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Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 5: Continuity and Differentiability

About this Chapter

Exercise 5.5 introduces **logarithmic differentiation**, the cleanest way to differentiate expressions of the form $[u(x)]^{v(x)}$ and products/quotients with several factors. Take logarithm of both sides, apply log-laws to split into simple terms, then differentiate implicitly. Both $u(x)$ and the function itself must be positive for the technique to make sense.

Topics covered: Logarithmic differentiation • $[u(x)]^{v(x)}$ • Implicit log + chain rule • Comparison of methods

Quick Formula Sheet

The trick:
If $y = [u(x)]^{v(x)}$ take log to get $\log y = v(x) \log u(x)$, then differentiate.

Log rules:
$\log(ab) = \log a + \log b$, $\log(a/b) = \log a - \log b$, $\log a^n = n \log a$.

Product splitting:
$\log\left(\prod f_i\right) = \sum \log f_i$ converts a long product into a sum, much easier to differentiate.

Exercise 5.5

Q 5.1 **Differentiate** $\cos x \cdot \cos 2x \cdot \cos 3x$ **with respect to** x.

SOLUTION

Concept used. Take logarithm of the product and use $\log(abc) = \log a + \log b + \log c$ before differentiating. (Implicit differentiation of $\log y$.)

Step 1. Let $y = \cos x \cos 2x \cos 3x$. Take log:

$$\log y = \log \cos x + \log \cos 2x + \log \cos 3x.$$

Step 2. Differentiate both sides w.r.t. x. Recall $\frac{d}{dx}\log\cos(kx) = -k\tan(kx)$:

$$\frac{1}{y}\frac{dy}{dx} = -\tan x - 2\tan 2x - 3\tan 3x.$$

Step 3. Multiply both sides by y:

$$\frac{dy}{dx} = -y\big(\tan x + 2\tan 2x + 3\tan 3x\big) = -\cos x\cos 2x\cos 3x\big(\tan x + 2\tan 2x + 3\tan 3x\big).$$

Final Answer: $\frac{dy}{dx} = -\cos x\cos 2x\cos 3x\,(\tan x + 2\tan 2x + 3\tan 3x)$.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Bombay*

Quick reading. Three-factor product; log-split before differentiating.

Step 1. $\log y = \sum \log\cos(kx)$.

Step 2. $y'/y = -\sum k\tan(kx)$ for $k = 1, 2, 3$.

Step 3. Multiply by y.

Final Answer: $-y(\tan x + 2\tan 2x + 3\tan 3x)$.

Q 5.2 **Differentiate** $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$ **with respect to** x.

SOLUTION

Concept used. Logarithmic differentiation; $\log\sqrt{f} = \frac{1}{2}\log f$ and $\log$ converts products/quotients into sums/differences.

Step 1. Let $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$. Take $\log$:

$$\log y = \frac{1}{2}\big[\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)\big].$$

Step 2. Differentiate both sides:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right].$$

Step 3. Solve for dy/dx:

$$\frac{dy}{dx} = \frac{y}{2}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right].$$

Final Answer: $\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right].$

EXPERT'S SOLUTION *: Sneha Iyer, Ph.D Mathematics, IIT Delhi*

Strategic angle. Five-factor product/quotient under a square root; log-split is the only sane method.

Step 1. Take $\log$, use square-root prefactor $1/2$.

Step 2. Each $\log(x-a)$ derivative is $1/(x-a)$.

Step 3. Multiply by y to recover dy/dx.

Final Answer: $\frac{y}{2}\sum(\pm)(x-a_i)^{-1}$.

Q 5.3 Differentiate $(\log x)^{\cos x}$ with respect to x.

SOLUTION

Concept used. Variable-base-variable-exponent: take $\log$ of both sides to convert to a product.

Step 1. Let $y = (\log x)^{\cos x}$. Take $\log$:

$$\log y = \cos x \cdot \log(\log x).$$

Step 2. Differentiate (product rule on right; chain on $\log\log x$):

$$\frac{1}{y}\frac{dy}{dx} = -\sin x \cdot \log(\log x) + \cos x \cdot \frac{1}{\log x} \cdot \frac{1}{x}.$$

Step 3. Solve:

$$\frac{dy}{dx} = y\left[\frac{\cos x}{x\log x} - \sin x \cdot \log(\log x)\right].$$

Substitute y:

$$\frac{dy}{dx} = (\log x)^{\cos x}\left[\frac{\cos x}{x\log x} - \sin x \cdot \log(\log x)\right].$$

Final Answer: $\frac{dy}{dx} = (\log x)^{\cos x}\left[\frac{\cos x}{x\log x} - \sin x \log(\log x)\right]$.

EXPERT'S SOLUTION *: Vivaan Gupta, M.Tech CS, IIT Madras*

Quick reading. Variable base $\log x$ raised to variable exponent $\cos x$.

Step 1. $\log y = \cos x \log(\log x)$.

Step 2. Differentiate using product + chain (twice).

Step 3. Multiply back by y.

Final Answer: $(\log x)^{\cos x}\big[\cos x/(x\log x) - \sin x \log(\log x)\big]$.

Q 5.4 Differentiate $x^x - 2^{\sin x}$ with respect to x.

SOLUTION

Concept used. Differentiate term-by-term. Each term is of the form a^b with a or b varying; logarithmic differentiation handles both.

Step 1. **Let** $u = x^x$**. Take** log: $\log u = x \log x$. Differentiate:

$$\frac{1}{u}\frac{du}{dx} = \log x + x \cdot \frac{1}{x} = \log x + 1.$$

Hence $\frac{du}{dx} = x^x(1 + \log x)$.

Step 2. **Let** $v = 2^{\sin x}$**. Take** log: $\log v = \sin x \cdot \log 2$. Differentiate:

$$\frac{1}{v}\frac{dv}{dx} = \cos x \cdot \log 2.$$

Hence $\frac{dv}{dx} = 2^{\sin x}\cos x \log 2$.

Step 3. Combine:

$$\frac{d}{dx}(x^x - 2^{\sin x}) = x^x(1 + \log x) - 2^{\sin x}\cos x \log 2.$$

Final Answer: $\frac{d}{dx}(x^x - 2^{\sin x}) = x^x(1 + \log x) - 2^{\sin x}\cos x \log 2$.

☞ **Exam Tip**

x^x comes up so often that the formula $\frac{d}{dx}x^x = x^x(1 + \log x)$ is worth memorising. It uses both x's of the base and the exponent.

EXPERT'S SOLUTION : *Aanya Mehta, Ph.D Pure Mathematics, IISc Bangalore*

Structural observation. Two unrelated a^b terms; log-differentiate each.

Step 1. $x^x \Rightarrow x^x(1 + \log x)$.

Step 2. $2^{\sin x} \Rightarrow 2^{\sin x} \cos x \log 2$.

Final Answer: $x^x(1 + \log x) - 2^{\sin x} \cos x \log 2$.

Q 5.5 **Differentiate** $(x + 3)^2(x + 4)^3(x + 5)^4$ **with respect to** x.

SOLUTION

Concept used. Logarithmic differentiation on a three-factor product with integer exponents.

Step 1. Let $y = (x + 3)^2(x + 4)^3(x + 5)^4$. Take log:

$$\log y = 2 \log(x + 3) + 3 \log(x + 4) + 4 \log(x + 5).$$

Step 2. Differentiate:

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x + 3} + \frac{3}{x + 4} + \frac{4}{x + 5}.$$

Step 3. Multiply by y:

$$\frac{dy}{dx} = (x + 3)^2(x + 4)^3(x + 5)^4 \left[\frac{2}{x + 3} + \frac{3}{x + 4} + \frac{4}{x + 5}\right].$$

Final Answer: $\frac{dy}{dx} = (x + 3)^2(x + 4)^3(x + 5)^4 \left[\frac{2}{x + 3} + \frac{3}{x + 4} + \frac{4}{x + 5}\right]$.

EXPERT'S SOLUTION : *Priya Singh, M.Sc Applied Mathematics, IIT Kanpur*

Quick reading. Log-product splits exponents into coefficients.

Step 1. $\log y = \sum n_i \log(x + a_i)$.

Step 2. $y'/y = \sum n_i/(x+a_i)$.

Step 3. Multiply by y.

Final Answer: $y\left(\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right)$.

Q 5.6 Differentiate $\left(x+\frac{1}{x}\right)^x + x^{1+1/x}$ with respect to x.

SOLUTION

Concept used. Two variable-exponent terms; log-differentiate each.

Step 1. **Let** $u = (x+1/x)^x$. $\log u = x\log(x+1/x)$. Differentiate:

$$\frac{1}{u}\frac{du}{dx} = \log\left(x+\frac{1}{x}\right) + x\cdot\frac{1}{x+1/x}\cdot\left(1-\frac{1}{x^2}\right).$$

Simplify the second term. Multiply numerator and denominator by x:

$$\frac{x(1-1/x^2)}{x+1/x} = \frac{x-1/x}{x+1/x} = \frac{x^2-1}{x^2+1}.$$

Hence

$$\frac{du}{dx} = u\left[\log\left(x+\frac{1}{x}\right) + \frac{x^2-1}{x^2+1}\right].$$

Step 2. **Let** $v = x^{1+1/x}$. $\log v = (1+1/x)\log x$. Differentiate (product rule):

$$\frac{1}{v}\frac{dv}{dx} = -\frac{1}{x^2}\cdot\log x + \left(1+\frac{1}{x}\right)\cdot\frac{1}{x} = -\frac{\log x}{x^2} + \frac{1}{x} + \frac{1}{x^2}.$$

Combine over x^2:

$$= \frac{-\log x + x + 1}{x^2} = \frac{x+1-\log x}{x^2}.$$

Hence

$$\frac{dv}{dx} = x^{1+1/x}\cdot\frac{x+1-\log x}{x^2}.$$

Step 3. Add:

$$\frac{dy}{dx} = \left(x+\frac{1}{x}\right)^x\left[\log\left(x+\frac{1}{x}\right) + \frac{x^2-1}{x^2+1}\right] + \frac{x^{1+1/x}(x+1-\log x)}{x^2}.$$

Final Answer: $\frac{dy}{dx} = \left(x+\frac{1}{x}\right)^x\left[\log(x+1/x) + \frac{x^2-1}{x^2+1}\right] + x^{1+1/x}\cdot\frac{x+1-\log x}{x^2}$.

EXPERT'S SOLUTION : *Karan Joshi, M.Sc Mathematics, IIT Bombay*

Strategic angle. Two log-differentiations followed by simplification.

Step 1. For u: $\log u = x\log(x+1/x)$; derivative simplifies to $u[\log(x+1/x)+(x^2-1)/(x^2+1)]$.

Step 2. For v: $\log v = (1+1/x)\log x$; derivative is $v(x+1-\log x)/x^2$.

Step 3. Sum.

Final Answer: See main.

Q 5.7 **Differentiate $(\log x)^x + x^{\log x}$ with respect to x, $x > 1$.**

SOLUTION

Concept used. Variable-exponent terms; log-differentiate each.

Step 1. **Let $u = (\log x)^x$.** $\log u = x\log(\log x)$. Differentiate:

$$\frac{1}{u}\frac{du}{dx} = \log(\log x) + x\cdot\frac{1}{\log x}\cdot\frac{1}{x} = \log(\log x) + \frac{1}{\log x}.$$

Hence

$$\frac{du}{dx} = (\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right].$$

Step 2. **Let $v = x^{\log x}$.** $\log v = (\log x)(\log x) = (\log x)^2$. Differentiate:

$$\frac{1}{v}\frac{dv}{dx} = 2\log x\cdot\frac{1}{x} = \frac{2\log x}{x}.$$

Hence

$$\frac{dv}{dx} = x^{\log x}\cdot\frac{2\log x}{x} = \frac{2\log x}{x}\cdot x^{\log x}.$$

Step 3. Sum:

$$\frac{dy}{dx} = (\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right] + \frac{2\log x}{x}\cdot x^{\log x}.$$

Final Answer: $\dfrac{dy}{dx} = (\log x)^x[\log(\log x) + 1/\log x] + \dfrac{2\log x}{x}\,x^{\log x}$.

EXPERT'S SOLUTION : *Diya Bhat, Ph.D Mathematics, IIT Delhi*

Structural observation. Note $\log(x^{\log x}) = (\log x)^2$ - a quadratic in $\log x$.

Step 1. $u'/u = \log(\log x) + 1/\log x$.

Step 2. $v'/v = 2\log x/x$.

Final Answer: See main.

Q 5.8 **Differentiate $(\sin x)^x + \sin^{-1}\sqrt{x}$ with respect to x.**

SOLUTION

Concept used. Log-differentiate the first term; chain-rule the second.

Step 1. **Let $u = (\sin x)^x$. $\log u = x\log(\sin x)$.** Differentiate:

$$\frac{1}{u}\frac{du}{dx} = \log\sin x + x\cdot\frac{\cos x}{\sin x} = \log\sin x + x\cot x.$$

Hence

$$\frac{du}{dx} = (\sin x)^x(\log\sin x + x\cot x).$$

Step 2. **Let $v = \sin^{-1}\sqrt{x}$.** Chain rule:

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(1-x)}}.$$

Step 3. Add:

$$\frac{dy}{dx} = (\sin x)^x(\log\sin x + x\cot x) + \frac{1}{2\sqrt{x(1-x)}}.$$

Final Answer: $\frac{dy}{dx} = (\sin x)^x(\log\sin x + x\cot x) + \frac{1}{2\sqrt{x(1-x)}}.$

EXPERT'S SOLUTION : *Yash Nair, M.Tech CS, IIT Madras*

Quick reading. First term: log-diff; second: chain rule.

Step 1. $(\sin x)^x \Rightarrow$ log-diff gives $(\sin x)^x(\log\sin x + x\cot x)$.

Step 2. $\sin^{-1}\sqrt{x}$: outer $1/\sqrt{1-x}$, inner $1/(2\sqrt{x})$.

Final Answer: See main.

Q 5.9 **Differentiate $x^{\sin x} + (\sin x)^{\cos x}$ with respect to x.**

SOLUTION

Concept used. Both terms variable-exponent; log-differentiate each.

Step 1. **Let $u = x^{\sin x}$.** $\log u = \sin x \cdot \log x$.

$$\frac{1}{u}\frac{du}{dx} = \cos x \cdot \log x + \sin x \cdot \frac{1}{x} = \cos x \log x + \frac{\sin x}{x}.$$

Hence

$$\frac{du}{dx} = x^{\sin x}\left[\cos x \log x + \frac{\sin x}{x}\right].$$

Step 2. **Let $v = (\sin x)^{\cos x}$.** $\log v = \cos x \log \sin x$.

$$\frac{1}{v}\frac{dv}{dx} = -\sin x \cdot \log \sin x + \cos x \cdot \frac{\cos x}{\sin x} = -\sin x \log \sin x + \frac{\cos^2 x}{\sin x}.$$

Hence

$$\frac{dv}{dx} = (\sin x)^{\cos x}\left[-\sin x \log \sin x + \frac{\cos^2 x}{\sin x}\right].$$

Step 3. Sum:

$$\frac{dy}{dx} = x^{\sin x}\left[\cos x \log x + \frac{\sin x}{x}\right] + (\sin x)^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x \log \sin x\right].$$

Final Answer: $\frac{dy}{dx} = x^{\sin x}\left[\cos x \log x + \frac{\sin x}{x}\right] + (\sin x)^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x \log \sin x\right]$.

EXPERT'S SOLUTION : *Aditi Banerjee, M.Sc Mathematics, ISI Kolkata*

Strategic angle. Twice log-differentiate.

Step 1. $u' = u[\cos x \log x + \sin x / x]$.

Step 2. $v' = v[\cos^2 x / \sin x - \sin x \log \sin x]$.

Final Answer: $u' + v'$.

Q 5.10 **Differentiate $x^{x \cos x} + \frac{x^2+1}{x^2-1}$ with respect to x.**

SOLUTION

Concept used. Log-differentiate the first term; quotient-rule the second.

Step 1. **Let** $u = x^{x\cos x}$**.** $\log u = (x\cos x)\log x$. Differentiate (product rule on $x\cos x$, product rule for $(\cdot)\log x$):

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x\cos x)\cdot\log x + x\cos x\cdot\frac{1}{x}.$$

Now $\dfrac{d}{dx}(x\cos x) = \cos x - x\sin x$. Hence

$$\frac{1}{u}\frac{du}{dx} = (\cos x - x\sin x)\log x + \cos x.$$

Therefore

$$\frac{du}{dx} = x^{x\cos x}\left[(\cos x - x\sin x)\log x + \cos x\right].$$

Step 2. **Let** $v = (x^2+1)/(x^2-1)$**.** Quotient rule with numerator derivative $2x$, denominator derivative $2x$:

$$\frac{dv}{dx} = \frac{2x(x^2-1) - (x^2+1)(2x)}{(x^2-1)^2}$$
$$= \frac{2x\left[(x^2-1) - (x^2+1)\right]}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}.$$

Step 3. Add:

$$\frac{dy}{dx} = x^{x\cos x}\left[(\cos x - x\sin x)\log x + \cos x\right] - \frac{4x}{(x^2-1)^2}.$$

Final Answer: $\dfrac{dy}{dx} = x^{x\cos x}\left[(\cos x - x\sin x)\log x + \cos x\right] - \dfrac{4x}{(x^2-1)^2}.$

EXPERT'S SOLUTION *: Tara Pillai, Ph.D Mathematics, IIT Delhi*

Structural observation. Two unrelated derivatives summed.

Step 1. Log-diff first term: $u' = u[(\cos x - x\sin x)\log x + \cos x]$.

Step 2. Quotient rule second term: $-4x/(x^2-1)^2$.

Final Answer: Sum as in main.

Q 5.11 **Differentiate** $(x\cos x)^x + (x\sin x)^{1/x}$ **with respect to** x**.**

SOLUTION

Concept used. Both terms variable-exponent and variable-base; log-differentiate each.

Step 1. **Let** $u = (x\cos x)^x$. $\log u = x\log(x\cos x) = x[\log x + \log\cos x]$. Differentiate:

$$\frac{1}{u}\frac{du}{dx} = \log(x\cos x) + x\left[\frac{1}{x} + \frac{-\sin x}{\cos x}\right] = \log(x\cos x) + 1 - x\tan x.$$

Hence

$$\frac{du}{dx} = (x\cos x)^x[1 + \log(x\cos x) - x\tan x].$$

Step 2. **Let** $v = (x\sin x)^{1/x}$. $\log v = \frac{1}{x}\log(x\sin x) = \frac{1}{x}[\log x + \log\sin x]$. Differentiate:

$$\frac{1}{v}\frac{dv}{dx} = -\frac{1}{x^2}\log(x\sin x) + \frac{1}{x}\left[\frac{1}{x} + \cot x\right].$$

Simplify:

$$= -\frac{\log(x\sin x)}{x^2} + \frac{1}{x^2} + \frac{\cot x}{x} = \frac{1-\log(x\sin x)}{x^2} + \frac{\cot x}{x}.$$

Hence

$$\frac{dv}{dx} = (x\sin x)^{1/x}\left[\frac{1-\log(x\sin x)}{x^2} + \frac{\cot x}{x}\right].$$

Step 3. Sum:

$$\frac{dy}{dx} = (x\cos x)^x[1+\log(x\cos x)-x\tan x]+(x\sin x)^{1/x}\left[\frac{1-\log(x\sin x)}{x^2} + \frac{\cot x}{x}\right].$$

Final Answer: See above expression.

EXPERT'S SOLUTION *: Krishna Rao, M.Sc Mathematics, IIT Bombay*

Strategic angle. Log-differentiate both terms; tedious but mechanical.

Step 1. For u: $\log u = x\log(x\cos x) \Rightarrow$ result.

Step 2. For v: $\log v = (1/x)\log(x\sin x) \Rightarrow$ result.

Final Answer: Sum as in main.

Q 5.12 **Find** $\frac{dy}{dx}$ **if** $x^y + y^x = 1$.

SOLUTION

Concept used. Both terms are variable-base, variable-exponent. Introduce $u = x^y$ and $v = y^x$, then $u + v = 1$ gives $du/dx + dv/dx = 0$. Log-differentiate each.

Step 1. **Let $u = x^y$.** $\log u = y \log x$. Differentiate:

$$\frac{1}{u}\frac{du}{dx} = \frac{dy}{dx}\log x + y \cdot \frac{1}{x}.$$

Hence

$$\frac{du}{dx} = x^y\left[\log x \cdot \frac{dy}{dx} + \frac{y}{x}\right].$$

Step 2. **Let $v = y^x$.** $\log v = x \log y$. Differentiate:

$$\frac{1}{v}\frac{dv}{dx} = \log y + x \cdot \frac{1}{y} \cdot \frac{dy}{dx}.$$

Hence

$$\frac{dv}{dx} = y^x\left[\log y + \frac{x}{y}\frac{dy}{dx}\right].$$

Step 3. Set $\frac{du}{dx} + \frac{dv}{dx} = 0$:

$$x^y \log x \frac{dy}{dx} + \frac{yx^y}{x} + y^x \log y + \frac{xy^x}{y}\frac{dy}{dx} = 0.$$

Step 4. Group dy/dx:

$$\left[x^y \log x + \frac{xy^x}{y}\right]\frac{dy}{dx} = -\left[\frac{yx^y}{x} + y^x \log y\right].$$

Simplify (multiply through by xy to clear fractions, or just write directly):

$$\frac{dy}{dx} = -\frac{yx^y/x + y^x \log y}{x^y \log x + xy^x/y} = -\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}.$$

Final Answer: $\frac{dy}{dx} = -\frac{y \cdot x^{y-1} + y^x \log y}{x^y \log x + x \cdot y^{x-1}}$.

EXPERT'S SOLUTION : *Meera Chatterjee, M.Sc Mathematics, ISI Kolkata*

Structural observation. Sum of two variable-base-exponent terms equal to a constant.

Step 1. Compute u', v' separately by log-diff.

Step 2. Set $u' + v' = 0$, solve for y'.

Final Answer: See main.

Q 5.13 **Find $\frac{dy}{dx}$ if $y^x = x^y$.**

SOLUTION

Concept used. Take log of both sides, then differentiate implicitly.

Step 1. $y^x = x^y \Rightarrow x\log y = y\log x$.

Step 2. Differentiate both sides w.r.t. x. Left:

$$\log y + x\cdot\frac{1}{y}\frac{dy}{dx} = \log y + \frac{x}{y}\frac{dy}{dx}.$$

Right:

$$\frac{dy}{dx}\log x + y\cdot\frac{1}{x} = \log x\cdot\frac{dy}{dx} + \frac{y}{x}.$$

Step 3. Equate and group dy/dx:

$$\log y + \frac{x}{y}\frac{dy}{dx} = \log x\cdot\frac{dy}{dx} + \frac{y}{x}.$$

$$\left(\frac{x}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x} - \log y.$$

Step 4. Solve:

$$\frac{dy}{dx} = \frac{y/x - \log y}{x/y - \log x} = \frac{y(y - x\log y)}{x(x - y\log x)}.$$

(Multiply numerator and denominator by xy.)

Final Answer: $\frac{dy}{dx} = \frac{y(y - x\log y)}{x(x - y\log x)}$.

EXPERT'S SOLUTION *: Rohit Verma, M.Sc Mathematics, ISI Kolkata*

Quick reading. Take log first to convert exponential equality to a linear-in-logs equation.

Step 1. $x\log y = y\log x$.

Step 2. Implicit-differentiate.

Step 3. Solve for y'.

Final Answer: $y(y - x\log y)/[x(x - y\log x)]$.

Q 5.14 **Find** $\frac{dy}{dx}$ **if** $(\cos x)^y = (\cos y)^x$.

SOLUTION

Concept used. Take log, then differentiate.

Step 1. $y \log \cos x = x \log \cos y$.

Step 2. Differentiate:

$$\frac{dy}{dx} \log \cos x + y \cdot \frac{-\sin x}{\cos x} = \log \cos y + x \cdot \frac{-\sin y}{\cos y} \cdot \frac{dy}{dx}.$$

$$\log \cos x \cdot \frac{dy}{dx} - y \tan x = \log \cos y - x \tan y \cdot \frac{dy}{dx}.$$

Step 3. Group dy/dx:

$$(\log \cos x + x \tan y)\frac{dy}{dx} = \log \cos y + y \tan x.$$

Step 4. Solve:

$$\frac{dy}{dx} = \frac{\log \cos y + y \tan x}{\log \cos x + x \tan y}.$$

Final Answer: $\frac{dy}{dx} = \frac{\log \cos y + y \tan x}{\log \cos x + x \tan y}$.

EXPERT'S SOLUTION *: Pranav Kapoor, M.Sc Mathematics, IIT Bombay*

Strategic angle. Symmetric in $(x, y) \leftrightarrow (y, x)$ except for the y vs x multipliers; log first.

Step 1. $y \log \cos x = x \log \cos y$ after log.

Step 2. Differentiate and isolate y'.

Final Answer: $(\log \cos y + y \tan x)/(\log \cos x + x \tan y)$.

Q 5.15 **Find** $\frac{dy}{dx}$ **if** $xy = e^{(x-y)}$.

SOLUTION

Concept used. Take log of both sides.

Step 1. $\log(xy) = x - y$, i.e. $\log x + \log y = x - y$.

Step 2. Differentiate:

$$\frac{1}{x}+\frac{1}{y}\frac{dy}{dx}=1-\frac{dy}{dx}.$$

Step 3. Group dy/dx:

$$\frac{1}{y}\frac{dy}{dx}+\frac{dy}{dx}=1-\frac{1}{x}\implies\frac{dy}{dx}\left(\frac{1}{y}+1\right)=\frac{x-1}{x}.$$

$$\frac{dy}{dx}\cdot\frac{1+y}{y}=\frac{x-1}{x}.$$

Step 4. Solve:

$$\frac{dy}{dx}=\frac{y(x-1)}{x(1+y)}.$$

Final Answer: $\dfrac{dy}{dx}=\dfrac{y(x-1)}{x(y+1)}$.

EXPERT'S SOLUTION *: Aditya Patel, M.Sc Mathematics, IIT Bombay*

Quick reading. Log linearises the exponential; implicit-diff then solves.

Step 1. $\log x+\log y=x-y$.

Step 2. Differentiate, isolate y'.

Final Answer: $y(x-1)/[x(y+1)]$.

Q 5.16 **Find the derivative of the function given by $f(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f'(1)$.**

SOLUTION

Concept used. Log-differentiation on a four-factor product.

Step 1. Take log:

$$\log f(x)=\log(1+x)+\log(1+x^2)+\log(1+x^4)+\log(1+x^8).$$

Step 2. Differentiate:

$$\frac{f'(x)}{f(x)}=\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}.$$

Step 3. Multiply by $f(x)$:

$$f'(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right].$$

Step 4. **Evaluate at $x = 1$.** First $f(1) = 2 \cdot 2 \cdot 2 \cdot 2 = 16$. Each fraction becomes $1/2, 2/2, 4/2, 8/2 = 1/2 + 1 + 2 + 4 = 7.5$. Therefore

$$f'(1) = 16 \times 7.5 = 120.$$

Final Answer: $f'(x) = f(x)\left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right]$ and $f'(1) = 120$.

EXPERT'S SOLUTION *: Ishita Reddy, Ph.D Mathematics, IIT Delhi*

Strategic angle. Geometric structure: $(1+x)(1+x^2)(1+x^4)(1+x^8) = \frac{x^{16}-1}{x-1}$ (sum of geometric series in disguise).

Step 1. Alternative: $f(x) = (x^{16}-1)/(x-1)$, so $f(1)$ requires l'Hopital or limit, giving 16.

Step 2. Log-diff method above yields $f'(1) = 120$ directly.

Final Answer: $f'(1) = 120$.

Q 5.17 **Differentiate $(x^2 - 5x + 8)(x^3 + 7x + 9)$ in three ways: (i) by product rule, (ii) by expanding to a single polynomial, (iii) by logarithmic differentiation. Do they all give the same answer?**

SOLUTION

Concept used. Three methods to differentiate the same product. They must agree.
(i) Product rule. Let $u = x^2 - 5x + 8$, $v = x^3 + 7x + 9$. Then $u' = 2x - 5$, $v' = 3x^2 + 7$.

$$\frac{dy}{dx} = u'v + uv' = (2x-5)(x^3+7x+9) + (x^2-5x+8)(3x^2+7).$$

Expand both products:

$$\begin{aligned}(2x-5)(x^3+7x+9) &= 2x^4+14x^2+18x-5x^3-35x-45\\ &= 2x^4-5x^3+14x^2-17x-45,\\ (x^2-5x+8)(3x^2+7) &= 3x^4+7x^2-15x^3-35x+24x^2+56\\ &= 3x^4-15x^3+31x^2-35x+56.\end{aligned}$$

Add:
$$\frac{dy}{dx} = 5x^4-20x^3+45x^2-52x+11.$$

(ii) Expand first. Multiply out:

$$\begin{aligned}y &= (x^2-5x+8)(x^3+7x+9)\\ &= x^5+7x^3+9x^2-5x^4-35x^2-45x+8x^3+56x+72\\ &= x^5-5x^4+15x^3-26x^2+11x+72.\end{aligned}$$

Differentiate:
$$\frac{dy}{dx} = 5x^4-20x^3+45x^2-52x+11.$$

(iii) Logarithmic. $\log y = \log(x^2-5x+8)+\log(x^3+7x+9)$:

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x-5}{x^2-5x+8}+\frac{3x^2+7}{x^3+7x+9}.$$

Multiply by y:

$$\frac{dy}{dx} = y\left[\frac{2x-5}{x^2-5x+8}+\frac{3x^2+7}{x^3+7x+9}\right] = (2x-5)(x^3+7x+9)+(x^2-5x+8)(3x^2+7),$$

which is exactly the product-rule expression, equal to $5x^4-20x^3+45x^2-52x+11$.

> **Final Answer:** All three give $\frac{dy}{dx} = 5x^4-20x^3+45x^2-52x+11$. Yes, all the same answer.

EXPERT'S SOLUTION *: Ishaan Pillai, B.Tech CSE, IIT Roorkee*

Why this matters. The exercise shows that the three rules (product, polynomial, log) are equivalent for differentiating products.

Step 1. Verify by computing the same polynomial via three routes.

Step 2. Expansion route is cleanest computationally.

> **Final Answer:** Same answer: $5x^4-20x^3+45x^2-52x+11$.

Q 5.18 **If u, v and w are functions of x, show that**

$$\frac{d}{dx}(u \cdot v \cdot w) = \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}$$

in two ways: first by repeated application of product rule, second by logarithmic differentiation.

SOLUTION

Concept used. Product rule: $(fg)' = f'g + fg'$; logarithmic differentiation: $\log(uvw) = \log u + \log v + \log w$, then differentiate.

(i) Repeated product rule. Group as $(uv) \cdot w$. First,

$$\frac{d}{dx}(uv) = u'v + uv'.$$

Apply product rule again:

$$\frac{d}{dx}((uv) \cdot w) = (uv)'w + (uv)w' = (u'v + uv')w + uvw' = u'vw + uv'w + uvw'.$$

(ii) Logarithmic. Assume $u, v, w > 0$. Let $y = uvw$:

$$\log y = \log u + \log v + \log w.$$

Differentiate:

$$\frac{1}{y}\frac{dy}{dx} = \frac{u'}{u} + \frac{v'}{v} + \frac{w'}{w}.$$

Multiply by $y = uvw$:

$$\frac{dy}{dx} = uvw \cdot \frac{u'}{u} + uvw \cdot \frac{v'}{v} + uvw \cdot \frac{w'}{w} = u'vw + uv'w + uvw'.$$

Both methods give the same expression.

Final Answer: $\frac{d}{dx}(uvw) = u'vw + uv'w + uvw'$.

EXPERT'S SOLUTION *: Kavya Singh, M.Sc Mathematics, IIT Bombay*

Why this matters. The three-factor product rule generalises to n factors by the same routes; this exercise nails down the pattern.

Step 1. Method 1: Apply product rule twice.

Step 2. Method 2: Take log, differentiate, multiply by y.

Final Answer: $u'vw + uv'w + uvw'$.

Key Takeaways

- For $[u(x)]^{v(x)}$, take $\log$ of both sides; the technique handles variable base AND variable exponent simultaneously.
- Log converts products into sums and powers into multipliers, making complicated chains routine.
- Implicit equations involving $x^y, y^x, e^{f(x,y)}$ are simplified by taking $\log$ before differentiating.

End of Exercise 5.5