Collegedunia NCERT Solutions

Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 2: Inverse Trigonometric Functions

About this Chapter

Exercise 2.1 of Class 12th Mathematics asks for **principal values** of inverse trigonometric functions. Each inverse trig function has a restricted range (its **principal value branch**) that makes the original trig function one-one. To find $\sin^{-1}(a)$ we look for the unique angle y in $[-\pi/2, \pi/2]$ with $\sin y = a$. The same idea applies to $\cos^{-1}$, $\tan^{-1}$, $\cot^{-1}$, $\sec^{-1}$, cosec^{-1}, each with its own principal range.

Topics covered: Principal value branches • Standard reference angles • Quadrant sign rules • Inverse trig identities

Quick Formula Sheet

Principal value ranges:

$\sin^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\cos^{-1} : [-1, 1] \to [0, \pi]$

$\tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$

$\cot^{-1} : \mathbb{R} \to (0, \pi)$

$\sec^{-1} : \mathbb{R} \setminus (-1, 1) \to [0, \pi] \setminus \{\frac{\pi}{2}\}$

$\text{cosec}^{-1} : \mathbb{R} \setminus (-1, 1) \to [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$

Useful sign rules:

$\sin^{-1}(-x) = -\sin^{-1} x$

$\cos^{-1}(-x) = \pi - \cos^{-1} x$

$\tan^{-1}(-x) = -\tan^{-1} x$

Exercise 2.1

Q 2.1 **Find the principal value of** $\sin^{-1}\left(-\frac{1}{2}\right)$**.**

SOLUTION

Concept used. The inverse sine function $\sin^{-1}$ takes a real number $a \in [-1, 1]$ and returns the *unique* angle y in the principal value branch $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin y = a$. We write

$$\sin^{-1}(a) = y \iff \sin y = a, \quad y \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right].$$

For a negative argument we also use the odd-function identity $\sin^{-1}(-x) = -\sin^{-1}(x)$, valid for every $x \in [-1, 1]$, because sine itself is an odd function: $\sin(-\theta) = -\sin\theta$.

☞ **Reference angle**

The standard reference value here is $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.

Step 1. Set $y = \sin^{-1}\left(-\frac{1}{2}\right)$. By the definition of $\sin^{-1}$, this means

$$\sin y = -\tfrac{1}{2}, \qquad y \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right].$$

Step 2. Use the odd-function rule to pull the minus sign out:

$$\sin^{-1}\left(-\tfrac{1}{2}\right) = -\sin^{-1}\left(\tfrac{1}{2}\right).$$

Step 3. Find $\sin^{-1}\left(\frac{1}{2}\right)$. We need the angle $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $\sin\theta = \frac{1}{2}$. The standard value $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

Step 4. Substitute back:

$$y = -\sin^{-1}\left(\tfrac{1}{2}\right) = -\tfrac{\pi}{6}.$$

Step 5. Check membership in the principal range: $-\frac{\pi}{6}$ lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ ✓; and $\sin\left(-\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$ ✓.

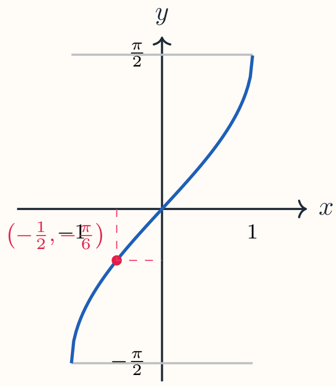
Final Answer: $\sin^{-1}\left(-\dfrac{1}{2}\right) = -\dfrac{\pi}{6}$

✗ Common Mistake

A common slip is to write the answer as $\frac{7\pi}{6}$ or $\frac{11\pi}{6}$: both have sine equal to $-\frac{1}{2}$ but neither lies inside $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so neither is the *principal* value. Always check that your candidate angle sits inside the principal range.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Bombay*

Picture-first. The graph of $y = \sin^{-1} x$ is the dark portion of the sine curve reflected across $y = x$, with x on $[-1, 1]$ and y on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We just read off the height $-\frac{1}{2}$ on the x-axis and lift it to the curve.



Concept used. The graph of $\sin^{-1}$ on $[-1,1]$ passes through the points $\left(\frac{1}{2}, \frac{\pi}{6}\right)$ and, by symmetry of the odd function through the origin, also through $\left(-\frac{1}{2}, -\frac{\pi}{6}\right)$.

Step 1. Mark $x = -\frac{1}{2}$ on the x-axis. The graph value at this point is the y we want.

Step 2. Symmetry of $\sin^{-1}$ across the origin sends $\left(\frac{1}{2}, \frac{\pi}{6}\right) \mapsto \left(-\frac{1}{2}, -\frac{\pi}{6}\right)$.

Step 3. Hence $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$. Verify: $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ and $-\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Why this matters. Visualising the graph makes the principal range a feature you can *see* (the curve never leaves the horizontal strip $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$), so you never accidentally pick a stray-quadrant angle.

Final Answer: $-\dfrac{\pi}{6}$

Q 2.2 Find the principal value of $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$.

SOLUTION

Concept used. The inverse cosine function $\cos^{-1}$ takes $a \in [-1,1]$ and returns the unique angle y in the principal value branch $[0,\pi]$ with $\cos y = a$. Formally,

$$\cos^{-1}(a) = y \iff \cos y = a, \quad y \in [0,\pi].$$

The principal range $[0,\pi]$ covers exactly the first and second quadrants where cosine is positive, then zero, then negative.

> ☞ **Reference angle**
> Standard reference: $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.

Step 1. Set $y = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$. By definition this means

$$\cos y = \frac{\sqrt{3}}{2}, \qquad y \in [0,\pi].$$

Step 2. Recall the standard value

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}.$$

This is a memorised result from the special triangle with angles $30°, 60°, 90°$ and side ratios $1 : \sqrt{3} : 2$.

Step 3. Check that $\frac{\pi}{6}$ lies in the principal range: $0 \le \frac{\pi}{6} \le \pi$ ✓. So $y = \frac{\pi}{6}$.

Final Answer: $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\pi}{6}$

EXPERT'S SOLUTION : *Priya Iyer, M.Sc Mathematics, ISI Kolkata*

Quick reading. Cosine is positive only in the first quadrant of the principal range $[0, \pi]$, so the answer must be an acute angle. We just match the value $\frac{\sqrt{3}}{2}$ against the $30°$-$60°$-$90°$ triangle.

Concept used. The acute angle with cosine $= \frac{\sqrt{3}}{2}$ is $30°$, i.e. $\frac{\pi}{6}$ radian.

Step 1. In a 30-60-90 triangle, the side opposite $30°$ has length 1, opposite $60°$ has length $\sqrt{3}$, and the hypotenuse is 2.

Step 2. $\cos 30° = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{\sqrt{3}}{2}$.

Step 3. Convert to radians: $30° = \frac{\pi}{6}$. Verify $\frac{\pi}{6} \in [0, \pi]$ ✓.

Why this matters. Knowing the three first-quadrant reference angles $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$ and their sine/cosine values handles roughly 80% of principal-value computations in this exercise.

Final Answer: $\dfrac{\pi}{6}$

Q 2.3 **Find the principal value of $\operatorname{cosec}^{-1}(2)$.**

SOLUTION

Concept used. The inverse cosecant function takes $a \in \mathbb{R} \setminus (-1, 1)$ and returns the unique angle y in the principal value branch $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$ with $\operatorname{cosec} y = a$. Equivalently, using $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$,

$$\operatorname{cosec}^{-1}(a) = y \iff \sin y = \tfrac{1}{a}, \quad y \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right] \setminus \{0\}.$$

Step 1. Set $y = \operatorname{cosec}^{-1}(2)$, so

$$\operatorname{cosec} y = 2 \iff \sin y = \tfrac{1}{2}, \qquad y \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right] \setminus \{0\}.$$

Step 2. The standard value $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ gives a candidate $y = \frac{\pi}{6}$.

Step 3. Check membership in the principal range: $\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\frac{\pi}{6} \neq 0$ ✓.

Step 4. Verify: $\operatorname{cosec}\left(\frac{\pi}{6}\right) = \dfrac{1}{\sin(\pi/6)} = \dfrac{1}{1/2} = 2$ ✓.

Final Answer: $\operatorname{cosec}^{-1}(2) = \dfrac{\pi}{6}$

☞ Exam Tip

For cosec^{-1} and $\sec^{-1}$, always reduce to a sine or cosine equation: $\sin y = 1/a$ or $\cos y = 1/a$. The reciprocal trick removes the need to memorise separate principal-value tables.

EXPERT'S SOLUTION : *Vivaan Gupta, M.Tech CS, IIT Madras*

Strategic angle. Replace cosec by $1/\sin$ at sight. That collapses the problem to a familiar $\sin^{-1}$ lookup.

Concept used. $\text{cosec}^{-1} a$ and $\sin^{-1}(1/a)$ share the same principal-range definition for $|a| \geq 1$, because the principal range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$ of cosec^{-1} is exactly the principal range of $\sin^{-1}$ with $y = 0$ removed.

Step 1. Write $\text{cosec}^{-1}(2) = \sin^{-1}\left(\frac{1}{2}\right)$.

Step 2. From Q1's reference work, $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

Step 3. Confirm $\frac{\pi}{6} \neq 0$ ✓, so it lies in the cosec^{-1} principal range.

Why this matters. The reciprocal identity $\text{cosec}^{-1} a = \sin^{-1}(1/a)$ (for $|a| \geq 1$) lets you reuse the sine table you already memorised.

Final Answer: $\frac{\pi}{6}$

Q 2.4 Find the principal value of $\tan^{-1}(-\sqrt{3})$.

SOLUTION

Concept used. The inverse tangent function takes any real a and returns the unique angle y in the open principal-value branch $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with $\tan y = a$. Formally,

$$\tan^{-1}(a) = y \iff \tan y = a, \quad y \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right).$$

For negative arguments we use the odd-function rule $\tan^{-1}(-x) = -\tan^{-1}(x)$, valid for every real x, because $\tan(-\theta) = -\tan\theta$.

Step 1. Set $y = \tan^{-1}(-\sqrt{3})$, so

$$\tan y = -\sqrt{3}, \qquad y \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right).$$

Step 2. Use the odd-function identity:

$$\tan^{-1}(-\sqrt{3}) = -\tan^{-1}(\sqrt{3}).$$

Step 3. Find $\tan^{-1}(\sqrt{3})$. The standard value is $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$, and $\frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$.

Step 4. Substitute back to get $y = -\tan^{-1}(\sqrt{3}) = -\frac{\pi}{3}$.

Step 5. Check: $-\frac{\pi}{3}$ lies in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ✓, and $\tan\left(-\frac{\pi}{3}\right) = -\tan\left(\frac{\pi}{3}\right) = -\sqrt{3}$ ✓.

Final Answer: $\tan^{-1}(-\sqrt{3}) = -\dfrac{\pi}{3}$

EXPERT'S SOLUTION *: Arjun Mehta, Ph.D Mathematics, IIT Delhi*

Strategic angle. Tangent is positive in quadrant 1 and negative in quadrant 4 (which corresponds to the negative half of the open principal range). So a negative argument always gives a negative angle in $(-\pi/2, 0)$.

Step 1. Reference: $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$.

Step 2. Sign flip via the odd-function rule: $\tan^{-1}(-\sqrt{3}) = -\tan^{-1}(\sqrt{3}) = -\frac{\pi}{3}$.

Step 3. Verify membership: $-\frac{\pi}{3}$ lies between $-\frac{\pi}{2}$ and 0, so inside the principal open interval ✓.

Why this matters. Quadrant sign rules let you handle every negative-argument inverse-trig question in two lines: reduce to a positive standard angle, then flip the sign per the relevant sign rule.

Final Answer: $-\dfrac{\pi}{3}$

Q 2.5 Find the principal value of $\cos^{-1}\left(-\dfrac{1}{2}\right)$.

SOLUTION

Concept used. $\cos^{-1}$ has principal range $[0, \pi]$. For a negative argument we use the identity

$$\cos^{-1}(-x) = \pi - \cos^{-1}(x), \qquad x \in [-1, 1].$$

This follows from $\cos(\pi - \theta) = -\cos\theta$: if $\cos\theta = x$ with $\theta \in [0, \pi]$, then $\cos(\pi - \theta) = -x$ and $\pi - \theta \in [0, \pi]$ too, so $\pi - \theta$ is the principal value of $\cos^{-1}(-x)$.

Step 1. Set $y = \cos^{-1}\left(-\frac{1}{2}\right)$, so

$$\cos y = -\tfrac{1}{2}, \qquad y \in [0, \pi].$$

Step 2. Apply the negative-argument identity with $x = \frac{1}{2}$:

$$\cos^{-1}\left(-\tfrac{1}{2}\right) = \pi - \cos^{-1}\left(\tfrac{1}{2}\right).$$

Step 3. Compute $\cos^{-1}\left(\frac{1}{2}\right)$. We know $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $\frac{\pi}{3} \in [0, \pi]$, so $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

Step 4. Substitute:

$$y = \pi - \frac{\pi}{3} = \frac{3\pi - \pi}{3} = \frac{2\pi}{3}.$$

Step 5. Verify: $\cos\left(\frac{2\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$ ✓and $\frac{2\pi}{3} \in [0, \pi]$ ✓.

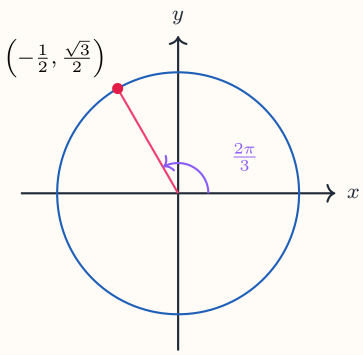
Final Answer: $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

✗ Common Mistake

Beware: $\cos^{-1}$ is *not* an odd function, so $\cos^{-1}(-x) \neq -\cos^{-1}(x)$. The correct identity is $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$. Confusing the two changes the answer from $\frac{2\pi}{3}$ to $-\frac{\pi}{3}$, which is not even in the principal range $[0, \pi]$.

EXPERT'S SOLUTION : *Aanya Kapoor, M.Sc Applied Mathematics, IIT Kanpur*

Picture-first. On the unit circle, the point with x-coordinate $-\frac{1}{2}$ in the upper half-plane is at $120°$. That is exactly $\frac{2\pi}{3}$ radians, and it sits in the principal range $[0, \pi]$.



Concept used. On the unit circle, $\cos\theta$ is the x-coordinate of the point at angle θ from the positive x-axis. The principal range of $\cos^{-1}$ corresponds to the upper half-circle, $\theta \in [0, \pi]$.

Step 1. We want the $\theta \in [0, \pi]$ whose x-coordinate on the unit circle is $-\frac{1}{2}$.

Step 2. In the upper half-circle, $x = -\frac{1}{2}$ occurs at $\theta = 120° = \frac{2\pi}{3}$.

Step 3. So $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$.

Why this matters. The unit-circle picture replaces five identities with one diagram. Memorise the eight standard upper-half points and every $\cos^{-1}$ in this exercise becomes immediate.

Final Answer: $\frac{2\pi}{3}$

Q 2.6 **Find the principal value of $\tan^{-1}(-1)$.**

SOLUTION

Concept used. $\tan^{-1}$ has principal range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and is an odd function: $\tan^{-1}(-x) = -\tan^{-1}(x)$.

Step 1. Set $y = \tan^{-1}(-1)$, so

$$\tan y = -1, \qquad y \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right).$$

Step 2. Use the odd-function rule:

$$\tan^{-1}(-1) = -\tan^{-1}(1).$$

Step 3. Reference value: $\tan\left(\frac{\pi}{4}\right) = 1$, and $\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so $\tan^{-1}(1) = \frac{\pi}{4}$.

Step 4. Hence $y = -\frac{\pi}{4}$. Verify $\tan\left(-\frac{\pi}{4}\right) = -1$ and $-\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ✓.

Final Answer: $\tan^{-1}(-1) = -\dfrac{\pi}{4}$

EXPERT'S SOLUTION *: Riya Nair, B.Tech CSE, IIT Roorkee*

Quick reading. Tangent equals -1 at $\theta = -45°$ (in the principal open interval) and also at $\theta = 135°$ (which is not). Pick the one in $(-\pi/2, \pi/2)$.

Step 1. Convert $45°$ to radians: $\frac{\pi}{4}$.

Step 2. Sign rule: $\tan^{-1}(-1) = -\tan^{-1}(1) = -\frac{\pi}{4}$.

Step 3. $-\frac{\pi}{4}$ lies in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ✓.

Why this matters. The pattern "$\tan^{-1}$ of a negative number = negative angle in $(-\pi/2, 0)$" works for every negative input. No new identity is needed.

Final Answer: $-\dfrac{\pi}{4}$

Q 2.7 **Find the principal value of $\sec^{-1}\left(\dfrac{2}{\sqrt{3}}\right)$.**

SOLUTION

Concept used. The inverse secant function takes $a \in \mathbb{R} \setminus (-1, 1)$ and returns the unique angle y in the principal value branch $[0, \pi] \setminus \{\frac{\pi}{2}\}$ with $\sec y = a$. Using $\sec\theta = \frac{1}{\cos\theta}$,

$$\sec^{-1}(a) = y \iff \cos y = \tfrac{1}{a}, \quad y \in [0, \pi] \setminus \{\tfrac{\pi}{2}\}.$$

Step 1. Set $y = \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$, so

$$\sec y = \tfrac{2}{\sqrt{3}} \iff \cos y = \tfrac{\sqrt{3}}{2}, \qquad y \in [0, \pi] \setminus \{\tfrac{\pi}{2}\}.$$

Here we used $\dfrac{1}{2/\sqrt{3}} = \dfrac{\sqrt{3}}{2}$.

Step 2. Reference: $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, and $\frac{\pi}{6} \in [0, \pi] \setminus \{\frac{\pi}{2}\}$.

Step 3. So $y = \frac{\pi}{6}$. Verify: $\sec\left(\frac{\pi}{6}\right) = \dfrac{1}{\cos(\pi/6)} = \dfrac{1}{\sqrt{3}/2} = \dfrac{2}{\sqrt{3}}$ ✓.

Final Answer: $\sec^{-1}\left(\dfrac{2}{\sqrt{3}}\right) = \dfrac{\pi}{6}$

EXPERT'S SOLUTION *: Karan Bhat, Ph.D Pure Mathematics, IISc Bangalore*

Strategic angle. The reciprocal identity $\sec^{-1}(a) = \cos^{-1}(1/a)$ for $|a| \geq 1$ converts this to a $\cos^{-1}$ lookup.

Step 1. $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

Step 2. From Q2, $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$.

Step 3. Membership check: $\frac{\pi}{6} \neq \frac{\pi}{2}$ ✓.

Why this matters. Every $\sec^{-1}$ reduces to a $\cos^{-1}$, every $\operatorname{cosec}^{-1}$ reduces to a $\sin^{-1}$. Two reciprocal identities cover four functions.

Final Answer: $\dfrac{\pi}{6}$

Q 2.8 **Find the principal value of $\cot^{-1}(\sqrt{3})$.**

SOLUTION

Concept used. The inverse cotangent function takes any real a and returns the unique angle y in the open principal-value branch $(0, \pi)$ with $\cot y = a$. Using $\cot\theta = \frac{\cos\theta}{\sin\theta}$ or $\cot\theta = \frac{1}{\tan\theta}$ (when $\tan\theta$ is non-zero),

$$\cot^{-1}(a) = y \iff \cot y = a, \quad y \in (0, \pi).$$

> ☞ **Reference angle**
> $\cot\left(\frac{\pi}{6}\right) = \frac{\cos(\pi/6)}{\sin(\pi/6)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}.$

Step 1. Set $y = \cot^{-1}(\sqrt{3})$, so

$$\cot y = \sqrt{3}, \qquad y \in (0, \pi).$$

Step 2. Reference: $\cot\left(\frac{\pi}{6}\right) = \sqrt{3}$ (see the recall box above), and $\frac{\pi}{6} \in (0, \pi)$.

Step 3. Hence $y = \frac{\pi}{6}$.

> **Final Answer:** $\cot^{-1}(\sqrt{3}) = \dfrac{\pi}{6}$

EXPERT'S SOLUTION : *Sneha Verma, M.Sc Mathematics, IIT Bombay*

Strategic angle. Convert cotangent to tangent: if $\cot\theta = \sqrt{3}$, then $\tan\theta = \frac{1}{\sqrt{3}}$, an even more familiar reference value.

Concept used. $\cot\theta = 1/\tan\theta$ when $\tan\theta$ is non-zero. Inside $(0, \pi)$, $\cot$ is positive on $(0, \pi/2)$, zero at $\pi/2$, negative on $(\pi/2, \pi)$.

Step 1. From $\cot y = \sqrt{3}$, we get $\tan y = \frac{1}{\sqrt{3}}$.

Step 2. Reference: $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$, and $\frac{\pi}{6} \in (0, \pi)$.

Step 3. Therefore $y = \frac{\pi}{6}$.

Why this matters. $\cot^{-1}$ has a quirk: its principal range is $(0, \pi)$, *not* $(-\pi/2, \pi/2)$. So $\cot^{-1}$ of a negative number gives an angle in $(\pi/2, \pi)$, not a negative angle. Remember this when handling negative inputs.

> **Final Answer:** $\dfrac{\pi}{6}$

Q 2.9 **Find the principal value of** $\cos^{-1}\left(-\dfrac{1}{\sqrt{2}}\right)$.

SOLUTION

Concept used. Same as Q5: principal range $[0, \pi]$, and for a negative argument

$$\cos^{-1}(-x) = \pi - \cos^{-1}(x).$$

Step 1. Set $y = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$, so

$$\cos y = -\frac{1}{\sqrt{2}}, \qquad y \in [0, \pi].$$

Step 2. Apply the negative-argument rule with $x = \frac{1}{\sqrt{2}}$:

$$y = \pi - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right).$$

Step 3. Reference: $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ and $\frac{\pi}{4} \in [0, \pi]$, so $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$.

Step 4. Substitute:

$$y = \pi - \frac{\pi}{4} = \frac{4\pi - \pi}{4} = \frac{3\pi}{4}.$$

Step 5. Verify: $\cos\left(\frac{3\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ ✓and $\frac{3\pi}{4} \in [0, \pi]$ ✓.

Final Answer: $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$

EXPERT'S SOLUTION : *Aditya Joshi, M.Sc Mathematics, ISI Kolkata*

Quick reading. Cosine is $-\frac{1}{\sqrt{2}}$ at $135°$ in the upper half-plane: that is $\frac{3\pi}{4}$ radians, comfortably inside $[0, \pi]$.

Step 1. Reference: $\cos 45° = \frac{1}{\sqrt{2}}$.

Step 2. Reflect about the y-axis (i.e., apply $\theta \mapsto \pi - \theta$):
$\cos(180° - 45°) = -\cos 45° = -\frac{1}{\sqrt{2}}$.

Step 3. So the answer is $135° = \frac{3\pi}{4}$.

Why this matters. A negative cosine value always pushes the principal answer into the second quadrant of $[0, \pi]$: angles strictly between $\frac{\pi}{2}$ and π.

Final Answer: $\frac{3\pi}{4}$

Q 2.10 **Find the principal value of** $\operatorname{cosec}^{-1}(-\sqrt{2})$.

SOLUTION

Concept used. $\operatorname{cosec}^{-1}$ has principal range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$ and is odd: $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}(x)$ for $|x| \geq 1$. This is because cosec itself is odd.

Step 1. Set $y = \operatorname{cosec}^{-1}(-\sqrt{2})$, so

$$\operatorname{cosec} y = -\sqrt{2} \iff \sin y = -\frac{1}{\sqrt{2}}, \quad y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}.$$

Step 2. Use the odd-function rule:

$$\operatorname{cosec}^{-1}(-\sqrt{2}) = -\operatorname{cosec}^{-1}(\sqrt{2}).$$

Step 3. Compute $\operatorname{cosec}^{-1}(\sqrt{2})$. We need $\sin y' = \frac{1}{\sqrt{2}}$ with $y' \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$. Reference: $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, so $\operatorname{cosec}^{-1}(\sqrt{2}) = \frac{\pi}{4}$.

Step 4. Substitute:

$$y = -\frac{\pi}{4}.$$

Step 5. Verify: $\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ so $\operatorname{cosec}\left(-\frac{\pi}{4}\right) = -\sqrt{2}$ ✓, and $-\frac{\pi}{4}$ lies in the principal range ✓.

Final Answer: $\operatorname{cosec}^{-1}(-\sqrt{2}) = -\dfrac{\pi}{4}$

♥ Symmetry pays off

Among the six inverse-trig functions, three are odd ($\sin^{-1}$, $\tan^{-1}$, $\operatorname{cosec}^{-1}$) and three are not ($\cos^{-1}$, $\cot^{-1}$, $\sec^{-1}$). The three odd ones reduce negative-input problems by a sign flip; the other three need the $\pi - \theta$ rule. Knowing which group a function belongs to halves the work.

EXPERT'S SOLUTION *: Pranav Chatterjee, M.Sc Applied Mathematics, IIT Kanpur*

Strategic angle. Convert $\operatorname{cosec}^{-1}$ to $\sin^{-1}$ first, then apply the odd-function rule on $\sin^{-1}$.

Step 1. $\operatorname{cosec}^{-1}(-\sqrt{2}) = \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ (reciprocal identity).

Step 2. By the odd-function rule for $\sin^{-1}$: $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

Step 3. Reference: $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$.

Step 4. Therefore $y = -\frac{\pi}{4}$. Confirm $-\frac{\pi}{4} \neq 0$ and lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ ✓.

Why this matters. Stacking identities (reciprocal + odd) is faster than memorising one identity per function, and is much harder to misremember.

Final Answer: $-\dfrac{\pi}{4}$

Q 2.11 **Find the value of** $\tan^{-1}(1) + \cos^{-1}\left(-\dfrac{1}{2}\right) + \sin^{-1}\left(-\dfrac{1}{2}\right)$.

SOLUTION

Concept used. Each of the three inverse trig values takes the principal value from its own principal range. Compute each separately and add. The three principal ranges are

$$\tan^{-1}\colon (-\tfrac{\pi}{2}, \tfrac{\pi}{2}), \quad \cos^{-1}\colon [0, \pi], \quad \sin^{-1}\colon [-\tfrac{\pi}{2}, \tfrac{\pi}{2}].$$

Step 1. Compute $\tan^{-1}(1)$. Reference: $\tan(\pi/4) = 1$ and $\pi/4 \in (-\pi/2, \pi/2)$, so

$$\tan^{-1}(1) = \tfrac{\pi}{4}.$$

Step 2. Compute $\cos^{-1}(-\tfrac{1}{2})$. By Q5,

$$\cos^{-1}(-\tfrac{1}{2}) = \pi - \cos^{-1}(\tfrac{1}{2}) = \pi - \tfrac{\pi}{3} = \tfrac{2\pi}{3}.$$

Step 3. Compute $\sin^{-1}(-\tfrac{1}{2})$. By Q1,

$$\sin^{-1}(-\tfrac{1}{2}) = -\sin^{-1}(\tfrac{1}{2}) = -\tfrac{\pi}{6}.$$

Step 4. Add the three results. Bring them to a common denominator 12:

$$\tfrac{\pi}{4} = \tfrac{3\pi}{12}, \quad \tfrac{2\pi}{3} = \tfrac{8\pi}{12}, \quad -\tfrac{\pi}{6} = -\tfrac{2\pi}{12}.$$

Sum:

$$\tfrac{3\pi}{12} + \tfrac{8\pi}{12} - \tfrac{2\pi}{12} = \tfrac{3+8-2}{12}\pi = \tfrac{9\pi}{12} = \tfrac{3\pi}{4}.$$

Final Answer: $\tan^{-1}(1) + \cos^{-1}\left(-\dfrac{1}{2}\right) + \sin^{-1}\left(-\dfrac{1}{2}\right) = \dfrac{3\pi}{4}$

EXPERT'S SOLUTION *: Ananya Reddy, Ph.D Mathematics, IIT Delhi*

Strategic angle. Use the standard identity $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ for $x \in [-1, 1]$ to collapse two of the three terms into a single $\frac{\pi}{2}$.

Concept used. For every $x \in [-1, 1]$, the sine and cosine inverses of the same number sum to $\frac{\pi}{2}$, because the angle $\sin^{-1} x$ and the angle $\cos^{-1} x$ are complementary inside the unit-circle picture.

Step 1. Group cleverly: keep $\tan^{-1}(1)$ aside, then

$$\cos^{-1}(-\tfrac{1}{2}) + \sin^{-1}(-\tfrac{1}{2}) = \tfrac{\pi}{2} \quad \text{(complement identity with } x = -\tfrac{1}{2}\text{)}.$$

Step 2. Compute $\tan^{-1}(1) = \frac{\pi}{4}$.

Step 3. Add: total $= \frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4} + \frac{2\pi}{4} = \frac{3\pi}{4}$.

Why this matters. Spotting the complement pair $\sin^{-1} x + \cos^{-1} x$ saves two reference-angle lookups and one common-denominator addition. This identity is one of the four major inverse-trig "complement identities" you should keep at the front of your mind:

$$\sin^{-1} x + \cos^{-1} x = \tan^{-1} x + \cot^{-1} x = \sec^{-1} x + \text{cosec}^{-1} x = \tfrac{\pi}{2}.$$

Final Answer: $\dfrac{3\pi}{4}$

Q 2.12 **Find the value of** $\cos^{-1}\left(\dfrac{1}{2}\right) + 2\sin^{-1}\left(\dfrac{1}{2}\right)$.

SOLUTION

Concept used. Each inverse function returns the principal value: $\cos^{-1}$ into $[0, \pi]$ and $\sin^{-1}$ into $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The factor 2 multiplies the principal value of $\sin^{-1}(\frac{1}{2})$ (it does *not* mean $\sin^{-1}(2 \cdot \frac{1}{2})$).

Step 1. Compute $\cos^{-1}(\frac{1}{2})$. Reference: $\cos(\frac{\pi}{3}) = \frac{1}{2}$ and $\frac{\pi}{3} \in [0, \pi]$. So

$$\cos^{-1}(\tfrac{1}{2}) = \tfrac{\pi}{3}.$$

Step 2. Compute $\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$ (from Q1's reference work).

Step 3. Multiply the second result by 2:

$$2\sin^{-1}(\tfrac{1}{2}) = 2 \cdot \tfrac{\pi}{6} = \tfrac{2\pi}{6} = \tfrac{\pi}{3}.$$

Step 4. Add:

$$\tfrac{\pi}{3} + \tfrac{\pi}{3} = \tfrac{2\pi}{3}.$$

Final Answer: $\cos^{-1}\left(\dfrac{1}{2}\right) + 2\sin^{-1}\left(\dfrac{1}{2}\right) = \dfrac{2\pi}{3}$

✗ Common Mistake

$2\sin^{-1}(x)$ is *not* the same as $\sin^{-1}(2x)$. Here $2\sin^{-1}(\frac{1}{2}) = \frac{\pi}{3}$ but $\sin^{-1}(1) = \frac{\pi}{2}$. Always evaluate the inner inverse first, then multiply.

EXPERT'S SOLUTION *: Yash Singh, M.Tech CS, IIT Madras*

Structural observation. The expression equals $\cos^{-1} x + 2\sin^{-1} x$ at $x = \frac{1}{2}$. Using $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, this rewrites as $\frac{\pi}{2} + \sin^{-1} x$, which is faster than evaluating each term separately.

Step 1. Write

$$\cos^{-1} x + 2\sin^{-1} x = (\cos^{-1} x + \sin^{-1} x) + \sin^{-1} x = \tfrac{\pi}{2} + \sin^{-1} x.$$

Step 2. Substitute $x = \frac{1}{2}$:

$$= \tfrac{\pi}{2} + \sin^{-1}\left(\tfrac{1}{2}\right) = \tfrac{\pi}{2} + \tfrac{\pi}{6}.$$

Step 3. Common denominator 6:

$$\tfrac{\pi}{2} = \tfrac{3\pi}{6}, \qquad \tfrac{3\pi}{6} + \tfrac{\pi}{6} = \tfrac{4\pi}{6} = \tfrac{2\pi}{3}.$$

Why this matters. The same complement identity $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ keeps paying off. Algebra before substitution is usually the fastest path.

Final Answer: $\dfrac{2\pi}{3}$

Q 2.13 **If $\sin^{-1} x = y$, then**

(A) $0 \le y \le \pi$ **(B)** $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$ **(C)** $0 < y < \pi$ **(D)** $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$.

SOLUTION

Concept used. The principal value branch of $\sin^{-1}$ is the *closed* interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. This is the range of the function $\sin^{-1}: [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ specified in the NCERT table at the end of Section 2.2.

Step 1. By definition of the principal value, if $y = \sin^{-1} x$ then y lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Step 2. The endpoints are achieved: $\sin^{-1}(1) = \frac{\pi}{2}$ and $\sin^{-1}(-1) = -\frac{\pi}{2}$. So the interval is *closed*, not open.

Step 3. Option (A) $0 \le y \le \pi$ is the range of $\cos^{-1}$, not $\sin^{-1}$ $\times$.

Step 4. Option (C) $0 < y < \pi$ is an open subset of the $\cos^{-1}$ range and never equals the

$\sin^{-1}$ range $\times$.

Step 5. Option (D) $-\frac{\pi}{2} < y < \frac{\pi}{2}$ is open and excludes the endpoints $\pm\frac{\pi}{2}$ $\times$.

Step 6. Option (B) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ matches exactly. ✓

> **Final Answer:** Option (B): $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$.

EXPERT'S SOLUTION : *Diya Pillai, M.Sc Mathematics, IIT Bombay*

Quick reading. The question tests one fact: the principal range of $\sin^{-1}$. The only ambiguity is closed-vs-open. Since $\sin^{-1}(\pm 1) = \pm\frac{\pi}{2}$ are perfectly valid, the endpoints belong, so the interval is closed.

Step 1. Eliminate ranges that start at 0: those belong to $\cos^{-1}$. Cuts options (A) and (C).

Step 2. Between closed (B) and open (D): test the endpoint. $\sin^{-1}(1) = \frac{\pi}{2}$ exists and is well-defined, so $\frac{\pi}{2}$ is attained. Therefore the interval is closed.

Step 3. Answer: (B).

Why this matters. Closed vs open boundaries matter for domains in calculus (a principal branch with $\sec^{-1}$ excludes $\pi/2$, $\operatorname{cosec}^{-1}$ excludes 0). Memorise which endpoints belong.

> **Final Answer:** (B)

Q 2.14 $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ **is equal to**

(A) π **(B)** $-\dfrac{\pi}{3}$ **(C)** $\dfrac{\pi}{3}$ **(D)** $\dfrac{2\pi}{3}$.

SOLUTION

Concept used. Compute each term in its own principal range:

$$\tan^{-1}\colon \mathbb{R} \to \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right), \quad \sec^{-1}\colon \mathbb{R} \setminus (-1, 1) \to [0, \pi] \setminus \{\tfrac{\pi}{2}\}.$$

For the second term we use the negative-argument identity

$$\sec^{-1}(-x) = \pi - \sec^{-1}(x), \qquad |x| \ge 1,$$

which follows from $\sec(\pi - \theta) = -\sec\theta$ and is analogous to the $\cos^{-1}$ identity (because $\sec^{-1}$ shares its principal range $[0, \pi] \setminus \{\pi/2\}$ with $\cos^{-1}$ minus the $\pi/2$ point).

Step 1. Compute $\tan^{-1}\sqrt{3}$. Reference: $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$ and $\frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. So

$$\tan^{-1}\sqrt{3} = \frac{\pi}{3}.$$

Step 2. Compute $\sec^{-1}(-2)$. Apply the negative-argument rule with $x = 2$:

$$\sec^{-1}(-2) = \pi - \sec^{-1}(2).$$

Step 3. Compute $\sec^{-1}(2)$. This means $\cos y = \frac{1}{2}$ with $y \in [0, \pi] \setminus \{\frac{\pi}{2}\}$, so

$$\sec^{-1}(2) = \cos^{-1}\left(\tfrac{1}{2}\right) = \frac{\pi}{3}.$$

Step 4. Back-substitute:

$$\sec^{-1}(-2) = \pi - \frac{\pi}{3} = \frac{3\pi - \pi}{3} = \frac{2\pi}{3}.$$

Step 5. Subtract:

$$\tan^{-1}\sqrt{3} - \sec^{-1}(-2) = \frac{\pi}{3} - \frac{2\pi}{3} = \frac{\pi - 2\pi}{3} = -\frac{\pi}{3}.$$

Final Answer: Option (B): $\tan^{-1}\sqrt{3} - \sec^{-1}(-2) = -\dfrac{\pi}{3}$.

☞ Exam Tip

For MCQs of the form "find f^{-1}(positive) $\pm$ g^{-1}(negative)": do the positive term first (a one-step lookup), then handle the negative argument with the relevant $\pi - \theta$ identity.

EXPERT'S SOLUTION : *Ishaan Banerjee, M.Tech CS, IIT Madras*

Strategic angle. The two reference angles are both $\frac{\pi}{3}$ (since $\tan(\pi/3) = \sqrt{3}$ and $\cos(\pi/3) = 1/2$). That is the cleanest possible numerical setup.

Step 1. $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$.

Step 2. For $\sec^{-1}(-2)$: $\sec y = -2 \Rightarrow \cos y = -\frac{1}{2}$, with $y \in [0, \pi] \setminus \{\frac{\pi}{2}\}$. From Q5, $\cos^{-1}(-1/2) = \frac{2\pi}{3}$. So $\sec^{-1}(-2) = \frac{2\pi}{3}$.

Step 3. Difference: $\frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$.

Step 4. This matches option (B).

Why this matters. $\sec^{-1}(a) = \cos^{-1}(1/a)$ for $|a| \geq 1$ is a one-step bridge that lets you reuse the $\cos^{-1}$ table for every $\sec^{-1}$ problem. The same trick (replace $\sec^{-1}$ by $\cos^{-1}$ of the reciprocal) keeps numerical work to a minimum.

Final Answer: (B)

Key Takeaways

- Each inverse trigonometric function returns the unique angle in its principal value branch, not just any angle whose trig value matches.
- Memorise the six principal ranges: closed brackets for $\sin^{-1}$, $\cos^{-1}$, $\sec^{-1}$; open brackets for $\tan^{-1}$, $\cot^{-1}$; and $\{0\}$ removed for $\operatorname{cosec}^{-1}$, $\{\pi/2\}$ removed for $\sec^{-1}$.
- Negative-argument identities split by parity: odd functions ($\sin^{-1}$, $\tan^{-1}$, $\operatorname{cosec}^{-1}$) flip sign; the others use $\pi - \theta$.
- The complement identity $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ (and analogues for tan/cot, sec/cosec) is the single most useful relation in the chapter.

End of Exercise 2.1