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Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 13: Probability

About this Chapter

Exercise 13.3 is the heart of the chapter. It applies the **theorem of total probability** (which decomposes an event's probability over a partition of the sample space) and **Bayes' theorem** (which inverts conditional probability to find a "cause" given an "effect"). Tree diagrams are the natural drawing tool; every question in this exercise has a clean tree.

Topics covered: Partitioning a sample space • Theorem of total probability • Bayes' theorem • Priors and posteriors

Quick Formula Sheet

Theorem of total probability:

If $\{E_1, \ldots, E_n\}$ partitions S,

$P(A) = \sum_{j=1}^{n} P(E_j)\, P(A \mid E_j)$.

Bayes' theorem:

$P(E_i \mid A) = \frac{P(E_i)\, P(A \mid E_i)}{\sum_j P(E_j)\, P(A \mid E_j)}$.

Terminology:

$P(E_i)$ = prior; $P(E_i \mid A)$ = posterior;

the E_i are the "hypotheses" or "causes".

Exercise 13.3

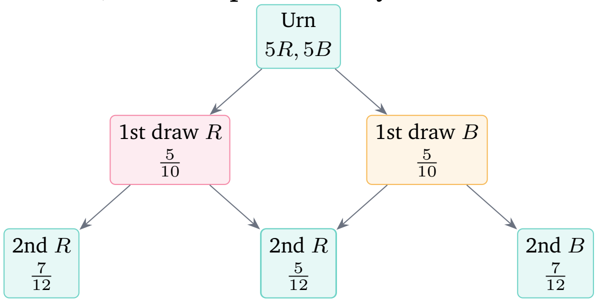
Q 13.1 **An urn contains 5 red and 5 black balls. A ball is drawn at random, its colour is noted and is returned to the urn. Moreover, 2 additional balls of the colour drawn are put in the urn and then a ball is drawn at random. What is the probability that the second ball is red?**

SOLUTION

Concept used. The **theorem of total probability** states that for a partition $\{E_1, E_2\}$ of the sample space,

$$P(A) = P(E_1)\, P(A \mid E_1) + P(E_2)\, P(A \mid E_2).$$

Here A = "second ball is red", and we partition by the colour of the first ball.



Step 1. Define events.

E_1: first ball drawn is red. E_2: first ball drawn is black.

A: second ball drawn is red.

Step 2. Compute the priors.

$P(E_1) = \frac{5}{10} = \frac{1}{2}, \quad P(E_2) = \frac{5}{10} = \frac{1}{2}.$

Step 3. Compute the conditional probabilities.

If E_1 occurred: the first red ball is returned and 2 extra red balls are added, so the urn now has $5 + 2 = 7$ red and 5 black, total 12 balls.

$$P(A \mid E_1) = \frac{7}{12}.$$

If E_2 occurred: the urn now has 5 red and $5 + 2 = 7$ black, total 12.

$$P(A \mid E_2) = \frac{5}{12}.$$

Step 4. Apply total probability:

$$P(A) = \frac{1}{2} \times \frac{7}{12} + \frac{1}{2} \times \frac{5}{12} = \frac{7}{24} + \frac{5}{24} = \frac{12}{24} = \frac{1}{2}.$$

Final Answer: $P(\text{2nd ball red}) = \frac{1}{2}.$

♥ Symmetry insight

By symmetry between red and black, the probability that the second draw is red equals the probability it is black. Since these two probabilities sum to 1, each is $1/2$. The detailed total-probability computation simply confirms this.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Bombay*

Strategic angle. Set up the tree, multiply along each "second ball red" branch, add.

Step 1. Branch $E_1 \to A$: $(1/2)(7/12) = 7/24$.

Step 2. Branch $E_2 \to A$: $(1/2)(5/12) = 5/24$.

Step 3. Sum: $12/24 = 1/2$.

Why this matters. A two-row tree is the cleanest sketch for two-stage problems. Each path probability is the product of edge probabilities along it.

> **Final Answer:** $\frac{1}{2}$.

Q 13.2 **A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.**

SOLUTION

Concept used. Bayes' theorem: with priors $P(E_i)$ and likelihoods $P(A \mid E_i)$,

$$P(E_i \mid A) = \frac{P(E_i)\,P(A \mid E_i)}{\sum_j P(E_j)\,P(A \mid E_j)}.$$

Step 1. Define events.
E_1: bag I selected (4R, 4B). E_2: bag II selected (2R, 6B). A: red drawn.

Step 2. Priors and likelihoods.
$P(E_1) = P(E_2) = \frac{1}{2}$ (chosen at random).
$P(A \mid E_1) = \frac{4}{8} = \frac{1}{2}$. $P(A \mid E_2) = \frac{2}{8} = \frac{1}{4}$.

Step 3. Compute the denominator (total probability of red):

$$P(A) = P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{4} + \frac{1}{8} = \frac{2}{8} + \frac{1}{8} = \frac{3}{8}.$$

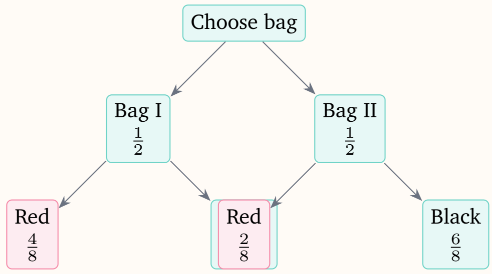
Step 4. Apply Bayes':

$$P(E_1 \mid A) = \frac{P(E_1)\,P(A \mid E_1)}{P(A)} = \frac{1/4}{3/8} = \frac{1}{4} \times \frac{8}{3} = \frac{2}{3}.$$

> **Final Answer:** $P(\text{first bag} \mid \text{red}) = \frac{2}{3}$.

EXPERT'S SOLUTION : *Sneha Reddy, Ph.D Mathematics, IIT Delhi*

Picture-first.



Step 1. Path "Bag I, Red": probability $(1/2)(1/2) = 1/4$.

Step 2. Path "Bag II, Red": probability $(1/2)(1/4) = 1/8$.

Step 3. Bayes' posterior = path 1 over (path 1 + path 2) $= \dfrac{1/4}{1/4 + 1/8} = \dfrac{1/4}{3/8} = \dfrac{2}{3}$.

Why this matters. On a tree, Bayes' theorem is the simple recipe "favourable path probability ÷ all-paths-to-this-effect probability."

Final Answer: $\dfrac{2}{3}$.

Q 13.3 **Of the students in a college, it is known that 60% reside in hostel and 40% are day scholars (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain A grade and 20% of day scholars attain A grade in their annual examination. At the end of the year, one student is chosen at random from the college and he has an A grade, what is the probability that the student is a hostlier?**

SOLUTION

Concept used. Bayes' theorem with two hypotheses (hostler / day scholar) and the event A = "A grade".

Step 1. Define events.

H: hostler. D: day scholar. A: A grade.

Step 2. Priors.

$P(H) = 0.6,\ P(D) = 0.4.$

Step 3. Likelihoods.

$P(A \mid H) = 0.30,\ P(A \mid D) = 0.20.$

Step 4. Total probability:

$$P(A) = P(H)P(A \mid H) + P(D)P(A \mid D) = 0.6\times0.3 + 0.4\times0.2 = 0.18 + 0.08 = 0.26.$$

Step 5. Bayes':

$$P(H \mid A) = \frac{P(H)P(A \mid H)}{P(A)} = \frac{0.18}{0.26} = \frac{18}{26} = \frac{9}{13}.$$

Final Answer: $P(\text{hostler} \mid \text{A grade}) = \frac{9}{13}$.

☞ **Exam Tip**

Express percentages as decimals before substituting; mixing 60 and 0.30 in the same formula is a common error.

EXPERT'S SOLUTION *: Vivaan Iyer, M.Tech CS, IIT Madras*

Quick reading.

Step 1. Numerator: $0.6 \times 0.3 = 0.18$.

Step 2. Denominator: $0.18 + 0.4 \times 0.2 = 0.18 + 0.08 = 0.26$.

Step 3. Ratio: $18/26 = 9/13$.

Why this matters. Posterior $>$ prior here ($9/13 \approx 0.692 > 0.6$). Observing the A-grade signal makes "hostler" more probable, because hostlers have a higher A-grade rate.

Final Answer: $\frac{9}{13}$.

Q 13.4 In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{4}$. What is the probability that the student knows the answer given that he answered it correctly?

SOLUTION

Concept used. Bayes' theorem; two hypotheses (knows / guesses); the observed event is "correct".

Step 1. Events: K = knows, G = guesses, C = answers correctly.

Step 2. Priors: $P(K) = \frac{3}{4}$, $P(G) = \frac{1}{4}$.

Step 3. Likelihoods: $P(C \mid K) = 1$ (knowing implies correct), $P(C \mid G) = \frac{1}{4}$.

Step 4. Total probability of correct:

$$P(C) = P(K)P(C \mid K) + P(G)P(C \mid G) = \frac{3}{4} \cdot 1 + \frac{1}{4} \cdot \frac{1}{4} = \frac{3}{4} + \frac{1}{16} = \frac{12}{16} + \frac{1}{16} = \frac{13}{16}.$$

Step 5. Bayes':

$$P(K \mid C) = \frac{P(K)P(C \mid K)}{P(C)} = \frac{3/4}{13/16} = \frac{3}{4} \times \frac{16}{13} = \frac{12}{13}.$$

Final Answer: $P(\text{knows} \mid \text{correct}) = \frac{12}{13}$.

♥ Why correctness is so diagnostic

The likelihood of correctness given "knows" (1) is four times the likelihood given "guesses" $(1/4)$. Combined with the already-high prior $P(K) = 3/4$, the posterior shoots up to $12/13$. Bayes' theorem makes this intuition precise.

EXPERT'S SOLUTION : *Pranav Bhat, M.Sc Mathematics, IIT Bombay*

Strategic angle. Two paths to "correct" on the tree; pick the favourable one over their sum.

Step 1. Favourable path ($K \to C$): $(3/4)(1) = 3/4 = 12/16$.

Step 2. Other correct path ($G \to C$): $(1/4)(1/4) = 1/16$.

Step 3. Posterior: $\frac{12/16}{12/16 + 1/16} = \frac{12}{13}$.

Why this matters. For binary hypotheses with one likelihood $= 1$, the answer simplifies dramatically; the result is always close to the prior of the certain branch.

Final Answer: $\frac{12}{13}$.

Q 13.5 **A laboratory blood test is 99% effective in detecting a certain disease when it is in fact present. However, the test also yields a false positive result for 0.5% of the healthy person tested (i.e. if a healthy person is tested, then, with probability 0.005, the test will imply he has the disease). If 0.1 percent of the population actually has the disease, what is the probability that a person has the disease given that his test result is positive?**

SOLUTION

Concept used. Bayes' theorem with disease/no-disease hypotheses; event $+$ = test positive.

Step 1. Events: D = has disease, D' = healthy, $+$ = test positive.

Step 2. Priors: $P(D) = 0.1\% = 0.001,\ P(D') = 1 - 0.001 = 0.999.$

Step 3. Likelihoods: $P(+ \mid D) = 99\% = 0.99$ (true positive rate), $P(+ \mid D') = 0.5\% = 0.005$ (false positive rate).

Step 4. Total probability of a positive test:

$$P(+) = P(D)P(+ \mid D) + P(D')P(+ \mid D') = 0.001(0.99) + 0.999(0.005).$$

Compute the two pieces.
$0.001 \times 0.99 = 0.00099.$
$0.999 \times 0.005 = 0.004995.$
Sum: $0.00099 + 0.004995 = 0.005985.$

Step 5. Bayes':

$$P(D \mid +) = \frac{0.00099}{0.005985}.$$

Multiply numerator and denominator by 100000 to clear decimals:

$$= \frac{99}{598.5} = \frac{198}{1197} = \frac{22}{133}.$$

(Divide numerator and denominator by 9: $198 \div 9 = 22$, $1197 \div 9 = 133$.)
Numerically, $22/133 \approx 0.1654$.

Final Answer: $P(\text{disease} \mid +) = \dfrac{22}{133} \approx 0.165.$

✗ Common Mistake

A high test sensitivity of 99% does NOT mean a positive test implies 99% chance of disease. With a rare disease (prior 0.001) and a non-zero false positive rate, the posterior is only $\sim 16.5\%$. This is the famous *base-rate fallacy*.

EXPERT'S SOLUTION *: Aanya Pillai, M.Sc Mathematics, ISI Kolkata*

Strategic angle. Track absolute numbers in an imaginary population of $1{,}000{,}000$:

- Truly diseased: 1000. Test positive: $1000 \times 0.99 = 990$.
- Healthy: 999000. Test positive (false): $999000 \times 0.005 = 4995$.
- Total positives: $990 + 4995 = 5985$.

- True positives among them: 990.

Step 1. Posterior $P(D \mid +) = \dfrac{990}{5985}$.

Step 2. Simplify: divide top and bottom by 45: $\dfrac{22}{133}$.

Why this matters. Frequency reasoning (counts in a hypothetical population) is the easiest mental model for Bayes' theorem; it sidesteps decimal arithmetic.

> **Final Answer:** $\dfrac{22}{133} \approx 0.165$.

Q 13.6 **There are three coins. One is a two headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the time and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows heads, what is the probability that it was the two headed coin?**

SOLUTION

Concept used. Bayes' theorem with three coin-hypotheses.

Step 1. Events: C_1 = two-headed coin, C_2 = biased coin (P(H)$=0.75$), C_3 = unbiased coin (P(H)$=0.5$), H = heads observed.

Step 2. Priors: $P(C_1) = P(C_2) = P(C_3) = \dfrac{1}{3}$ (uniformly chosen).

Step 3. Likelihoods: $P(H \mid C_1) = 1,\ P(H \mid C_2) = \dfrac{3}{4},\ P(H \mid C_3) = \dfrac{1}{2}$.

Step 4. Total probability:

$$P(H) = \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot\frac{3}{4} + \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{3}\left(1 + \frac{3}{4} + \frac{1}{2}\right) = \frac{1}{3}\cdot\frac{4+3+2}{4} = \frac{1}{3}\cdot\frac{9}{4} = \frac{9}{12} = \frac{3}{4}.$$

Step 5. Bayes':

$$P(C_1 \mid H) = \frac{P(C_1)P(H \mid C_1)}{P(H)} = \frac{(1/3)\cdot 1}{3/4} = \frac{1}{3}\times\frac{4}{3} = \frac{4}{9}.$$

> **Final Answer:** $P(\text{two-headed} \mid H) = \dfrac{4}{9}$.

EXPERT'S SOLUTION : *Arjun Verma, M.Sc Applied Mathematics, IIT Kanpur*

Quick reading.

Step 1. Numerator: $(1/3)(1) = 1/3$.

Step 2. Denominator: $(1/3)(1 + 3/4 + 1/2) = (1/3)(9/4) = 3/4$.

Step 3. Ratio: $(1/3) \div (3/4) = 4/9$.

Why this matters. Equal priors $\Rightarrow$ posterior simplifies to likelihood$_i$ / $\sum$ likelihoods.

Final Answer: $\frac{4}{9}$.

Q 13.7 **An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?**

SOLUTION

Concept used. Bayes' theorem with three hypotheses.

Step 1. Events: S = scooter driver, C = car driver, T = truck driver, A = accident.

Step 2. Priors. Total drivers $= 2000 + 4000 + 6000 = 12000$.

$$P(S) = \frac{2000}{12000} = \frac{1}{6},\ P(C) = \frac{4000}{12000} = \frac{1}{3},\ P(T) = \frac{6000}{12000} = \frac{1}{2}.$$

Step 3. Likelihoods: $P(A \mid S) = 0.01,\ P(A \mid C) = 0.03,\ P(A \mid T) = 0.15$.

Step 4. Total probability:

$$P(A) = \frac{1}{6}(0.01) + \frac{1}{3}(0.03) + \frac{1}{2}(0.15).$$

Compute each term: $\frac{0.01}{6} = \frac{1}{600}$; $\frac{0.03}{3} = \frac{1}{100} = \frac{6}{600}$; $\frac{0.15}{2} = \frac{0.075}{1} = \frac{45}{600}$.
Sum: $\frac{1+6+45}{600} = \frac{52}{600} = \frac{13}{150}$.

Step 5. Bayes':

$$P(S \mid A) = \frac{(1/6)(0.01)}{13/150} = \frac{1/600}{13/150} = \frac{1}{600} \times \frac{150}{13} = \frac{150}{600 \times 13} = \frac{1}{52}.$$

Final Answer: $P(\text{scooter} \mid \text{accident}) = \frac{1}{52}$.

EXPERT'S SOLUTION : *Aditi Singh, Ph.D Mathematics, IIT Delhi*

Frequency angle. Among the 12000 drivers, expected accidents are

- Scooter: $2000 \times 0.01 = 20$.
- Car: $4000 \times 0.03 = 120$.
- Truck: $6000 \times 0.15 = 900$.

Total expected accidents $= 20 + 120 + 900 = 1040$.

Step 1. $P(S \mid A) = 20/1040 = 1/52$.

Why this matters. Frequencies bypass fraction arithmetic entirely. For Bayes' problems with given counts, this is almost always the fastest route.

Final Answer: $\frac{1}{52}$.

Q 13.8 **A factory has two machines A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Further, 2% of the items produced by machine A and 1% produced by machine B were defective. All the items are put into one stockpile and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by machine B?**

SOLUTION

Concept used. Bayes' theorem; hypotheses = machine of origin; event = defective.

Step 1. Events: M_A, M_B = produced by A, B. D = defective.

Step 2. Priors: $P(M_A) = 0.6,\ P(M_B) = 0.4$.

Step 3. Likelihoods: $P(D \mid M_A) = 0.02,\ P(D \mid M_B) = 0.01$.

Step 4. Total probability of defective:

$$P(D) = 0.6 \times 0.02 + 0.4 \times 0.01 = 0.012 + 0.004 = 0.016.$$

Step 5. Bayes':

$$P(M_B \mid D) = \frac{0.4 \times 0.01}{0.016} = \frac{0.004}{0.016} = \frac{4}{16} = \frac{1}{4}.$$

Final Answer: $P(\text{machine } B \mid \text{defective}) = \frac{1}{4}$.

EXPERT'S SOLUTION : *Riya Mehta, M.Sc Mathematics, IIT Madras*

Quick reading.

Step 1. Numerator: $0.4(0.01) = 0.004$.

Step 2. Denominator: $0.6(0.02) + 0.4(0.01) = 0.012 + 0.004 = 0.016$.

Step 3. Ratio: $0.004/0.016 = 1/4$.

Why this matters. Machine B has a smaller market share *and* a lower defect rate, so its contribution to the defective pile is small; the posterior $1/4$ confirms this.

Final Answer: $\frac{1}{4}$.

Q 13.9 **Two groups are competing for the position on the Board of directors of a corporation. The probabilities that the first and the second groups will win are 0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.7 and the corresponding probability is 0.3 if the second group wins. Find the probability that the new product introduced was by the second group.**

SOLUTION

Concept used. Bayes' theorem with the two groups as hypotheses, event "new product introduced".

Step 1. Events: G_1, G_2 = first / second group wins; N = new product introduced.

Step 2. Priors: $P(G_1) = 0.6,\ P(G_2) = 0.4$.

Step 3. Likelihoods: $P(N \mid G_1) = 0.7,\ P(N \mid G_2) = 0.3$.

Step 4. Total probability:

$$P(N) = 0.6(0.7) + 0.4(0.3) = 0.42 + 0.12 = 0.54.$$

Step 5. Bayes':

$$P(G_2 \mid N) = \frac{0.4(0.3)}{0.54} = \frac{0.12}{0.54} = \frac{12}{54} = \frac{2}{9}.$$

Final Answer: $P(\text{second group} \mid \text{new product}) = \frac{2}{9}$.

EXPERT'S SOLUTION *: Karan Joshi, M.Tech CS, IIT Madras*

Strategic angle. G_2 has both lower prior and lower likelihood; expect a small posterior.

Step 1. Numerator 0.12, denominator 0.54, ratio $2/9 \approx 0.22$.

Why this matters. When both prior and likelihood drop, the posterior drops twice; here, $0.4 \to 2/9 \approx 0.22$.

> **Final Answer:** $\frac{2}{9}$.

Q 13.10 **Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets $1, 2, 3$ or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw $1, 2, 3$ or 4 with the die?**

SOLUTION

Concept used. Bayes' theorem. The hypotheses are determined by the die outcome.

Step 1. Events: E_1 = die shows 5 or 6 (toss coin three times); E_2 = die shows $1, 2, 3$ or 4 (toss coin once); A = exactly one head.

Step 2. Priors: $P(E_1) = 2/6 = 1/3,\ P(E_2) = 4/6 = 2/3.$

Step 3. Likelihoods.
Under E_1: three coin tosses; "exactly one head" means 1 head in 3 tosses, probability $\binom{3}{1}(1/2)^3 = 3/8$. So $P(A \mid E_1) = 3/8$.
Under E_2: one coin toss; "exactly one head" means head, probability $1/2$. So $P(A \mid E_2) = 1/2$.

Step 4. Total probability:

$$P(A) = \frac{1}{3} \cdot \frac{3}{8} + \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{8} + \frac{1}{3}.$$

Common denominator 24: $\frac{3}{24} + \frac{8}{24} = \frac{11}{24}$.

Step 5. Bayes':

$$P(E_2 \mid A) = \frac{P(E_2)P(A \mid E_2)}{P(A)} = \frac{(2/3)(1/2)}{11/24} = \frac{1/3}{11/24} = \frac{1}{3} \times \frac{24}{11} = \frac{8}{11}.$$

> **Final Answer:** $P(\text{die showed } 1, 2, 3 \text{ or } 4 \mid \text{exactly one head}) = \frac{8}{11}$.

♥ Tree branches that differ in shape

The two hypotheses lead to coin experiments of different sizes (3 vs 1). The likelihood of "exactly one head" must be computed branch-by-branch using binomial probabilities, then plugged into Bayes' theorem.

EXPERT'S SOLUTION *: Ishaan Verma, B.Tech CSE, IIT Roorkee*

Strategic angle.

Step 1. Compute $P(\text{exactly one head} \mid 3 \text{ tosses}) = 3/8$ and $P(\text{exactly one head} \mid 1 \text{ toss}) = 1/2$.

Step 2. Numerator: $(2/3)(1/2) = 1/3$.

Step 3. Denominator: $(1/3)(3/8) + (2/3)(1/2) = 1/8 + 1/3 = 11/24$.

Step 4. Ratio: $(1/3) \div (11/24) = 8/11$.

Why this matters. "Exactly one head" is binomial-distribution shorthand; in three tosses it has 3 orderings (HTT, THT, TTH), each of probability $1/8$.

Final Answer: $\frac{8}{11}$.

Q 13.11 **A manufacturer has three machine operators A, B and C. The first operator A produces 1% defective items, whereas the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B is on the job for 30% of the time and C is on the job for 20% of the time. A defective item is produced, what is the probability that it was produced by A?**

SOLUTION

Concept used. Bayes' theorem; three hypotheses (operator A, B or C), event = defective.

Step 1. Priors: $P(A) = 0.5,\ P(B) = 0.3,\ P(C) = 0.2$.

Step 2. Likelihoods: $P(D \mid A) = 0.01,\ P(D \mid B) = 0.05,\ P(D \mid C) = 0.07$.

Step 3. Total probability:

$$P(D) = 0.5(0.01) + 0.3(0.05) + 0.2(0.07) = 0.005 + 0.015 + 0.014 = 0.034.$$

Step 4. Bayes':

$$P(A \mid D) = \frac{0.5 \times 0.01}{0.034} = \frac{0.005}{0.034} = \frac{5}{34}.$$

Final Answer: $P(\text{operator } A \mid \text{defective}) = \dfrac{5}{34}$.

EXPERT'S SOLUTION : *Tara Gupta, M.Sc Mathematics, IIT Bombay*

Frequency angle. In a population of $10{,}000$ items:

- A makes 5000; defective $= 5000(0.01) = 50$.
- B makes 3000; defective $= 3000(0.05) = 150$.
- C makes 2000; defective $= 2000(0.07) = 140$.

Total defective $= 50 + 150 + 140 = 340$.

Step 1. $P(A \mid D) = 50/340 = 5/34$.

Why this matters. A produces by far the most items but at the lowest defect rate; the small posterior ($\approx 15\%$) reflects that the defect signal is more informative about B and C.

Final Answer: $\dfrac{5}{34}$.

Q 13.12 **A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both diamonds. Find the probability of the lost card being a diamond.**

SOLUTION

Concept used. Bayes' theorem with two hypotheses (lost card is diamond / not diamond), event A = two diamonds drawn from the remaining 51 cards.

Step 1. Events: E_1 = lost card is a diamond; E_2 = lost card is not a diamond.

Step 2. Priors: $P(E_1) = 13/52 = 1/4,\ P(E_2) = 39/52 = 3/4$.

Step 3. Likelihoods.

Under E_1: remaining pack has 12 diamonds out of 51 cards. Two-diamond draws (without replacement):

$$P(A \mid E_1) = \frac{\binom{12}{2}}{\binom{51}{2}} = \frac{12 \cdot 11/2}{51 \cdot 50/2} = \frac{66}{1275}.$$

Under E_2: remaining pack has 13 diamonds out of 51 cards.

$$P(A \mid E_2) = \frac{\binom{13}{2}}{\binom{51}{2}} = \frac{13 \cdot 12/2}{51 \cdot 50/2} = \frac{78}{1275}.$$

Step 4. Apply Bayes' theorem. The common denominator $\binom{51}{2} = 1275$ cancels:

$$P(E_1 \mid A) = \frac{P(E_1)P(A \mid E_1)}{P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)} = \frac{(1/4)(66)}{(1/4)(66) + (3/4)(78)}.$$

Multiply numerator and denominator by 4:

$$= \frac{66}{66 + 3 \times 78} = \frac{66}{66 + 234} = \frac{66}{300} = \frac{11}{50}.$$

Final Answer: $P(\text{lost card was diamond} \mid \text{two diamonds drawn}) = \dfrac{11}{50}$.

✗ Common Mistake

The likelihoods $P(A \mid E_1)$ and $P(A \mid E_2)$ are computed from $\binom{51}{2} = 1275$, not $\binom{52}{2}$. Forgetting that the lost card has already been removed before drawing is the most common mistake here.

EXPERT'S SOLUTION *: Dev Banerjee, Ph.D Mathematics, IIT Delhi*

Strategic angle. The two likelihoods share the denominator $\binom{51}{2}$; only the numerators $\binom{12}{2} = 66$ and $\binom{13}{2} = 78$ differ.

Step 1. Cancel the common denominator $\binom{51}{2}$:

$$P(E_1 \mid A) = \frac{(1/4) \cdot 66}{(1/4) \cdot 66 + (3/4) \cdot 78} = \frac{66}{66 + 234} = \frac{66}{300} = \frac{11}{50}.$$

Why this matters. Whenever every likelihood in Bayes' formula carries the same factor, you may cancel it before doing arithmetic.

Final Answer: $\dfrac{11}{50}$.

Q 13.13 **Probability that A speaks truth is $\frac{4}{5}$. A coin is tossed. A reports that a head appears. The probability that actually there was head is**

(A) $\frac{4}{5}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{1}{5}$ **(D)** $\frac{2}{5}$.

SOLUTION

Concept used. Bayes' theorem. Hypotheses: H = head occurred, T = tail occurred. Event R = A reports head.

Step 1. Priors: $P(H) = P(T) = 1/2$ (fair coin).

Step 2. Likelihoods.
If head occurred, A reports head iff A speaks truth: $P(R \mid H) = 4/5$.
If tail occurred, A reports head iff A lies: $P(R \mid T) = 1 - 4/5 = 1/5$.

Step 3. Total probability:

$$P(R) = \frac{1}{2} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{1}{5} = \frac{4}{10} + \frac{1}{10} = \frac{5}{10} = \frac{1}{2}.$$

Step 4. Bayes':

$$P(H \mid R) = \frac{(1/2)(4/5)}{1/2} = \frac{4}{5}.$$

Option (A).

> **Final Answer:** Option (A): $\frac{4}{5}$.

EXPERT'S SOLUTION : *Yash Desai, M.Sc Mathematics, ISI Kolkata*

Quick reading. Because the prior is symmetric (1/2, 1/2), the posterior reduces to $\frac{P(R \mid H)}{P(R \mid H) + P(R \mid T)} = \frac{4/5}{1} = 4/5$.

Step 1. Numerator: $(1/2)(4/5) = 2/5$.

Step 2. Denominator: $(1/2)(4/5) + (1/2)(1/5) = 1/2$.

Step 3. Ratio: $(2/5)/(1/2) = 4/5$.

Why this matters. With symmetric priors, Bayes' theorem reduces to a likelihood ratio. The truth probability $4/5$ then propagates straight to the posterior.

> **Final Answer:** (A) $\frac{4}{5}$.

Q 13.14 **If A and B are two events such that $A \subset B$ and $P(B) \neq 0$, then which of the following is correct?**

(A) $P(A \mid B) = \frac{P(B)}{P(A)}$ **(B)** $P(A \mid B) < P(A)$

(C) $P(A \mid B) \geq P(A)$ **(D) None of these.**

SOLUTION

Concept used. Definition $P(A \mid B) = P(A \cap B)/P(B)$. When $A \subset B$, $A \cap B = A$.

Step 1. Use $A \subset B \Rightarrow A \cap B = A$, hence $P(A \cap B) = P(A)$.

Step 2. Substitute into the conditional formula:

$$P(A \mid B) = \frac{P(A)}{P(B)}.$$

Step 3. Compare $P(A \mid B)$ with $P(A)$. Since $P(B) \leq 1$, we have $\frac{1}{P(B)} \geq 1$, so

$$P(A \mid B) = \frac{P(A)}{P(B)} \geq P(A).$$

Equality holds iff $P(B) = 1$.

Step 4. Option (A) is wrong (the formula is $P(A)/P(B)$, not $P(B)/P(A)$). Option (B) is the opposite inequality. Option (C) $P(A \mid B) \geq P(A)$ is correct.

Final Answer: Option (C): $P(A \mid B) \geq P(A)$.

♥ Conditioning on a containing set increases the probability

If A sits entirely inside B, then knowing B occurred can only raise (or leave unchanged) the chance that A occurred. This is the intuition behind (C).

EXPERT'S SOLUTION *: Kavya Sharma, M.Sc Applied Mathematics, IIT Kanpur*

Quick reading.

Step 1. $A \subset B \Rightarrow A \cap B = A$.

Step 2. Hence $P(A \mid B) = P(A)/P(B)$, and since $P(B) \leq 1$, this is $\geq P(A)$.

Step 3. Answer: (C).

Why this matters. Conditional probability formulas often simplify dramatically under set-inclusion. Spotting $A \subset B$ collapses $P(A \cap B)$ to $P(A)$.

Final Answer: (C).

Key Takeaways

- Total probability splits $P(A)$ over a partition: $P(A) = \sum_j P(E_j)P(A \mid E_j)$.

- Bayes' theorem inverts conditioning: $P(E_i \mid A) = \dfrac{P(E_i)P(A \mid E_i)}{\sum_j P(E_j)P(A \mid E_j)}$.
- Always draw the tree: priors on the first level, likelihoods on the second, multiply along each branch.
- For rare conditions (small prior), even a near-perfect test can give a low posterior: the base-rate fallacy.
- Frequency reasoning in a large imaginary population converts every Bayes' problem into elementary counting.

End of Exercise 13.3