Collegedunia NCERT Solutions

Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 1: Relations and Functions

About this Chapter

Exercise 1.2 sharpens your reading of **one-one (injective)** and **onto (surjective)** functions, and the combined notion of a **bijection**. The 12 problems span concrete maps on $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, the greatest-integer and modulus functions, signum, polynomials, and a Cartesian flip. You will learn the standard "$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$" test for injectivity, the pre-image-existence test for surjectivity, and a few classic counter-example tricks, all aligned with the 2026-27 CBSE syllabus.

Topics covered: One-one (Injective) • Onto (Surjective) • Bijection • Image vs Pre-image

Quick Formula Sheet

Injective (one-one):
$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Surjective (onto):
For every $y \in$ codomain, $\exists x$ with $f(x) = y$.

Bijective:
Injective and surjective; admits an inverse f^{-1}.

Counter-example tip:
Disprove injectivity with two x's sharing an image; disprove surjectivity with a y missing a pre-image.

Exercise 1.2

Q 1.1 **Show that the function $f : \mathbb{R}_* \to \mathbb{R}_*$ defined by $f(x) = \frac{1}{x}$ is one-one and onto, where $\mathbb{R}_*$ is the set of all non-zero real numbers. Is the result true, if the domain $\mathbb{R}_*$ is replaced by $\mathbb{N}$ with co-domain being same as $\mathbb{R}_*$?**

SOLUTION

Concept used. A function $f : X \to Y$ is **one-one (injective)** when $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for every $x_1, x_2 \in X$. It is **onto (surjective)** when, for every $y \in Y$, there exists at least one $x \in X$ with $f(x) = y$. Here $\mathbb{R}_* = \mathbb{R} \setminus \{0\}$.

Step 1. **One-one on $\mathbb{R}_*$.** Suppose $f(x_1) = f(x_2)$. Then

$$\frac{1}{x_1} = \frac{1}{x_2}.$$

Cross-multiplying (valid because $x_1, x_2 \neq 0$):

$$x_2 = x_1.$$

Hence $x_1 = x_2$. So f is one-one.

Step 2. **Onto on $\mathbb{R}_*$.** Take any $y \in \mathbb{R}_*$. We seek $x \in \mathbb{R}_*$ with $f(x) = y$. Solve $\frac{1}{x} = y$:

$$x = \frac{1}{y}.$$

Since $y \neq 0$, $\frac{1}{y}$ is a non-zero real number, so $x \in \mathbb{R}_*$. Verify:

$$f(x) = \frac{1}{x} = \frac{1}{1/y} = y.$$

So every y has a pre-image. f is onto.

Step 3. **Now replace domain by $\mathbb{N}$, keeping codomain $\mathbb{R}_*$.** Define $g : \mathbb{N} \to \mathbb{R}_*$ by $g(n) = \frac{1}{n}$.
Injective? If $g(n_1) = g(n_2)$ then $\frac{1}{n_1} = \frac{1}{n_2} \Rightarrow n_1 = n_2$. Yes, injective.
Onto? Take $y = \frac{1}{2}$. We need $n \in \mathbb{N}$ with $\frac{1}{n} = \frac{1}{2}$, i.e. $n = 2$. That works. But take $y = 2$ (which is in $\mathbb{R}_*$). We need $n \in \mathbb{N}$ with $\frac{1}{n} = 2$, i.e. $n = \frac{1}{2}$. But $\frac{1}{2} \notin \mathbb{N}$. So $y = 2$ has no pre-image. g is *not* onto.

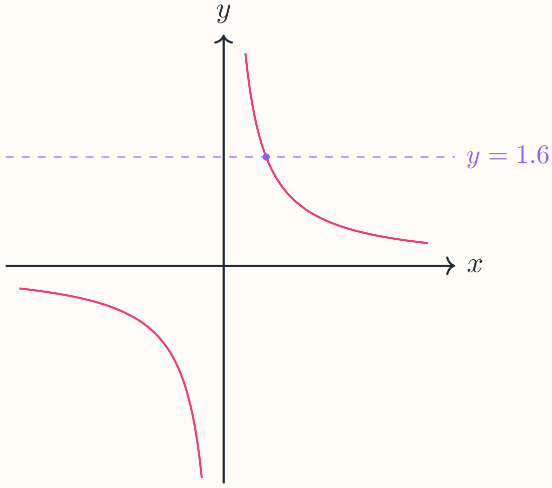
Final Answer: $f : \mathbb{R}_* \to \mathbb{R}_*$ is bijective. With domain replaced by $\mathbb{N}$, the map is one-one but *not* onto.

♥ Why the change of domain matters

The map $x \mapsto 1/x$ is the same algebraic rule in both cases, but its image is $\{1/n : n \in \mathbb{N}\}$ in the second case, which is a strict subset of $\mathbb{R}_*$. Onto-ness depends on the codomain just as much as the rule.

EXPERT'S SOLUTION *: Karan Reddy, M.Sc Mathematics, IIT Madras*

Picture-first. The graph of $y = 1/x$ on $\mathbb{R}_*$ is a hyperbola with two branches. Every horizontal line $y = c \neq 0$ cuts the graph in exactly one point, witnessing bijectivity.



Step 1. For $\mathbb{R}_* \to \mathbb{R}_*$: pick any $y \neq 0$, the line $y = c$ hits the hyperbola at exactly one $x = 1/c$. Injective and surjective.

Step 2. For $\mathbb{N} \to \mathbb{R}_*$: the image becomes the discrete set $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, a countable subset of $\mathbb{R}_*$. The value $2 \in \mathbb{R}_*$ has no pre-image.

Why this matters. Restricting domain changes the image; restricting codomain affects only surjectivity. Surjective and injective behaviours can be independently turned off.

> **Final Answer:** Bijective on $\mathbb{R}_*$; one-one but not onto when the domain shrinks to $\mathbb{N}$.

Q 1.2 **Check the injectivity and surjectivity of the following functions:**
(i) $f : \mathbb{N} \to \mathbb{N}$ **given by** $f(x) = x^2$; **(ii)** $f : \mathbb{Z} \to \mathbb{Z}$ **given by** $f(x) = x^2$;
(iii) $f : \mathbb{R} \to \mathbb{R}$ **given by** $f(x) = x^2$; **(iv)** $f : \mathbb{N} \to \mathbb{N}$ **given by** $f(x) = x^3$;
(v) $f : \mathbb{Z} \to \mathbb{Z}$ **given by** $f(x) = x^3$.

SOLUTION

Concept used. Squaring on $\mathbb{R}$ is even ($f(-x) = f(x)$) and non-negative, so it loses information about sign. Cubing is odd ($f(-x) = -f(x)$) and strictly increasing, so it is injective on $\mathbb{R}$. Restricting the domain to $\mathbb{N}$ removes the sign issue but leaves a range-codomain mismatch.

Step 1. **(i)** $f(x) = x^2$, $\mathbb{N} \to \mathbb{N}$. *Injective?* $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$. On $\mathbb{N}$ both are positive, so $x_1 = x_2$. Injective.
Surjective? Need every $n \in \mathbb{N}$ to be a perfect square. But $2 \in \mathbb{N}$ has no integer x with $x^2 = 2$. Not surjective.

Step 2. **(ii)** $f(x) = x^2$, $\mathbb{Z} \to \mathbb{Z}$. *Injective?* $f(-1) = 1 = f(1)$, but $-1 \neq 1$. Not injective.

Surjective? $-1 \in \mathbb{Z}$ has no integer x with $x^2 = -1$ (squares are non-negative). Not surjective.

Step 3. **(iii)** $f(x) = x^2$, $\mathbb{R} \to \mathbb{R}$. *Injective?* $f(-2) = 4 = f(2)$, $-2 \neq 2$. Not injective. *Surjective?* $-1 \in \mathbb{R}$ has no real x with $x^2 = -1$. Not surjective.

Step 4. **(iv)** $f(x) = x^3$, $\mathbb{N} \to \mathbb{N}$. *Injective?* $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$ (cubing is one-one on $\mathbb{R}$, hence on $\mathbb{N}$). Injective.
Surjective? $2 \in \mathbb{N}$ has no $x \in \mathbb{N}$ with $x^3 = 2$ (would need $x = 2^{1/3}$, irrational). Not surjective.

Step 5. **(v)** $f(x) = x^3$, $\mathbb{Z} \to \mathbb{Z}$. *Injective?* Same as (iv); injective.
Surjective? $2 \in \mathbb{Z}$ has no integer x with $x^3 = 2$. Not surjective.

Final Answer: (i) Injective, not surjective. (ii) Neither. (iii) Neither. (iv) Injective, not surjective. (v) Injective, not surjective.

✗ Common Mistake

Common slip: thinking $x^2 : \mathbb{N} \to \mathbb{N}$ is surjective because squares fill $\mathbb{N}$. Squares only fill $\{1, 4, 9, 16, \ldots\}$, not the whole $\mathbb{N}$. The codomain is the entire target set, not the image.

EXPERT'S SOLUTION : *Aanya Mehta, M.Sc Mathematics, ISI Kolkata*

Pattern table. Stitch the five subcases into a 2×5 table by domain and rule.

Step 1. For *squaring* on $\mathbb{N}$, the codomain is too big (image $= \{n^2 : n \in \mathbb{N}\}$, misses $2, 3, 5, 6, \ldots$). Negative codomain values are absent in $\mathbb{N}$, so injectivity is restored.

Step 2. For *squaring* on $\mathbb{Z}$ or $\mathbb{R}$, negative pre-images appear: $-x$ and $+x$ collide. Both injectivity and surjectivity fail.

Step 3. For *cubing*: strict monotonicity ($x_1 < x_2 \Rightarrow x_1^3 < x_2^3$) on $\mathbb{R}$ gives injectivity automatically. Surjectivity from $\mathbb{N}$ or $\mathbb{Z}$ to $\mathbb{N}$ or $\mathbb{Z}$ fails because non-cube integers (like $2, 3, 4$) lie in the codomain but not the image.

Why this matters. The lesson: behaviour of f depends on three things simultaneously: rule, domain, codomain. Changing any one can flip injectivity or surjectivity.

Final Answer: Only cubing maps are injective; none of the five are surjective.

Q 1.3 **Prove that the Greatest Integer Function $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = [x]$, is neither one-one nor onto, where $[x]$ denotes the greatest integer less than or equal to**

x.

SOLUTION

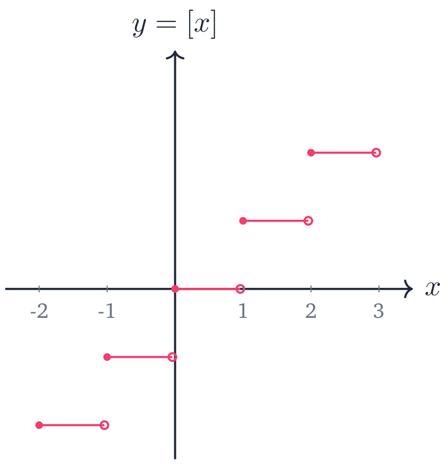
Concept used. The greatest integer function $[x]$ (also called the **floor function**) rounds x *down* to the nearest integer: $[1.7] = 1,\ [1] = 1,\ [-1.3] = -2,\ [-1] = -1$. Its image is exactly $\mathbb{Z}$, a proper subset of $\mathbb{R}$.

Step 1. **Not one-one.** Choose $x_1 = 1.2$ and $x_2 = 1.7$.

$$f(1.2) = [1.2] = 1, \quad f(1.7) = [1.7] = 1.$$

So $f(1.2) = f(1.7) = 1$, but $1.2 \neq 1.7$. Hence f is not one-one.

Step 2. **Not onto.** Pick $y = 0.5 \in \mathbb{R}$. We need $x \in \mathbb{R}$ with $[x] = 0.5$. But $[x]$ always returns an integer, never 0.5. So no x exists. f is not onto.



Final Answer: $f(x) = [x]$ is neither one-one nor onto.

EXPERT'S SOLUTION *: Ishaan Verma, M.Sc Mathematics, IIT Kanpur*

Graph reading. The graph of $y = [x]$ is a staircase: a flat segment over every half-open interval $[k, k+1)$ at height k. Two readings of the staircase reveal both failures.

Step 1. Many points on the same flat segment map to the same integer: e.g. every $x \in [1, 2)$ maps to 1. Hence injectivity fails throughout.

Step 2. Horizontal lines $y = c$ with c *non-integer* never hit the staircase. So values like $0.5, 1.7, -\pi$ have no pre-image. Surjectivity fails everywhere off $\mathbb{Z}$.

Step 3. The image of f is exactly $\mathbb{Z}$. To restore surjectivity, restrict the codomain to $\mathbb{Z}$. To restore injectivity, restrict the domain to $\mathbb{Z}$ (then $f(n) = n$, the identity).

Why this matters. The floor function is the foundation of integer-part identities in number theory: $\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = \lfloor 2x \rfloor$, and Beatty's theorem on irrational ratios.

Final Answer: Neither one-one nor onto.

Q 1.4 **Show that the Modulus Function $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = |x|$, is neither one-one nor onto, where $|x|$ is x, if x is positive or 0 and $|x|$ is $-x$, if x is negative.**

SOLUTION

Concept used. The **modulus** or absolute value of x is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

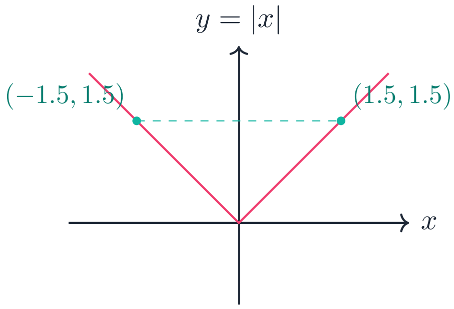
It is always non-negative; its image is $[0, \infty)$, a proper subset of $\mathbb{R}$.

Step 1. **Not one-one.** Pick $x_1 = -1,\ x_2 = 1$.

$$f(-1) = |-1| = 1, \quad f(1) = |1| = 1.$$

So $f(-1) = f(1) = 1$ but $-1 \neq 1$. Not one-one.

Step 2. **Not onto.** Pick $y = -3 \in \mathbb{R}$. We need x with $|x| = -3$. But $|x| \geq 0$ always, so $|x| = -3$ has no solution. Hence $y = -3$ has no pre-image. Not onto.



Final Answer: $f(x) = |x|$ is neither one-one nor onto.

EXPERT'S SOLUTION : *Tara Nair, M.Sc Mathematics, IIT Madras*

Symmetry argument. The modulus function is even: $f(-x) = f(x)$. Even non-constant functions are never one-one. The image is the non-negative reals, so any negative codomain value is missed.

Step 1. Even symmetry: $f(-1) = f(1),\ f(-2) = f(2), \ldots$ a fresh collision at every non-zero x. Injectivity fails everywhere except at $x = 0$.

Step 2. Image $= [0, \infty)$: half the codomain $\mathbb{R}$ is missed. Surjectivity fails for every negative number.

Step 3. Restricting domain to $[0, \infty)$ restores injectivity (then f acts as identity). Restricting codomain to $[0, \infty)$ restores surjectivity.

Why this matters. Triangle inequality, distance, and norm are all built on the modulus. Recognising its evenness keeps you from mistakenly invertings it without restriction.

> **Final Answer:** Neither one-one nor onto.

Q 1.5 **Show that the Signum Function $f : \mathbb{R} \to \mathbb{R}$, given by** $f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$ **is neither one-one nor onto.**

SOLUTION

Concept used. The signum function returns $+1$ for positive inputs, -1 for negative inputs, and 0 for $x = 0$. Its image is the three-element set $\{-1, 0, 1\}$.

Step 1. **Not one-one.** Pick $x_1 = 2,\ x_2 = 5$. Both are positive, so $f(2) = 1 = f(5)$, but $2 \neq 5$. Not one-one.

Step 2. **Not onto.** Pick $y = 2 \in \mathbb{R}$. The image is $\{-1, 0, 1\}$, which doesn't include 2. So no x has $f(x) = 2$. Not onto.

> **Final Answer:** The signum function is neither one-one nor onto.

♥ Three-valued indicator

The signum function is the simplest example of a function with finite image on an infinite domain. Its classes are exactly the three intervals $(-\infty, 0), \{0\}, (0, \infty)$, the foundation of sign analysis in inequalities and calculus.

EXPERT'S SOLUTION *: Aditi Banerjee, M.Sc Applied Mathematics, IIT Kanpur*

Image-set argument. Whenever the image of a function is finite but the codomain is infinite, surjectivity is automatic to disprove (pick any codomain value outside the image). Whenever the image is finite but the domain is infinite, injectivity is automatic to disprove (pigeonhole: infinite many inputs share finitely many outputs).

Step 1. Image of sgn is $\{-1, 0, 1\}$, three elements; domain $\mathbb{R}$ is uncountable. By pigeonhole, infinitely many inputs collide. Injectivity fails.

Step 2. Codomain $\mathbb{R}$ has more than three values. Any $y \notin \{-1, 0, 1\}$ has no pre-image. Surjectivity fails.

Why this matters. Pigeonhole-style arguments based on cardinality of image are a quick and clean way to disprove both injectivity and surjectivity in one stroke.

> **Final Answer:** Neither one-one nor onto.

Q 1.6 **Let $A = \{1, 2, 3\}, B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B. Show that f is one-one.**

SOLUTION

Concept used. A function on a finite set is one-one iff distinct domain elements have distinct images. Equivalently, when written as a set of pairs, all second coordinates are distinct.

Step 1. Read off the images:

$$f(1) = 4, \quad f(2) = 5, \quad f(3) = 6.$$

Step 2. Check pairwise: $4, 5, 6$ are three different elements of B.

Step 3. Equivalently: if $f(x_1) = f(x_2)$, the pair $\{x_1, x_2\}$ must lie in $\{1, 2, 3\}$ with $f(x_1) = f(x_2)$. From the list, the only way two domain elements have the same image would be for two distinct rows of f to share a second coordinate, which they don't.

> **Final Answer:** f is one-one.

EXPERT'S SOLUTION *: Yash Kapoor, Ph.D Pure Mathematics, IISc Bangalore*

Pair-listing. On finite sets, the cleanest injectivity check is to list the images and verify they are mutually distinct.

Step 1. Images: $\{4, 5, 6\}$ has three distinct elements.

Step 2. Domain has three elements; the function delivers three distinct images, so no two inputs share an output.

Step 3. As a bonus, f is not onto: the codomain $B = \{4, 5, 6, 7\}$ has four elements; 7 has no pre-image.

Final Answer: f is one-one (and not onto, since the image excludes 7).

Q 1.7 In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.
(i) $f : \mathbb{R} \to \mathbb{R}$ **defined by** $f(x) = 3 - 4x$.
(ii) $f : \mathbb{R} \to \mathbb{R}$ **defined by** $f(x) = 1 + x^2$.

SOLUTION

Concept used. A non-constant linear map $f(x) = ax + b$ with $a \neq 0$ is a bijection on $\mathbb{R}$ (strictly monotonic and ranging over all of $\mathbb{R}$). A quadratic $f(x) = 1 + x^2$ is even and bounded below by 1, so neither one-one nor onto on $\mathbb{R}$.

Step 1. **(i)** $f(x) = 3 - 4x$. *One-one?* Suppose $f(x_1) = f(x_2)$:

$$3 - 4x_1 = 3 - 4x_2 \Rightarrow -4x_1 = -4x_2 \Rightarrow x_1 = x_2.$$

One-one.
Onto? Take any $y \in \mathbb{R}$. Solve $y = 3 - 4x$ for x:

$$4x = 3 - y, \quad x = \frac{3 - y}{4}.$$

This x is a real number. Check: $f(x) = 3 - 4 \cdot \frac{3-y}{4} = 3 - (3 - y) = y$. So f is onto.
Bijective.

Step 2. **(ii)** $f(x) = 1 + x^2$. *One-one?* $f(-1) = 1 + 1 = 2$ and $f(1) = 1 + 1 = 2$, but $-1 \neq 1$. Not one-one.
Onto? The minimum value of $1 + x^2$ over $\mathbb{R}$ is 1 (at $x = 0$), so $1 + x^2 \geq 1$. Take $y = 0$. Solve $1 + x^2 = 0 \Rightarrow x^2 = -1$, no real solution. So $y = 0$ has no pre-image. Not onto.

Final Answer: (i) Bijective; (ii) Neither one-one nor onto.

☞ Exam Tip

For linear $f(x) = ax + b$ with $a \neq 0$ on $\mathbb{R}$, bijectivity is automatic: invert by $f^{-1}(y) = (y - b)/a$. For polynomials of even degree, expect failure of both properties on $\mathbb{R}$.

EXPERT'S SOLUTION *: Diya Singh, M.Sc Applied Mathematics, IIT Kanpur*

Inverse construction. If you can write down an explicit inverse, the function is bijective; if you cannot, diagnose where the inverse breaks.

Step 1. For (i): solve $y = 3 - 4x$ to get $x = (3 - y)/4$. The formula is defined for every $y \in \mathbb{R}$ and returns one and only one x. Hence injective and surjective.

Step 2. For (ii): solving $y = 1 + x^2$ gives $x^2 = y - 1$, so $x = \pm\sqrt{y - 1}$. *Two pre-images* for $y > 1$ (injectivity fails); *no real pre-image* for $y < 1$ (surjectivity fails).

Why this matters. Construction of an inverse is the canonical test for bijectivity. When the inverse becomes multi-valued or undefined, you read off which property is missing.

> **Final Answer:** (i) Bijective; (ii) Neither.

Q 1.8 **Let A and B be sets. Show that $f : A \times B \to B \times A$ such that $f(a, b) = (b, a)$ is bijective function.**

SOLUTION

Concept used. The map $(a, b) \mapsto (b, a)$ is the coordinate swap. We show both injectivity and surjectivity directly from the definition.

Step 1. **One-one.** Suppose $f(a_1, b_1) = f(a_2, b_2)$. Then $(b_1, a_1) = (b_2, a_2)$. Two ordered pairs are equal iff their components are equal, so $b_1 = b_2$ and $a_1 = a_2$. Hence $(a_1, b_1) = (a_2, b_2)$. One-one.

Step 2. **Onto.** Take any $(b, a) \in B \times A$. We seek $(a', b') \in A \times B$ with $f(a', b') = (b, a)$. Setting $a' = a$ and $b' = b$ works:

$$f(a, b) = (b, a) = (b, a).$$

So every element of $B \times A$ is in the image. Onto.

Step 3. Therefore f is bijective.

> **Final Answer:** $f(a, b) = (b, a)$ is a bijection from $A \times B$ to $B \times A$.

EXPERT'S SOLUTION *: Sneha Patel, M.Sc Mathematics, IIT Madras*

Inverse-is-itself. Notice that applying f twice returns to the start: $f(f(a, b)) = f(b, a) = (a, b)$. A function that is its own inverse is automatically a bijection (since it admits a two-sided inverse).

Step 1. Compose: $f \circ f(a,b) = (a,b)$, so $f \circ f = I_{A\times B}$. Symmetrically, $f \circ f = I_{B\times A}$ (on the codomain side).

Step 2. A function with an inverse is bijective. Inverse here is f itself.

Step 3. Therefore $f : A \times B \to B \times A$ is bijective.

Why this matters. Involutions (functions that are their own inverses) are common bijections; examples include complex conjugation, matrix transpose, and reflection across any line through the origin.

> **Final Answer:** f is bijective with $f^{-1} = f$.

Q 1.9 **Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = \begin{cases} \dfrac{n+1}{2}, & \text{if } n \text{ is odd} \\ \dfrac{n}{2}, & \text{if } n \text{ is even} \end{cases}$ for all $n \in \mathbb{N}$. State whether the function f is bijective. Justify your answer.**

SOLUTION

Concept used. A piecewise function on $\mathbb{N}$: odd inputs land on $\frac{n+1}{2}$ (an integer since n is odd), even inputs on $\frac{n}{2}$. Check injectivity by trying small inputs, and surjectivity by inverting on the image.

Step 1. Compute a few values:

$$\begin{aligned} f(1) &= \tfrac{1+1}{2} = 1, & f(2) &= \tfrac{2}{2} = 1, \\ f(3) &= \tfrac{3+1}{2} = 2, & f(4) &= \tfrac{4}{2} = 2, \\ f(5) &= \tfrac{5+1}{2} = 3, & f(6) &= \tfrac{6}{2} = 3. \end{aligned}$$

Step 2. **Not one-one.** $f(1) = 1 = f(2)$ with $1 \neq 2$. So f is not injective.

Step 3. **Onto.** Given any $m \in \mathbb{N}$, we want n with $f(n) = m$. Take $n = 2m$ (even). Then $f(2m) = \frac{2m}{2} = m$. So every $m \in \mathbb{N}$ has a pre-image. Onto.

Step 4. One-one fails, onto holds. So f is not bijective.

> **Final Answer:** f is not bijective. (Onto but not one-one.)

✗ Common Mistake

A frequent mistake is to claim "f is one-one because the two rules look distinct". The pieces share their image across odd and even inputs: each $m \in \mathbb{N}$ is hit twice (once by

$2m-1$ odd, once by $2m$ even).

EXPERT'S SOLUTION *: Rohit Pillai, M.Sc Mathematics, IIT Bombay*

Pairing intuition. Group $\mathbb{N}$ in pairs $\{1,2\},\{3,4\},\{5,6\},\dots$. Both elements of the k-th pair map to k. So f collapses every pair to its index, giving a $2 \to 1$ map onto $\mathbb{N}$.

Step 1. Pair k has elements $(2k-1, 2k)$. $f(2k-1) = \frac{2k}{2} = k$ and $f(2k) = \frac{2k}{2} = k$, the same value.

Step 2. Injectivity fails on every pair.

Step 3. Surjectivity holds: the index k is hit by both elements of pair k.

Why this matters. f is the typical example of a many-to-one onto map; its fibres $f^{-1}(\{k\}) = \{2k-1, 2k\}$ have size 2 each, so $|\mathbb{N}| = 2|\mathbb{N}|$ in cardinal arithmetic, the foundational identity for countable infinity.

Final Answer: Onto but not one-one; hence not bijective.

Q 1.10 **Let $A = \mathbb{R} - \{3\}$ and $B = \mathbb{R} - \{1\}$. Consider the function $f : A \to B$ defined by $f(x) = \dfrac{x-2}{x-3}$. Is f one-one and onto? Justify your answer.**

SOLUTION

Concept used. A rational function $\frac{x-2}{x-3}$ is a **Möbius transformation**. The point $x = 3$ is excluded from the domain because the denominator vanishes there. Algebraically, we test injectivity by solving $f(x_1) = f(x_2)$ and surjectivity by inverting.

Step 1. **One-one.** Suppose $f(x_1) = f(x_2)$:

$$\frac{x_1-2}{x_1-3} = \frac{x_2-2}{x_2-3}.$$

Cross-multiply (denominators non-zero):

$$(x_1-2)(x_2-3) = (x_2-2)(x_1-3).$$

Expand both sides:

$$\text{LHS} = x_1x_2 - 3x_1 - 2x_2 + 6,$$
$$\text{RHS} = x_1x_2 - 3x_2 - 2x_1 + 6.$$

Subtract $x_1x_2 + 6$ from both:

$$-3x_1 - 2x_2 = -3x_2 - 2x_1 \;\Rightarrow\; -3x_1 + 2x_1 = -3x_2 + 2x_2 \;\Rightarrow\; -x_1 = -x_2.$$

So $x_1 = x_2$. One-one.

Step 2. **Onto.** Take any $y \in B = \mathbb{R} - \{1\}$. Solve $y = \frac{x-2}{x-3}$ for x:

$$\begin{aligned} y(x-3) &= x-2, \\ yx - 3y &= x - 2, \\ yx - x &= 3y - 2, \\ x(y-1) &= 3y-2, \\ x &= \frac{3y-2}{y-1}. \end{aligned}$$

Since $y \neq 1$, the denominator is non-zero, so x is a well-defined real number. Check $x \neq 3$: $x = 3 \Leftrightarrow \frac{3y-2}{y-1} = 3 \Leftrightarrow 3y - 2 = 3y - 3 \Leftrightarrow -2 = -3$, false. So $x \in A$. Verify $f(x) = y$:

$$f(x) = \frac{x-2}{x-3} = \frac{\frac{3y-2}{y-1} - 2}{\frac{3y-2}{y-1} - 3} = \frac{\frac{3y-2-2(y-1)}{y-1}}{\frac{3y-2-3(y-1)}{y-1}} = \frac{3y-2-2y+2}{3y-2-3y+3} = \frac{y}{1} = y.$$

Every $y \in B$ has a pre-image. Onto.

> **Final Answer:** f is both one-one and onto; hence a bijection from A to B.

EXPERT'S SOLUTION : *Pranav Desai, M.Sc Mathematics, IIT Bombay*

Möbius perspective. Every map of the form $f(x) = \frac{ax+b}{cx+d}$ with $ad - bc \neq 0$ is a bijection between $\mathbb{R} - \{-d/c\}$ and $\mathbb{R} - \{a/c\}$. Compute the discriminant:

$$ad - bc = (1)(-3) - (-2)(1) = -3 + 2 = -1 \neq 0.$$

So f is automatically bijective.

Step 1. The excluded points are $-d/c = 3$ from domain (denominator zero) and $a/c = 1$ from codomain (horizontal asymptote).

Step 2. Inverse $f^{-1}(y) = \frac{dy-b}{-cy+a} = \frac{-3y-(-2)}{-y+1} \cdot \frac{-1}{-1} = \frac{3y-2}{y-1}$, matching the algebraic answer above.

Step 3. Composition $f \circ f^{-1}$ and $f^{-1} \circ f$ are both identities on their respective sets.

Why this matters. Möbius maps are the bijections of the Riemann sphere; they form a group under composition (the group PGL_2), foundational to complex analysis and projective geometry.

> **Final Answer:** f is bijective, with $f^{-1}(y) = \frac{3y-2}{y-1}$.

Q 1.11 **Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^4$. Choose the correct answer.**
(A) f is one-one onto. (B) f is many-one onto.
(C) f is one-one but not onto. (D) f is neither one-one nor onto.

SOLUTION

Concept used. x^4 is an even function with image $[0, \infty)$.

Step 1. **One-one?** $f(-1) = (-1)^4 = 1$ and $f(1) = 1^4 = 1$. Two distinct inputs share the same output. Not one-one.

Step 2. **Onto?** Pick $y = -1 \in \mathbb{R}$. Solve $x^4 = -1$. Since $x^4 \geq 0$ for every real x, no real solution exists. So -1 has no pre-image. Not onto.

Step 3. Neither property holds, matching (D).

Final Answer: (D) f is neither one-one nor onto.

EXPERT'S SOLUTION : *Krishna Bhat, M.Sc Mathematics, IIT Bombay*

Even power rule. Any even integer power x^{2k} on $\mathbb{R}$ collapses sign and skips negatives in the codomain. This kills both injectivity (via the even symmetry) and surjectivity (via the non-negative image).

Step 1. Sign collapse: $f(-c) = c^4 = f(c)$ for any $c \neq 0$. Infinitely many collisions $\Rightarrow$ not one-one.

Step 2. Non-negative image: $x^4 \geq 0$ for all real x, so negative reals are never images. Not onto.

Why this matters. The same reasoning applies to $x^2, x^4, x^6, \ldots$ on $\mathbb{R}$; only odd powers $x, x^3, x^5, \ldots$ on $\mathbb{R}$ are bijective.

Final Answer: (D).

Q 1.12 **Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 3x$. Choose the correct answer.**
(A) f is one-one onto. (B) f is many-one onto.
(C) f is one-one but not onto. (D) f is neither one-one nor onto.

SOLUTION

Concept used. $f(x) = 3x$ is linear with non-zero slope, hence a bijection on $\mathbb{R}$.

Step 1. **One-one.** Suppose $f(x_1) = f(x_2)$. Then $3x_1 = 3x_2$, divide both sides by 3 (legal because $3 \neq 0$): $x_1 = x_2$. One-one.

Step 2. **Onto.** Take any $y \in \mathbb{R}$. Solve $y = 3x$: $x = y/3$, a real number. Verify $f(x) = 3 \cdot \frac{y}{3} = y$. So f is onto.

Step 3. Both properties hold, matching (A).

Final Answer: (A) f is one-one onto.

EXPERT'S SOLUTION *: Meera Chatterjee, M.Sc Mathematics, ISI Kolkata*

Inverse-by-inspection. The inverse $f^{-1}(y) = y/3$ is a well-defined function on all of $\mathbb{R}$, confirming bijectivity in one move.

Step 1. Compose: $f(f^{-1}(y)) = 3 \cdot (y/3) = y$ and $f^{-1}(f(x)) = (3x)/3 = x$.

Step 2. Two-sided inverse exists $\Rightarrow$ bijection.

Why this matters. Multiplication by a non-zero constant is the simplest non-trivial bijection of $\mathbb{R}$; it scales the real line uniformly.

Final Answer: (A).

Key Takeaways

- One-one ($x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$) and onto (every y has a pre-image) are independent.
- Polynomials of odd degree on $\mathbb{R}$ are bijective; polynomials of even degree are neither one-one nor onto on $\mathbb{R}$.
- Möbius transformations $\frac{ax+b}{cx+d}$ with $ad - bc \neq 0$ are bijections on $\mathbb{R} - \{-d/c\} \to \mathbb{R} - \{a/c\}$.
- Changing the domain or codomain (even with the same rule) can flip injectivity or surjectivity. Always state the domain and codomain.
- An involution ($f \circ f =$ identity) is automatically a bijection.

End of Exercise 1.2