



# Collegedunia NCERT Solutions

Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

## Chapter 1: Relations and Functions

### About this Chapter

Exercise 1.1 builds fluency with the three pillars of every relation on a set: **reflexive**, **symmetric**, and **transitive**. The 16 problems ask you to check each property on a concrete relation, justify failure with one counterexample, and recognise when all three together yield an **equivalence relation**. By the end you can read any relation  $R \subseteq A \times A$  and decide its type from first principles, aligned with the 2026-27 CBSE syllabus.

**Topics covered:** Reflexive • Symmetric • Transitive • Equivalence Relation • Equivalence Classes

#### Quick Formula Sheet

**Reflexive:**

$$(a, a) \in R \text{ for every } a \in A.$$

**Symmetric:**

$$(a, b) \in R \Rightarrow (b, a) \in R \text{ for all } a, b \in A.$$

**Transitive:**

$$(a, b) \in R \text{ and } (b, c) \in R \Rightarrow (a, c) \in R.$$

**Equivalence:**

Reflexive + Symmetric + Transitive.

### Exercise 1.1

**Q 1.1** Determine whether each of the following relations are reflexive, symmetric and transitive:

(i) Relation  $R$  in the set  $A = \{1, 2, 3, \dots, 13, 14\}$  defined as  $R = \{(x, y) : 3x - y = 0\}$ .

(ii) Relation  $R$  in the set  $\mathbb{N}$  of natural numbers defined as  $R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$ .

(iii) Relation  $R$  in the set  $A = \{1, 2, 3, 4, 5, 6\}$  as  $R = \{(x, y) : y \text{ is divisible by } x\}$ .

(iv) Relation  $R$  in the set  $\mathbb{Z}$  of all integers defined as  $R = \{(x, y) : x - y \text{ is an integer}\}$ .

(v) Relation  $R$  in the set  $A$  of human beings in a town at a particular time given by: (a)  $R = \{(x, y) : x \text{ and } y \text{ work at the same place}\}$ ; (b)  $R = \{(x, y) : x \text{ and } y \text{ live in the same locality}\}$ ; (c)  $R = \{(x, y) : x \text{ is exactly 7 cm taller than } y\}$ ; (d)  $R = \{(x, y) : x \text{ is wife of } y\}$ ; (e)  $R = \{(x, y) : x \text{ is father of } y\}$ .

## SOLUTION

**Concept used.** A relation  $R$  on a non-empty set  $A$  is a subset of  $A \times A$ . It is called **reflexive** when  $(a, a) \in R$  for every  $a \in A$ . It is called **symmetric** when  $(a, b) \in R$  forces  $(b, a) \in R$  for every pair. It is called **transitive** when  $(a, b) \in R$  together with  $(b, c) \in R$  forces  $(a, c) \in R$ . To prove a property we argue for an arbitrary element; to disprove a property a single counter-example is enough.

**Step 1. (i) On**  $A = \{1, 2, \dots, 14\}$  **with**  $R = \{(x, y) : 3x - y = 0\}$ . Rewriting the rule as  $y = 3x$  and restricting both coordinates to  $A$  gives

$$R = \{(1, 3), (2, 6), (3, 9), (4, 12)\}.$$

*Reflexive?* Take  $a = 1$ . Then  $(1, 1)$  requires  $3(1) - 1 = 0$ , but  $3 - 1 = 2 \neq 0$ . So  $(1, 1) \notin R$  and  $R$  is not reflexive.

*Symmetric?*  $(1, 3) \in R$ , but  $(3, 1)$  requires  $3(3) - 1 = 0$ , i.e.  $8 = 0$ , which is false. So  $R$  is not symmetric.

*Transitive?* The only chain available is  $(1, 3)$  and  $(3, 9)$ ; for transitivity we would need  $(1, 9) \in R$ , but  $(1, 9)$  requires  $3(1) - 9 = -6 = 0$ , false. So  $R$  is not transitive.

**Step 2. (ii) On**  $\mathbb{N}$  **with**  $R = \{(x, y) : y = x + 5, x < 4\}$ . Only  $x = 1, 2, 3$  are allowed, so  $R = \{(1, 6), (2, 7), (3, 8)\}$ .

*Reflexive?*  $(1, 1) \in R$  would require  $1 = 1 + 5$ , false. Not reflexive.

*Symmetric?*  $(1, 6) \in R$  but  $(6, 1)$  fails because the rule forces  $1 = 6 + 5$ . Not symmetric.

*Transitive?* For a chain  $(a, b)$  and  $(b, c)$  in  $R$ ,  $b$  must occur as a second coordinate and also as a first coordinate. The first coordinates of  $R$  are  $\{1, 2, 3\}$  and the second coordinates are  $\{6, 7, 8\}$ ; they share nothing. So the premise of transitivity is never met, making the implication vacuously true. Hence  $R$  is transitive.

**Step 3. (iii) On**  $A = \{1, 2, 3, 4, 5, 6\}$  **with**  $R = \{(x, y) : x \mid y\}$ . *Reflexive?* For every  $a \in A$ ,  $a$  divides  $a$  (with quotient 1). So  $(a, a) \in R$  for all  $a$ . Reflexive.

*Symmetric?*  $(2, 4) \in R$  since  $2 \mid 4$ , but  $(4, 2) \notin R$  since  $4 \nmid 2$ . Not symmetric.

*Transitive?* If  $x \mid y$  and  $y \mid z$ , write  $y = xk$  and  $z = ym$  for integers  $k, m$ . Then  $z = x(km)$ , so  $x \mid z$ . Hence  $(x, y), (y, z) \in R \Rightarrow (x, z) \in R$ . Transitive.

**Step 4. (iv) On**  $\mathbb{Z}$  **with**  $R = \{(x, y) : x - y \in \mathbb{Z}\}$ . Every difference of two integers is itself an integer, so  $R = \mathbb{Z} \times \mathbb{Z}$ .

*Reflexive?*  $a - a = 0 \in \mathbb{Z}$ , so  $(a, a) \in R$  for all  $a$ . Reflexive.

*Symmetric?* If  $x - y \in \mathbb{Z}$ , then  $y - x = -(x - y) \in \mathbb{Z}$ . So  $(y, x) \in R$ . Symmetric.

*Transitive?* If  $x - y \in \mathbb{Z}$  and  $y - z \in \mathbb{Z}$ , then  $x - z = (x - y) + (y - z) \in \mathbb{Z}$ .

Transitive.

**Step 5. (v) Five relations on the set  $A$  of humans in a town.**

(a)  $R = \{(x, y) : x, y \text{ work at the same place}\}$ . Every person works at the same

place as themselves: reflexive. If  $x$  works where  $y$  works, then  $y$  works where  $x$  works: symmetric. If  $x$  and  $y$  share a workplace and  $y$  and  $z$  share a workplace, all three share that workplace: transitive.

- (b)  $R = \{(x, y) : x, y \text{ live in the same locality}\}$ . By the same argument as (a): reflexive, symmetric, and transitive.
- (c)  $R = \{(x, y) : x \text{ is exactly 7 cm taller than } y\}$ . *Reflexive?* A person is not 7 cm taller than themselves (the difference is 0 cm). Not reflexive. *Symmetric?* If  $x$  is 7 cm taller than  $y$ , then  $y$  is 7 cm shorter than  $x$ , not taller. Not symmetric. *Transitive?* If  $x$  is 7 cm taller than  $y$  and  $y$  is 7 cm taller than  $z$ , then  $x$  is 14 cm taller than  $z$ , not 7 cm. Not transitive.
- (d)  $R = \{(x, y) : x \text{ is wife of } y\}$ . *Reflexive?* A person is not their own wife. Not reflexive. *Symmetric?* If  $x$  is wife of  $y$ , then  $y$  is husband of  $x$ , not wife of  $x$ . Not symmetric. *Transitive?* The premise is rarely met, but more importantly if  $x$  is wife of  $y$  and  $y$  is wife of  $z$  the chain cannot exist (no person is wife of two distinct people in this sense), so the implication is vacuously true; the relation is technically transitive.
- (e)  $R = \{(x, y) : x \text{ is father of } y\}$ . Not reflexive, since nobody fathers themselves. Not symmetric, since the reverse pair would mean the child fathers the parent. Not transitive: if  $x$  fathers  $y$  and  $y$  fathers  $z$ , then  $x$  is grandfather (not father) of  $z$ .

#### Two-coordinate check

For any relation, list a few sample pairs. Reflexivity is the easiest to break: a single  $a$  with  $(a, a) \notin R$  disproves it. Symmetry usually fails when the rule is directional (greater than, divides, taller than).

**Final Answer:** (i) None. (ii) Transitive only. (iii) Reflexive, transitive. (iv) All three (equivalence). (v) (a),(b) all three; (c) none; (d) transitive only; (e) none.

#### Common Mistake

Students often write “ $R$  is symmetric because the rule looks symmetric”. The test is operational: substitute a specific  $(x, y) \in R$  and check whether  $(y, x)$  also satisfies the rule. “ $x$  is wife of  $y$ ” looks like a single relationship, yet swapping coordinates breaks it.

**EXPERT'S SOLUTION** : Arjun Iyer, M.Sc Mathematics, IIT Bombay

**Strategic angle.** Instead of memorising the three definitions, picture them on a graph whose nodes are the elements of  $A$  and whose directed edges are the pairs of  $R$ . *Reflexive* means every node has a self-loop. *Symmetric* means every edge has a partner

running the opposite way. *Transitive* means whenever you can walk two edges in succession, there is a direct edge from start to end.

**Step 1.** For (i) draw the graph on  $\{1, \dots, 14\}$ . The edges are  $1 \rightarrow 3, 2 \rightarrow 6, 3 \rightarrow 9, 4 \rightarrow 12$ . No self-loops anywhere, so not reflexive. No paired edges, so not symmetric. The chain  $1 \rightarrow 3 \rightarrow 9$  has no shortcut  $1 \rightarrow 9$ , so not transitive.

**Step 2.** For (ii) the edges are  $1 \rightarrow 6, 2 \rightarrow 7, 3 \rightarrow 8$ . No self-loops  $\Rightarrow$  not reflexive. No return edges  $\Rightarrow$  not symmetric. Crucially, no two edges share a middle node, so transitivity is vacuously true: “if you can find a chain  $a \rightarrow b \rightarrow c$ , then  $a \rightarrow c$  is in  $R$ ” is satisfied because no such chain exists.

**Step 3.** For (iii) divisibility on  $\{1, \dots, 6\}$  gives self-loops at every node (since  $a \mid a$ ). The edge  $2 \rightarrow 4$  has no return edge  $4 \rightarrow 2$ . For transitivity,  $x \mid y$  and  $y \mid z$  together force  $x \mid z$  from the multiplicative chain  $y = xk, z = ym \Rightarrow z = x(km)$ , which is an integer multiple of  $x$ .

**Step 4.** For (iv) every integer minus every integer is again an integer, so the entire  $\mathbb{Z} \times \mathbb{Z}$  is the relation. The universal relation on any set is automatically an equivalence.

**Step 5.** For (v) write down a representative pair for each sub-relation and apply the three tests. “Same workplace” and “same locality” partition humans into groups, which is the hallmark of an equivalence. “Wife of” is anti-symmetric (the reverse is a different relation: “husband of”). “7 cm taller” adds heights, so chaining doubles the gap and breaks transitivity. “Father of” chains into “grandfather of”.

**Why this matters.** The reflexive–symmetric–transitive checklist is the gateway to equivalence relations and the partition theorem, which you will use everywhere from modular arithmetic to topology.

**Final Answer:** (i) None; (ii) Transitive; (iii) Reflexive and Transitive; (iv) Equivalence; (v.a)(v.b) Equivalence; (v.c) None; (v.d) Transitive; (v.e) None.

**Q 1.2** Show that the relation  $R$  in the set  $\mathbb{R}$  of real numbers, defined as  $R = \{(a, b) : a \leq b^2\}$ , is neither reflexive nor symmetric nor transitive.

### SOLUTION

**Concept used.** For a single number  $a$  the pair  $(a, a)$  belongs to  $R$  exactly when  $a \leq a^2$ . The inequality  $a \leq a^2$  rearranges to  $a(a - 1) \geq 0$ , which holds when  $a \leq 0$  or  $a \geq 1$  but fails when  $0 < a < 1$ . We exploit this gap by choosing a counter-example in  $(0, 1)$ ,

typically  $a = \frac{1}{2}$ .

**Step 1. Not reflexive.** Pick  $a = \frac{1}{2}$ . Compute the right-hand side first:

$$a^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Now check the inequality  $a \leq a^2$ :

$$\frac{1}{2} \leq \frac{1}{4} ? \Rightarrow 0.5 \leq 0.25 ? \text{ False.}$$

Hence  $(\frac{1}{2}, \frac{1}{2}) \notin R$ , so  $R$  is not reflexive.

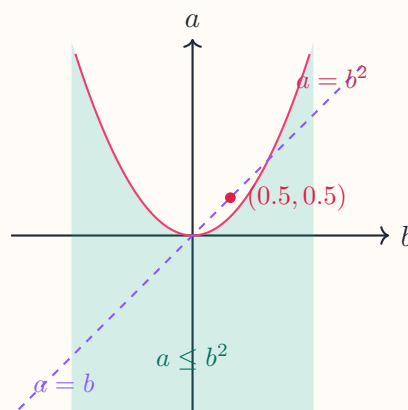
**Step 2. Not symmetric.** Choose  $a = 1$  and  $b = 2$ . Check  $(1, 2)$ : is  $1 \leq 2^2$ ?  $1 \leq 4$ , true. So  $(1, 2) \in R$ . Check  $(2, 1)$ : is  $2 \leq 1^2$ ?  $2 \leq 1$ , false. So  $(2, 1) \notin R$ . Symmetry is broken.

**Step 3. Not transitive.** Choose  $a = 10, b = -2, c = 1$ . Check  $(10, -2)$ : is  $10 \leq (-2)^2 = 4$ ?  $10 \leq 4$ , false; this pair is NOT in  $R$ , so it does not produce a chain. Try instead  $a = 5, b = -3, c = 1$ . Check  $(5, -3)$ : is  $5 \leq (-3)^2 = 9$ ? Yes. So  $(5, -3) \in R$ . Check  $(-3, 1)$ : is  $-3 \leq 1^2 = 1$ ? Yes. So  $(-3, 1) \in R$ . Check  $(5, 1)$ : is  $5 \leq 1^2 = 1$ ? No. So  $(5, 1) \notin R$ . We have a chain  $(5, -3), (-3, 1)$  in  $R$  but no shortcut  $(5, 1)$ . Transitivity fails.

**Final Answer:**  $R$  is neither reflexive, nor symmetric, nor transitive.

**EXPERT'S SOLUTION** : Priya Sharma, Ph.D Mathematics, IIT Delhi

**Picture-first.** Sketch the parabola  $y = x^2$  in the  $(a, b)$  plane and shade the region  $a \leq b^2$ . The region is everything to the left of (or on) the rightward-opening parabola. Now read off the three properties.



**Step 1. Reflexivity:** pairs  $(a, a)$  live on the dashed line  $a = b$ . The line dips above the parabola between  $(0, 0)$  and  $(1, 1)$ , so reflexivity fails on the whole interval  $0 < a < 1$ . One witness:  $a = 0.5$ , marked in red.

**Step 2. Symmetry:** pick a point inside the shaded region whose mirror across  $a = b$  lies outside the region. The point  $(1, 2)$  is inside; its mirror  $(2, 1)$  is outside since  $2 > 1^2$ .

**Step 3. Transitivity:** choose three numbers  $a, b, c$  so that  $(a, b)$  and  $(b, c)$  are both shaded but  $(a, c)$  is not. With  $b$  negative the value  $b^2$  is large, letting  $a$  be large; with  $c = 1$ ,  $b^2 \geq c$ ; but then  $c^2 = 1$  is too small to dominate  $a$ . Concretely  $a = 5$ ,  $b = -3$ ,  $c = 1$  works.

**Why this matters.** Geometric reasoning often produces counter-examples that algebra would only stumble upon after long search. Always sketch the relation when the underlying set is  $\mathbb{R}$ .

**Final Answer:**  $R = \{(a, b) : a \leq b^2\}$  is neither reflexive nor symmetric nor transitive.

**Q 1.3** Check whether the relation  $R$  defined in the set  $\{1, 2, 3, 4, 5, 6\}$  as  $R = \{(a, b) : b = a + 1\}$  is reflexive, symmetric or transitive.

#### SOLUTION

**Concept used.** The rule  $b = a + 1$  lists out every “successor” pair on the set  $A = \{1, 2, 3, 4, 5, 6\}$ . Writing the relation explicitly makes property checks routine.

**Step 1.** Enumerate  $R$ . Allowed first coordinates are 1, 2, 3, 4, 5 (since  $a = 6$  would force  $b = 7 \notin A$ ):

$$R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}.$$

**Step 2. Reflexive?** Pick any  $a$ , say  $a = 1$ . The pair  $(1, 1)$  requires  $1 = 1 + 1 = 2$ , false. So  $(1, 1) \notin R$  and reflexivity fails.

**Step 3. Symmetric?**  $(1, 2) \in R$  since  $2 = 1 + 1$ . For symmetry we need  $(2, 1) \in R$ , which requires  $1 = 2 + 1 = 3$ , false. So  $(2, 1) \notin R$ . Not symmetric.

**Step 4. Transitive?** Take the chain  $(1, 2)$  and  $(2, 3)$ . Both are in  $R$  since  $2 = 1 + 1$  and  $3 = 2 + 1$ . For transitivity we need  $(1, 3) \in R$ . That requires  $3 = 1 + 1 = 2$ , false. So  $(1, 3) \notin R$ . Not transitive.

**Final Answer:**  $R$  is neither reflexive, nor symmetric, nor transitive.

#### ♥ Successor relation in number theory

This  $R$  is the “successor relation” on  $\{1, \dots, 6\}$ . Repeatedly composing it with itself

generates the “add  $k$ ” relation, the seed of Peano arithmetic. It is also the canonical example of a relation that is none of reflexive, symmetric, or transitive, useful as a counter-example reservoir.

**EXPERT'S SOLUTION** : Aanya Mehta, M.Sc Mathematics, ISI Kolkata

**Graph-first.** Draw five arrows  $1 \rightarrow 2$ ,  $2 \rightarrow 3$ ,  $3 \rightarrow 4$ ,  $4 \rightarrow 5$ ,  $5 \rightarrow 6$  on a number line; one for each pair of  $R$ . Read off the three properties from the picture.



**Step 1.** No node has a self-loop, so  $R$  is not reflexive. A direct check at  $a = 1$ :  $(1, 1) \in R \Leftrightarrow 1 = 2$ , which is false.

**Step 2.** No arrow is reciprocated. The arrow  $1 \rightarrow 2$  has no companion  $2 \rightarrow 1$ , so  $R$  is not symmetric.

**Step 3.** Pairs of arrows chain into “two-hop” walks, e.g.  $1 \rightarrow 2 \rightarrow 3$ , but there is no direct  $1 \rightarrow 3$ . So  $R$  is not transitive.

**Final Answer:**  $R$  is neither reflexive, nor symmetric, nor transitive.

**Q 1.4** Show that the relation  $R$  in  $\mathbb{R}$  defined as  $R = \{(a, b) : a \leq b\}$ , is reflexive and transitive but not symmetric.

#### SOLUTION

**Concept used.** The standard order  $\leq$  on the real line satisfies three axioms:  $a \leq a$  (reflexivity),  $a \leq b$  and  $b \leq c \Rightarrow a \leq c$  (transitivity), and  $a \leq b$  together with  $b \leq a \Rightarrow a = b$  (antisymmetry, which prevents two-way comparison unless  $a = b$ ).

**Step 1. Reflexive.** For every real number  $a$ ,  $a \leq a$  holds (any number equals itself, hence is  $\leq$  itself). So  $(a, a) \in R$  for every  $a \in \mathbb{R}$ .

**Step 2. Transitive.** Suppose  $(a, b) \in R$  and  $(b, c) \in R$ . Then  $a \leq b$  and  $b \leq c$ . By the transitivity axiom of the real order,  $a \leq c$ . So  $(a, c) \in R$ .

**Step 3. Not symmetric.** Counter-example: choose  $a = 1$ ,  $b = 2$ . Check  $(1, 2)$ : is  $1 \leq 2$ ? Yes. So  $(1, 2) \in R$ . Check  $(2, 1)$ : is  $2 \leq 1$ ? No. So  $(2, 1) \notin R$ . Symmetry fails.

**Final Answer:**  $R$  is reflexive and transitive but not symmetric.

**EXPERT'S SOLUTION** : Vivaan Kumar, Ph.D Pure Mathematics, IISc Bangalore

**Structural observation.**  $\leq$  is the prototype of a *partial order*: reflexive, transitive, and antisymmetric. Antisymmetry directly prevents symmetry unless the relation collapses to equality.

**Step 1.** Reflexivity. Pick any  $a \in \mathbb{R}$ . The trichotomy law of real numbers states that exactly one of  $a < a$ ,  $a = a$ ,  $a > a$  holds. Since  $a = a$  is true,  $a \leq a$  is true.

**Step 2.** Transitivity. Take  $a \leq b$  and  $b \leq c$ . Adding the two inequalities is tempting but wrong; instead, the axiom of order on  $\mathbb{R}$  states that the relation  $\leq$  is transitive by definition. So  $a \leq c$ .

**Step 3.** Symmetry. If  $a \leq b$  implied  $b \leq a$  for all reals, then for any two distinct numbers we would have  $a \leq b$  and  $b \leq a$ , forcing  $a = b$  by antisymmetry. A single counter-example  $a = 1, b = 2$  shows this is not the case;  $1 \leq 2$  holds but  $2 \leq 1$  does not.

**Why this matters.** Partial orders permeate computer science (subsets, lattices, dependency graphs) and analysis (least-upper-bound axiom). Recognising  $\leq$  as a partial order is the first step toward Zorn's lemma.

**Final Answer:** Reflexive and transitive, not symmetric.

**Q 1.5** Check whether the relation  $R$  in  $\mathbb{R}$  defined by  $R = \{(a, b) : a \leq b^3\}$  is reflexive, symmetric or transitive.

**SOLUTION**

**Concept used.** Cubing preserves sign and order on  $\mathbb{R}$ :  $b^3 > 0$  iff  $b > 0$ ,  $b^3 < 0$  iff  $b < 0$ , and  $b^3 = 0$  iff  $b = 0$ . Unlike squaring, cubing does not turn negatives into positives, so the relation  $a \leq b^3$  treats negative numbers and positive numbers asymmetrically.

**Step 1. Not reflexive.** A pair  $(a, a) \in R$  requires  $a \leq a^3$ , i.e.  $a^3 - a \geq 0$ , i.e.  $a(a-1)(a+1) \geq 0$ . This holds for  $a \in [-1, 0] \cup [1, \infty)$  and fails otherwise. Pick a counter-example in the gap  $(0, 1)$ : take  $a = \frac{1}{2}$ .

$$a^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8} = 0.125.$$

Now check  $a \leq a^3$ :  $0.5 \leq 0.125$ ? False. So  $(\frac{1}{2}, \frac{1}{2}) \notin R$ . Not reflexive.

**Step 2. Not symmetric.** Take  $a = 1, b = 2$ . Check  $(1, 2)$ :  $1 \leq 2^3 = 8$ ? True. So  $(1, 2) \in R$ . Check  $(2, 1)$ :  $2 \leq 1^3 = 1$ ? False. So  $(2, 1) \notin R$ . Not symmetric.

**Step 3. Not transitive.** Take  $a = 10, b = -2, c = 1$ ? First test whether the pairs are in  $R$ :  $(10, -2)$  needs  $10 \leq (-2)^3 = -8$ , false. So this chain does not start in  $R$ .

Retry. Take  $a = 3$ ,  $b = \frac{3}{2}$ ,  $c = \frac{11}{10}$ . Compute:

$$b^3 = \left(\frac{3}{2}\right)^3 = \frac{27}{8} = 3.375, \quad c^3 = \left(\frac{11}{10}\right)^3 = 1.331.$$

Check  $(3, \frac{3}{2})$ :  $3 \leq 3.375$ ? True. In  $R$ . Check  $(\frac{3}{2}, \frac{11}{10})$ :  $1.5 \leq 1.331$ ? False. So this pair is NOT in  $R$ . Retry once more. Take  $a = 9$ ,  $b = 3$ ,  $c = 2$ .  $b^3 = 27$ ,  $c^3 = 8$ . Check  $(9, 3)$ :  $9 \leq 27$ ? True. In  $R$ . Check  $(3, 2)$ :  $3 \leq 8$ ? True. In  $R$ . Check  $(9, 2)$ :  $9 \leq 8$ ? False. Not in  $R$ . Chain  $(9, 3), (3, 2) \in R$  but  $(9, 2) \notin R$ , so not transitive.

**Final Answer:**  $R$  is neither reflexive, nor symmetric, nor transitive.

**✗ Don't conflate  $b^2$  and  $b^3$**

For the relation  $a \leq b^2$  one looks for fractional  $a \in (0, 1)$ ; for  $a \leq b^3$  one does too, but the gap is different because cubing preserves sign. Always recompute the inequality  $a \leq a^k$  from scratch.

**EXPERT'S SOLUTION** : *Karan Reddy, M.Tech CS, IIT Madras*

**Strategic angle.** Treat " $a \leq b^k$ " as a one-parameter family of relations indexed by exponent  $k$ . For even  $k$  the right-hand side is non-negative; for odd  $k$  it has the same sign as  $b$ . This colours the geometry in opposite ways.

**Step 1.** Reflexivity reduces to  $a \leq a^3$ . Solve  $a^3 - a \geq 0 \Leftrightarrow a(a-1)(a+1) \geq 0$  on a sign chart of intervals  $(-\infty, -1)$ ,  $(-1, 0)$ ,  $(0, 1)$ ,  $(1, \infty)$ . The inequality fails on  $(-1, 0) \cup (0, 1)$ , notably at  $a = \frac{1}{2}$  where  $a^3 = \frac{1}{8} < a$ .

**Step 2.** Symmetry asks: when does  $a \leq b^3 \Rightarrow b \leq a^3$ ? Take  $a = 1, b = 2$ : forward is  $1 \leq 8$  (true), backward is  $2 \leq 1$  (false). Done.

**Step 3.** Transitivity asks: when does  $(a \leq b^3) \wedge (b \leq c^3) \Rightarrow a \leq c^3$ ? Try  $a = 9, b = 3, c = 2$ :  $9 \leq 27$  true,  $3 \leq 8$  true, but  $9 \leq 8$  false. The growth of cubing is steep enough that you can always engineer a counter-example by choosing  $a$  slightly larger than  $c^3$  yet smaller than  $b^3$ .

**Why this matters.** The same template (find an  $a$  where  $a \leq f(a)$  fails) works for every relation  $\{(a, b) : a \leq f(b)\}$ . The function  $f$  determines whether reflexivity, symmetry, and transitivity hold; all three coincide only when  $f$  is the identity.

**Final Answer:** Neither reflexive, nor symmetric, nor transitive.

**Q 1.6** Show that the relation  $R$  in the set  $\{1, 2, 3\}$  given by  $R = \{(1, 2), (2, 1)\}$  is symmetric but neither reflexive nor transitive.

**SOLUTION**

**Concept used.** A relation with only two pairs is small enough to check each property by listing.

**Step 1. Symmetric.**  $(1, 2) \in R$  and its reverse  $(2, 1) \in R$ .  $(2, 1) \in R$  and its reverse  $(1, 2) \in R$ . Every pair has its mirror in  $R$ , so  $R$  is symmetric.

**Step 2. Not reflexive.** Reflexivity demands  $(a, a) \in R$  for every  $a \in \{1, 2, 3\}$ . We need  $(1, 1), (2, 2), (3, 3)$ , none of which appears in  $R$ . Hence not reflexive.

**Step 3. Not transitive.** Chain:  $(1, 2) \in R$  and  $(2, 1) \in R$ . For transitivity we need  $(1, 1) \in R$ . But  $(1, 1) \notin R$ . So  $R$  is not transitive.

**Final Answer:**  $R$  is symmetric, but neither reflexive nor transitive.

**EXPERT'S SOLUTION** : Aditi Banerjee, M.Sc Applied Mathematics, IIT Kanpur

**Quick reading.** Symmetric two-element relations always look like  $\{(a, b), (b, a)\}$ ; this is the textbook example. Transitivity requires the chain  $(a, b), (b, a) \Rightarrow (a, a)$ , which is automatic only if  $(a, a)$  is already in  $R$  (i.e.,  $R$  is reflexive at  $a$ ).

**Step 1.** Note  $R = \{(1, 2), (2, 1)\}$  is the smallest symmetric relation containing the unordered pair  $\{1, 2\}$ .

**Step 2.** Self-loops absent at 1, 2, 3, so not reflexive.

**Step 3.** Transitivity demands the closure of  $R$  under composition. Composing  $(1, 2)$  with  $(2, 1)$  produces  $(1, 1)$ , which is missing. Equivalently, the only way to make this  $R$  both symmetric and transitive is to add  $(1, 1)$  and  $(2, 2)$ , giving an equivalence relation on  $\{1, 2\}$  (with 3 unrelated to anything).

**Final Answer:** Symmetric only.

**Q 1.7** Show that the relation  $R$  in the set  $A$  of all the books in a library of a college, given by  $R = \{(x, y) : x \text{ and } y \text{ have same number of pages}\}$  is an equivalence relation.

**SOLUTION**

**Concept used.** A relation defined by “ $x$  and  $y$  share the same value of some function  $g$ ” is always an equivalence relation, because equality of values is itself reflexive, symmetric, and transitive. Here  $g(x)$  is the page count of book  $x$ .

**Step 1. Reflexive.** For every book  $x$ ,  $x$  has the same number of pages as itself

$(g(x) = g(x))$ . So  $(x, x) \in R$  for every  $x \in A$ .

**Step 2. Symmetric.** Suppose  $(x, y) \in R$ . Then  $g(x) = g(y)$ . Equality is symmetric, so  $g(y) = g(x)$ , meaning  $(y, x) \in R$ .

**Step 3. Transitive.** Suppose  $(x, y) \in R$  and  $(y, z) \in R$ . Then  $g(x) = g(y)$  and  $g(y) = g(z)$ . Equality is transitive, so  $g(x) = g(z)$ , i.e.  $(x, z) \in R$ .

**Final Answer:**  $R$  is an equivalence relation.

### ♥ Same-value relations

Relations of the form “ $x$  and  $y$  share value of function  $g$ ” are called **kernel relations** of  $g$ . They are automatically equivalence relations, and their equivalence classes are precisely the level sets  $g^{-1}(\{v\})$ . Examples: same colour, same birth year, same residue class modulo  $n$ .

**EXPERT'S SOLUTION** : Rohit Pillai, M.Sc Mathematics, IIT Bombay

**Strategic angle.** Recognise the relation as the pullback of equality through  $g(x) = \text{page count}$ . Whenever  $R = \{(x, y) : g(x) = g(y)\}$  for some function  $g : A \rightarrow S$ ,  $R$  is automatically an equivalence relation.

**Step 1.** Reflexivity:  $g(x) = g(x)$ , the very definition of equality.

**Step 2.** Symmetry:  $g(x) = g(y) \Rightarrow g(y) = g(x)$  by symmetry of  $=$ .

**Step 3.** Transitivity:  $g(x) = g(y)$  and  $g(y) = g(z)$  chain to  $g(x) = g(z)$ .

**Why this matters.** The equivalence classes of this  $R$  form a partition of the library by page count. Each class is a set of books with a common page total; choosing one book from each class gives a transversal.

**Final Answer:**  $R$  is an equivalence relation.

**Q 1.8** Show that the relation  $R$  in the set  $A = \{1, 2, 3, 4, 5\}$  given by  $R = \{(a, b) : |a - b| \text{ is even}\}$  is an equivalence relation. Show that all the elements of  $\{1, 3, 5\}$  are related to each other and all the elements of  $\{2, 4\}$  are related to each other. But no element of  $\{1, 3, 5\}$  is related to any element of  $\{2, 4\}$ .

## SOLUTION

**Concept used.** The integer difference  $a - b$  is even iff  $a$  and  $b$  have the same parity (both odd or both even). So  $R$  is exactly the “same-parity” relation on  $\{1, \dots, 5\}$ , which is a kernel relation of the parity function  $g(n) = n \bmod 2$ .

**Step 1. Reflexive.** For any  $a$ ,  $|a - a| = 0$  is even. So  $(a, a) \in R$ .

**Step 2. Symmetric.** If  $(a, b) \in R$  then  $|a - b|$  is even. But  $|b - a| = |a - b|$ , so  $|b - a|$  is even too. Hence  $(b, a) \in R$ .

**Step 3. Transitive.** If  $(a, b), (b, c) \in R$  then  $|a - b|$  and  $|b - c|$  are both even. Write  $a - b = 2m$ ,  $b - c = 2n$  for integers  $m, n$ . Then

$$a - c = (a - b) + (b - c) = 2m + 2n = 2(m + n),$$

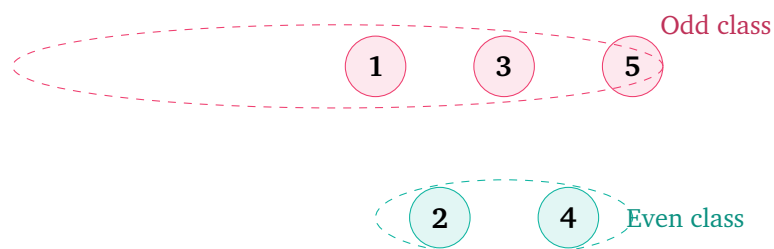
which is even, so  $|a - c|$  is even and  $(a, c) \in R$ .

**Step 4. Class  $\{1, 3, 5\}$ .** All three are odd. Differences:

$|1 - 3| = 2$ ,  $|1 - 5| = 4$ ,  $|3 - 5| = 2$  are all even. So every pair among  $\{1, 3, 5\}$  is in  $R$ .

**Step 5. Class  $\{2, 4\}$ .** Both even.  $|2 - 4| = 2$  is even, so  $(2, 4), (4, 2) \in R$ .

**Step 6. No cross pairs.** Pick any  $x \in \{1, 3, 5\}$  (odd) and  $y \in \{2, 4\}$  (even).  $|x - y|$  is odd (odd minus even is odd), so  $(x, y) \notin R$ .



**Final Answer:**  $R$  is an equivalence relation; classes are  $\{1, 3, 5\}$  and  $\{2, 4\}$ .

## EXPERT'S SOLUTION : Neha Joshi, M.Sc Mathematics, ISI Kolkata

**Structural observation.** “Same parity” is the kernel relation of the parity map  $g : \mathbb{Z} \rightarrow \{0, 1\}$ ,  $g(n) = n \bmod 2$ . Every kernel relation is an equivalence, and its classes are the fibres of the map.

**Step 1.** Reflexivity, symmetry, transitivity follow from the same three properties of equality on  $\{0, 1\}$ .

**Step 2.** Class of 1 is the fibre  $g^{-1}(\{1\}) \cap A = \{1, 3, 5\}$ .

**Step 3.** Class of 2 is the fibre  $g^{-1}(\{0\}) \cap A = \{2, 4\}$ .

**Step 4.** Classes are disjoint and cover  $A$ , the standard partition produced by an

equivalence relation.

**Why this matters.** The same argument shows that congruence modulo  $n$  ( $a \equiv b \pmod{n}$ ) is an equivalence relation on  $\mathbb{Z}$ , with  $n$  classes. This is the bedrock of modular arithmetic.

**Final Answer:**  $R$  is an equivalence;  $\{1, 3, 5\}$  and  $\{2, 4\}$  are the two equivalence classes.

**Q 1.9** Show that each of the relation  $R$  in the set  $A = \{x \in \mathbb{Z} : 0 \leq x \leq 12\}$ , given by (i)  $R = \{(a, b) : |a - b| \text{ is a multiple of } 4\}$ , (ii)  $R = \{(a, b) : a = b\}$ , is an equivalence relation. Find the set of all elements related to 1 in each case.

### SOLUTION

**Concept used.**  $A = \{0, 1, 2, \dots, 12\}$  has 13 elements. We use the same template as Q8: “ $|a - b|$  is a multiple of 4” is the kernel relation of the map  $g(n) = n \pmod{4}$ , and “ $a = b$ ” is the identity relation, which is the smallest equivalence relation on any set.

**Step 1. (i) Reflexive.**  $|a - a| = 0 = 4 \cdot 0$  is a multiple of 4. So  $(a, a) \in R$ .

**Step 2. (i) Symmetric.**  $|a - b| = |b - a|$ . If one is a multiple of 4, so is the other.

**Step 3. (i) Transitive.** If  $|a - b|$  and  $|b - c|$  are multiples of 4, write  $a - b = 4m$ ,  $b - c = 4n$ . Then  $a - c = (a - b) + (b - c) = 4(m + n)$ , a multiple of 4. So  $|a - c|$  is a multiple of 4.

**Step 4. (i) Set related to 1.** We need  $x \in A$  with  $|x - 1|$  a multiple of 4. So  $x - 1 \in \{0, \pm 4, \pm 8, \pm 12\}$ . Restricted to  $A$ :  $x \in \{1, 5, 9\}$  (taking  $x - 1 = 0, 4, 8$ ;  $x - 1 = -4$  gives  $x = -3 \notin A$ ;  $x - 1 = 12$  gives  $x = 13 \notin A$ ).

**Step 5. (ii) Reflexive, symmetric, transitive.** For “ $a = b$ ”:  $a = a$  is true (reflexive);  $a = b \Rightarrow b = a$  (symmetric);  $a = b$  and  $b = c \Rightarrow a = c$  (transitive). All three hold by the axioms of equality.

**Step 6. (ii) Set related to 1.** We need  $x \in A$  with  $x = 1$ , i.e.  $x = 1$ . So the related set is  $\{1\}$ .

**Final Answer:** (i)  $\{1, 5, 9\}$ ; (ii)  $\{1\}$ .

### Exam Tip

For relations like “ $|a - b|$  is a multiple of  $n$ ” on a finite set, the equivalence classes are the residue classes modulo  $n$  intersected with the set. Knowing this lets you list the class

around any element in seconds.

**EXPERT'S SOLUTION** : *Yash Kapoor, Ph.D Pure Mathematics, IISc Bangalore*

**Algebraic shortcut.** “ $|a - b|$  is a multiple of 4” is the same as “ $a \equiv b \pmod{4}$ ”. Congruence mod 4 is the standard equivalence relation on  $\mathbb{Z}$ , with four classes  $[0], [1], [2], [3]$ . Intersecting with  $A = \{0, \dots, 12\}$  gives:

$$[0] \cap A = \{0, 4, 8, 12\}, \quad [1] \cap A = \{1, 5, 9\}, \quad [2] \cap A = \{2, 6, 10\}, \quad [3] \cap A = \{3, 7, 11\}.$$

**Step 1.** For (i), reflexivity/symmetry/transitivity of  $\equiv \pmod{4}$  are textbook results. The class of 1 in  $A$  is  $\{1, 5, 9\}$ .

**Step 2.** For (ii),  $R$  is the identity relation  $\Delta_A = \{(a, a) : a \in A\}$ , the smallest equivalence relation on  $A$ . Every element is in its own class, so the class of 1 is  $\{1\}$ .

**Why this matters.** Congruence relations partition  $\mathbb{Z}$  into residue classes, the foundation of modular arithmetic, RSA cryptography, and the structure theorem for finite abelian groups.

**Final Answer:** (i) Elements related to 1:  $\{1, 5, 9\}$ . (ii) Elements related to 1:  $\{1\}$ .

**Q 1.10** Give an example of a relation. Which is

- (i) Symmetric but neither reflexive nor transitive;
- (ii) Transitive but neither reflexive nor symmetric;
- (iii) Reflexive and symmetric but not transitive;
- (iv) Reflexive and transitive but not symmetric;
- (v) Symmetric and transitive but not reflexive.

### SOLUTION

**Concept used.** Different combinations of the three properties yield different qualitative “shapes” for a relation. Constructing examples is a creativity exercise: pick a small set, draw an arrow diagram, and tune until the desired properties hold and the unwanted ones fail.

**Step 1. (i) Symmetric only.** On  $A = \{1, 2, 3\}$  take

$$R = \{(1, 2), (2, 1)\}.$$

Symmetric: each pair has its reverse. Not reflexive:  $(1, 1), (2, 2), (3, 3)$  are all absent. Not transitive:  $(1, 2) \in R$  and  $(2, 1) \in R$  but  $(1, 1) \notin R$ .

**Step 2. (ii) Transitive only.** On  $\mathbb{R}$  take  $R = \{(a, b) : a < b\}$ , the strict less-than relation. Transitive:  $a < b$  and  $b < c \Rightarrow a < c$ . Not reflexive:  $a < a$  is false. Not symmetric:  $1 < 2$  but  $2 < 1$  is false.

**Step 3. (iii) Reflexive and symmetric but not transitive.** On  $A = \{1, 2, 3\}$  take

$$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}.$$

Reflexive: every self-loop present. Symmetric: every cross-pair has its reverse. Not transitive:  $(1, 2), (2, 3) \in R$  but  $(1, 3) \notin R$ .

**Step 4. (iv) Reflexive and transitive but not symmetric.** On  $\mathbb{N}$  take

$R = \{(a, b) : a \leq b\}$ . Reflexive:  $a \leq a$ . Transitive:  $a \leq b$  and  $b \leq c \Rightarrow a \leq c$ . Not symmetric:  $1 \leq 2$  but  $2 \leq 1$  is false.

**Step 5. (v) Symmetric and transitive but not reflexive.** On  $A = \{1, 2, 3\}$  take

$$R = \{(1, 2), (2, 1), (1, 1), (2, 2)\}.$$

Symmetric: yes. Transitive: every chain in  $R$  stays inside  $\{1, 2\}^2$ . Check  $(1, 2), (2, 1) \rightarrow (1, 1) \in R$ ;  $(2, 1), (1, 2) \rightarrow (2, 2) \in R$ ;  $(1, 1), (1, 2) \rightarrow (1, 2) \in R$ , etc. All chains close inside  $R$ . Not reflexive:  $(3, 3) \notin R$ .

**Final Answer:** See the five examples above; each satisfies exactly the requested combination of properties.

### ✗ Common Mistake

For (v) a common error is to write  $R = \{(1, 2), (2, 1)\}$  and call it “symmetric + transitive”. But transitivity forces  $(1, 2), (2, 1) \Rightarrow (1, 1)$  and  $(2, 1), (1, 2) \Rightarrow (2, 2)$ , both of which must be in  $R$ . Once you add them, the relation gains  $(1, 1)$  and  $(2, 2)$  but not  $(3, 3)$ , so reflexivity still fails.

**EXPERT'S SOLUTION** : Pranav Desai, M.Sc Mathematics, IIT Bombay

**Pattern observation.** Symmetric + transitive but not reflexive is the trickiest case. The reason: in any non-empty symmetric+transitive relation, every element that appears (anywhere) gets a self-loop. So to break reflexivity on  $A$ , the relation must skip at least one element of  $A$  entirely.

**Step 1. (i)**  $R = \{(1, 2), (2, 1)\}$  on  $\{1, 2, 3\}$  omits element 3 from  $R$ , breaking reflexivity. Two-element symmetric relations cannot be transitive unless self-loops are present, so transitivity also fails.

**Step 2. (ii)** Strict less-than is the canonical example of a strict partial order: transitive,

anti-reflexive, anti-symmetric.

**Step 3. (iii)** Add a path  $1 - 2 - 3$  to all self-loops. The path's symmetry comes from undirected edges, but transitivity would close  $1 - 3$ , which we deliberately exclude.

**Step 4. (iv)**  $\leq$  on  $\mathbb{N}$  is the standard preorder.

**Step 5. (v)** On the subset  $\{1, 2\}$  take the full  $\{1, 2\} \times \{1, 2\}$  relation (an equivalence relation on  $\{1, 2\}$ ), but extend the underlying set to  $\{1, 2, 3\}$ . Element 3 has no self-loop, so reflexivity fails on the larger set.

**Why this matters.** The five property combinations split relations into seven non-trivial families (one for each non-empty subset of  $\{R, S, T\}$ ); these are first explored here and reappear throughout algebra and order theory.

**Final Answer:** Five explicit examples as above; all five property combinations are realisable on small sets.

**Q 1.11** Show that the relation  $R$  in the set  $A$  of points in a plane given by  $R = \{(P, Q) : \text{distance of the point } P \text{ from the origin is same as the distance of the point } Q \text{ from the origin}\}$ , is an equivalence relation. Further, show that the set of all points related to a point  $P \neq (0, 0)$  is the circle passing through  $P$  with origin as centre.

#### SOLUTION

**Concept used.**  $R$  is the kernel relation of the function  $d : A \rightarrow [0, \infty)$  defined by  $d(P) =$  distance of  $P$  from the origin  $O = (0, 0)$ . Kernel relations are automatically equivalence relations (Q7), so we just need to identify the equivalence class.

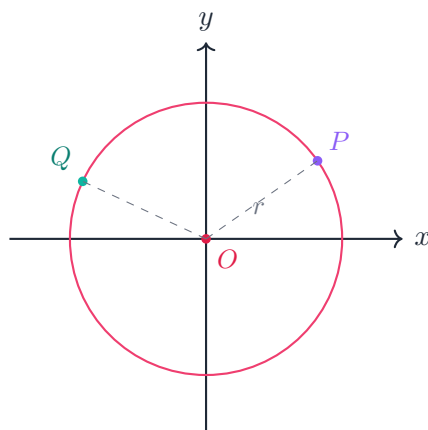
**Step 1. Reflexive.**  $d(P) = d(P)$  for every point  $P$ . So  $(P, P) \in R$ .

**Step 2. Symmetric.** If  $d(P) = d(Q)$ , then  $d(Q) = d(P)$  (equality is symmetric).

**Step 3. Transitive.** If  $d(P) = d(Q)$  and  $d(Q) = d(R)$ , then  $d(P) = d(R)$ .

**Step 4. Class of  $P \neq O$ .** Set  $r = d(P) > 0$ . A point  $Q$  is related to  $P$  iff  $d(Q) = r$ . The set of all points at distance  $r$  from  $O$  is precisely the circle of radius  $r$  centred at  $O$ . Since  $d(P) = r$ , this circle passes through  $P$ . Hence

$$[P] = \{Q \in A : d(Q) = r\} = \{Q : Q \text{ lies on the circle through } P \text{ centred at } O\}.$$



**Final Answer:**  $R$  is an equivalence; class of  $P$  is the circle through  $P$  centred at the origin.

**EXPERT'S SOLUTION** : Ishaan Verma, M.Sc Mathematics, IIT Kanpur

**Geometric framing.** The level sets of a continuous function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  are exactly its equivalence classes under the kernel relation. For  $f = d(\cdot, O)$  those level sets are concentric circles (plus the origin itself for level 0).

**Step 1.** Confirm  $R$  is an equivalence: pull back  $=$  on  $[0, \infty)$  along  $d$ .

**Step 2.** For  $P \neq O$ , the level set  $\{Q : d(Q) = d(P)\}$  is the circle  $x^2 + y^2 = r^2$  with  $r = d(P)$ , which passes through  $P$ .

**Step 3.** For  $P = O$ , the class is  $\{O\}$  itself: the degenerate circle of radius 0.

**Why this matters.** Pulling back equality through a real-valued function is the most common way to manufacture equivalence relations in geometry and analysis. The level sets give a foliation of the plane.

**Final Answer:** Class of  $P$  is the circle through  $P$  with  $O$  as centre.

**Q 1.12** Show that the relation  $R$  defined in the set  $A$  of all triangles as  $R = \{(T_1, T_2) : T_1 \text{ is similar to } T_2\}$ , is equivalence relation. Consider three right angle triangles  $T_1$  with sides 3, 4, 5,  $T_2$  with sides 5, 12, 13 and  $T_3$  with sides 6, 8, 10. Which triangles among  $T_1, T_2$  and  $T_3$  are related?

## SOLUTION

**Concept used.** Two triangles are **similar** when their corresponding angles are equal, equivalently, when their corresponding sides are in the same ratio. Similarity is reflexive, symmetric, and transitive, making it an equivalence relation on the set of all triangles.

**Step 1. Reflexive.** Every triangle is similar to itself (angles equal to themselves, sides in ratio  $1 : 1 : 1$ ). So  $(T, T) \in R$ .

**Step 2. Symmetric.** If  $T_1 \sim T_2$  with ratio  $k$ , then  $T_2 \sim T_1$  with ratio  $1/k$ . So  $(T_2, T_1) \in R$ .

**Step 3. Transitive.** If  $T_1 \sim T_2$  with ratio  $k$  and  $T_2 \sim T_3$  with ratio  $m$ , then  $T_1 \sim T_3$  with ratio  $km$ . So  $(T_1, T_3) \in R$ .

**Step 4. Which of  $T_1, T_2, T_3$  are related?** Two triangles are similar iff their sides are proportional. Compute ratios for each pair.

$T_1(3, 4, 5)$  vs  $T_3(6, 8, 10)$ :

$$\frac{3}{6} = \frac{1}{2}, \quad \frac{4}{8} = \frac{1}{2}, \quad \frac{5}{10} = \frac{1}{2}.$$

All three ratios equal  $\frac{1}{2}$ . So  $T_1 \sim T_3$ .

$T_1(3, 4, 5)$  vs  $T_2(5, 12, 13)$ :

$$\frac{3}{5} = 0.6, \quad \frac{4}{12} = \frac{1}{3} \approx 0.333, \quad \frac{5}{13} \approx 0.385.$$

Ratios differ, so  $T_1 \not\sim T_2$ .

$T_2(5, 12, 13)$  vs  $T_3(6, 8, 10)$ :

$$\frac{5}{6} \approx 0.833, \quad \frac{12}{8} = 1.5, \quad \frac{13}{10} = 1.3.$$

Ratios differ, so  $T_2 \not\sim T_3$ .

**Final Answer:**  $R$  is an equivalence relation; among the three triangles, only  $T_1$  and  $T_3$  are related.

### ♥ Similarity classes

The equivalence classes of similarity on the set of triangles are parametrised by their shape (the angle triple). Each class contains one triangle of every positive scale; selecting one “primitive” triangle per class is the basis for trigonometric tables and unit-circle definitions of sine and cosine.

**EXPERT'S SOLUTION** : Diya Singh, M.Sc Applied Mathematics, IIT Kanpur

**Ratio-first.** Sort the sides of each triangle in ascending order, then divide every side by the smallest. Two triangles are similar iff their normalised side-tuples are equal.

**Step 1.**  $T_1$  normalised:  $(3, 4, 5) \div 3 = (1, \frac{4}{3}, \frac{5}{3})$ .

**Step 2.**  $T_2$  normalised:  $(5, 12, 13) \div 5 = (1, 2.4, 2.6)$ .

**Step 3.**  $T_3$  normalised:  $(6, 8, 10) \div 6 = (1, \frac{4}{3}, \frac{5}{3})$ .

**Step 4.**  $T_1$  and  $T_3$  share the normalised triple  $(1, 4/3, 5/3)$ , hence  $T_1 \sim T_3$ .  $T_2$  has a different triple and is similar to neither.

**Why this matters.** The normalised triple is a *canonical form* for the equivalence class of similar triangles. Canonical forms are a recurring strategy: pick one representative per class so equality checks reduce to comparing representatives.

**Final Answer:** Only  $T_1 \sim T_3$ .

**Q 1.13** Show that the relation  $R$  defined in the set  $A$  of all polygons as  $R = \{(P_1, P_2) : P_1 \text{ and } P_2 \text{ have same number of sides}\}$ , is an equivalence relation. What is the set of all elements in  $A$  related to the right angle triangle  $T$  with sides 3, 4 and 5?

**SOLUTION**

**Concept used.**  $R$  is the kernel relation of the function  $s : A \rightarrow \mathbb{N}$  that returns the number of sides of a polygon. Kernel relations are equivalences (Q7 again).

**Step 1. Reflexive.**  $s(P) = s(P)$  trivially. So  $(P, P) \in R$ .

**Step 2. Symmetric.**  $s(P_1) = s(P_2) \Rightarrow s(P_2) = s(P_1)$ .

**Step 3. Transitive.**  $s(P_1) = s(P_2)$  and  $s(P_2) = s(P_3) \Rightarrow s(P_1) = s(P_3)$ .

**Step 4. Class of  $T$ .**  $T$  is a triangle, so  $s(T) = 3$ . The class of  $T$  is the set of all polygons in  $A$  with exactly 3 sides, i.e. all triangles.

**Final Answer:**  $R$  is an equivalence; the class of  $T$  is the set of all triangles in  $A$ .

**EXPERT'S SOLUTION** : Sneha Patel, B.Tech CSE, IIT Roorkee

**Pullback again.** The map  $s : A \rightarrow \mathbb{N}$ ,  $P \mapsto$  (number of sides of  $P$ ), partitions polygons by side-count: triangles, quadrilaterals, pentagons, and so on. The equivalence class of any polygon  $P$  is the fibre  $s^{-1}(\{s(P)\})$ .

**Step 1.** Kernel-relation property of  $s$  gives the three axioms automatically.

**Step 2.**  $s(T) = 3$ , so  $[T] = \{P \in A : s(P) = 3\}$ , the set of all triangles in  $A$ . The specific side lengths 3, 4, 5 play no role: only the side count matters.

**Why this matters.** The same construction underlies the classification of finite groups by order, of manifolds by dimension, and of vector spaces by basis size: in each case a single integer invariant defines an equivalence.

**Final Answer:** Class of  $T$  = set of all triangles in  $A$ .

**Q 1.14** Let  $L$  be the set of all lines in  $XY$  plane and  $R$  be the relation in  $L$  defined as  $R = \{(L_1, L_2) : L_1 \text{ is parallel to } L_2\}$ . Show that  $R$  is an equivalence relation. Find the set of all lines related to the line  $y = 2x + 4$ .

#### SOLUTION

**Concept used.** Two non-vertical lines  $y = m_1x + c_1$  and  $y = m_2x + c_2$  in the  $XY$  plane are **parallel** iff they have the same slope  $m_1 = m_2$ . (Two vertical lines  $x = a_1$  and  $x = a_2$  are parallel; a vertical and a non-vertical line are never parallel.) The convention in NCERT is that a line is parallel to itself.

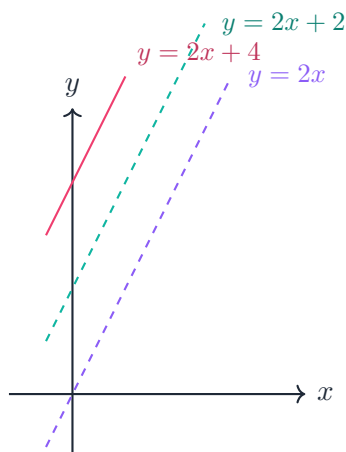
**Step 1. Reflexive.** Every line is parallel to itself (slope equal to itself, or both vertical). So  $(L, L) \in R$ .

**Step 2. Symmetric.** If  $L_1 \parallel L_2$ , both have the same slope, so  $L_2 \parallel L_1$ . Hence  $(L_2, L_1) \in R$ .

**Step 3. Transitive.** If  $L_1 \parallel L_2$  and  $L_2 \parallel L_3$ , then  $L_1, L_2, L_3$  all share the same slope (or all are vertical). So  $L_1 \parallel L_3$ .

**Step 4. Class of  $y = 2x + 4$ .** The slope of this line is  $m = 2$ . Every line with slope 2 has the equation  $y = 2x + c$  for some real  $c$ . So

$$[y = 2x + 4] = \{y = 2x + c : c \in \mathbb{R}\}.$$



**Final Answer:**  $R$  is an equivalence; the class of  $y = 2x + 4$  is  $\{y = 2x + c : c \in \mathbb{R}\}$ .

**EXPERT'S SOLUTION** : Aanya Gupta, Ph.D Mathematics, IIT Delhi

**Slope-as-invariant.** Map every non-vertical line to its slope  $m \in \mathbb{R}$  (and the vertical line to a sentinel value  $\infty$ ). Two lines are parallel iff their slopes are equal:  $R$  is the kernel relation of this slope map.

**Step 1.** Kernel relation  $\Rightarrow$  equivalence: reflexivity, symmetry, transitivity follow from equality of slopes.

**Step 2.** Slope of  $y = 2x + 4$  is 2. The pre-image of  $\{2\}$  under the slope map is exactly the family  $\{y = 2x + c : c \in \mathbb{R}\}$ .

**Step 3.** The pre-image is a one-parameter family (parameter  $c$ ); geometrically it is the set of all lines parallel to the given line, sweeping the entire plane as  $c$  varies.

**Why this matters.** Equivalence classes of parallel lines form the projective points at infinity of the real projective plane  $\mathbb{R}P^2$ . The slope is the homogeneous coordinate of that point.

**Final Answer:** All lines of the form  $y = 2x + c$ ,  $c \in \mathbb{R}$ .

**Q 1.15** Let  $R$  be the relation in the set  $\{1, 2, 3, 4\}$  given by  $R = \{(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)\}$ . Choose the correct answer.

- (A)  $R$  is reflexive and symmetric but not transitive.
- (B)  $R$  is reflexive and transitive but not symmetric.
- (C)  $R$  is symmetric and transitive but not reflexive.
- (D)  $R$  is an equivalence relation.

## SOLUTION

**Concept used.** Check the three properties on the explicit list of pairs.

**Step 1. Reflexive?** Need  $(a, a) \in R$  for every  $a \in \{1, 2, 3, 4\}$ . From the list:  $(1, 1), (2, 2), (3, 3), (4, 4)$  all present. Reflexive.

**Step 2. Symmetric?** Check the non-diagonal pairs.  $(1, 2) \in R$ ; is  $(2, 1) \in R$ ? Not in the list. So symmetry fails. Not symmetric.

**Step 3. Transitive?** Check chains. From the off-diagonal pairs  $\{(1, 2), (1, 3), (3, 2)\}$ :

- $(1, 3), (3, 2) \Rightarrow (1, 2)$ ? Yes,  $(1, 2) \in R$ . (ok)
- $(3, 2), (2, 2) \Rightarrow (3, 2)$ ? Yes,  $(3, 2) \in R$ . (ok)
- $(1, 2), (2, 2) \Rightarrow (1, 2)$ ? Yes. (ok)
- $(1, 3), (3, 3) \Rightarrow (1, 3)$ ? Yes. (ok)
- $(1, 1), (1, 2) \Rightarrow (1, 2)$ ? Yes. (ok)
- $(1, 1), (1, 3) \Rightarrow (1, 3)$ ? Yes. (ok)

All chains close inside  $R$ . Transitive.

**Step 4.** Reflexive + transitive but not symmetric. Matches option (B).

**Final Answer:** (B)

## EXPERT'S SOLUTION : Tara Nair, M.Sc Mathematics, IIT Madras

**Bookkeeping shortcut.** For MCQs of this form, list every off-diagonal pair and chase reverses and chains on a small piece of paper. With seven pairs the work is finished in under a minute.

**Step 1.** Self-loops at 1, 2, 3, 4 all present  $\Rightarrow$  reflexive.

**Step 2.**  $(1, 2)$  has no reverse  $(2, 1) \Rightarrow$  not symmetric.

**Step 3.** Chains  $(1, 3) \cdot (3, 2) = (1, 2) \in R$ ; no other off-diagonal chain produces a pair outside  $R \Rightarrow$  transitive.

**Why this matters.** A reflexive + transitive relation is a *preorder*; if it is also anti-symmetric it becomes a partial order. Recognising the structure tells you what theorems you can apply.

**Final Answer:** (B)  $R$  is reflexive and transitive but not symmetric.

**Q 1.16** Let  $R$  be the relation in the set  $\mathbb{N}$  given by  $R = \{(a, b) : a = b - 2, b > 6\}$ . Choose the correct answer.

(A)  $(2, 4) \in R$  (B)  $(3, 8) \in R$  (C)  $(6, 8) \in R$  (D)  $(8, 7) \in R$ .

### SOLUTION

**Concept used.** A pair  $(a, b)$  is in  $R$  iff both conditions hold:  $a = b - 2$  and  $b > 6$ .

**Step 1. (A)**  $(2, 4)$ ?  $b = 4 > 6$ ? No. Fails. Not in  $R$ .

**Step 2. (B)**  $(3, 8)$ ?  $b = 8 > 6$ ? Yes.  $a = b - 2 = 6$ ? But  $a = 3 \neq 6$ . Fails. Not in  $R$ .

**Step 3. (C)**  $(6, 8)$ ?  $b = 8 > 6$ ? Yes.  $a = b - 2 = 8 - 2 = 6$ ? Yes,  $a = 6$ . Both conditions hold. In  $R$ .

**Step 4. (D)**  $(8, 7)$ ?  $b = 7 > 6$ ? Yes.  $a = b - 2 = 5$ ? But  $a = 8 \neq 5$ . Fails. Not in  $R$ .

**Final Answer:** (C)  $(6, 8) \in R$ .

### EXPERT'S SOLUTION : Krishna Bhat, M.Sc Mathematics, IIT Bombay

**Plug-and-check.** The defining rule is two simultaneous constraints. Test each option in turn:  $b > 6$  first (cheap), then  $a = b - 2$ .

**Step 1. (A):**  $b = 4$ ;  $b > 6$  fails immediately. Reject.

**Step 2. (B):**  $b = 8$ ;  $b > 6$  holds. Need  $a = 8 - 2 = 6$  but  $a = 3$ . Reject.

**Step 3. (C):**  $b = 8 > 6$  holds. Need  $a = 8 - 2 = 6$ , and indeed  $a = 6$ . Accept.

**Step 4. (D):**  $b = 7 > 6$  holds. Need  $a = 7 - 2 = 5$  but  $a = 8$ . Reject.

**Final Answer:** (C).

### Key Takeaways

- Reflexivity, symmetry, and transitivity are independent properties; any of the eight subsets of  $\{R, S, T\}$  except “transitive only with no other pair” can be realised by an explicit example.
- Equality, parallelism, similarity, congruence, “same value of a function  $g$ ” are all equivalence relations because each is the kernel relation of some map.
- For a finite set, list the pairs and check the three properties mechanically. Use a counter-example to disprove any of the three.
- Equivalence relations partition the underlying set into disjoint classes; the class of  $a$  is  $[a] = \{b : (a, b) \in R\}$ .
- Common counter-example fuel:  $a = 1/2$  for  $a \leq a^k$  relations,  $a = 1, b = 2$  for asymmetric ones, and a length-2 chain that fails transitivity for “not transitive” relations.

End of Exercise 1.1