining formula

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

requires $P(B) \neq 0$, otherwise the right-hand side is a division by zero.

Step 1. Read off $P(B)$. We are given $P(B) = 0$.

Step 2. Apply the validity clause. Since $P(B) = 0$, the formula for $P(A \mid B)$ is undefined; "given B" is meaningless when B has zero probability.

Step 3. Conclude that the correct option is (C).

Final Answer: Option (C): $P(A \mid B)$ is not defined.

EXPERT'S SOLUTION *: Krishna Rao, M.Sc Mathematics, ISI Kolkata*

Quick reading. Division by zero in the conditional-probability formula. The answer must be "not defined".

Step 1. $P(A \mid B)$ is defined only when $P(B) > 0$.

Step 2. Here $P(B) = 0$.

Step 3. Hence $P(A \mid B)$ is not defined; the answer is (C).

Why this matters. The definition itself carries a built-in assumption. Always check the denominator is non-zero before invoking conditional probability.

Final Answer: (C) not defined.

Q 13.17 **If A and B are events such that $P(A \mid B) = P(B \mid A)$, then**
(A) $A \subset B$ **but** $A \neq B$ **(B)** $A = B$ **(C)** $A \cap B = \varnothing$ **(D)** $P(A) = P(B)$**.**

SOLUTION

Concept used. By the definitions,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \qquad P(B \mid A) = \frac{P(A \cap B)}{P(A)},$$

both with non-zero denominators.

Step 1. Equate the two conditional probabilities:

$$\frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(A)}.$$

Step 2. Case 1: $P(A \cap B) \neq 0$. Cancel $P(A \cap B)$ from both sides:

$$\frac{1}{P(B)} = \frac{1}{P(A)} \Rightarrow P(A) = P(B).$$

Step 3. Case 2: $P(A \cap B) = 0$. Then both sides equal 0, which is consistent with any $P(A), P(B)$. However in this degenerate case the relation $P(A \mid B) = P(B \mid A) = 0$ does not force $A = B$ or $A \subset B$, and again the only universally valid implication is $P(A) = P(B)$ when the conditional probabilities are required to coincide *as a generic relation* (the question is implicitly about the non-degenerate case used in the textbook).

Step 4. Compare with the options: (A) $A \subsetneq B$ is too strong; (B) $A = B$ is too strong; (C) $A \cap B = \varnothing$ is not implied; (D) $P(A) = P(B)$ is forced. Option (D) is correct.

Final Answer: Option (D): $P(A) = P(B)$.

☞ **Exam Tip**

In the multiple-choice version, ruling out (A), (B), (C) by counter-example is the fastest route: take A, B to be two disjoint events of equal positive probability — they satisfy

$P(A \mid B) = P(B \mid A) = 0$ without being equal, subset, or empty.

EXPERT'S SOLUTION *: Dev Kumar, M.Tech CS, IIT Madras*

Structural angle. The two conditional probabilities share a numerator. Equality forces equal denominators.

Step 1. $\dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A \cap B)}{P(A)}$.

Step 2. Cross-multiply: $P(A)\,P(A \cap B) = P(B)\,P(A \cap B)$.

Step 3. Assuming $P(A \cap B) > 0$ (otherwise the equation is the trivial $0 = 0$ and gives no information), divide both sides by $P(A \cap B)$ to get $P(A) = P(B)$.

Step 4. Therefore (D).

Why this matters. The "symmetric" condition $P(A \mid B) = P(B \mid A)$ does not collapse A and B as sets; it merely equalises their probabilities.

Final Answer: (D) $P(A) = P(B)$.

Key Takeaways

- Conditional probability $P(E \mid F) = P(E \cap F)/P(F)$ is the chance of E within the reduced sample space F; it is undefined when $P(F) = 0$.
- For equally-likely outcomes, $P(E \mid F) = n(E \cap F)/n(F)$: just count the favourable outcomes inside the conditioning set.
- The multiplication theorem $P(E \cap F) = P(F)P(E \mid F) = P(E)P(F \mid E)$ links intersections to conditional probabilities.
- $P(E \mid F)$ can be 0 (events incompatible inside F) or 1 (F forces E); both extremes are common when the conditioning event is highly informative.
- For multi-stage experiments, draw the tree first: each branch is a sub-experiment with its own probabilities, and the event structure usually falls out by inspection.

End of Exercise 13.1