Collegedunia NCERT Solutions

Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 1: Relations and Functions

About this Chapter

The Miscellaneous Exercise on Chapter 1 pulls together every thread of the chapter: injective and surjective maps, equivalence relations on $\mathcal{P}(X)$, counting one-to-one and onto functions, and equality of functions defined on a four-element set. The seven problems span a creative range and are perfect revision before examinations. All solutions follow the 2026-27 CBSE syllabus.

Topics covered: Bijection on $\mathbb{R}$ • Injectivity • Equivalence on $\mathcal{P}(X)$ • Counting Onto Maps • MCQs

Quick Formula Sheet

Injective: $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Surjective: $\forall\, y \in$ codomain, $\exists\, x$ with $f(x) = y$.

Equivalence: Reflexive + Symmetric + Transitive.

Onto maps $\{1, \ldots, n\} \to$ itself: Equals $n!$ (each onto map on a finite set of size n is a permutation).

Miscellaneous Exercise

Q 1.1 **Show that the function $f : \mathbb{R} \to \{x \in \mathbb{R} : -1 < x < 1\}$ defined by $f(x) = \dfrac{x}{1+|x|}$, $x \in \mathbb{R}$, is one one and onto function.**

SOLUTION

Concept used. The codomain $(-1, 1)$ is the open unit interval. The function $f(x) = \frac{x}{1+|x|}$ is an odd, strictly increasing map from $\mathbb{R}$ onto $(-1, 1)$. Because $|x|$ behaves differently for $x \geq 0$ and $x < 0$, treat the two pieces separately.

Step 1. Split by sign.

$$f(x) = \begin{cases} \dfrac{x}{1+x}, & x \geq 0, \\ \dfrac{x}{1-x}, & x < 0. \end{cases}$$

Note $f(0) = 0$.

Step 2. One-one on each piece. *Case* $x \geq 0$. Suppose $f(x_1) = f(x_2)$ for $x_1, x_2 \geq 0$:

$$\frac{x_1}{1+x_1} = \frac{x_2}{1+x_2}.$$

Cross-multiply: $x_1(1+x_2) = x_2(1+x_1)$, i.e. $x_1 + x_1x_2 = x_2 + x_1x_2$, so $x_1 = x_2$.

Case $x < 0$. Suppose $f(x_1) = f(x_2)$ for $x_1, x_2 < 0$:

$$\frac{x_1}{1-x_1} = \frac{x_2}{1-x_2}.$$

Cross-multiply: $x_1(1-x_2) = x_2(1-x_1)$, i.e. $x_1 - x_1x_2 = x_2 - x_1x_2$, so $x_1 = x_2$.

Step 3. No collision across pieces. If $x_1 \geq 0$ then $f(x_1) = \frac{x_1}{1+x_1} \geq 0$. If $x_2 < 0$ then $f(x_2) = \frac{x_2}{1-x_2} < 0$ (negative numerator, positive denominator). So images on the two pieces have opposite signs and never coincide except at 0. Hence f is one-one on the whole of $\mathbb{R}$.

Step 4. Onto $(-1, 1)$**.** Take any $y \in (-1, 1)$.

Subcase $0 \leq y < 1$. Solve $y = \frac{x}{1+x}$ for $x \geq 0$:

$$y(1+x) = x \;\Rightarrow\; y + yx = x \;\Rightarrow\; y = x - yx = x(1-y) \;\Rightarrow\; x = \frac{y}{1-y}.$$

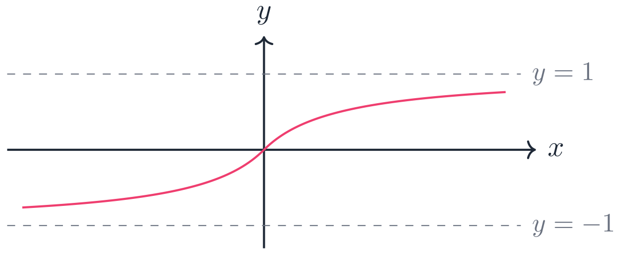
Since $0 \leq y < 1$, $1 - y > 0$, so $x = \frac{y}{1-y} \geq 0$, valid.

Subcase $-1 < y < 0$. Solve $y = \frac{x}{1-x}$ for $x < 0$:

$$y(1-x) = x \;\Rightarrow\; y - yx = x \;\Rightarrow\; y = x + yx = x(1+y) \;\Rightarrow\; x = \frac{y}{1+y}.$$

Since $-1 < y < 0$, $1 + y > 0$ and $y < 0$, so $x = \frac{y}{1+y} < 0$, valid.

In both subcases, x exists in $\mathbb{R}$ with $f(x) = y$. Onto.



Final Answer: f is one-one and onto.

♥ Compactifying $\mathbb{R}$

This f is the classical homeomorphism between $\mathbb{R}$ and the open interval $(-1, 1)$. It shows that $\mathbb{R}$ is topologically the same as a bounded open interval, a starting point for the one-point compactification of the real line.

EXPERT'S SOLUTION *: Arjun Iyer, M.Sc Mathematics, IIT Bombay*

Inverse formula. The cleanest proof of bijectivity is an explicit two-sided inverse.

Step 1. Define $g : (-1, 1) \to \mathbb{R}$ by $g(y) = \frac{y}{1-|y|}$.

Step 2. Compute $f(g(y))$. With $y \geq 0$: $g(y) = \frac{y}{1-y} \geq 0$ and $|g(y)| = \frac{y}{1-y}$, so

$$f(g(y)) = \frac{g(y)}{1 + |g(y)|} = \frac{y/(1-y)}{1 + y/(1-y)} = \frac{y/(1-y)}{(1-y+y)/(1-y)} = \frac{y/(1-y)}{1/(1-y)} = y.$$

With $y < 0$ the computation is analogous and yields y.

Step 3. Compute $g(f(x))$. With $x \geq 0$: $f(x) = \frac{x}{1+x} \geq 0$ and $|f(x)| = \frac{x}{1+x}$, so

$$g(f(x)) = \frac{f(x)}{1 - |f(x)|} = \frac{x/(1+x)}{1 - x/(1+x)} = \frac{x/(1+x)}{1/(1+x)} = x.$$

With $x < 0$, the computation mirrors and yields x.

Step 4. f has a two-sided inverse g, so f is a bijection.

Why this matters. Constructing an explicit inverse is the gold standard for proving bijectivity. It also hands you the inverse function for free, useful in calculus (compute derivative of f^{-1} via implicit differentiation).

Final Answer: f is bijective with $f^{-1}(y) = \frac{y}{1-|y|}$.

Q 1.2 Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is injective.

SOLUTION

Concept used. A function f is injective iff $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. The cube function is strictly increasing on $\mathbb{R}$ because its derivative $f'(x) = 3x^2 \geq 0$ (equal to 0 only at $x = 0$, but the function is strictly increasing across 0 since the cube preserves sign).

Step 1. Suppose $f(x_1) = f(x_2)$, i.e. $x_1^3 = x_2^3$.

Step 2. Take cube roots:

$$(x_1^3)^{1/3} = (x_2^3)^{1/3} \;\Rightarrow\; x_1 = x_2.$$

The cube root is a well-defined real function on all of $\mathbb{R}$ (unlike the square root, it accepts negatives).

Step 3. Algebraic check (without cube roots): $x_1^3 = x_2^3 \Leftrightarrow x_1^3 - x_2^3 = 0$. Factor:

$$x_1^3 - x_2^3 = (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0.$$

The second factor is a quadratic form in x_1, x_2: complete the square in x_1:

$$x_1^2 + x_1x_2 + x_2^2 = \left(x_1 + \frac{x_2}{2}\right)^2 + \frac{3x_2^2}{4} \geq 0,$$

with equality only when $x_1 + \frac{x_2}{2} = 0$ and $x_2 = 0$, i.e. when $x_1 = x_2 = 0$. So when x_1, x_2 are not both zero, the quadratic factor is strictly positive, forcing $x_1 - x_2 = 0$.

Final Answer: $f(x) = x^3$ is injective on $\mathbb{R}$.

EXPERT'S SOLUTION *: Vivaan Kumar, Ph.D Pure Mathematics, IISc Bangalore*

Monotonicity argument. A strictly monotonic function is automatically injective, because $x_1 < x_2$ forces $f(x_1) < f(x_2)$ (or the reverse for decreasing), so distinct inputs have distinct outputs.

Step 1. Show $f(x) = x^3$ is strictly increasing on $\mathbb{R}$. Take $x_1 < x_2$; we want $x_1^3 < x_2^3$.
Case $0 \leq x_1 < x_2$: multiplying the chain $x_1 < x_2$ by the positive number x_2^2 gives $x_1x_2^2 < x_2^3$; similarly $x_1^3 < x_1x_2^2$ (multiply by positive x_1). Chain: $x_1^3 \leq x_1x_2^2 < x_2^3$ if $x_1 > 0$, and $x_1^3 = 0 < x_2^3$ if $x_1 = 0$.
Case $x_1 < x_2 \leq 0$: both negative, x_1^3 and x_2^3 are negative. By odd symmetry, $-x_1^3 = (-x_1)^3$ and $-x_2^3 = (-x_2)^3$. Apply the positive case to $-x_2 < -x_1$.
Case $x_1 < 0 \leq x_2$: $x_1^3 < 0 \leq x_2^3$.

Step 2. Strictly increasing $\Rightarrow$ injective.

Why this matters. Monotonicity is the cleanest sufficient condition for injectivity of a continuous function. It also tells you the inverse exists and is itself continuous (and monotonic).

Final Answer: $f(x) = x^3$ is injective.

Q 1.3 **Given a non empty set X, consider $\mathcal{P}(X)$ which is the set of all subsets of X. Define the relation R in $\mathcal{P}(X)$ as follows: For subsets A, B in $\mathcal{P}(X)$, ARB if and only if $A \subset B$. Is R an equivalence relation on $\mathcal{P}(X)$? Justify your answer.**

SOLUTION

Concept used. $\mathcal{P}(X)$ is the **power set** of X: the collection of every subset of X, including $\emptyset$ and X itself. The relation $ARB \Leftrightarrow A \subset B$ (proper or improper inclusion; in the NCERT convention $\subset$ allows equality unless stated otherwise) needs to be tested on the three equivalence axioms.

Step 1. **Reflexive.** For any $A \in \mathcal{P}(X)$, $A \subset A$ holds (every element of A is an element of A). So ARA.

Step 2. **Symmetric?** Take $X = \{1, 2\}$. Let $A = \{1\}$ and $B = \{1, 2\}$. Then $A \subset B$ (every element of A is in B), so ARB. But $B \not\subset A$, because $2 \in B$ and $2 \notin A$. So BRA fails. *Not symmetric.*

Step 3. **Transitive.** Suppose ARB and BRC. Then $A \subset B$ and $B \subset C$. Take any $x \in A$. Then $x \in B$ (since $A \subset B$), then $x \in C$ (since $B \subset C$). So every $x \in A$ is in C, i.e. $A \subset C$. Transitive.

Step 4. Since symmetry fails, R is not an equivalence relation.

> **Final Answer:** R is reflexive and transitive but not symmetric; hence not an equivalence relation.

♥ Partial order on the power set

The subset relation on $\mathcal{P}(X)$ is reflexive, transitive, and anti-symmetric ($A \subset B$ and $B \subset A \Rightarrow A = B$). This makes $(\mathcal{P}(X), \subset)$ a **partial order**, in fact a complete lattice with $\emptyset$ as bottom and X as top.

EXPERT'S SOLUTION *: Priya Sharma, Ph.D Mathematics, IIT Delhi*

Counter-example first. Symmetry of $\subset$ would mean every $A \subset B$ forces $B \subset A$; combined with anti-symmetry this would force $A = B$. So subset is symmetric only on equal pairs, which is too narrow to call symmetric.

Step 1. Pick the smallest non-trivial $X = \{1\}$. Then $\mathcal{P}(X) = \{\emptyset, \{1\}\}$. The pair $(\emptyset, \{1\})$ has $\emptyset \subset \{1\}$ but $\{1\} \not\subset \emptyset$. Symmetry fails on this two-element example.

Step 2. Reflexivity and transitivity follow from the definition of $\subset$: "every element of ... is in ...".

Why this matters. Subset partial orders generalise to Boolean lattices, divisibility lattices, and topology (open sets ordered by inclusion). The non-symmetric nature is what makes them *partial orders* rather than equivalence relations.

> **Final Answer:** Reflexive and transitive, not symmetric, so not an equivalence.

Q 1.4 **Find the number of all onto functions from the set $\{1, 2, 3, \ldots, n\}$ to itself.**

SOLUTION

Concept used. On a finite set A of size n, a function $f : A \to A$ is onto iff it is one-one (both follow from $|A|$ being finite: range size equals domain size, and surjectivity forces every output to appear, leaving no room for collisions). So counting onto self-maps equals counting permutations of n symbols.

Step 1. Let $A = \{1, 2, \ldots, n\}$. An onto map $f : A \to A$ has $\text{Image}(f) = A$, which has n elements.

Step 2. Since the domain has n elements and the image has n elements, the map is a bijection (no two domain elements share an image, else the image would have fewer than n elements).

Step 3. Counting bijections $f : A \to A$ is the same as counting permutations of n symbols, which is $n!$.

> **Final Answer:** Number of onto functions from $\{1, 2, \ldots, n\}$ to itself $= n!$.

☞ **Exam Tip**

The finite-set equivalence "onto $\Leftrightarrow$ one-one" fails on infinite sets. On $\mathbb{N}$ the map $f(n) = 2n$ is one-one but not onto, and $f(n) = \lceil n/2 \rceil$ is onto but not one-one. This finite/infinite contrast is a popular MCQ trap.

EXPERT'S SOLUTION *: Aanya Gupta, Ph.D Mathematics, IIT Delhi*

Counting by choices. An onto self-map on n symbols must hit every output exactly once, so we are placing n distinct outputs into n slots.

Step 1. Slot 1 (input $= 1$): choose its image freely from $\{1, \ldots, n\}$. There are n choices.

Step 2. Slot 2 (input $= 2$): its image must differ from slot 1's image (else two slots share an output and the map is not onto, since one output would be missed). $n - 1$ choices.

Step 3. Continue: slot k has $n - k + 1$ choices.

Step 4. Total: $n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 = n!$.

Why this matters. The symmetric group S_n has exactly $n!$ elements; this exercise computes its order. The connection between bijections, permutations, and symmetry groups runs throughout algebra and combinatorics.

> **Final Answer:** $n!$.

Q 1.5 **Let $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$ and $f, g : A \to B$ be functions defined by $f(x) = x^2 - x,\ x \in A$ and $g(x) = 2\left|x - \frac{1}{2}\right| - 1,\ x \in A$. Are f and g equal? Justify your answer.**
(Hint: One may note that two functions $f : A \to B$ and $g : A \to B$ such that $f(a) = g(a)$ for all $a \in A$, are called equal functions.)

SOLUTION

Concept used. Two functions on the same domain and codomain are **equal** iff they agree pointwise: $f(a) = g(a)$ for every a. Verify by computing both at each of the four points of A.

Step 1. Compute f at each $a \in A$.

$$\begin{aligned} f(-1) &= (-1)^2 - (-1) = 1 + 1 = 2, \\ f(0) &= 0^2 - 0 = 0, \\ f(1) &= 1^2 - 1 = 1 - 1 = 0, \\ f(2) &= 2^2 - 2 = 4 - 2 = 2. \end{aligned}$$

Step 2. Compute g at each $a \in A$.

$$\begin{aligned} g(-1) &= 2\left|-1 - \tfrac{1}{2}\right| - 1 = 2 \cdot \tfrac{3}{2} - 1 = 3 - 1 = 2, \\ g(0) &= 2\left|0 - \tfrac{1}{2}\right| - 1 = 2 \cdot \tfrac{1}{2} - 1 = 1 - 1 = 0, \\ g(1) &= 2\left|1 - \tfrac{1}{2}\right| - 1 = 2 \cdot \tfrac{1}{2} - 1 = 1 - 1 = 0, \\ g(2) &= 2\left|2 - \tfrac{1}{2}\right| - 1 = 2 \cdot \tfrac{3}{2} - 1 = 3 - 1 = 2. \end{aligned}$$

Step 3. Compare:

$$f(-1) = 2 = g(-1), \quad f(0) = 0 = g(0), \quad f(1) = 0 = g(1), \quad f(2) = 2 = g(2).$$

Agreement at every input.

Final Answer: Yes, $f = g$ on A.

EXPERT'S SOLUTION *: Aditi Banerjee, M.Sc Applied Mathematics, IIT Kanpur*

Algebraic identification. Try to rewrite g in the same form as f. Since $|x - \frac{1}{2}|$ is the distance from $\frac{1}{2}$, multiplying by 2 gives $|2x - 1|$. So

$$g(x) = |2x - 1| - 1.$$

For integer x, $2x - 1$ is non-zero (odd). On $A = \{-1, 0, 1, 2\}$: $2x - 1$ equals $-3, -1, 1, 3$

respectively.

Step 1. Tabulate $|2x-1|-1$ at A: at $x=-1$ get $|-3|-1=2$. At $x=0$, $|-1|-1=0$. At $x=1$, $|1|-1=0$. At $x=2$, $|3|-1=2$.

Step 2. Tabulate $f(x)=x^2-x=x(x-1)$ at A: at $x=-1$, $(-1)(-2)=2$. At $x=0$, 0. At $x=1$, 0. At $x=2$, $(2)(1)=2$.

Step 3. Identical tables. Hence f and g agree on A, so $f=g$.

Why this matters. Two functions can be defined by very different formulas yet equal on a chosen finite domain; the formulas need only coincide pointwise. This is the basis of interpolation: many polynomials pass through the same finite set of points.

Final Answer: $f=g$ on $A=\{-1,0,1,2\}$.

Q 1.6 **Let $A=\{1,2,3\}$. Then the number of relations containing $(1,2)$ and $(1,3)$ which are reflexive and symmetric but not transitive is:**
(A) 1 (B) 2 (C) 3 (D) 4.

SOLUTION

Concept used. A relation on $A=\{1,2,3\}$ is a subset of $A\times A$ (9 ordered pairs in total). We need it to: (a) contain $(1,2)$ and $(1,3)$, (b) be reflexive: contain $(1,1),(2,2),(3,3)$, (c) be symmetric: contain reverses, so also $(2,1),(3,1)$, (d) NOT be transitive.

Step 1. **Mandatory pairs.** Reflexivity forces $(1,1),(2,2),(3,3)\in R$. Given pairs $(1,2),(1,3)$. Symmetry forces $(2,1),(3,1)$. So R must contain

$$S=\{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1)\}.$$

Step 2. **Remaining pairs.** The only pairs not yet forced are $(2,3)$ and $(3,2)$. By symmetry, they must be added together or not at all. Two options: add both, or add neither.

Step 3. **Test transitivity.**

Option 1: add neither. Take $R=S$. Check transitivity: $(2,1)\in R$ and $(1,3)\in R$, so transitivity requires $(2,3)\in R$. But $(2,3)\notin R$. So R is not transitive. Valid.

Option 2: add both $(2,3)$ and $(3,2)$. Then $S\cup\{(2,3),(3,2)\}$ contains $7+2=9$ pairs, i.e. all of $A\times A$, the universal relation, which is transitive (every chain closes). So this option fails the "not transitive" requirement. Invalid.

Step 4. Only Option 1 is valid. So the count is 1.

Final Answer: (A) 1.

EXPERT'S SOLUTION : *Neha Joshi, M.Sc Mathematics, ISI Kolkata*

Closure intuition. Adding $(2,3)$ (and forced $(3,2)$) makes the relation universal on $\{1,2,3\}$, which is automatically transitive. So the only way to avoid transitivity is to omit $\{(2,3),(3,2)\}$ and leave the missing two-step pre-condition unfilled.

Step 1. Start with S (the seven mandatory pairs). Check transitivity: chain $(2,1)\cdot(1,3)=(2,3)$ is missing from S, so S is not transitive.

Step 2. Adding $(2,3)$ alone violates symmetry. Adding $(2,3)+(3,2)$ restores symmetry but completes the universal relation, which is transitive. So no other option avoids both pitfalls.

Why this matters. Building relations with prescribed properties is a packed exercise in implication closure: each axiom forces new pairs, and the question is whether closure ever overshoots into the property you want to avoid.

Final Answer: (A) 1.

Q 1.7 **Let $A=\{1,2,3\}$. Then the number of equivalence relations containing $(1,2)$ is:**
(A) 1 (B) 2 (C) 3 (D) 4.

SOLUTION

Concept used. An equivalence relation on A corresponds bijectively to a **partition** of A. We need every equivalence relation that contains the pair $(1,2)$; equivalently, every partition of $\{1,2,3\}$ in which 1 and 2 lie in the same block.

Step 1. List partitions of $\{1,2,3\}$ with 1 and 2 together.

Partition P_1: $\{\{1,2\},\{3\}\}$. Class of $\{1,2\}$, class of $\{3\}$. The relation has $(1,1),(2,2),(3,3),(1,2),(2,1)$.

Partition P_2: $\{\{1,2,3\}\}$. One single block: the universal relation, which contains every pair, including $(1,2)$.

Step 2. Are there others? A partition with 1 and 2 in the same block must put 3 either with them (giving P_2) or alone (giving P_1). No third option.

Step 3. Hence exactly two equivalence relations contain $(1,2)$.

Final Answer: (B) 2.

EXPERT'S SOLUTION : *Ananya Reddy, M.Sc Mathematics, IIT Kanpur*

Partition correspondence. The fundamental bijection: equivalence relations on a finite set $A \leftrightarrow$ partitions of A. To force $(1,2) \in R$ is to force $1, 2$ into the same block.

Step 1. Position of 3: with $\{1,2\}$ (giving the full block $\{1,2,3\}$), or in its own singleton $\{3\}$. Two cases.

Step 2. Total partitions of $A = \{1,2,3\}$ is the Bell number $B_3 = 5$: $\{\{1\},\{2\},\{3\}\}$, $\{\{1,2\},\{3\}\}$, $\{\{1,3\},\{2\}\}$, $\{\{2,3\},\{1\}\}$, $\{\{1,2,3\}\}$. Of these, exactly two place 1 and 2 together.

Why this matters. The partition–equivalence bijection is one of the cleanest examples of mathematical duality. Counting equivalence relations on a finite set reduces to counting partitions, encoded by Bell numbers.

Final Answer: (B) 2.

Key Takeaways

- Bijectivity is most cleanly proved by constructing an explicit two-sided inverse.
- For finite sets of size n, onto $\Leftrightarrow$ one-one $\Leftrightarrow$ bijective, and the count of self-bijections is $n!$.
- Equivalence relations on a finite set correspond bijectively to partitions; the number of equivalence relations equals the Bell number.
- Subset relation $\subset$ on a power set is a partial order, not an equivalence (symmetry fails).
- Two functions are equal iff they agree pointwise on the domain. Different-looking formulas can define equal functions on a small domain.

End of Miscellaneous Exercise