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Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 5: Continuity and Differentiability

About this Chapter

Exercise 5.2 introduces the **chain rule** for composites of two or more functions. Every question is a direct application: identify the outer and inner functions, differentiate the outer function at the inner expression, and multiply by the derivative of the inner function. The last two problems use the left-hand/right-hand derivative test to check **differentiability** at a corner of the modulus function and at integer points of the greatest-integer function.

Topics covered: Chain rule • Composites of two functions • Composites of three functions • LHD/RHD at a corner

Quick Formula Sheet

Chain rule (two layers):
If $y = f(g(x))$ then $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$.

Standard derivatives:
$\frac{d}{dx}\sin x = \cos x$, $\frac{d}{dx}\cos x = -\sin x$, $\frac{d}{dx}\tan x = \sec^2 x$, $\frac{d}{dx}\sec x = \sec x \tan x$.

Differentiability:
f is differentiable at c iff $\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h}$, both finite.

Exercise 5.2

Q 5.1 **Differentiate** $\sin(x^2+5)$ **with respect to** x.

SOLUTION

Concept used. **Chain rule.** If $y = f(g(x))$, then $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$. Here $f(u) = \sin u$ with $f'(u) = \cos u$, and $g(x) = x^2 + 5$ with $g'(x) = 2x$.

Step 1. Set $u = x^2 + 5$ so that $y = \sin u$.

Step 2. Differentiate the outer:

$$\frac{dy}{du} = \cos u = \cos(x^2 + 5).$$

Step 3. Differentiate the inner:

$$\frac{du}{dx} = \frac{d}{dx}(x^2 + 5) = 2x + 0 = 2x.$$

Step 4. Multiply by the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos(x^2 + 5) \cdot 2x = 2x\cos(x^2 + 5).$$

Final Answer: $\frac{dy}{dx} = 2x\cos(x^2 + 5)$.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Bombay*

Quick reading. Outer is sine; inner is the polynomial $x^2 + 5$ with derivative $2x$.

Step 1. $\frac{d}{dx}\sin(\square) = \cos(\square) \cdot \frac{d\square}{dx}$.

Step 2. Apply with $\square = x^2 + 5$, derivative $2x$.

Final Answer: $2x\cos(x^2 + 5)$.

Q 5.2 Differentiate $\cos(\sin x)$ with respect to x.

SOLUTION

Concept used. Chain rule applied to a cosine-of-sine composite. $\frac{d}{dx}\cos u = -\sin u \cdot \frac{du}{dx}$; here $u = \sin x$, so $\frac{du}{dx} = \cos x$.

Step 1. Let $u = \sin x$, $y = \cos u$.

Step 2. $\frac{dy}{du} = -\sin u = -\sin(\sin x)$.

Step 3. $\frac{du}{dx} = \cos x$.

Step 4. Chain rule:

$$\frac{dy}{dx} = -\sin(\sin x) \cdot \cos x.$$

Final Answer: $\frac{dy}{dx} = -\cos x \cdot \sin(\sin x)$.

EXPERT'S SOLUTION *: Sneha Iyer, Ph.D Mathematics, IIT Delhi*

Structural observation. Two trig layers; differentiate each in order.

Step 1. Outer derivative: $-\sin(\square)$ at $\square = \sin x$.

Step 2. Inner derivative: $\cos x$.

Step 3. Product: $-\sin(\sin x)\cos x$.

Final Answer: $-\cos x \sin(\sin x)$.

Q 5.3 Differentiate $\sin(ax + b)$ with respect to x.

SOLUTION

Concept used. Chain rule with outer $\sin$ and inner linear function $ax + b$.

Step 1. Let $u = ax + b$ so that $y = \sin u$.

Step 2. $\frac{dy}{du} = \cos u = \cos(ax + b)$.

Step 3. $\frac{du}{dx} = a$.

Step 4. Chain rule:

$$\frac{dy}{dx} = \cos(ax + b) \cdot a = a\cos(ax + b).$$

Final Answer: $\frac{dy}{dx} = a\cos(ax + b)$.

EXPERT'S SOLUTION *: Vivaan Gupta, M.Tech CS, IIT Madras*

Quick reading. The chain rule scales the cosine by the constant a.

Step 1. Derivative of $\sin(\text{linear}) = a\cos(\text{same linear})$.

Final Answer: $a\cos(ax+b)$.

Q 5.4 **Differentiate $\sec(\tan(\sqrt{x}))$ with respect to x.**

SOLUTION

Concept used. Chain rule applied to a composite of *three* functions: $\sec$, $\tan$, $\sqrt{\cdot}$. The derivative of $\sec u$ is $\sec u \tan u$; the derivative of $\tan v$ is $\sec^2 v$; the derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$.

Step 1. Let $w=\sqrt{x}$, $v=\tan w$, $y=\sec v$. Then $y=\sec(\tan(\sqrt{x}))$.

Step 2. $\frac{dy}{dv}=\sec v\tan v=\sec(\tan\sqrt{x})\tan(\tan\sqrt{x})$.

Step 3. $\frac{dv}{dw}=\sec^2 w=\sec^2(\sqrt{x})$.

Step 4. $\frac{dw}{dx}=\frac{1}{2\sqrt{x}}$.

Step 5. Chain rule (three layers):

$$\frac{dy}{dx}=\frac{dy}{dv}\cdot\frac{dv}{dw}\cdot\frac{dw}{dx}=\sec(\tan\sqrt{x})\cdot\tan(\tan\sqrt{x})\cdot\sec^2(\sqrt{x})\cdot\frac{1}{2\sqrt{x}}.$$

Final Answer: $\frac{dy}{dx}=\frac{\sec(\tan\sqrt{x})\cdot\tan(\tan\sqrt{x})\cdot\sec^2\sqrt{x}}{2\sqrt{x}}$.

☞ Exam Tip

For a composite of k functions, the derivative is a product of k derivatives. Peel from the outside in, multiplying each layer's derivative by the next inner function's derivative.

EXPERT'S SOLUTION *: Aanya Mehta, Ph.D Pure Mathematics, IISc Bangalore*

Strategic angle. Three-layer chain: $\sec\to\tan\to\sqrt{\ }$.

Step 1. Layer 1: $\sec(\square)'=\sec\square\tan\square$.

Step 2. Layer 2: $\tan(\square)'=\sec^2\square$.

Step 3. Layer 3: $\sqrt{x}'=1/(2\sqrt{x})$.

Step 4. Multiply all three with $\square=\tan\sqrt{x}, \sqrt{x}, x$ respectively.

Final Answer: $\dfrac{\sec(\tan\sqrt{x})\tan(\tan\sqrt{x})\sec^2\sqrt{x}}{2\sqrt{x}}$.

Q 5.5 **Differentiate $\dfrac{\sin(ax+b)}{\cos(cx+d)}$ with respect to x.**

SOLUTION

Concept used. **Quotient rule.** If $y = u/v$ then $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$. Combined with the chain rule for $u = \sin(ax+b)$ and $v = \cos(cx+d)$.

Step 1. Let $u = \sin(ax+b)$ and $v = \cos(cx+d)$.

Step 2. Derivatives of numerator and denominator (chain rule):

$$u' = a\cos(ax+b), \qquad v' = -c\sin(cx+d).$$

Step 3. Quotient rule:

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{a\cos(ax+b)\cos(cx+d) - \sin(ax+b)\big(-c\sin(cx+d)\big)}{\cos^2(cx+d)}.$$

Step 4. Simplify the numerator:

$$= \frac{a\cos(ax+b)\cos(cx+d) + c\sin(ax+b)\sin(cx+d)}{\cos^2(cx+d)}.$$

Final Answer: $\dfrac{dy}{dx} = \dfrac{a\cos(ax+b)\cos(cx+d) + c\sin(ax+b)\sin(cx+d)}{\cos^2(cx+d)}$.

EXPERT'S SOLUTION *: Priya Singh, M.Sc Applied Mathematics, IIT Kanpur*

Quick reading. Quotient rule with chain-rule numerator/denominator derivatives.

Step 1. $u = \sin(ax+b)$, $u' = a\cos(ax+b)$.

Step 2. $v = \cos(cx+d)$, $v' = -c\sin(cx+d)$.

Step 3. $(u/v)' = (u'v - uv')/v^2$ as written.

Final Answer: $\dfrac{a\cos(ax+b)\cos(cx+d) + c\sin(ax+b)\sin(cx+d)}{\cos^2(cx+d)}$.

Q 5.6 **Differentiate $\cos x^3 \cdot \sin^2(x^5)$ with respect to x.**

SOLUTION

Concept used. Product rule: $(uv)' = u'v + uv'$. Each factor needs the chain rule.

Step 1. Let $u = \cos(x^3)$ and $v = \sin^2(x^5) = (\sin(x^5))^2$.

Step 2. Derivative of u: chain rule, outer $\cos$, inner x^3.

$$u' = -\sin(x^3) \cdot \frac{d}{dx}(x^3) = -3x^2 \sin(x^3).$$

Step 3. Derivative of v: write $v = (\sin(x^5))^2$. Chain rule three layers (square, sine, x^5):

$$v' = 2\sin(x^5) \cdot \cos(x^5) \cdot 5x^4 = 10x^4 \sin(x^5)\cos(x^5).$$

Step 4. Product rule:

$$\frac{dy}{dx} = u'v + uv' = -3x^2 \sin(x^3)\sin^2(x^5) + 10x^4 \cos(x^3)\sin(x^5)\cos(x^5).$$

Final Answer: $\frac{dy}{dx} = -3x^2 \sin(x^3)\sin^2(x^5) + 10x^4 \cos(x^3)\sin(x^5)\cos(x^5)$.

EXPERT'S SOLUTION *: Karan Joshi, M.Sc Mathematics, IIT Bombay*

Structural observation. Product of two trig composites; differentiate by parts then layer-by-layer.

Step 1. $u' = -3x^2 \sin(x^3)$.

Step 2. $v = \sin^2(x^5)$, $v' = 2\sin(x^5)\cos(x^5) \cdot 5x^4$ using $\sin(2\theta) = 2\sin\theta\cos\theta$ this equals $5x^4 \sin(2x^5)$.

Step 3. Sum:

$$-3x^2 \sin(x^3)\sin^2(x^5) + 5x^4 \cos(x^3)\sin(2x^5).$$

Final Answer: Equivalent form: $-3x^2 \sin(x^3)\sin^2(x^5) + 5x^4 \cos(x^3)\sin(2x^5)$.

Q 5.7 **Differentiate $2\sqrt{\cot(x^2)}$ with respect to x.**

SOLUTION

Concept used. Chain rule three layers: outer $\sqrt{\cdot}$, middle $\cot$, inner x^2. The derivative of $\sqrt{u}$ is $\frac{1}{2\sqrt{u}}$; the derivative of $\cot v$ is $-\csc^2 v$.

Step 1. Let $w = x^2$, $v = \cot w$, $y = 2\sqrt{v}$.

Step 2. $\frac{dy}{dv} = 2 \cdot \frac{1}{2\sqrt{v}} = \frac{1}{\sqrt{v}} = \frac{1}{\sqrt{\cot(x^2)}}$.

Step 3. $\frac{dv}{dw} = -\csc^2 w = -\csc^2(x^2)$.

Step 4. $\frac{dw}{dx} = 2x$.

Step 5. Chain rule:

$$\frac{dy}{dx} = \frac{1}{\sqrt{\cot(x^2)}} \cdot (-\csc^2(x^2)) \cdot 2x = \frac{-2x\csc^2(x^2)}{\sqrt{\cot(x^2)}}.$$

Final Answer: $\frac{dy}{dx} = -\frac{2x\csc^2(x^2)}{\sqrt{\cot(x^2)}}$.

EXPERT'S SOLUTION *: Pranav Kapoor, M.Sc Mathematics, IIT Bombay*

Quick reading. Square-root strips the 2 at the front; $\cot$ gives a $-\csc^2$; x^2 gives $2x$.

Step 1. Derivative of $2\sqrt{u}$ is $u'/\sqrt{u}$.

Step 2. Here $u = \cot(x^2)$, $u' = -\csc^2(x^2) \cdot 2x$.

Step 3. Combine to $-2x\csc^2(x^2)/\sqrt{\cot(x^2)}$.

Final Answer: $-2x\csc^2(x^2)/\sqrt{\cot(x^2)}$.

Q 5.8 Differentiate $\cos(\sqrt{x})$ with respect to x.

SOLUTION

Concept used. Chain rule with outer $\cos$ and inner $\sqrt{x}$. $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$.

Step 1. Let $u = \sqrt{x}$, $y = \cos u$.

Step 2. $\frac{dy}{du} = -\sin u = -\sin\sqrt{x}$.

Step 3. $\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$.

Step 4. Chain rule:

$$\frac{dy}{dx} = -\sin\sqrt{x} \cdot \frac{1}{2\sqrt{x}} = -\frac{\sin\sqrt{x}}{2\sqrt{x}}.$$

Final Answer: $\dfrac{dy}{dx} = -\dfrac{\sin\sqrt{x}}{2\sqrt{x}}$.

EXPERT'S SOLUTION *: Diya Bhat, Ph.D Mathematics, IIT Delhi*

Quick reading. Two-layer composite; product of $-\sin\sqrt{x}$ and $1/(2\sqrt{x})$.

Step 1. Outer: $-\sin\sqrt{x}$.

Step 2. Inner: $1/(2\sqrt{x})$.

Step 3. Product: $-\sin\sqrt{x}/(2\sqrt{x})$.

Final Answer: $-\sin\sqrt{x}/(2\sqrt{x})$.

Q 5.9 **Prove that the function f given by $f(x) = |x-1|$, $x \in \mathbb{R}$, is not differentiable at $x = 1$.**

SOLUTION

Concept used. f is differentiable at c iff

$$\lim_{h\to 0^-} \frac{f(c+h) - f(c)}{h} \quad \text{and} \quad \lim_{h\to 0^+} \frac{f(c+h) - f(c)}{h}$$

both exist (finite) and are equal. We compute LHD and RHD at $c = 1$.

Step 1. $f(1) = |1-1| = 0$. For h near 0:

$$f(1+h) = |1+h-1| = |h|.$$

Step 2. **LHD at** 1. For $h \to 0^-$, $h < 0$ so $|h| = -h$:

$$\lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^-} \frac{|h| - 0}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1.$$

Step 3. **RHD at** 1. For $h \to 0^+$, $h > 0$ so $|h| = h$:

$$\lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1.$$

Step 4. LHD $= -1$ but RHD $= 1$, so the two one-sided derivatives are unequal. Hence f is not differentiable at $x = 1$.

> **Final Answer:** $f(x) = |x-1|$ is not differentiable at $x = 1$ (corner with LHD $= -1$ and RHD $= 1$).

♥ Corners kill differentiability

A modulus function has a corner at the inner expression's zero; the graph turns sharply. The left and right tangent slopes differ in sign, so no single tangent line exists. Continuous, yes, but not differentiable.

EXPERT'S SOLUTION *: Yash Nair, M.Tech CS, IIT Madras*

Picture-first. The graph of $y = |x-1|$ is a V-shape with the vertex at $(1, 0)$; slope -1 to the left of 1 and $+1$ to the right.

Step 1. Left-side slope -1, right-side slope $+1$.

Step 2. Slopes unequal $\Rightarrow$ derivative does not exist at $x = 1$.

> **Final Answer:** Not differentiable at $x = 1$.

Q 5.10 Prove that the greatest integer function defined by $f(x) = [x]$, $0 < x < 3$, is not differentiable at $x = 1$ and $x = 2$.

SOLUTION

Concept used. A function that is discontinuous at c cannot be differentiable at c (Theorem 3 of the chapter: every differentiable function is continuous). We show f is discontinuous at $x = 1$ and $x = 2$; this rules out differentiability automatically. (Alternatively, the LHD/RHD limits are computed and shown unequal.)

Step 1. **At $x = 1$.** $f(1) = [1] = 1$. For $h \to 0^-$ (so $1 + h \in (0, 1)$), $[1 + h] = 0$:

$$\lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^-} \frac{0-1}{h} = \lim_{h\to 0^-} \frac{-1}{h} = +\infty.$$

The LHD is not finite, hence f is not differentiable at $x = 1$. (One can also note: f is discontinuous at $x = 1$, hence not differentiable there.)

Step 2. **At $x = 2$.** $f(2) = [2] = 2$. For $h \to 0^-$ (so $1 + h < 2$ but > 1), $[2 + h] = 1$:

$$\lim_{h\to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0^-} \frac{1-2}{h} = \lim_{h\to 0^-} \frac{-1}{h} = +\infty.$$

Again the LHD diverges; f is not differentiable at $x = 2$.

Step 3. In both cases the function jumps by 1 as x crosses the integer, so the difference quotient $(f(c+h) - f(c))/h$ blows up. Hence differentiability fails.

> **Final Answer:** $f(x) = [x]$ is not differentiable at $x = 1$ and $x = 2$ (jump discontinuities at both points).

EXPERT'S SOLUTION : *Aditi Banerjee, M.Sc Mathematics, ISI Kolkata*

Strategic angle. A discontinuity automatically kills differentiability. So we just verify the jump.

Step 1. LHL of $[x]$ at $x = 1$ is 0; value is 1. Discontinuous; not differentiable.

Step 2. LHL of $[x]$ at $x = 2$ is 1; value is 2. Discontinuous; not differentiable.

> **Final Answer:** Not differentiable at $x = 1, 2$.

Key Takeaways

- The chain rule peels a composite from the outside in: $\frac{d}{dx} f(g(h(x))) = f'(g(h)) \cdot g'(h) \cdot h'(x)$.
- Combine chain rule with the quotient and product rules for fractions and products of composites.
- Differentiability requires LHD $=$ RHD, both finite. Modulus functions fail at the vertex of the V; greatest-integer fails at every integer.

End of Exercise 5.2