Collegedunia NCERT Solutions

Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 2: Inverse Trigonometric Functions

About this Chapter

Exercise 2.2 of Class 12th Mathematics drills the **identities of inverse trigonometric functions** that follow from the **principal value branches** fixed in Section 2.2. The questions fall into four buckets: proving the multiple-angle identities, writing composite expressions in simplest form via clever substitutions like $x = \tan\theta$ or $x = \cos 2\theta$, evaluating mixed expressions using the complementary identity $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, and applying $f^{-1}(f(x)) = x$ *only inside the principal range*.

Topics covered: Multiple-angle identities • Half-angle substitutions • Complementary identities • $f^{-1}(f(x))$ on principal branch

Quick Formula Sheet

Multiple-angle identities:

$3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$

$3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$

Substitution helpers:

$x = \tan\theta$: $\sqrt{1 + x^2} = \sec\theta$

$x = \sec\theta$: $\sqrt{x^2 - 1} = \tan\theta$

$x = \cos 2\theta$: $\sqrt{1 - x} = \sqrt{2}\sin\theta$

Complement / sum identities:

$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

Exercise 2.2

Q 2.1 **Prove that** $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$ **for** $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

SOLUTION

Concept used. The triple-angle formula for sine states

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

This is derived from $\sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$ and the double-angle formulas. Combined with the principal-range identity

$$\sin^{-1}(\sin u) = u \quad \text{provided } u \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right],$$

it gives the inverse-trig identity asked here. The condition $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ is precisely what guarantees $3\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ in the step where we apply this principal-range identity.

Step 1. **Substitute.** Let $x = \sin\theta$. By the definition of $\sin^{-1}$, this is equivalent to $\theta = \sin^{-1} x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Step 2. **Restrict θ.** Since $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, $\sin\theta \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ which forces $\theta \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$ (because $\sin^{-1}$ is increasing and $\sin^{-1}(\pm\frac{1}{2}) = \pm\frac{\pi}{6}$).

Step 3. **Hence 3θ stays in the principal range.** From $\theta \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$, multiply by 3 to get $3\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. This is the key bound that lets us apply $\sin^{-1}(\sin\cdot)$ later.

Step 4. **Apply the triple-angle formula.** Compute $3x - 4x^3$:
$3x - 4x^3 = 3\sin\theta - 4\sin^3\theta = \sin 3\theta$.

Step 5. **Apply $\sin^{-1}$ on both sides.** Take $\sin^{-1}$: $\sin^{-1}(3x - 4x^3) = \sin^{-1}(\sin 3\theta) = 3\theta$, where the last equality uses $3\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Step 6. **Recover the right-hand side.** Since $\theta = \sin^{-1} x$, $3\theta = 3\sin^{-1} x$, so $\sin^{-1}(3x - 4x^3) = 3\sin^{-1} x$.

Final Answer: $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$ for $x \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$.

♥ Why the bound $|x| \le \frac{1}{2}$ matters

The triple-angle identity for sine holds for *all* real θ, but the inverse-sine identity needs 3θ to stay in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. That forces θ to lie in $\left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$, i.e. $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Outside this range, $\sin^{-1}(\sin 3\theta) \ne 3\theta$: the answer folds back to a coterminal angle and the identity fails.

EXPERT'S SOLUTION *: Aanya Sharma, M.Sc Mathematics, IIT Bombay*

Strategic angle. Read the identity as a statement about the angle, not the value: tripling the principal-value angle is the same as taking $\sin^{-1}$ of the triple-angle value, *as long as the tripled angle still lives in the principal range*.

Concept used. Same triple-angle expansion $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ and same principal-range condition for $\sin^{-1} \circ \sin$ to be the identity.

Step 1. Set $\theta = \sin^{-1} x$, so $\sin\theta = x$ and $\theta \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$ when $|x| \le \frac{1}{2}$.

Step 2. Use the triple-angle formula: $\sin 3\theta = 3\sin\theta - 4\sin^3\theta = 3x - 4x^3$.

Step 3. Since $|3\theta| \le \frac{\pi}{2}$, we may invert: $\sin^{-1}(\sin 3\theta) = 3\theta = 3\sin^{-1} x$.

Step 4. Therefore $\sin^{-1}(3x - 4x^3) = 3\sin^{-1} x$.

Why this matters. The exact analogue $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$ (Q2 below) uses the same template: triple-angle expansion plus a domain check that keeps the multiplied angle inside the principal range.

Final Answer: $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

Q 2.2 **Prove that $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$ for $x \in \left[\frac{1}{2}, 1\right]$.**

SOLUTION

Concept used. The triple-angle formula for cosine is

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

This is derived from $\cos 3\theta = \cos(2\theta + \theta)$ via the double-angle and addition formulas. We combine it with the principal-range identity $\cos^{-1}(\cos u) = u$ for $u \in [0, \pi]$. The bound $x \in \left[\frac{1}{2}, 1\right]$ guarantees 3θ stays inside $[0, \pi]$.

Step 1. **Substitute.** Let $x = \cos\theta$. By the definition of $\cos^{-1}$, equivalently $\theta = \cos^{-1} x \in [0, \pi]$.

Step 2. **Restrict θ.** Since $x \in \left[\frac{1}{2}, 1\right]$ and $\cos^{-1}$ is decreasing, $\cos^{-1}(1) = 0$ and $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$, hence $\theta \in \left[0, \frac{\pi}{3}\right]$.

Step 3. **So 3θ stays in $[0, \pi]$.** Multiply $\theta \in \left[0, \frac{\pi}{3}\right]$ by 3 to get $3\theta \in [0, \pi]$.

Step 4. **Apply the triple-angle formula.** Compute $4x^3 - 3x = 4\cos^3\theta - 3\cos\theta = \cos 3\theta$.

Step 5. **Apply $\cos^{-1}$.** Since $3\theta \in [0, \pi]$, $\cos^{-1}(4x^3 - 3x) = \cos^{-1}(\cos 3\theta) = 3\theta = 3\cos^{-1} x$.

Final Answer: $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$ for $x \in \left[\frac{1}{2}, 1\right]$.

✗ Common Mistake

The bound here is $\left[\frac{1}{2}, 1\right]$, not $\left[-\frac{1}{2}, \frac{1}{2}\right]$. The principal range of $\cos^{-1}$ is $[0, \pi]$, so we need $3\theta \le \pi$, i.e. $\theta \le \frac{\pi}{3}$, i.e. $\cos\theta \ge \frac{1}{2}$. The lower bound $x = \frac{1}{2}$ comes from this, not from any sign condition.

EXPERT'S SOLUTION : *Vivaan Iyer, Ph.D Mathematics, IIT Delhi*

Structural observation. The proof mirrors Q1 structure for structure. Only the principal range changes from $[-\pi/2, \pi/2]$ (for $\sin^{-1}$) to $[0, \pi]$ (for $\cos^{-1}$), and the triple-angle polynomial changes from $3x - 4x^3$ to $4x^3 - 3x$.

Step 1. Put $\theta = \cos^{-1} x$, so $\cos\theta = x$ and, with $x \in [\frac{1}{2}, 1]$, $\theta \in [0, \frac{\pi}{3}]$.

Step 2. Triple-angle: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta = 4x^3 - 3x$.

Step 3. Since $3\theta \in [0, \pi]$, the inversion is clean: $\cos^{-1}(\cos 3\theta) = 3\theta$.

Step 4. Conclude $\cos^{-1}(4x^3 - 3x) = 3\theta = 3\cos^{-1} x$.

Why this matters. Whenever you see a multiple-angle inverse-trig identity, ask first: *what range of the original variable keeps the multiplied angle in the principal range?* That range is exactly the hypothesis of the identity.

Final Answer: $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$ on $\left[\frac{1}{2}, 1\right]$.

Q 2.3 **Write** $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$, $x \neq 0$**, in simplest form.**

SOLUTION

Concept used. The substitution $x = \tan\theta$ is the canonical way to simplify any inverse-tangent expression containing $\sqrt{1+x^2}$, because that radical becomes $\sec\theta$ and trig identities collapse the algebra. We also need the half-angle identities $1 - \cos\theta = 2\sin^2(\theta/2)$ and $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$.

Step 1. **Substitute.** Let $x = \tan\theta$ with $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $\sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = \sec\theta$, with the positive root because $\sec\theta > 0$ on the principal range of $\tan^{-1}$.

Step 2. **Rewrite the argument.** Substitute $x = \tan\theta$ and $\sqrt{1+x^2} = \sec\theta$:

$$\frac{\sqrt{1+x^2}-1}{x} = \frac{\sec\theta - 1}{\tan\theta} = \frac{\frac{1}{\cos\theta} - 1}{\frac{\sin\theta}{\cos\theta}} = \frac{1-\cos\theta}{\sin\theta}.$$

(Multiplied top and bottom by $\cos\theta$.)

Step 3. **Apply half-angle identities.**

$$\frac{1-\cos\theta}{\sin\theta} = \frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = \tan\left(\frac{\theta}{2}\right).$$

Step 4. **Apply** $\tan^{-1}$. The half-angle $\frac{\theta}{2}$ lies in $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, well inside the principal range, so $\tan^{-1}(\tan(\theta/2)) = \theta/2$.

Step 5. **Back-substitute** $\theta = \tan^{-1} x$. $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{1}{2}\tan^{-1} x$.

Final Answer: $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \dfrac{1}{2}\tan^{-1} x$.

EXPERT'S SOLUTION *: Arjun Patel, M.Sc Mathematics, IIT Bombay*

Strategic angle. See $\sqrt{1+x^2}$, write $x = \tan\theta$. That single move turns every term into a sine or cosine, and the half-angle pair $1-\cos\theta = 2\sin^2(\theta/2)$ takes you the rest of the way.

Step 1. Put $x = \tan\theta$, so $\sqrt{1+x^2} = \sec\theta$.

Step 2. Argument becomes $\dfrac{\sec\theta - 1}{\tan\theta} = \dfrac{1-\cos\theta}{\sin\theta} = \tan(\theta/2)$.

Step 3. So the expression equals $\tan^{-1}(\tan(\theta/2)) = \theta/2 = \frac{1}{2}\tan^{-1} x$.

Why this matters. Many "simplest form" problems hinge on seeing one substitution. Build a small mental catalogue: $\sqrt{1+x^2} \Rightarrow x = \tan\theta$; $\sqrt{1-x^2} \Rightarrow x = \sin\theta$; $\sqrt{x^2-1} \Rightarrow x = \sec\theta$; $\sqrt{a^2-x^2} \Rightarrow x = a\sin\theta$.

Final Answer: $\dfrac{1}{2}\tan^{-1} x$.

Q 2.4 **Write** $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$, $0 < x < \pi$, **in simplest form.**

SOLUTION

Concept used. The half-angle identities $1-\cos x = 2\sin^2(x/2)$ and $1+\cos x = 2\cos^2(x/2)$. Their ratio reduces to $\tan^2(x/2)$, and the principal-range identity $\tan^{-1}(\tan u) = u$ for $u \in (-\pi/2, \pi/2)$ finishes the work.

Step 1. **Rewrite the radicand.** Substitute the half-angle identities:

$$\frac{1-\cos x}{1+\cos x} = \frac{2\sin^2(x/2)}{2\cos^2(x/2)} = \tan^2(x/2).$$

Step 2. **Take the square root.** Since $0 < x < \pi$, we have $0 < x/2 < \pi/2$, so $\tan(x/2) > 0$. Hence $\sqrt{\tan^2(x/2)} = \tan(x/2)$.

Step 3. **Apply** $\tan^{-1}$. Because $x/2 \in (0, \pi/2) \subset (-\pi/2, \pi/2)$, $\tan^{-1}(\tan(x/2)) = x/2$.

Final Answer: $\tan^{-1}\left(\sqrt{\dfrac{1-\cos x}{1+\cos x}}\right) = \dfrac{x}{2}$.

☞ **Quick recall**

$\sqrt{a^2} = |a|$ in general. The given range $0 < x < \pi$ ensures $\tan(x/2) > 0$ so the absolute-value bars drop. If the range were different we would have to be careful with the sign.

EXPERT'S SOLUTION *: Pranav Singh, M.Sc Applied Mathematics, IIT Kanpur*

Quick reading. Recognise the ratio $\frac{1-\cos x}{1+\cos x}$ instantly as $\tan^2(x/2)$. The square root and the inverse tangent then cancel each other on the principal range.

Step 1. $\dfrac{1-\cos x}{1+\cos x} = \tan^2(x/2)$ by the half-angle ratio identity.

Step 2. For $0 < x < \pi$, $\tan(x/2) > 0$, so the radical equals $\tan(x/2)$.

Step 3. $\tan^{-1}(\tan(x/2)) = x/2$ since $x/2 \in (0, \pi/2)$ lies in the principal range.

Why this matters. The half-angle ratio identity is one of the highest-yield identities in this exercise: it shows up again in Q9 (with $\sqrt{1 \pm \sin x}$) and in the Misc Q9.

Final Answer: $\dfrac{x}{2}$.

Q 2.5 **Write** $\tan^{-1}\left(\dfrac{\cos x - \sin x}{\cos x + \sin x}\right)$, $-\dfrac{\pi}{4} < x < \dfrac{3\pi}{4}$, **in simplest form.**

SOLUTION

Concept used. Divide numerator and denominator by $\cos x$ to expose $\tan x$, then apply the tangent subtraction formula $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$ with $A = \frac{\pi}{4}$ (so $\tan A = 1$). The principal-range identity $\tan^{-1}(\tan u) = u$ for $u \in (-\pi/2, \pi/2)$ finishes the work.

Step 1. Divide top and bottom by $\cos x$. $\dfrac{\cos x - \sin x}{\cos x + \sin x} = \dfrac{1 - \tan x}{1 + \tan x}$.

Step 2. Recognise the tangent of a difference. Write $1 = \tan(\pi/4)$:

$$\frac{1-\tan x}{1+\tan x} = \frac{\tan(\pi/4) - \tan x}{1 + \tan(\pi/4)\tan x} = \tan\left(\tfrac{\pi}{4} - x\right).$$

Step 3. Check principal-range membership. As x runs through $\left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$, $u = \frac{\pi}{4} - x$ runs through $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$: $x = -\frac{\pi}{4}$ gives $u = \frac{\pi}{2}$ and $x = \frac{3\pi}{4}$ gives $u = -\frac{\pi}{2}$, with the open interval inside.

Step 4. **Apply** $\tan^{-1}$**.** $\tan^{-1}(\tan(\pi/4 - x)) = \pi/4 - x$.

Final Answer: $\tan^{-1}\left(\dfrac{\cos x - \sin x}{\cos x + \sin x}\right) = \dfrac{\pi}{4} - x.$

☞ Exam Tip

Whenever you see $\dfrac{A \pm B}{A \mp B}$ inside $\tan^{-1}$, dividing by A and matching against the tangent-sum or tangent-difference formula is almost always the right first move.

EXPERT'S SOLUTION *: Karan Mehta, M.Tech CS, IIT Madras*

Strategic angle. The given fraction is the tangent of a difference angle. Spot the 1 as $\tan(\pi/4)$ and the rest is mechanical.

Step 1. Divide top and bottom by $\cos x$ to get $\dfrac{1 - \tan x}{1 + \tan x}$.

Step 2. Replace 1 by $\tan(\pi/4)$: argument equals $\tan(\pi/4 - x)$.

Step 3. Check $\pi/4 - x \in (-\pi/2, \pi/2)$ for $x \in (-\pi/4, 3\pi/4)$.

Step 4. Simplified form: $\pi/4 - x$.

Why this matters. The same recipe (divide, then match a known $\tan$ formula) handles the related expression $\frac{\cos x + \sin x}{\cos x - \sin x}$ to give $\frac{\pi}{4} + x$.

Final Answer: $\dfrac{\pi}{4} - x.$

Q 2.6 **Write** $\tan^{-1}\left(\dfrac{x}{\sqrt{a^2 - x^2}}\right)$, $|x| < a$, **in simplest form.**

SOLUTION

Concept used. The trigonometric substitution $x = a\sin\theta$ converts $\sqrt{a^2 - x^2}$ to $a\cos\theta$, after which the argument of $\tan^{-1}$ reduces to $\tan\theta$.

Step 1. **Substitute.** Let $x = a\sin\theta$ with $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then $\sin\theta = x/a$ and $\theta = \sin^{-1}(x/a)$.

Step 2. **Simplify the radical.** Since $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos\theta \geq 0$, so $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\sqrt{1 - \sin^2\theta} = a\cos\theta$.

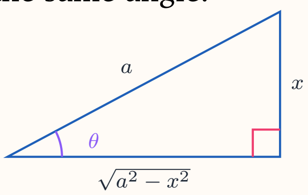
Step 3. **Rewrite the argument.** $\dfrac{x}{\sqrt{a^2 - x^2}} = \dfrac{a\sin\theta}{a\cos\theta} = \tan\theta.$

Step 4. **Apply** $\tan^{-1}$**.** The condition $|x| < a$ gives $|\sin\theta| < 1$, hence $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, exactly the principal range of $\tan^{-1}$: $\tan^{-1}(\tan\theta) = \theta = \sin^{-1}(x/a)$.

> **Final Answer:** $\tan^{-1}\left(\dfrac{x}{\sqrt{a^2 - x^2}}\right) = \sin^{-1}\left(\dfrac{x}{a}\right).$

EXPERT'S SOLUTION *: Aditi Verma, M.Sc Mathematics, ISI Kolkata*

Picture-first. Draw a right triangle with hypotenuse a and opposite side x: the adjacent side is $\sqrt{a^2 - x^2}$ and the acute angle θ satisfies $\sin\theta = x/a$ and $\tan\theta = x/\sqrt{a^2 - x^2}$. So both inverse expressions name the same angle.



Concept used. In any right triangle, $\tan\theta = \dfrac{\text{opp}}{\text{adj}}$ and $\sin\theta = \dfrac{\text{opp}}{\text{hyp}}$. Reading two different ratios for the same acute angle produces the identity.

Step 1. Place the right angle as shown; mark θ at the left vertex.

Step 2. Compute $\sin\theta = x/a$, so $\theta = \sin^{-1}(x/a)$.

Step 3. Compute $\tan\theta = x/\sqrt{a^2 - x^2}$, so $\theta = \tan^{-1}\left(x/\sqrt{a^2 - x^2}\right)$ as well.

Step 4. Equate the two expressions for θ.

Why this matters. Geometric pictures of inverse-trig identities are usually faster than algebraic substitutions, and they make the domain $|x| < a$ visually obvious (the triangle must close, so $|x| \le a$).

> **Final Answer:** $\sin^{-1}\left(\dfrac{x}{a}\right).$

Q 2.7 **Write** $\tan^{-1}\left(\dfrac{3a^2x - x^3}{a^3 - 3ax^2}\right)$, $a > 0$, $\dfrac{-a}{\sqrt{3}} < x < \dfrac{a}{\sqrt{3}}$, **in simplest form.**

SOLUTION

Concept used. The substitution $x = a\tan\theta$ turns the given rational expression into the tangent of 3θ, via the triple-angle formula $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$. The bounds on x are crafted so that 3θ stays inside the principal range of $\tan^{-1}$.

Step 1. **Substitute.** Let $x = a\tan\theta$ with $\theta = \tan^{-1}(x/a)$. The given range $\frac{-a}{\sqrt{3}} < x < \frac{a}{\sqrt{3}}$ gives $|\tan\theta| < \frac{1}{\sqrt{3}}$, hence $|\theta| < \frac{\pi}{6}$ and $|3\theta| < \frac{\pi}{2}$.

Step 2. **Compute the numerator.** With $x = a\tan\theta$,
$3a^2x - x^3 = 3a^2(a\tan\theta) - (a\tan\theta)^3 = a^3(3\tan\theta - \tan^3\theta)$.

Step 3. **Compute the denominator.**
$a^3 - 3ax^2 = a^3 - 3a(a\tan\theta)^2 = a^3 - 3a^3\tan^2\theta = a^3(1 - 3\tan^2\theta)$.

Step 4. **Form the ratio.** Cancel the common factor a^3:
$$\frac{3a^2x - x^3}{a^3 - 3ax^2} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \tan 3\theta.$$

Step 5. **Apply $\tan^{-1}$.** Since $|3\theta| < \frac{\pi}{2}$,
$$\tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right) = \tan^{-1}(\tan 3\theta) = 3\theta = 3\tan^{-1}(x/a).$$

Final Answer: $\tan^{-1}\left(\dfrac{3a^2x - x^3}{a^3 - 3ax^2}\right) = 3\tan^{-1}\left(\dfrac{x}{a}\right)$.

♥ Pattern

This problem packages the tangent triple-angle formula in disguise. Any time a rational function looks like $\frac{3p-p^3}{1-3p^2}$ in some natural variable p, suspect $\tan 3\theta$ with $p = \tan\theta$.

EXPERT'S SOLUTION : *Riya Joshi, B.Tech CSE, IIT Roorkee*

Structural observation. Once you spot the $3p - p^3$ over $1 - 3p^2$ shape, the rest is bookkeeping.

Step 1. Factor the powers of a: numerator $= a^3(3p - p^3)$, denominator $= a^3(1 - 3p^2)$, where $p = x/a$.

Step 2. Cancel a^3. Recognise the result as $\tan 3\theta$ with $p = \tan\theta$.

Step 3. Bound: $|p| < \frac{1}{\sqrt{3}}$ gives $|3\theta| < \frac{\pi}{2}$.

Step 4. Hence the expression simplifies to $3\theta = 3\tan^{-1} p = 3\tan^{-1}(x/a)$.

Why this matters. The given bound is not arbitrary: it is exactly the largest range of x that keeps $3\tan^{-1}(x/a)$ inside the principal open interval. Outside this range, an extra $\pm\pi$ appears.

Final Answer: $3\tan^{-1}\left(\frac{x}{a}\right)$.

Q 2.8 **Find the value of** $\tan^{-1}\left(2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right)$.

SOLUTION

Concept used. Work from the innermost inverse outwards. First evaluate $\sin^{-1}(1/2)$ to a principal angle, then take its double-angle cosine, then multiply by 2, then take $\tan^{-1}$.

Step 1. **Innermost inverse.** $\sin^{-1}(1/2) = \pi/6$ (since $\sin(\pi/6) = 1/2$ and $\pi/6 \in [-\pi/2, \pi/2]$).

Step 2. **Double it.** $2\sin^{-1}(1/2) = 2 \cdot \pi/6 = \pi/3$.

Step 3. **Take the cosine.** $\cos(\pi/3) = 1/2$.

Step 4. **Multiply by** 2. $2\cos(2\sin^{-1}(1/2)) = 2 \cdot 1/2 = 1$.

Step 5. **Outermost inverse.** $\tan^{-1}(1)$. Reference: $\tan(\pi/4) = 1$ and $\pi/4 \in (-\pi/2, \pi/2)$, so $\tan^{-1}(1) = \pi/4$.

Final Answer: $\tan^{-1}\left(2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right) = \frac{\pi}{4}$.

EXPERT'S SOLUTION *: Diya Banerjee, Ph.D Pure Mathematics, IISc Bangalore*

Quick reading. Peel from the inside, one operation at a time. Each step shrinks the expression by exactly one symbol.

Step 1. $\sin^{-1}(1/2) = \pi/6$.

Step 2. $2 \cdot \pi/6 = \pi/3$.

Step 3. $\cos(\pi/3) = 1/2$.

Step 4. $2 \cdot 1/2 = 1$.

Step 5. $\tan^{-1}(1) = \pi/4$.

Why this matters. Nested inverse-trig expressions are mostly careful arithmetic. The hardest part is keeping each intermediate value inside its principal range, which here is automatic because every input lies in $[-1, 1]$.

Final Answer: $\frac{\pi}{4}$.

Q 2.9 **Find the value of** $\tan\left(\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2}+\frac{1}{2}\cos^{-1}\frac{1-y^2}{1+y^2}\right)$ **for** $|x|<1$, $y>0$, $xy<1$.

SOLUTION

Concept used. Two important inverse-trig identities. For $|x| \le 1$, $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x$. For $y \ge 0$, $\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right) = 2\tan^{-1}y$. Both follow by setting $x = \tan A$ or $y = \tan B$ and using the double-angle formulas $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$ and $\cos 2B = \frac{1-\tan^2 B}{1+\tan^2 B}$. We also use $\tan(A+B) = \dfrac{\tan A + \tan B}{1-\tan A\tan B}$.

Step 1. **Simplify the two inverse terms.** $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x$ and $\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right) = 2\tan^{-1}y$.

Step 2. **Multiply each by $\frac{1}{2}$.** $\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \tan^{-1}x$, $\frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right) = \tan^{-1}y$.

Step 3. **Add inside the outer** $\tan$**.** The expression becomes $\tan(\tan^{-1}x + \tan^{-1}y)$.

Step 4. **Apply the tangent sum formula.** Let $A = \tan^{-1}x$, $B = \tan^{-1}y$, so $\tan A = x$, $\tan B = y$. Then $\tan(A+B) = \dfrac{\tan A + \tan B}{1-\tan A\tan B} = \dfrac{x+y}{1-xy}$. The condition $xy<1$ keeps the denominator positive and non-zero.

Final Answer: $\tan\left(\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2}+\frac{1}{2}\cos^{-1}\frac{1-y^2}{1+y^2}\right) = \dfrac{x+y}{1-xy}$.

☞ **Identity recap**

$\frac{2x}{1+x^2}$ is $\sin 2A$ and $\frac{1-x^2}{1+x^2}$ is $\cos 2A$ whenever $x = \tan A$. These two "double-angle in disguise" forms appear repeatedly in this chapter.

EXPERT'S SOLUTION *: Krishna Gupta, M.Sc Mathematics, IIT Bombay*

Strategic angle. Replace each inverse-trig expression by $2\tan^{-1}$(base var) at sight; the halves collapse, and the outer $\tan$ of a sum is one formula away from the final answer.

Step 1. Identities: $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x$, $\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right) = 2\tan^{-1}y$.

Step 2. Halves clear: argument $= \tan^{-1} x + \tan^{-1} y$.

Step 3. Tangent of sum: $\tan(\tan^{-1} x + \tan^{-1} y) = \dfrac{x+y}{1-xy}$, valid because $xy < 1$.

Why this matters. The compact result $\dfrac{x+y}{1-xy}$ is exactly the formula for $\tan^{-1} x + \tan^{-1} y$ when $xy < 1$. So the whole expression is just a re-encoding of that sum identity in disguise.

Final Answer: $\dfrac{x+y}{1-xy}$.

Q 2.10 **Find the value of** $\sin^{-1}\left(\sin \dfrac{2\pi}{3}\right)$**.**

SOLUTION

Concept used. The identity $\sin^{-1}(\sin u) = u$ holds only when $u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. For u outside the principal range we use $\sin(\pi - u) = \sin u$ to find an equivalent angle inside the principal range.

Step 1. **Check the inner angle.** Compare $\frac{2\pi}{3} = \frac{4\pi}{6}$ with $\frac{\pi}{2} = \frac{3\pi}{6}$: $\frac{2\pi}{3} > \frac{\pi}{2}$. Not in the principal range.

Step 2. **Find an equivalent angle.** Apply $\sin(\pi - u) = \sin u$:
$\sin(2\pi/3) = \sin(\pi - 2\pi/3) = \sin(\pi/3)$.

Step 3. **Check.** $\pi/3 \in [-\pi/2, \pi/2]$ ✓.

Step 4. **Apply** $\sin^{-1}$**.** $\sin^{-1}(\sin(2\pi/3)) = \sin^{-1}(\sin(\pi/3)) = \pi/3$.

Final Answer: $\sin^{-1}\left(\sin \dfrac{2\pi}{3}\right) = \dfrac{\pi}{3}$.

✗ Common Mistake

A very common slip is to write $\sin^{-1}(\sin(2\pi/3)) = 2\pi/3$. That is wrong because $2\pi/3$ is not in $[-\pi/2, \pi/2]$. The principal value is $\pi/3$, the only angle in the principal range whose sine agrees.

EXPERT'S SOLUTION *: Tara Desai, M.Tech CS, IIT Madras*

Picture-first. On the unit circle, $\sin(2\pi/3)$ is the y-coordinate at $120°$, namely $\sqrt{3}/2$. The principal angle whose sine is $\sqrt{3}/2$ is $60° = \pi/3$.

Step 1. $\sin(2\pi/3) = \sqrt{3}/2$.

Step 2. $\sin^{-1}(\sqrt{3}/2) = \pi/3$, and $\pi/3 \in [-\pi/2, \pi/2]$.

Step 3. Therefore $\sin^{-1}(\sin(2\pi/3)) = \pi/3$.

Why this matters. For any u in the second quadrant, the shortcut $\sin^{-1}(\sin u) = \pi - u$ gives the principal answer directly. Here $\pi - 2\pi/3 = \pi/3$.

> **Final Answer:** $\dfrac{\pi}{3}$.

Q 2.11 **Find the value of** $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right)$.

SOLUTION

Concept used. $\tan^{-1}(\tan u) = u$ only when $u \in (-\pi/2, \pi/2)$. For u outside this open interval, use the periodicity $\tan(u - \pi) = \tan u$ to shift u into the principal range.

Step 1. **Check.** Compare $3\pi/4$ with $\pi/2 = 2\pi/4$: $3\pi/4 > \pi/2$, so $3\pi/4$ is not in the principal range.

Step 2. **Shift by π.** Apply $\tan(u - \pi) = \tan u$: $\tan(3\pi/4) = \tan(3\pi/4 - \pi) = \tan(-\pi/4)$.

Step 3. **Check.** $-\pi/2 < -\pi/4 < \pi/2$ ✓.

Step 4. **Apply $\tan^{-1}$.** $\tan^{-1}(\tan(3\pi/4)) = \tan^{-1}(\tan(-\pi/4)) = -\pi/4$.

> **Final Answer:** $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right) = -\dfrac{\pi}{4}$.

EXPERT'S SOLUTION *: Ishaan Chatterjee, M.Sc Mathematics, ISI Kolkata*

Strategic angle. For $\tan^{-1}(\tan u)$, only adding integer multiples of π preserves the value. Subtract one π from $3\pi/4$ to land in the open principal interval.

Step 1. Compute $\tan(3\pi/4) = -1$.

Step 2. Principal $\tan^{-1}(-1) = -\pi/4$.

Step 3. Done: $\tan^{-1}(\tan(3\pi/4)) = -\pi/4$.

Why this matters. General rule: if $u \in (\pi/2, \pi)$, then $\tan^{-1}(\tan u) = u - \pi$. Memorise this and the periodicity step takes one line.

Final Answer: $-\dfrac{\pi}{4}$.

Q 2.12 **Find the value of** $\tan\left(\sin^{-1}\dfrac{3}{5} + \cot^{-1}\dfrac{3}{2}\right)$.

SOLUTION

Concept used. Right-triangle reading of $\sin^{-1}$ and $\cot^{-1}$. If $\sin^{-1}(3/5) = A$, then in a right triangle with opposite 3 and hypotenuse 5, the adjacent side is $\sqrt{5^2 - 3^2} = 4$, giving $\tan A = 3/4$. If $\cot^{-1}(3/2) = B$, then $\cot B = 3/2$, so $\tan B = 2/3$. The tangent sum formula $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ finishes the work.

Step 1. **Find** $\tan A$. From $A = \sin^{-1}(3/5)$: $\sin A = 3/5$, $\cos A = 4/5$ (positive in $[0, \pi/2)$), so $\tan A = \dfrac{\sin A}{\cos A} = \dfrac{3/5}{4/5} = \dfrac{3}{4}$.

Step 2. **Find** $\tan B$. From $B = \cot^{-1}(3/2)$: $\cot B = 3/2$, hence $\tan B = 2/3$.

Step 3. **Apply tangent sum.** $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\,\tan B} = \dfrac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{3}{4}\cdot\frac{2}{3}}$.

Step 4. **Simplify the numerator.** Common denominator 12: $\frac{3}{4} + \frac{2}{3} = \frac{9}{12} + \frac{8}{12} = \frac{17}{12}$.

Step 5. **Simplify the denominator.** $1 - \frac{3}{4}\cdot\frac{2}{3} = 1 - \frac{6}{12} = 1 - \frac{1}{2} = \frac{1}{2}$.

Step 6. **Divide.** $\tan(A+B) = \dfrac{17/12}{1/2} = \dfrac{17}{12}\cdot 2 = \dfrac{17}{6}$.

Final Answer: $\tan\left(\sin^{-1}\dfrac{3}{5} + \cot^{-1}\dfrac{3}{2}\right) = \dfrac{17}{6}$.

EXPERT'S SOLUTION *: Aanya Rao, M.Sc Applied Mathematics, IIT Kanpur*

Picture-first. Two right triangles with leg ratios $(3, 4, 5)$ and $(2, 3, \sqrt{13})$ encode both inverse expressions; the tangent of the sum follows from the addition formula.

Step 1. Triangle 1 $(3, 4, 5)$: $\sin A = 3/5$, $\tan A = 3/4$.

Step 2. Triangle 2 $(2, 3, \sqrt{13})$: $\cot B = 3/2$, $\tan B = 2/3$.

Step 3. Tangent of sum: $\tan(A + B) = \dfrac{3/4 + 2/3}{1 - (3/4)(2/3)} = \dfrac{17/12}{1/2} = \dfrac{17}{6}$.

Why this matters. Right-triangle pictures make "$\tan(\sin^{-1} \frac{a}{c})$"-style values almost automatic. Pick the two legs that match the ratio, find the third from Pythagoras, read off the desired ratio.

Final Answer: $\dfrac{17}{6}$.

Q 2.13 $\cos^{-1}\left(\cos \dfrac{7\pi}{6}\right)$ **is equal to**

(A) $\dfrac{7\pi}{6}$ **(B)** $\dfrac{5\pi}{6}$ **(C)** $\dfrac{\pi}{3}$ **(D)** $\dfrac{\pi}{6}$.

SOLUTION

Concept used. $\cos^{-1}(\cos u) = u$ only when $u \in [0, \pi]$. For u outside this range, use $\cos(2\pi - u) = \cos u$ to find an equivalent angle in $[0, \pi]$.

Step 1. **Check.** Is $7\pi/6$ in $[0, \pi]$? Compare $7\pi/6 > \pi = 6\pi/6$. No.

Step 2. **Find equivalent angle.** Apply $\cos(2\pi - u) = \cos u$:
$\cos(7\pi/6) = \cos(2\pi - 7\pi/6) = \cos((12\pi - 7\pi)/6) = \cos(5\pi/6)$.

Step 3. **Check.** $0 \leq 5\pi/6 \leq \pi$ ✓.

Step 4. **Apply $\cos^{-1}$.** $\cos^{-1}(\cos(7\pi/6)) = \cos^{-1}(\cos(5\pi/6)) = 5\pi/6$. Matches option (B).

Final Answer: Option (B): $\dfrac{5\pi}{6}$.

EXPERT'S SOLUTION : *Pooja Kumar, M.Sc Mathematics, IIT Bombay*

Quick reading. For $u \in (\pi, 2\pi)$, the principal version of u under $\cos^{-1} \circ \cos$ is $2\pi - u$. Apply directly: $2\pi - 7\pi/6 = 5\pi/6$.

Step 1. $u = 7\pi/6$ is in $(\pi, 2\pi)$.

Step 2. Equivalent principal angle: $2\pi - 7\pi/6 = 5\pi/6$.

Step 3. Therefore $\cos^{-1}(\cos(7\pi/6)) = 5\pi/6$.

Why this matters. Two cases handle every $\cos^{-1}(\cos u)$: $u \in [0, \pi]$ gives u; $u \in (\pi, 2\pi)$

gives $2\pi - u$. Beyond $[0, 2\pi]$, first reduce modulo 2π.

Final Answer: (B)

Q 2.14 $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$ **is equal to**

(A) $\frac{1}{2}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{4}$ **(D)** 1.

SOLUTION

Concept used. Evaluate the inverse first to convert the whole expression into a single sine evaluation.

Step 1. Inner inverse. $\sin^{-1}(-1/2) = -\sin^{-1}(1/2) = -\pi/6$.

Step 2. Substitute. Expression becomes $\sin(\pi/3 - (-\pi/6)) = \sin(\pi/3 + \pi/6)$.

Step 3. Add fractions. Common denominator 6:
$\pi/3 + \pi/6 = 2\pi/6 + \pi/6 = 3\pi/6 = \pi/2$.

Step 4. Evaluate sin. $\sin(\pi/2) = 1$.

Final Answer: Option (D): 1.

EXPERT'S SOLUTION *: Ankit Kapoor, M.Tech CS, IIT Madras*

Strategic angle. Convert inverse to value, simplify the argument to a standard angle, read off the sine.

Step 1. $\sin^{-1}(-1/2) = -\pi/6$.

Step 2. Argument of outer sine: $\pi/3 + \pi/6 = \pi/2$.

Step 3. $\sin(\pi/2) = 1$.

Why this matters. The hidden simplification is that $\pi/3$ and $\pi/6$ add to exactly $\pi/2$. The question is engineered so the final sine is a famous value, which is the usual NCERT MCQ design.

Final Answer: (D)

Q 2.15 $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$ **is equal to**

(A) π **(B)** $-\frac{\pi}{2}$ **(C)** 0 **(D)** $2\sqrt{3}$.

SOLUTION

Concept used. The principal ranges: $\tan^{-1}: \mathbb{R} \to (-\pi/2, \pi/2)$ and $\cot^{-1}: \mathbb{R} \to (0, \pi)$. For a negative argument, $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$, $x \in \mathbb{R}$, following from $\cot(\pi - \theta) = -\cot\theta$.

Step 1. Compute $\tan^{-1}\sqrt{3}$. $\tan(\pi/3) = \sqrt{3}$ and $\pi/3 \in (-\pi/2, \pi/2)$, so $\tan^{-1}\sqrt{3} = \pi/3$.

Step 2. Compute $\cot^{-1}(\sqrt{3})$. $\cot(\pi/6) = \sqrt{3}$ and $\pi/6 \in (0, \pi)$, so $\cot^{-1}\sqrt{3} = \pi/6$.

Step 3. Compute $\cot^{-1}(-\sqrt{3})$. $\cot^{-1}(-\sqrt{3}) = \pi - \cot^{-1}\sqrt{3} = \pi - \pi/6 = 5\pi/6$.

Step 4. Subtract. $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3}) = \pi/3 - 5\pi/6$. Common denominator 6: $\pi/3 = 2\pi/6$, so $2\pi/6 - 5\pi/6 = -3\pi/6 = -\pi/2$.

> **Final Answer:** Option (B): $-\frac{\pi}{2}$.

✗ Common Mistake

The principal range of $\cot^{-1}$ is $(0, \pi)$, not $(-\pi/2, \pi/2)$. So $\cot^{-1}(-\sqrt{3})$ is positive $(= 5\pi/6)$, not negative. Always check the function's principal range before applying any sign rule.

EXPERT'S SOLUTION *: Sneha Bhat, Ph.D Mathematics, IIT Delhi*

Quick reading. $\tan^{-1}$ of a positive number is a small positive angle; $\cot^{-1}$ of a negative number is a large (obtuse) positive angle. The difference is therefore strongly negative, and the only negative option is $-\pi/2$.

Step 1. $\tan^{-1}\sqrt{3} = \pi/3$ (about $60°$).

Step 2. $\cot^{-1}(-\sqrt{3}) = 5\pi/6$ (about $150°$).

Step 3. Difference: $60° - 150° = -90° = -\pi/2$.

Why this matters. The shifted principal range of $\cot^{-1}$ is the single most common stumbling block in this exercise. Keep the table in front of mind: $\cot^{-1}: \mathbb{R} \to (0, \pi)$, never negative.

> **Final Answer:** (B)

Key Takeaways

- Multiple-angle identities $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$ and $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$ hold only on the x-range that keeps the multiplied angle in the principal range.

- Trig substitution dictionary: $\sqrt{1+x^2} \to x = \tan\theta$; $\sqrt{1-x^2} \to x = \sin\theta$; $\sqrt{x^2-1} \to x = \sec\theta$; $\sqrt{a^2-x^2} \to x = a\sin\theta$.
- The double-angle disguises $\frac{2x}{1+x^2} = \sin 2A$, $\frac{1-x^2}{1+x^2} = \cos 2A$ (with $x = \tan A$) collapse complex inverse-trig expressions to $\tan^{-1} x$.
- $\sin^{-1}(\sin u)$, $\cos^{-1}(\cos u)$, $\tan^{-1}(\tan u)$ equal u only on their respective principal ranges. Always check u first, then shift via supplementary or periodic identities if needed.

End of Exercise 2.2