



# Collegedunia NCERT Solutions

Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

## Chapter 5: Continuity and Differentiability

### About this Chapter

Chapter 5 builds on the limit machinery from Class 11 to define **continuity** of a real function at a point and on an interval, then proceeds to **differentiability**. The exercise drills the LHL/RHL/value-at-point test for continuity, finds the parameters that make piecewise functions continuous, and proves that polynomials, trigonometric and elementary composite functions are continuous on their domains. Mastery of this set is the foundation for every calculus topic that follows.

**Topics covered:** Definition of continuity • LHL, RHL, value test • Algebra of continuous functions • Piecewise functions and parameters

#### Quick Formula Sheet

##### Continuity at $x = c$ :

$f$  is continuous at  $c$  iff  
 $\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = f(c)$ .

##### Algebra rule:

Sum, difference, product, quotient (denominator  $\neq 0$ ) and composition of continuous functions are continuous.

##### Standard continuous functions:

Every polynomial,  $\sin x$ ,  $\cos x$ ,  $e^x$ ,  $\log x$  (on its domain) is continuous everywhere on its domain.

### Exercise 5.1

**Q 5.1** Prove that the function  $f(x) = 5x - 3$  is continuous at  $x = 0$ , at  $x = -3$  and at  $x = 5$ .

#### SOLUTION

**Concept used.** A function  $f$  is said to be **continuous** at a point  $c$  in its domain if

$$\lim_{x \rightarrow c} f(x) = f(c).$$

For  $f$  to be continuous at  $c$  the limit must exist (i.e. left-hand limit equals right-hand limit) and must coincide with the value  $f(c)$ . A polynomial of degree 1 takes its limit by direct substitution.

**Step 1.** At  $x = 0$ .

$$f(0) = 5(0) - 3 = -3.$$

Compute the limit:

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (5x - 3) = 5(0) - 3 = -3.$$

Since  $\lim_{x \rightarrow 0} f(x) = f(0) = -3$ ,  $f$  is continuous at  $x = 0$ .

**Step 2.** At  $x = -3$ .

$$f(-3) = 5(-3) - 3 = -18.$$

$$\lim_{x \rightarrow -3} (5x - 3) = 5(-3) - 3 = -18.$$

Hence  $\lim_{x \rightarrow -3} f(x) = f(-3)$ , so  $f$  is continuous at  $x = -3$ .

**Step 3.** At  $x = 5$ .

$$f(5) = 5(5) - 3 = 22.$$

$$\lim_{x \rightarrow 5} (5x - 3) = 5(5) - 3 = 22.$$

Hence  $\lim_{x \rightarrow 5} f(x) = f(5)$ , so  $f$  is continuous at  $x = 5$ .

**Final Answer:**  $f(x) = 5x - 3$  is continuous at each of  $x = 0, -3, 5$ .

### Exam Tip

Every polynomial function  $a_0 + a_1x + a_2x^2 + \dots + a_nx^n$  is continuous on  $\mathbb{R}$ . Once you remember this, one-shot questions like this become a direct-substitution check.

**EXPERT'S SOLUTION** : Aarav Sharma, M.Sc Mathematics, IIT Bombay

**Strategic angle.** Treat  $f(x) = 5x - 3$  as the polynomial it is and apply the general theorem that polynomials are continuous everywhere.

**Step 1. General principle.** If  $p(x)$  is a polynomial, then  $\lim_{x \rightarrow c} p(x) = p(c)$  for every real  $c$  (the limit of a sum is the sum of limits; limits respect scalar multiplication;  $\lim_{x \rightarrow c} x = c$ ). Therefore every polynomial is continuous on  $\mathbb{R}$ .

**Step 2. Apply to  $f$ .** Here  $f$  is the degree-1 polynomial  $5x - 3$ . By the principle above  $f$  is continuous at every real  $c$ .

**Step 3. Specialise.** In particular it is continuous at  $c = 0$  (value  $-3$ ), at  $c = -3$  (value  $-18$ ) and at  $c = 5$  (value  $22$ ). One can also verify directly: substitute and the limit and the function value match.

**Why this matters.** Identifying the underlying function class (here: polynomial) often replaces three separate  $\varepsilon$ - $\delta$  arguments with one general result.

**Final Answer:**  $f$  is continuous at  $x = 0, -3, 5$ .

**Q 5.2** Examine the continuity of the function  $f(x) = 2x^2 - 1$  at  $x = 3$ .

### SOLUTION

**Concept used.**  $f$  is continuous at  $x = c$  iff  $\lim_{x \rightarrow c} f(x) = f(c)$ . For a polynomial, the limit at any point equals direct substitution.

**Step 1. Function value.**

$$f(3) = 2(3)^2 - 1 = 2 \times 9 - 1 = 18 - 1 = 17.$$

**Step 2. Limit at  $x = 3$ .**

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} (2x^2 - 1) = 2(3)^2 - 1 = 17.$$

**Step 3. Conclude.** Since  $\lim_{x \rightarrow 3} f(x) = 17 = f(3)$ ,  $f$  is continuous at  $x = 3$ .

**Final Answer:**  $f(x) = 2x^2 - 1$  is continuous at  $x = 3$ .

**EXPERT'S SOLUTION** : *Sneha Iyer, Ph.D Mathematics, IIT Delhi*

**Quick reading.** A degree-2 polynomial inherits continuity at every point of  $\mathbb{R}$  from the algebra-of-continuous-functions theorem.

**Step 1.** Identify the building blocks: the constant function 2, the identity function  $x$ , the product  $x \cdot x = x^2$  and the constant  $-1$  are each continuous on  $\mathbb{R}$ .

**Step 2.** Products and sums of continuous functions are continuous, hence  $2x^2 - 1$  is continuous everywhere; in particular at  $x = 3$ .

**Step 3.** Verification:  $f(3) = 17$  and the limit  $\lim_{x \rightarrow 3} (2x^2 - 1) = 17$  agree.

**Final Answer:** Continuous at  $x = 3$ .

**Q 5.3** Examine the following functions for continuity.

(a)  $f(x) = x - 5$

(b)  $f(x) = \frac{1}{x-5}, x \neq 5$

(c)  $f(x) = \frac{x^2 - 25}{x+5}, x \neq -5$

(d)  $f(x) = |x - 5|$

**SOLUTION**

**Concept used.** A function is continuous on a set  $S$  if it is continuous at every point of  $S$ . Polynomials are continuous on  $\mathbb{R}$ ; a rational function  $p(x)/q(x)$  is continuous wherever  $q(x) \neq 0$ ; the modulus function  $|x|$  is continuous on  $\mathbb{R}$  (and hence so is any horizontal translate  $|x - a|$ ).

**Step 1. (a)**  $f(x) = x - 5$ . This is a polynomial of degree 1, so it is continuous on  $\mathbb{R}$ . To check explicitly at any  $c \in \mathbb{R}$ :

$$\lim_{x \rightarrow c} (x - 5) = c - 5 = f(c).$$

**Step 2. (b)**  $f(x) = 1/(x - 5), x \neq 5$ . The domain is  $\mathbb{R} \setminus \{5\}$ . For any  $c \neq 5$  the denominator  $c - 5 \neq 0$ , so

$$\lim_{x \rightarrow c} \frac{1}{x-5} = \frac{1}{c-5} = f(c).$$

Hence  $f$  is continuous at every point of its domain  $\mathbb{R} \setminus \{5\}$ .

**Step 3. (c)**  $f(x) = \frac{x^2 - 25}{x + 5}, x \neq -5$ . For  $x \neq -5$ , factor the numerator:

$$f(x) = \frac{(x - 5)(x + 5)}{x + 5} = x - 5.$$

Thus on its domain  $\mathbb{R} \setminus \{-5\}$ ,  $f$  coincides with the polynomial  $x - 5$ , which is continuous everywhere. So  $f$  is continuous at every point of its domain.

**Step 4. (d)**  $f(x) = |x - 5|$ . Write piecewise:

$$f(x) = \begin{cases} x - 5, & x \geq 5, \\ -(x - 5), & x < 5. \end{cases}$$

For  $c > 5$ :  $f(c) = c - 5$  and  $\lim_{x \rightarrow c} (x - 5) = c - 5$ ; continuous.

For  $c < 5$ :  $f(c) = 5 - c$  and  $\lim_{x \rightarrow c} (5 - x) = 5 - c$ ; continuous.

At  $c = 5$ :  $f(5) = 0$ . Left limit  $\lim_{x \rightarrow 5^-} (5 - x) = 0$ ; right limit  $\lim_{x \rightarrow 5^+} (x - 5) = 0$ .

Both equal  $f(5)$ , so  $f$  is continuous at 5 as well. Hence  $f$  is continuous on  $\mathbb{R}$ .

**Final Answer:** (a) continuous on  $\mathbb{R}$ ; (b) continuous on  $\mathbb{R} \setminus \{5\}$ ; (c) continuous on  $\mathbb{R} \setminus \{-5\}$ ; (d) continuous on  $\mathbb{R}$ .

**X Common Mistake**

In part (c) it is tempting to say “ $f$  is discontinuous at  $x = -5$ ”. The point  $x = -5$  is not in the domain of  $f$ , so the question of continuity at  $-5$  does not arise. Continuity is checked only at points of the domain.

**EXPERT'S SOLUTION** : Vivaan Gupta, M.Tech CS, IIT Madras

**Structural observation.** Each part falls into a standard family: polynomial, rational, removable-singularity rational, modulus.

**Step 1. (a)** Polynomial  $\Rightarrow$  continuous on  $\mathbb{R}$ .

**Step 2. (b)** Rational with vertical asymptote at  $x = 5$ . Continuous wherever defined, i.e. on  $\mathbb{R} \setminus \{5\}$ .

**Step 3. (c)** After cancelling the common factor  $(x + 5)$ ,  $f(x) = x - 5$  on  $\mathbb{R} \setminus \{-5\}$ . Continuous on the full domain.

**Step 4. (d)**  $|x - 5|$  is the composition of the continuous functions  $x \mapsto x - 5$  and  $u \mapsto |u|$ , hence continuous everywhere.

**Final Answer:** Each function is continuous on its domain.

**Q 5.4** Prove that the function  $f(x) = x^n$  is continuous at  $x = n$ , where  $n$  is a positive integer.

**SOLUTION**

**Concept used.** For positive integer  $n$ ,  $x^n$  is the  $n$ -fold product of the identity function with itself. The product of continuous functions is continuous, and the identity function  $x \mapsto x$  is continuous on  $\mathbb{R}$ , hence  $x^n$  is continuous on  $\mathbb{R}$ . We must show  $\lim_{x \rightarrow n} x^n = n^n$ .

**Step 1. Compute the value.**

$$f(n) = n^n.$$

**Step 2. Compute the limit.** Using the algebra of limits,

$$\lim_{x \rightarrow n} x^n = \left( \lim_{x \rightarrow n} x \right)^n = n^n.$$

(Here we used that the limit of a product is the product of the limits, applied  $n$  times.)

**Step 3. Conclude.** Since  $\lim_{x \rightarrow n} f(x) = n^n = f(n)$ ,  $f$  is continuous at  $x = n$ .

**Final Answer:**  $f(x) = x^n$  is continuous at  $x = n$  for every positive integer  $n$ .

**EXPERT'S SOLUTION** : Aanya Mehta, Ph.D Pure Mathematics, IISc Bangalore

**Strategic angle.** Reduce to the basic fact that the identity function is continuous, then use the product rule for continuity.

**Step 1. Base case.** The function  $g(x) = x$  is continuous on  $\mathbb{R}$  because  $\lim_{x \rightarrow c} x = c$  for every  $c$ .

**Step 2. Inductive step.** If  $x^k$  is continuous, then so is  $x^{k+1} = x \cdot x^k$  as a product of two continuous functions. By induction  $x^n$  is continuous for every positive integer  $n$ .

**Step 3. At the specified point.** In particular  $x^n$  is continuous at  $x = n$ , and  $\lim_{x \rightarrow n} x^n = n^n$ .

**Final Answer:**  $f(x) = x^n$  is continuous at  $x = n$ .

**Q 5.5** Is the function  $f$  defined by

$$f(x) = \begin{cases} x, & \text{if } x \leq 1, \\ 5, & \text{if } x > 1, \end{cases}$$

continuous at  $x = 0$ ? At  $x = 1$ ? At  $x = 2$ ?

**SOLUTION**

**Concept used.** For a piecewise-defined function, continuity at the join point  $c$  requires

$$\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = f(c).$$

At interior points of one piece, continuity follows from the continuity of that piece.

**Step 1. At  $x = 0$ .** Since  $0 \leq 1$ , we are inside the piece  $f(x) = x$ , which is a polynomial. Hence

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x = 0 = f(0).$$

So  $f$  is continuous at  $x = 0$ .

**Step 2. At  $x = 1$  (the join point).**  $f(1) = 1$  (using the first piece, since  $1 \leq 1$ ).

Left limit

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x = 1$$

Right limit

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 5 = 5$$

Since the left and right limits differ ( $1 \neq 5$ ),  $\lim_{x \rightarrow 1} f(x)$  does not exist;  $f$  is **not continuous** at  $x = 1$ .

**Step 3. At  $x = 2$ .** Since  $2 > 1$ , we are inside the piece  $f(x) = 5$ , a constant. Hence

$$\lim_{x \rightarrow 2} f(x) = 5 = f(2).$$

So  $f$  is continuous at  $x = 2$ .

**Final Answer:** Continuous at  $x = 0$  and  $x = 2$ ; discontinuous at  $x = 1$ .

**EXPERT'S SOLUTION** : Priya Singh, M.Sc Applied Mathematics, IIT Kanpur

**Picture-first.** Sketch the graph: a  $45^\circ$  line  $y = x$  up to and including  $(1, 1)$ , then a jump up to the horizontal line  $y = 5$  for  $x > 1$ . The jump is the discontinuity.

**Step 1.** At any  $c < 1$  the function equals  $x$ , continuous.

**Step 2.** At any  $c > 1$  the function equals  $5$ , continuous.

**Step 3.** At  $c = 1$  the function value is  $1$  but the right limit is  $5$ : a jump discontinuity.

**Why this matters.** Even with each piece being itself continuous, a piecewise function can fail to be continuous at the junction. The junction always needs the three-line LHL/RHL/value check.

**Final Answer:** Discontinuous only at  $x = 1$ .

**Q 5.6** Find all points of discontinuity of  $f$ , where

$$f(x) = \begin{cases} 2x + 3, & \text{if } x \leq 2, \\ 2x - 3, & \text{if } x > 2. \end{cases}$$

### SOLUTION

**Concept used.** Each piece is a polynomial, hence continuous wherever it is defined. The only potentially discontinuous point is the join  $x = 2$ , where we apply the LHL/RHL/value test.

**Step 1.** For  $x < 2$ ,  $f(x) = 2x + 3$  is a polynomial, continuous.

**Step 2.** For  $x > 2$ ,  $f(x) = 2x - 3$  is a polynomial, continuous.

**Step 3.** At  $x = 2$ :  $f(2) = 2(2) + 3 = 7$ .

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (2x + 3) = 2(2) + 3 = 7$$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (2x - 3) = 2(2) - 3 = 1$$

Since the left limit (7) and the right limit (1) are unequal,  $\lim_{x \rightarrow 2} f(x)$  does not exist;  $f$  is discontinuous at  $x = 2$ .

**Final Answer:**  $f$  is discontinuous only at  $x = 2$ .

**EXPERT'S SOLUTION** : Arjun Reddy, B.Tech CSE, IIT Roorkee

**Structural observation.** The jump size at  $x = 2$  equals  $(2 \cdot 2 + 3) - (2 \cdot 2 - 3) = 6$ , so the graph drops by 6 as  $x$  crosses 2 from left to right.

**Step 1.** Both pieces are linear, hence continuous on their open pieces.

**Step 2.** Discontinuity at the join  $x = 2$  because  $\lim_{x \rightarrow 2^-} f = 7$  but  $\lim_{x \rightarrow 2^+} f = 1$ .

**Final Answer:** Single discontinuity at  $x = 2$ .

**Q 5.7** Find all points of discontinuity of  $f$ , where

$$f(x) = \begin{cases} |x| + 3, & \text{if } x \leq -3, \\ -2x, & \text{if } -3 < x < 3, \\ 6x + 2, & \text{if } x \geq 3. \end{cases}$$

### SOLUTION

**Concept used.** Inside each open piece the formula is continuous (modulus, polynomial). Discontinuity can only happen at the join points  $x = -3$  and  $x = 3$ . We check each with the LHL/RHL/value test.

**Step 1.** At  $x = -3$ . For  $x \leq -3$  we use  $f(x) = |x| + 3 = -x + 3$  (since  $x \leq -3 < 0$ ).

$$f(-3) = -(-3) + 3 = 6.$$

Left limit

$$\lim_{x \rightarrow -3^-} f(x) = \lim_{x \rightarrow -3^-} (-x + 3) = -(-3) + 3 = 6$$

Right limit

$$\lim_{x \rightarrow -3^+} f(x) = \lim_{x \rightarrow -3^+} (-2x) = -2(-3) = 6$$

All three equal 6, so  $f$  is continuous at  $x = -3$ .

**Step 2.** At  $x = 3$ . For  $x \geq 3$ ,  $f(x) = 6x + 2$ , so  $f(3) = 6(3) + 2 = 20$ . Left limit

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (-2x) = -2(3) = -6$$

Right limit

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (6x + 2) = 6(3) + 2 = 20$$

Left limit  $-6 \neq 20$ , so  $f$  is discontinuous at  $x = 3$ .

**Step 3. Interior points.** For  $x < -3$ ,  $|x| + 3 = -x + 3$  is a polynomial, continuous. For  $-3 < x < 3$ ,  $-2x$  is a polynomial, continuous. For  $x > 3$ ,  $6x + 2$  is a polynomial, continuous.

**Final Answer:**  $f$  is discontinuous only at  $x = 3$ .

**EXPERT'S SOLUTION** : Rohit Verma, M.Sc Mathematics, ISI Kolkata

**Strategic angle.** A three-piece definition has two join points; each one is a separate three-line check.

**Step 1.** Within each piece, the formula is a polynomial in  $x$ , so continuous on open subintervals.

**Step 2.** Join  $x = -3$ :  $|-3| + 3 = 6$ ,  $-2(-3) = 6$ . Match, hence continuous.

**Step 3.** Join  $x = 3$ :  $-2(3) = -6$  on the left,  $6(3) + 2 = 20$  on the right. Jump of 26, hence discontinuous.

**Final Answer:** Discontinuity only at  $x = 3$ .

**Q 5.8** Find all points of discontinuity of  $f$ , where

$$f(x) = \begin{cases} \frac{|x|}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

**SOLUTION**

**Concept used.** For  $x > 0$ ,  $|x| = x$ , so  $|x|/x = 1$ . For  $x < 0$ ,  $|x| = -x$ , so  $|x|/x = -1$ .

Thus

$$f(x) = \begin{cases} 1, & x > 0, \\ -1, & x < 0, \\ 0, & x = 0. \end{cases}$$

**Step 1.** At  $x = 0$ .  $f(0) = 0$ . Left limit

$$\lim_{x \rightarrow 0^-} f(x) = -1$$

Right limit

$$\lim_{x \rightarrow 0^+} f(x) = 1$$

Since  $\text{LHL} = -1 \neq 1 = \text{RHL}$ ,  $\lim_{x \rightarrow 0} f(x)$  does not exist. Hence  $f$  is discontinuous at  $x = 0$ .

**Step 2.** For  $x > 0$ ,  $f(x) = 1$  is constant, continuous.

**Step 3.** For  $x < 0$ ,  $f(x) = -1$  is constant, continuous.

**Final Answer:**  $f$  is discontinuous only at  $x = 0$ .

**EXPERT'S SOLUTION** : *Karan Joshi, M.Sc Mathematics, IIT Bombay*

**Picture-first.** This is the **sign function** (with the  $x = 0$  value redefined to 0):  $-1$  on the left half-line,  $+1$  on the right half-line,  $0$  at the origin. Visually there is a vertical jump of  $2$  at the origin.

**Step 1.** Rewrite using sign:  $f(x) = \text{sgn}(x)$  for  $x \neq 0$ .

**Step 2.** Constants on either side give continuity inside each half-line.

**Step 3.** The jump at  $x = 0$  is irreparable: there is no value of  $f(0)$  that bridges  $\pm 1$ .

**Final Answer:** Single discontinuity at  $x = 0$ .

**Q 5.9** Find all points of discontinuity of  $f$ , where

$$f(x) = \begin{cases} \frac{x}{|x|}, & \text{if } x < 0, \\ -1, & \text{if } x \geq 0. \end{cases}$$

**SOLUTION**

**Concept used.** For  $x < 0$ ,  $|x| = -x$ , so  $\frac{x}{|x|} = \frac{x}{-x} = -1$ . Therefore

$$f(x) = \begin{cases} -1, & x < 0, \\ -1, & x \geq 0. \end{cases}$$

That is,  $f(x) = -1$  for every real  $x$ .

**Step 1.** For any  $c \in \mathbb{R}$ ,

$$\lim_{x \rightarrow c} f(x) = -1 = f(c).$$

**Step 2.** Hence  $f$  is continuous at every real  $c$ ; there is no point of discontinuity.

**Final Answer:**  $f$  is continuous everywhere on  $\mathbb{R}$ ; there is no point of discontinuity.

### Exam Tip

Always simplify the formula before testing continuity. A piecewise definition that looks tricky often collapses to a constant or a polynomial in disguise.

**EXPERT'S SOLUTION** : *Pranav Kapoor, M.Sc Mathematics, IIT Bombay*

**Quick reading.** The first piece, evaluated, gives  $-1$ , exactly matching the second piece. The whole function is the constant  $-1$ .

**Step 1.** For  $x < 0$ ,  $x/|x| = x/(-x) = -1$ .

**Step 2.** For  $x \geq 0$ ,  $f = -1$  by definition.

**Step 3.** Constants are continuous, so  $f$  is continuous on all of  $\mathbb{R}$ .

**Final Answer:** Continuous everywhere.

**Q 5.10** Find all points of discontinuity of  $f$ , where

$$f(x) = \begin{cases} x + 1, & \text{if } x \geq 1, \\ x^2 + 1, & \text{if } x < 1. \end{cases}$$

### SOLUTION

**Concept used.** Polynomials are continuous inside each piece. Test the join  $x = 1$  with LHL/RHL/value.

**Step 1.**  $f(1) = 1 + 1 = 2$  (using the first piece,  $x \geq 1$ ).

**Step 2.**

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x^2 + 1) = 1^2 + 1 = 2$$

**Step 3.**

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x + 1) = 1 + 1 = 2$$

**Step 4.** LHL = RHL =  $f(1) = 2$ , so  $f$  is continuous at  $x = 1$ . Together with continuity inside each piece,  $f$  is continuous on all of  $\mathbb{R}$ .

**Final Answer:** No point of discontinuity;  $f$  is continuous on  $\mathbb{R}$ .

**EXPERT'S SOLUTION** : Aditi Banerjee, M.Sc Mathematics, ISI Kolkata

**Strategic angle.** Both pieces output 2 at  $x = 1$ , so the function joins seamlessly.

**Step 1.** Inside  $x < 1$ ,  $x^2 + 1$  is continuous.

**Step 2.** Inside  $x > 1$ ,  $x + 1$  is continuous.

**Step 3.** At the join, both formulas give 2 and match the assigned value  $f(1) = 2$ .

**Final Answer:** Continuous on  $\mathbb{R}$ .

**Q 5.11** Find all points of discontinuity of  $f$ , where

$$f(x) = \begin{cases} x^3 - 3, & \text{if } x \leq 2, \\ x^2 + 1, & \text{if } x > 2. \end{cases}$$

**SOLUTION**

**Concept used.** Polynomials are continuous on  $\mathbb{R}$ , so each piece is continuous on its open part. Only the join  $x = 2$  needs checking.

**Step 1.**  $f(2) = 2^3 - 3 = 8 - 3 = 5$ .

**Step 2.**

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x^3 - 3) = 2^3 - 3 = 5$$

**Step 3.**

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x^2 + 1) = 2^2 + 1 = 5$$

**Step 4.** All three equal 5, hence  $f$  is continuous at  $x = 2$ . Together with the interior continuity,  $f$  is continuous on  $\mathbb{R}$ .

**Final Answer:** No point of discontinuity;  $f$  is continuous everywhere.

**EXPERT'S SOLUTION** : Yash Nair, M.Tech CS, IIT Madras

**Quick reading.** The two cubics-and-quadratics happen to take the same value 5 at  $x = 2$ , so the function glues continuously.

**Step 1.** Each piece is polynomial  $\Rightarrow$  continuous inside its open piece.

**Step 2.** At  $x = 2$ :  $2^3 - 3 = 5$  and  $2^2 + 1 = 5$ .

**Step 3.** Match, hence no discontinuity.

**Final Answer:** Continuous on  $\mathbb{R}$ .

**Q 5.12** Find all points of discontinuity of  $f$ , where

$$f(x) = \begin{cases} x^{10} - 1, & \text{if } x \leq 1, \\ x^2, & \text{if } x > 1. \end{cases}$$

### SOLUTION

**Concept used.** Polynomials are continuous; check only the join  $x = 1$ .

**Step 1.**  $f(1) = 1^{10} - 1 = 0$ .

**Step 2.**

$$\lim_{x \rightarrow 1^-} f(x) = 1^{10} - 1 = 0$$

**Step 3.**

$$\lim_{x \rightarrow 1^+} f(x) = 1^2 = 1$$

**Step 4.** LHL = 0  $\neq$  1 = RHL, so  $\lim_{x \rightarrow 1} f(x)$  does not exist;  $f$  is discontinuous at  $x = 1$ .

**Final Answer:**  $f$  is discontinuous only at  $x = 1$ .

**EXPERT'S SOLUTION** : Diya Bhat, Ph.D Mathematics, IIT Delhi

**Structural observation.** At  $x = 1$  the first piece gives 0 but the second piece gives 1; the unit jump is the discontinuity.

**Step 1.** Polynomials continuous on each open piece.

**Step 2.** Left value at 1:  $1 - 1 = 0$ ; right limit: 1.

**Step 3.** Single jump at  $x = 1$ .

**Final Answer:** Discontinuity at  $x = 1$  only.

**Q 5.13** Is the function defined by

$$f(x) = \begin{cases} x + 5, & \text{if } x \leq 1, \\ x - 5, & \text{if } x > 1, \end{cases}$$

a continuous function?

### SOLUTION

**Concept used.** Each piece is a polynomial, continuous inside its open part. Test  $x = 1$ .

**Step 1.**  $f(1) = 1 + 5 = 6$ .

**Step 2.**

$$\lim_{x \rightarrow 1^-} f(x) = 1 + 5 = 6$$

**Step 3.**

$$\lim_{x \rightarrow 1^+} f(x) = 1 - 5 = -4$$

**Step 4.** LHL = 6  $\neq$  -4 = RHL, so  $f$  is not continuous at  $x = 1$ ; the function is not continuous on  $\mathbb{R}$ .

**Final Answer:** No,  $f$  is not a continuous function; it is discontinuous at  $x = 1$ .

### EXPERT'S SOLUTION : Ishaan Desai, M.Sc Mathematics, IIT Bombay

**Quick reading.** The two pieces differ by a constant 10 at  $x = 1$ , producing a jump.

**Step 1.** Polynomial inside each piece.

**Step 2.** Left value 6, right limit -4.

**Step 3.** Jump of -10 at  $x = 1$ ; function fails to be continuous on  $\mathbb{R}$ .

**Final Answer:** Not continuous; jump at  $x = 1$ .

**Q 5.14** Discuss the continuity of the function  $f$ , where  $f$  is defined by

$$f(x) = \begin{cases} 3, & \text{if } 0 \leq x \leq 1, \\ 4, & \text{if } 1 < x < 3, \\ 5, & \text{if } 3 \leq x \leq 10. \end{cases}$$

### SOLUTION

**Concept used.** Domain is  $[0, 10]$ . Each piece is a constant, hence continuous in its open part. Test the join points  $x = 1$  and  $x = 3$  with LHL/RHL/value.

**Step 1.** At  $x = 1$ .  $f(1) = 3$  (from the piece  $0 \leq x \leq 1$ ).

LHL. (using  $f = 3$  to the left).

$$\lim_{x \rightarrow 1^-} f(x) = 3$$

RHL

$$\lim_{x \rightarrow 1^+} f(x) = 4$$

LHL  $\neq$  RHL, hence discontinuous at  $x = 1$ .

**Step 2.** At  $x = 3$ .  $f(3) = 5$  (from the piece  $3 \leq x \leq 10$ ).

LHL

$$\lim_{x \rightarrow 3^-} f(x) = 4$$

RHL

$$\lim_{x \rightarrow 3^+} f(x) = 5$$

LHL  $\neq$  RHL, hence discontinuous at  $x = 3$ .

**Step 3.** All other points (interior of the three intervals): the function is constant, hence continuous.

**Final Answer:**  $f$  is discontinuous at  $x = 1$  and  $x = 3$ ; continuous everywhere else on  $[0, 10]$ .

**EXPERT'S SOLUTION** : Tara Pillai, Ph.D Mathematics, IIT Delhi

**Picture-first.** A step function with two upward jumps: from 3 to 4 at  $x = 1$  and from 4 to 5 at  $x = 3$ .

**Step 1.** Constants are continuous inside each piece.

**Step 2.** Each join point exhibits a unit jump in the function value.

**Step 3.** Both joins are discontinuities; all other points are continuous.

**Final Answer:** Discontinuous at  $x = 1, 3$ .

**Q 5.15** Discuss the continuity of the function  $f$ , where  $f$  is defined by

$$f(x) = \begin{cases} 2x, & \text{if } x < 0, \\ 0, & \text{if } 0 \leq x \leq 1, \\ 4x, & \text{if } x > 1. \end{cases}$$

**SOLUTION**

**Concept used.** Each piece is a polynomial, continuous inside open intervals. Test the joins  $x = 0$  and  $x = 1$ .

**Step 1.** At  $x = 0$ .  $f(0) = 0$ .

LHL

$$\lim_{x \rightarrow 0^-} 2x = 0$$

RHL

$$\lim_{x \rightarrow 0^+} 0 = 0$$

All three equal 0, hence continuous at  $x = 0$ .

**Step 2.** At  $x = 1$ .  $f(1) = 0$  (using the second piece,  $0 \leq x \leq 1$ ).

LHL

$$\lim_{x \rightarrow 1^-} 0 = 0$$

RHL

$$\lim_{x \rightarrow 1^+} 4x = 4(1) = 4$$

LHL  $\neq$  RHL, hence discontinuous at  $x = 1$ .

**Step 3.** All other points:  $f$  is polynomial within open pieces, hence continuous.

**Final Answer:**  $f$  is discontinuous only at  $x = 1$ ; continuous everywhere else.

**EXPERT'S SOLUTION** : Krishna Rao, M.Sc Mathematics, IIT Bombay

**Strategic angle.** Two joins, but only the second one breaks continuity.

**Step 1.** At  $x = 0$ , the line  $y = 2x$  and the constant 0 meet at 0, so no break.

**Step 2.** At  $x = 1$ , the constant value 0 jumps to  $4(1) = 4$ , a break of 4 units.

**Final Answer:** Discontinuous only at  $x = 1$ .

**Q 5.16** Discuss the continuity of the function  $f$ , where  $f$  is defined by

$$f(x) = \begin{cases} -2, & \text{if } x \leq -1, \\ 2x, & \text{if } -1 < x \leq 1, \\ 2, & \text{if } x > 1. \end{cases}$$

**SOLUTION**

**Concept used.** Each piece is constant or linear, hence continuous in its open part. Check  $x = -1$  and  $x = 1$ .

**Step 1.** At  $x = -1$ .  $f(-1) = -2$ .

LHL

$$\lim_{x \rightarrow -1^-} (-2) = -2$$

RHL

$$\lim_{x \rightarrow -1^+} 2x = 2(-1) = -2$$

All equal  $-2$ , so continuous at  $x = -1$ .

**Step 2.** At  $x = 1$ .  $f(1) = 2(1) = 2$  (from the second piece,  $-1 < x \leq 1$ ).

LHL

$$\lim_{x \rightarrow 1^-} 2x = 2(1) = 2$$

RHL

$$\lim_{x \rightarrow 1^+} 2 = 2$$

All equal  $2$ , so continuous at  $x = 1$ .

**Step 3.** All other points are inside open pieces of polynomials, hence continuous.

**Final Answer:**  $f$  is continuous at every point of  $\mathbb{R}$ ; no discontinuity.

**EXPERT'S SOLUTION** : Meera Chatterjee, M.Sc Mathematics, ISI Kolkata

**Picture-first.** A horizontal segment at  $y = -2$  for  $x \leq -1$ , a line of slope 2 from  $(-1, -2)$  to  $(1, 2)$ , then a horizontal segment at  $y = 2$ . Continuous all the way.

**Step 1.** Constant pieces match the linear piece at both joins.

**Step 2.** LHL = RHL = value at  $x = -1$ : all  $-2$ .

**Step 3.** LHL = RHL = value at  $x = 1$ : all  $2$ .

**Final Answer:** Continuous on  $\mathbb{R}$ .

**Q5.17** Find the relationship between  $a$  and  $b$  so that the function  $f$  defined by

$$f(x) = \begin{cases} ax + 1, & \text{if } x \leq 3, \\ bx + 3, & \text{if } x > 3, \end{cases}$$

is continuous at  $x = 3$ .

**SOLUTION**

**Concept used.** Continuity at the join  $x = 3$  requires  $\text{LHL} = \text{RHL} = f(3)$ .

**Step 1.**  $f(3) = 3a + 1$  (using the first piece,  $x \leq 3$ ).

**Step 2.** LHL.

$$\lim_{x \rightarrow 3^-} (ax + 1) = 3a + 1$$

**Step 3.** RHL.

$$\lim_{x \rightarrow 3^+} (bx + 3) = 3b + 3$$

**Step 4.** Equate LHL and RHL:

$$3a + 1 = 3b + 3 \implies 3a - 3b = 2 \implies a - b = \frac{2}{3} \implies a = b + \frac{2}{3}.$$

**Final Answer:**  $a = b + \frac{2}{3}$ , i.e.  $3a - 3b = 2$ .

**EXPERT'S SOLUTION** : Siddharth Sharma, M.Sc Mathematics, IIT Bombay

**Strategic angle.** The function value  $3a + 1$  from the closed piece must equal the right limit  $3b + 3$ .

**Step 1.** Polynomial pieces, so continuity inside the open subintervals is automatic.

**Step 2.** Setting  $3a + 1 = 3b + 3$  gives the required relation  $a - b = 2/3$ .

**Final Answer:**  $a - b = 2/3$ .

**Q 5.18** For what value of  $\lambda$  is the function defined by

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \leq 0, \\ 4x + 1, & \text{if } x > 0, \end{cases}$$

continuous at  $x = 0$ ? What about continuity at  $x = 1$ ?

**SOLUTION**

**Concept used.** Match LHL, RHL and function value at  $x = 0$ . At  $x = 1$ , the function is single-valued by the formula  $4x + 1$ , continuous as a polynomial.

**Step 1.** At  $x = 0$ .  $f(0) = \lambda(0^2 - 2 \cdot 0) = 0$ .

LHL

$$\lim_{x \rightarrow 0^-} \lambda(x^2 - 2x) = \lambda(0 - 0) = 0$$

RHL

$$\lim_{x \rightarrow 0^+} (4x + 1) = 4(0) + 1 = 1$$

For continuity, LHL = RHL:  $0 = 1$ , which is impossible for any  $\lambda$ .

Therefore there is **no value of  $\lambda$**  making  $f$  continuous at  $x = 0$ .

**Step 2.** At  $x = 1$ . Since  $1 > 0$ ,  $f(x) = 4x + 1$  in a neighbourhood of 1, a polynomial.

Hence

$$\lim_{x \rightarrow 1} f(x) = 4(1) + 1 = 5 = f(1).$$

$f$  is continuous at  $x = 1$  for every  $\lambda$ .

**Final Answer:** No  $\lambda$  makes  $f$  continuous at  $x = 0$ . At  $x = 1$ ,  $f$  is continuous for every value of  $\lambda$ .

### ✗ Common Mistake

A common error is to write  $\lambda(0^2 - 2 \cdot 0) = \lambda$ . The substitution gives  $\lambda \cdot 0 = 0$ , independent of  $\lambda$ ; so the left side can never equal the right side 1.

**EXPERT'S SOLUTION** : Riya Kumar, M.Sc Mathematics, IIT Bombay

**Structural observation.** The left piece vanishes at  $x = 0$  for every  $\lambda$ . The right piece gives 1. The mismatch is parameter-independent.

**Step 1.** LHL at 0 is always 0 (for any  $\lambda$ ), since  $x^2 - 2x \rightarrow 0$ .

**Step 2.** RHL at 0 is 1, independent of  $\lambda$ .

**Step 3.** No  $\lambda$  can reconcile them. Continuity at  $x = 1$  holds trivially.

**Final Answer:** No  $\lambda$ ; continuous at 1 always.

**Q 5.19** Show that the function defined by  $g(x) = x - [x]$  is discontinuous at all integral points. Here  $[x]$  denotes the greatest integer less than or equal to  $x$ .

### SOLUTION

**Concept used.** The **greatest integer function**  $[x]$  is defined as the unique integer  $n$

such that  $n \leq x < n + 1$ . For any integer  $n$ :

$$[x] = \begin{cases} n - 1, & n - 1 \leq x < n, \\ n, & n \leq x < n + 1. \end{cases}$$

**Step 1.** Let  $n$  be an arbitrary integer. Evaluate  $g(n) = n - [n] = n - n = 0$ .

**Step 2. LHL at  $n$ .** For  $x$  slightly less than  $n$ ,  $[x] = n - 1$ , so  $g(x) = x - (n - 1)$ . Hence

$$\lim_{x \rightarrow n^-} g(x) = \lim_{x \rightarrow n^-} (x - (n - 1)) = n - (n - 1) = 1.$$

**Step 3. RHL at  $n$ .** For  $x$  slightly greater than  $n$  (but still  $< n + 1$ ),  $[x] = n$ , so  $g(x) = x - n$ . Hence

$$\lim_{x \rightarrow n^+} g(x) = \lim_{x \rightarrow n^+} (x - n) = n - n = 0.$$

**Step 4.** LHL = 1  $\neq$  0 = RHL, so  $\lim_{x \rightarrow n} g(x)$  does not exist; therefore  $g$  is discontinuous at every integer  $n$ .

**Final Answer:**  $g(x) = x - [x]$  is discontinuous at every integral point  $n \in \mathbb{Z}$ .

### ♥ Fractional part

The function  $\{x\} := x - [x]$  is called the **fractional part** of  $x$ . It satisfies  $0 \leq \{x\} < 1$  for every real  $x$  and is periodic with period 1. Its graph is a sequence of saw-tooth segments, each rising from 0 to almost 1 and resetting at the next integer.

**EXPERT'S SOLUTION** : Aditya Patel, M.Sc Mathematics, IIT Bombay

**Picture-first.** At every integer the saw-tooth crashes from just under 1 back down to 0, an irreparable jump.

**Step 1.**  $[x]$  is a step function increasing by 1 at every integer, so  $g(x) = x - [x]$  inherits a downward jump of 1 at every integer.

**Step 2.** Just before  $x = n$ :  $g(x) \rightarrow (n) - (n - 1) = 1$ .

**Step 3.** Just after  $x = n$ :  $g(x) \rightarrow n - n = 0$ .

**Step 4.** Unit downward jump at every integer;  $g$  is discontinuous at every integer.

**Final Answer:** Discontinuous at all integers.

**Q 5.20** Is the function defined by  $f(x) = x^2 - \sin x + 5$  continuous at  $x = \pi$ ?

### SOLUTION

**Concept used.** Sums, differences and products of continuous functions are continuous.  $x^2$  is a polynomial;  $\sin x$  is continuous on  $\mathbb{R}$ ; constants are continuous. Hence  $f$  is continuous on  $\mathbb{R}$ .

**Step 1. Value.**  $f(\pi) = \pi^2 - \sin \pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$ .

**Step 2. Limit.** Using algebra of limits,

$$\lim_{x \rightarrow \pi} f(x) = \lim_{x \rightarrow \pi} x^2 - \lim_{x \rightarrow \pi} \sin x + 5 = \pi^2 - \sin \pi + 5 = \pi^2 + 5.$$

**Step 3.**  $\lim_{x \rightarrow \pi} f(x) = f(\pi)$ , so  $f$  is continuous at  $x = \pi$ .

**Final Answer:** Yes,  $f$  is continuous at  $x = \pi$  with value  $\pi^2 + 5$ .

### EXPERT'S SOLUTION : Ananya Joshi, Ph.D Mathematics, IIT Delhi

**Quick reading.**  $f$  is built from three continuous building blocks; continuity is inherited.

**Step 1.**  $x^2$ ,  $\sin x$ ,  $5$  are continuous on  $\mathbb{R}$ .

**Step 2.** Algebra of continuous functions  $\Rightarrow f$  is continuous on  $\mathbb{R}$ .

**Step 3.** In particular at  $x = \pi$  with value  $\pi^2 + 5$ .

**Final Answer:** Continuous at  $\pi$ , value  $\pi^2 + 5$ .

**Q 5.21** Discuss the continuity of the following functions:

(a)  $f(x) = \sin x + \cos x$

(b)  $f(x) = \sin x - \cos x$

(c)  $f(x) = \sin x \cdot \cos x$

### SOLUTION

**Concept used.**  $\sin x$  and  $\cos x$  are continuous on  $\mathbb{R}$ . The sum, difference and product of two continuous functions is continuous.

**Step 1. (a)**  $\sin x + \cos x$  is a sum of two continuous functions, hence continuous on  $\mathbb{R}$ .

**Step 2. (b)**  $\sin x - \cos x$  is a difference of two continuous functions, hence continuous on  $\mathbb{R}$ .

**Step 3. (c)**  $\sin x \cdot \cos x$  is a product of two continuous functions, hence continuous on  $\mathbb{R}$ .

**Final Answer:** All three functions are continuous on the whole of  $\mathbb{R}$ .

**EXPERT'S SOLUTION** : Neha Mehta, M.Sc Mathematics, IIT Bombay

**Quick reading.** Three direct applications of the algebra-of-continuous-functions theorem.

**Step 1.**  $\sin x, \cos x$  continuous on  $\mathbb{R}$ .

**Step 2.** Sum, difference, product preserve continuity.

**Step 3.** Each of  $f(x)$  is continuous everywhere.

**Final Answer:** Continuous on  $\mathbb{R}$  in all three cases.

**Q 5.22** Discuss the continuity of the cosine, cosecant, secant and cotangent functions.

#### SOLUTION

**Concept used.**  $\sin x$  and  $\cos x$  are continuous on  $\mathbb{R}$ . A reciprocal  $1/g$  is continuous wherever  $g$  is continuous and non-zero. The quotient  $g/h$  is continuous wherever both  $g$  and  $h$  are continuous and  $h \neq 0$ .

**Step 1. Cosine**  $\cos x$ . For any  $c \in \mathbb{R}$ ,  $\lim_{x \rightarrow c} \cos x = \cos c$  (a standard limit). Hence  $\cos x$  is continuous on  $\mathbb{R}$ .

**Step 2. Cosecant**  $\csc x = 1/\sin x$ .  $\sin x = 0 \iff x = n\pi, n \in \mathbb{Z}$ . On the domain  $\mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$ ,  $\csc x$  is the reciprocal of a continuous, non-zero function, hence continuous.

**Step 3. Secant**  $\sec x = 1/\cos x$ .  $\cos x = 0 \iff x = (2n+1)\pi/2, n \in \mathbb{Z}$ . On the domain  $\mathbb{R} \setminus \{(2n+1)\pi/2 : n \in \mathbb{Z}\}$ ,  $\sec x$  is continuous.

**Step 4. Cotangent**  $\cot x = \cos x/\sin x$ . Same exclusion as  $\csc$ : continuous on  $\mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$ .

**Final Answer:**  $\cos x$ : continuous on  $\mathbb{R}$ .  $\csc x, \cot x$ : continuous on  $\mathbb{R} \setminus \{n\pi\}$ .  $\sec x$ : continuous on  $\mathbb{R} \setminus \{(2n+1)\pi/2\}$ .

**EXPERT'S SOLUTION** : Pooja Verma, M.Sc Mathematics, IIT Bombay

**Strategic angle.** Each trig reciprocal is continuous on the complement of its zero set.

**Step 1.**  $\cos x$  continuous on  $\mathbb{R}$  (a basic theorem).

**Step 2.**  $\csc x = 1/\sin x$  requires  $\sin x \neq 0$ ; bad set  $\{n\pi\}$ .

**Step 3.**  $\sec x = 1/\cos x$  requires  $\cos x \neq 0$ ; bad set  $\{(2n+1)\pi/2\}$ .

**Step 4.**  $\cot x = \cos x/\sin x$  requires  $\sin x \neq 0$ ; bad set  $\{n\pi\}$ .

**Final Answer:** All four continuous on their natural domains.

**Q 5.23** Find all points of discontinuity of  $f$ , where

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0, \\ x + 1, & \text{if } x \geq 0. \end{cases}$$

**SOLUTION**

**Concept used.** On  $x < 0$ ,  $\sin x/x$  is the quotient of two continuous functions with non-zero denominator, hence continuous. On  $x \geq 0$ ,  $x + 1$  is a polynomial, continuous. Check the join  $x = 0$ .

**Step 1.**  $f(0) = 0 + 1 = 1$  (using the second piece,  $x \geq 0$ ).

**Step 2. LHL at 0.** The standard limit  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$  holds for both one-sided approaches. Hence

$$\lim_{x \rightarrow 0^-} \frac{\sin x}{x} = 1.$$

**Step 3. RHL at 0.**

$$\lim_{x \rightarrow 0^+} (x + 1) = 0 + 1 = 1.$$

**Step 4.** All three equal 1, so  $f$  is continuous at  $x = 0$ . Combined with continuity inside each open piece,  $f$  is continuous on  $\mathbb{R}$ .

**Final Answer:** No point of discontinuity;  $f$  is continuous on  $\mathbb{R}$ .

**Standard limit**

$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ . This is established geometrically (squeeze theorem) and is the cornerstone of all derivative computations for trigonometric functions.

**EXPERT'S SOLUTION** : Ankit Sharma, M.Sc Mathematics, IIT Bombay

**Strategic angle.** The famous limit  $\sin x/x \rightarrow 1$  as  $x \rightarrow 0$  makes the two pieces glue at 0.

**Step 1.** Inside  $x < 0$ : quotient of continuous functions  $\sin x$  and  $x \neq 0$ .

**Step 2.** Inside  $x > 0$ : polynomial  $x + 1$ .

**Step 3.** At  $x = 0$ : LHL = 1, RHL = 1,  $f(0) = 1$ . Continuous.

**Final Answer:** Continuous on  $\mathbb{R}$ .

**Q 5.24** Determine if  $f$  defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

is a continuous function.

#### SOLUTION

**Concept used.** For  $x \neq 0$ ,  $f(x) = x^2 \sin(1/x)$  is a product of  $x^2$  (polynomial, continuous) and  $\sin(1/x)$  (composition of  $\sin$  and the continuous-on- $\mathbb{R} \setminus \{0\}$  function  $1/x$ ). Hence  $f$  is continuous on  $\mathbb{R} \setminus \{0\}$ . At  $x = 0$  we use the **Sandwich (Squeeze) theorem**: since  $-1 \leq \sin t \leq 1$  for all  $t$ , we have

$$-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2 \quad (x \neq 0).$$

**Step 1.** As  $x \rightarrow 0$ , both  $-x^2 \rightarrow 0$  and  $x^2 \rightarrow 0$ . By the Squeeze theorem,

$$\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0.$$

**Step 2.**  $f(0) = 0$  by definition.

**Step 3.** Since  $\lim_{x \rightarrow 0} f(x) = 0 = f(0)$ ,  $f$  is continuous at  $x = 0$ . Together with continuity for  $x \neq 0$ ,  $f$  is continuous on  $\mathbb{R}$ .

**Final Answer:** Yes,  $f$  is a continuous function on  $\mathbb{R}$ .

#### ♥ Bounded times small

This is the canonical example of “bounded  $\times$  small  $\rightarrow 0$ ”. Even though  $\sin(1/x)$  oscillates wildly near 0, the prefactor  $x^2$  kills the oscillation, dragging the product to 0.

**EXPERT'S SOLUTION** : Dev Iyer, Ph.D Mathematics, IIT Delhi

**Strategic angle.** Apply the squeeze theorem directly; everything else follows from algebra of continuous functions.

**Step 1.** Away from 0:  $x^2$  and  $\sin(1/x)$  are continuous, so is their product.

**Step 2.** Bounds:  $|x^2 \sin(1/x)| \leq x^2$ .

**Step 3.** As  $x \rightarrow 0$ ,  $x^2 \rightarrow 0$ , so by squeeze  $f(x) \rightarrow 0 = f(0)$ .

**Final Answer:** Continuous everywhere.

**Q 5.25** Examine the continuity of  $f$ , where  $f$  is defined by

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0, \\ -1, & \text{if } x = 0. \end{cases}$$

**SOLUTION**

**Concept used.** For  $x \neq 0$ ,  $\sin x - \cos x$  is the difference of two continuous functions, hence continuous. At  $x = 0$ , use LHL/RHL/value.

**Step 1.**  $f(0) = -1$ .

**Step 2. Limit at 0.**

$$\lim_{x \rightarrow 0} (\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1.$$

**Step 3.**  $\lim_{x \rightarrow 0} f(x) = -1 = f(0)$ , so  $f$  is continuous at  $x = 0$ .

**Step 4.** Outside 0,  $f$  is the difference of continuous functions, continuous on  $\mathbb{R} \setminus \{0\}$ .

**Final Answer:**  $f$  is continuous everywhere on  $\mathbb{R}$ .

**EXPERT'S SOLUTION** : Sanya Patel, M.Sc Applied Mathematics, IIT Kanpur

**Quick reading.** The assigned value at 0 matches the natural limit  $-1$ , so the function is continuous everywhere.

**Step 1.** Difference of continuous functions is continuous on  $\mathbb{R} \setminus \{0\}$ .

**Step 2.** Limit at 0 is  $\sin 0 - \cos 0 = -1 = f(0)$ .

**Final Answer:** Continuous on  $\mathbb{R}$ .

**Q 5.26** Find the values of  $k$  so that the function  $f$  is continuous at the indicated point:

$$f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2}, \\ 3, & \text{if } x = \frac{\pi}{2}, \end{cases} \quad \text{at } x = \frac{\pi}{2}.$$

### SOLUTION

**Concept used.** Continuity at  $\pi/2$  requires  $\lim_{x \rightarrow \pi/2} f(x) = f(\pi/2) = 3$ . We compute the limit using the substitution  $x = \pi/2 + h$ , then use  $\cos(\pi/2 + h) = -\sin h$  and the standard limit  $\sin h/h \rightarrow 1$ .

**Step 1.** Let  $h = x - \pi/2$ , so  $x \rightarrow \pi/2$  becomes  $h \rightarrow 0$ . Then

$$\pi - 2x = \pi - 2\left(\frac{\pi}{2} + h\right) = -2h, \quad \cos x = \cos\left(\frac{\pi}{2} + h\right) = -\sin h.$$

**Step 2.** Substitute:

$$\lim_{x \rightarrow \pi/2} \frac{k \cos x}{\pi - 2x} = \lim_{h \rightarrow 0} \frac{k(-\sin h)}{-2h} = \lim_{h \rightarrow 0} \frac{k \sin h}{2h} = \frac{k}{2} \lim_{h \rightarrow 0} \frac{\sin h}{h} = \frac{k}{2} \cdot 1 = \frac{k}{2}.$$

**Step 3.** Continuity demands  $\frac{k}{2} = 3$ , i.e.  $k = 6$ .

**Final Answer:**  $k = 6$ .

### EXPERT'S SOLUTION : *Ishita Reddy, Ph.D Mathematics, IIT Delhi*

**Strategic angle.** Indeterminate  $0/0$  at  $\pi/2$ : apply a shift to convert it into the standard  $\sin h/h$  limit.

**Step 1.** Shift  $x = \pi/2 + h$  gives numerator  $-k \sin h$  and denominator  $-2h$ .

**Step 2.** Ratio  $\rightarrow k/2$ .

**Step 3.** Set equal to 3:  $k = 6$ .

**Final Answer:**  $k = 6$ .

**Q 5.27** Find the values of  $k$  so that the function  $f$  is continuous at  $x = 2$ :

$$f(x) = \begin{cases} kx^2, & \text{if } x \leq 2, \\ 3, & \text{if } x > 2. \end{cases}$$

### SOLUTION

**Concept used.** Continuity at  $x = 2$  requires  $\text{LHL} = \text{RHL} = f(2)$ .

**Step 1.**  $f(2) = k(2)^2 = 4k$ .

**Step 2.** LHL.

$$\lim_{x \rightarrow 2^-} kx^2 = k(2)^2 = 4k$$

**Step 3.** RHL.

$$\lim_{x \rightarrow 2^+} 3 = 3$$

**Step 4.** Equate:  $4k = 3 \implies k = \frac{3}{4}$ .

**Final Answer:**  $k = \frac{3}{4}$ .

**EXPERT'S SOLUTION** : *Kavya Singh, M.Sc Mathematics, IIT Bombay*

**Quick reading.** Single equation in  $k$ :  $4k = 3$ .

**Step 1.** Left side at 2:  $4k$  (from  $kx^2$ ).

**Step 2.** Right side at 2: 3.

**Step 3.**  $k = 3/4$ .

**Final Answer:**  $k = 3/4$ .

**Q 5.28** Find the values of  $k$  so that the function  $f$  is continuous at  $x = \pi$ :

$$f(x) = \begin{cases} kx + 1, & \text{if } x \leq \pi, \\ \cos x, & \text{if } x > \pi. \end{cases}$$

**SOLUTION**

**Concept used.** Continuity at  $\pi$  requires  $\text{LHL} = \text{RHL} = f(\pi)$ .

**Step 1.**  $f(\pi) = k\pi + 1$ .

**Step 2.** LHL.

$$\lim_{x \rightarrow \pi^-} (kx + 1) = k\pi + 1$$

**Step 3.** RHL.

$$\lim_{x \rightarrow \pi^+} \cos x = \cos \pi = -1$$

**Step 4.** Equate  $\text{LHL} = \text{RHL}$ :

$$k\pi + 1 = -1 \implies k\pi = -2 \implies k = -\frac{2}{\pi}.$$

**Final Answer:**  $k = -\frac{2}{\pi}$ .

**EXPERT'S SOLUTION** : *Rahul Desai, M.Sc Mathematics, IIT Bombay*

**Structural observation.**  $\cos x$  at  $\pi$  equals  $-1$ . So the left expression  $k\pi + 1$  must also equal  $-1$ .

**Step 1.**  $\cos \pi = -1$ .

**Step 2.**  $k\pi + 1 = -1 \implies k = -2/\pi$ .

**Final Answer:**  $k = -2/\pi$ .

**Q 5.29** Find the values of  $k$  so that the function  $f$  is continuous at  $x = 5$ :

$$f(x) = \begin{cases} kx + 1, & \text{if } x \leq 5, \\ 3x - 5, & \text{if } x > 5. \end{cases}$$

**SOLUTION**

**Concept used.** Continuity at 5 requires  $\text{LHL} = \text{RHL} = f(5)$ .

**Step 1.**  $f(5) = 5k + 1$ .

**Step 2.** LHL.

$$\lim_{x \rightarrow 5^-} (kx + 1) = 5k + 1$$

**Step 3.** RHL.

$$\lim_{x \rightarrow 5^+} (3x - 5) = 3(5) - 5 = 10$$

**Step 4.** Equate:

$$5k + 1 = 10 \implies 5k = 9 \implies k = \frac{9}{5}.$$

**Final Answer:**  $k = \frac{9}{5}$ .

**EXPERT'S SOLUTION** : Ishaan Pillai, B.Tech CSE, IIT Roorkee

**Quick reading.** Match the linear pieces at  $x = 5$ .

**Step 1.** Right value =  $3(5) - 5 = 10$ .

**Step 2.** Left value =  $5k + 1$ .

**Step 3.**  $5k + 1 = 10 \implies k = 9/5$ .

**Final Answer:**  $k = 9/5$ .

**Q 5.30** Find the values of  $a$  and  $b$  such that the function defined by

$$f(x) = \begin{cases} 5, & \text{if } x \leq 2, \\ ax + b, & \text{if } 2 < x < 10, \\ 21, & \text{if } x \geq 10, \end{cases}$$

is a continuous function.

**SOLUTION**

**Concept used.** Continuity at the two join points  $x = 2$  and  $x = 10$  gives two linear equations in  $a$  and  $b$ .

**Step 1.** At  $x = 2$ .  $f(2) = 5$ . RHL:  $\lim_{x \rightarrow 2^+} (ax + b) = 2a + b$ . Continuity requires

$$2a + b = 5. \quad \dots(i)$$

**Step 2.** At  $x = 10$ .  $f(10) = 21$ . LHL:  $\lim_{x \rightarrow 10^-} (ax + b) = 10a + b$ . Continuity requires

$$10a + b = 21. \quad \dots(ii)$$

**Step 3.** Subtract (i) from (ii):

$$(10a + b) - (2a + b) = 21 - 5 \implies 8a = 16 \implies a = 2.$$

**Step 4.** Back-substitute into (i):

$$2(2) + b = 5 \implies b = 5 - 4 = 1.$$

**Final Answer:**  $a = 2, b = 1.$

**EXPERT'S SOLUTION** : Aarav Banerjee, M.Sc Mathematics, ISI Kolkata

**Strategic angle.** Three pieces, two joins, two unknowns: a  $2 \times 2$  linear system.

**Step 1.** Match at  $x = 2$ :  $2a + b = 5.$

**Step 2.** Match at  $x = 10$ :  $10a + b = 21.$

**Step 3.** Subtract:  $8a = 16, a = 2, b = 1.$

**Final Answer:**  $a = 2, b = 1.$

**Q 5.31** Show that the function defined by  $f(x) = \cos(x^2)$  is a continuous function.

#### SOLUTION

**Concept used.** The composition of two continuous functions is continuous: if  $g$  is continuous at  $c$  and  $h$  is continuous at  $g(c)$ , then  $h \circ g$  is continuous at  $c$ .

**Step 1.** Let  $g(x) = x^2$  and  $h(t) = \cos t$ . Then  $f = h \circ g$ , i.e.  $f(x) = h(g(x)) = \cos(x^2)$ .

**Step 2.**  $g(x) = x^2$  is a polynomial, hence continuous on  $\mathbb{R}$ .

**Step 3.**  $h(t) = \cos t$  is continuous on  $\mathbb{R}$ .

**Step 4.** By the composition theorem,  $f = h \circ g$  is continuous on  $\mathbb{R}$ .

**Final Answer:**  $f(x) = \cos(x^2)$  is continuous on  $\mathbb{R}$ .

**EXPERT'S SOLUTION** : Vivaan Joshi, M.Sc Mathematics, IIT Bombay

**Quick reading.** Polynomial inside, cosine outside, composition continuous.

**Step 1.** Inner:  $x^2$  continuous.

**Step 2.** Outer:  $\cos$  continuous.

**Step 3.** Composition: continuous.

**Final Answer:** Continuous on  $\mathbb{R}$ .

**Q 5.32** Show that the function defined by  $f(x) = |\cos x|$  is a continuous function.

### SOLUTION

**Concept used.**  $f = h \circ g$  where  $g(x) = \cos x$  (continuous on  $\mathbb{R}$ ) and  $h(t) = |t|$  (continuous on  $\mathbb{R}$ ). Composition of continuous functions is continuous.

**Step 1.**  $g(x) = \cos x$  is continuous on  $\mathbb{R}$ .

**Step 2.**  $h(t) = |t|$  is continuous on  $\mathbb{R}$  (as shown by the LHL = RHL argument at 0, and identity elsewhere).

**Step 3.** Therefore  $f(x) = |\cos x| = h(g(x))$  is continuous on  $\mathbb{R}$ .

**Final Answer:**  $f(x) = |\cos x|$  is continuous everywhere on  $\mathbb{R}$ .

**EXPERT'S SOLUTION** : Sneha Bhat, Ph.D Mathematics, IIT Delhi

**Structural observation.** Folding the cosine wave around the  $x$ -axis ( $|\cdot|$ ) does not break continuity.

**Step 1.**  $\cos x$  continuous;  $|\cdot|$  continuous.

**Step 2.** Composition is continuous; result lives in  $[0, 1]$ .

**Final Answer:** Continuous on  $\mathbb{R}$ .

**Q 5.33** Examine that  $\sin |x|$  is a continuous function.

### SOLUTION

**Concept used.** Composition of continuous functions is continuous. Here the inner function is  $g(x) = |x|$  and the outer is  $h(t) = \sin t$ .

**Step 1.**  $g(x) = |x|$  is continuous on  $\mathbb{R}$  (right-/left-limits at 0 both equal  $0 = g(0)$ ; elsewhere  $|x|$  agrees with the polynomial  $\pm x$ ).

**Step 2.**  $h(t) = \sin t$  is continuous on  $\mathbb{R}$ .

**Step 3.** By composition,  $f(x) = \sin |x| = h(g(x))$  is continuous on  $\mathbb{R}$ .

**Final Answer:**  $\sin |x|$  is a continuous function on  $\mathbb{R}$ .

**EXPERT'S SOLUTION** : Ananya Verma, M.Sc Mathematics, IIT Bombay

**Picture-first.** The graph is the standard  $\sin x$  for  $x \geq 0$ , reflected through the  $y$ -axis for  $x \leq 0$  (because  $|x|$  is even). It is continuous everywhere.

**Step 1.**  $|x|$  continuous on  $\mathbb{R}$ .

**Step 2.**  $\sin$  continuous on  $\mathbb{R}$ .

**Step 3.** Composition continuous.

**Final Answer:** Continuous on  $\mathbb{R}$ .

**Q 5.34** Find all the points of discontinuity of  $f$  defined by  $f(x) = |x| - |x + 1|$ .

**SOLUTION**

**Concept used.**  $|x|$  and  $|x + 1|$  are each continuous on  $\mathbb{R}$ , so their difference is continuous on  $\mathbb{R}$ .

**Step 1.**  $g(x) = |x|$  is continuous on  $\mathbb{R}$ .

**Step 2.**  $h(x) = |x + 1|$  is the composition of  $x \mapsto x + 1$  (continuous) and  $|\cdot|$  (continuous), so continuous on  $\mathbb{R}$ .

**Step 3.** The difference  $f(x) = g(x) - h(x)$  is continuous on  $\mathbb{R}$  as the difference of two continuous functions.

**Step 4.** Hence  $f$  has **no point of discontinuity**.

**Final Answer:**  $f(x) = |x| - |x + 1|$  has no point of discontinuity; it is continuous on  $\mathbb{R}$ .

 **Exam Tip**

The piecewise breakdown of  $f$  across  $x = -1$  and  $x = 0$  shows that  $f$  equals 1 on  $x \leq -1$ ,  $-2x - 1$  on  $-1 \leq x \leq 0$ , and  $-1$  on  $x \geq 0$ . At the joins both pieces match, confirming continuity.

**EXPERT'S SOLUTION** : *Karan Iyer, B.Tech Engineering Physics, IIT Bombay*

**Picture-first.** The graph is a horizontal segment at  $y = 1$  for  $x \leq -1$ , a line of slope  $-2$  from  $(-1, 1)$  to  $(0, -1)$ , then a horizontal segment at  $y = -1$  for  $x \geq 0$ . All three pieces meet without a jump.

**Step 1.** Piecewise:  $f(x) = 1$  for  $x \leq -1$ ;  $f(x) = -2x - 1$  for  $-1 \leq x \leq 0$ ;  $f(x) = -1$  for  $x \geq 0$ .

**Step 2.** Joins at  $x = -1$  (both sides give 1) and  $x = 0$  (both sides give  $-1$ ).

**Step 3.** Continuous throughout.

**Final Answer:** No point of discontinuity.

**Key Takeaways**

- A function  $f$  is continuous at  $c$  iff  $\lim_{x \rightarrow c^-} f = \lim_{x \rightarrow c^+} f = f(c)$ . Test piecewise functions only at the join points.
- Sums, differences, products and (non-zero-denominator) quotients of continuous functions are continuous; so are compositions.
- Every polynomial is continuous on  $\mathbb{R}$ ;  $\sin x$ ,  $\cos x$ ,  $|x|$ ,  $e^x$  and  $\log x$  (on their domains) are continuous;  $\csc$ ,  $\sec$ ,  $\cot$  are continuous on the complement of their zero-of-denominator sets.
- The greatest-integer function  $[x]$  jumps by 1 at every integer, hence  $x - [x]$  (the fractional part) is discontinuous at every integer.

End of Exercise 5.1