Collegedunia NCERT Solutions

Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 5: Continuity and Differentiability

About this Chapter

Exercise 5.6 covers **parametric differentiation**. When $x = f(t)$ and $y = g(t)$, the chain rule gives $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ provided $dx/dt \neq 0$. Most questions involve trig parameters and simplify cleanly using double-angle identities.

Topics covered: Parametric differentiation • Trig parameters • Half-angle simplifications • Hidden-relation proofs

Quick Formula Sheet

Parametric rule: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ when $dx/dt \neq 0$.

Double-angle identities: $\sin 2\theta = 2\sin\theta\cos\theta$, $1 - \cos 2\theta = 2\sin^2\theta$, $1 + \cos 2\theta = 2\cos^2\theta$.

Half-angle tangent: $\frac{\sin\theta}{1+\cos\theta} = \tan(\theta/2)$.

Exercise 5.6

Q 5.1 **Find $\frac{dy}{dx}$ if $x = 2at^2,\ y = at^4$.**

SOLUTION

Concept used. Parametric differentiation: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

Step 1. Differentiate w.r.t. t:

$$\frac{dx}{dt} = 4at, \qquad \frac{dy}{dt} = 4at^3.$$

Step 2. Divide:

$$\frac{dy}{dx} = \frac{4at^3}{4at} = t^2.$$

Final Answer: $\frac{dy}{dx} = t^2$.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Bombay*

Quick reading. Both derivatives are clean polynomials in t.

Step 1. $dx/dt = 4at$, $dy/dt = 4at^3$.

Step 2. Ratio: t^2.

Final Answer: t^2.

Q 5.2 **Find $\frac{dy}{dx}$ if $x = a\cos\theta,\ y = b\cos\theta$.**

SOLUTION

Concept used. Parametric differentiation. (Note: the NCERT text reads $y = b\cos\theta$; we follow it literally.)

Step 1. Differentiate:

$$\frac{dx}{d\theta} = -a\sin\theta, \qquad \frac{dy}{d\theta} = -b\sin\theta.$$

Step 2. Ratio:

$$\frac{dy}{dx} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}.$$

Final Answer: $\frac{dy}{dx} = \frac{b}{a}$ (a constant; the curve is a line).

EXPERT'S SOLUTION : *Sneha Iyer, Ph.D Mathematics, IIT Delhi*

Structural observation. Since $y = (b/a)x$, the parametric form encodes a straight line of slope b/a traced as θ varies.

Step 1. $y = (b/a)\,a\cos\theta = (b/a)x$.

Step 2. Slope of a line $y = mx$ is m.

Final Answer: b/a.

Q 5.3 **Find $\frac{dy}{dx}$ if $x = \sin t,\ y = \cos 2t$.**

SOLUTION

Concept used. Parametric differentiation with the chain rule on $\cos 2t$.

Step 1. Differentiate:

$$\frac{dx}{dt} = \cos t, \qquad \frac{dy}{dt} = -2\sin 2t = -4\sin t\cos t.$$

(Used $\sin 2t = 2\sin t\cos t$.)

Step 2. Ratio:

$$\frac{dy}{dx} = \frac{-4\sin t\cos t}{\cos t} = -4\sin t.$$

Final Answer: $\frac{dy}{dx} = -4\sin t.$

EXPERT'S SOLUTION : *Vivaan Gupta, M.Tech CS, IIT Madras*

Quick reading. Double-angle expansion of $\sin 2t$ cancels the $\cos t$ in the denominator.

Step 1. $dy/dt = -2\sin 2t = -4\sin t\cos t$.

Step 2. Divide by $dx/dt = \cos t$.

Final Answer: $-4\sin t$.

Q 5.4 **Find $\frac{dy}{dx}$ if $x = 4t,\ y = \frac{4}{t}$.**

SOLUTION

Concept used. Parametric differentiation.

Step 1. Differentiate:

$$\frac{dx}{dt} = 4, \qquad \frac{dy}{dt} = -\frac{4}{t^2}.$$

Step 2. Ratio:

$$\frac{dy}{dx} = \frac{-4/t^2}{4} = -\frac{1}{t^2}.$$

Final Answer: $\frac{dy}{dx} = -\frac{1}{t^2}$.

EXPERT'S SOLUTION : *Aanya Mehta, Ph.D Pure Mathematics, IISc Bangalore*

Picture-first. Eliminating t gives $xy = 16$, a hyperbola; the slope $-1/t^2$ is always negative.

Step 1. $t = x/4$ gives $y = 16/x$.

Step 2. Direct: $dy/dx = -16/x^2 = -1/t^2$.

Final Answer: $-1/t^2$.

Q 5.5 **Find $\frac{dy}{dx}$ if $x = \cos\theta - \cos 2\theta,\ y = \sin\theta - \sin 2\theta$.**

SOLUTION

Concept used. Parametric differentiation; the derivatives involve $\cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta$.

Step 1. Differentiate w.r.t. θ:

$$\frac{dx}{d\theta} = -\sin\theta + 2\sin 2\theta, \qquad \frac{dy}{d\theta} = \cos\theta - 2\cos 2\theta.$$

Step 2. Ratio:

$$\frac{dy}{dx} = \frac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}.$$

Final Answer: $\frac{dy}{dx} = \frac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$.

EXPERT'S SOLUTION : *Priya Singh, M.Sc Applied Mathematics, IIT Kanpur*

Quick reading. Direct differentiation; no algebraic simplification expected.

Step 1. $dx/d\theta = -\sin\theta + 2\sin 2\theta$.

Step 2. $dy/d\theta = \cos\theta - 2\cos 2\theta$.

Final Answer: Ratio as above.

Q 5.6 **Find** $\frac{dy}{dx}$ **if** $x = a(\theta - \sin\theta),\ y = a(1 + \cos\theta)$**.**

SOLUTION

Concept used. Parametric differentiation; simplify the ratio using half-angle identities $1 + \cos\theta = 2\cos^2(\theta/2)$ and $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$.

Step 1. Differentiate:

$$\frac{dx}{d\theta} = a(1 - \cos\theta), \qquad \frac{dy}{d\theta} = -a\sin\theta.$$

Step 2. Use half-angle identities:

$$1 - \cos\theta = 2\sin^2(\theta/2), \qquad \sin\theta = 2\sin(\theta/2)\cos(\theta/2).$$

Step 3. Ratio:

$$\frac{dy}{dx} = \frac{-a\sin\theta}{a(1 - \cos\theta)} = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = -\frac{\cos(\theta/2)}{\sin(\theta/2)} = -\cot\big(\theta/2\big).$$

Final Answer: $\frac{dy}{dx} = -\cot(\theta/2)$.

☞ Exam Tip

Half-angle identities are the lifeblood of parametric problems with $1 \pm \cos\theta$ or $\sin\theta$. Knowing $1 - \cos\theta = 2\sin^2(\theta/2)$ and the partner identity turns ratios into single trig functions of the half angle.

EXPERT'S SOLUTION *: Karan Joshi, M.Sc Mathematics, IIT Bombay*

Strategic angle. Reduce to half-angle form, then cancel.

Step 1. Numerator: $-2\sin(\theta/2)\cos(\theta/2)$.

Step 2. Denominator: $2\sin^2(\theta/2)$.

Step 3. Ratio: $-\cot(\theta/2)$.

Final Answer: $-\cot(\theta/2)$.

Q 5.7 **Find** $\frac{dy}{dx}$ **if** $x = \frac{\sin^3 t}{\sqrt{\cos 2t}},\ y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$.

SOLUTION

Concept used. Parametric differentiation; differentiate using quotient and chain rules, then simplify.

Step 1. Compute $\frac{dx}{dt}$. Write $x = \sin^3 t(\cos 2t)^{-1/2}$. Apply product rule:

$$\frac{dx}{dt} = 3\sin^2 t\cos t \cdot (\cos 2t)^{-1/2} + \sin^3 t \cdot \left(-\tfrac{1}{2}\right)(\cos 2t)^{-3/2}(-2\sin 2t).$$

Simplify the second term using $\sin 2t = 2\sin t\cos t$:

$$= \frac{3\sin^2 t\cos t}{\sqrt{\cos 2t}} + \frac{\sin^3 t \cdot 2\sin t\cos t}{(\cos 2t)^{3/2}}.$$

Combine over the common denominator $(\cos 2t)^{3/2}$:

$$= \frac{3\sin^2 t\cos t\cos 2t + 2\sin^4 t\cos t}{(\cos 2t)^{3/2}}.$$

Factor $\sin^2 t\cos t$ in the numerator:

$$= \frac{\sin^2 t\cos t(3\cos 2t + 2\sin^2 t)}{(\cos 2t)^{3/2}}.$$

Now $\cos 2t = 1 - 2\sin^2 t$ gives
$3\cos 2t + 2\sin^2 t = 3 - 6\sin^2 t + 2\sin^2 t = 3 - 4\sin^2 t$. Hence

$$\frac{dx}{dt} = \frac{\sin^2 t\cos t(3 - 4\sin^2 t)}{(\cos 2t)^{3/2}}.$$

Step 2. Similarly $\frac{dy}{dt}$. By symmetry (swap $\sin \leftrightarrow \cos$, with appropriate sign):

$$\frac{dy}{dt} = -\frac{\cos^2 t\sin t(3 - 4\cos^2 t)}{(\cos 2t)^{3/2}}$$
$$= -\frac{\cos^2 t\sin t\big(-(4\cos^2 t - 3)\big)}{(\cos 2t)^{3/2}}.$$

Note $3 - 4\cos^2 t = -(4\cos^2 t - 3)$.

Step 3. Take ratio (the $(\cos 2t)^{-3/2}$ factor cancels):

$$\frac{dy}{dx} = \frac{-\cos^2 t\sin t(3 - 4\cos^2 t)}{\sin^2 t\cos t(3 - 4\sin^2 t)} = -\frac{\cos t(3 - 4\cos^2 t)}{\sin t(3 - 4\sin^2 t)}.$$

Step 4. Recognise the triple-angle formulas: $\sin 3t = 3\sin t - 4\sin^3 t = \sin t(3 - 4\sin^2 t)$ and $\cos 3t = 4\cos^3 t - 3\cos t = -\cos t(3 - 4\cos^2 t)$. Hence

$$\frac{dy}{dx} = -\frac{-\cos 3t}{\sin 3t} = \frac{\cos 3t}{\sin 3t} = \cot 3t.$$

Wait, signs:
$-\cos t(3-4\cos^2 t) = -(\cos t \cdot 3 - 4\cos^3 t) = -3\cos t + 4\cos^3 t = \cos 3t$, so $\cos t(3-4\cos^2 t) = -\cos 3t$. Therefore

$$\frac{dy}{dx} = -\frac{-\cos 3t}{\sin 3t} = \frac{\cos 3t}{\sin 3t}.$$

But this contradicts the standard answer. Let us redo. From triple-angle formula:

$$\cos 3t = 4\cos^3 t - 3\cos t = \cos t(4\cos^2 t - 3) = -\cos t(3-4\cos^2 t).$$

Hence $\cos t(3-4\cos^2 t) = -\cos 3t$. Substitute:

$$\frac{dy}{dx} = -\frac{(-\cos 3t)}{\sin 3t} = \frac{\cos 3t}{\sin 3t} = \cot 3t.$$

Hmm, the expected NCERT answer is $-\cot 3t$ or $\tan 3t$ depending on convention. Recompute the sign of $\frac{dy}{dt}$ carefully:

$$\begin{aligned}\frac{dy}{dt} &= 3\cos^2 t(-\sin t)(\cos 2t)^{-1/2} + \cos^3 t(-1/2)(\cos 2t)^{-3/2}(-2\sin 2t)\\ &= \frac{-3\cos^2 t\sin t}{\sqrt{\cos 2t}} + \frac{\cos^3 t\cdot 2\sin t\cos t}{(\cos 2t)^{3/2}}\\ &= \frac{-3\cos^2 t\sin t\cos 2t + 2\sin t\cos^4 t}{(\cos 2t)^{3/2}}\\ &= \frac{\cos^2 t\sin t(-3\cos 2t + 2\cos^2 t)}{(\cos 2t)^{3/2}}.\end{aligned}$$

With $\cos 2t = 2\cos^2 t - 1$:
$-3(2\cos^2 t - 1) + 2\cos^2 t = -6\cos^2 t + 3 + 2\cos^2 t = 3 - 4\cos^2 t$. So

$$\frac{dy}{dt} = \frac{\cos^2 t\sin t(3-4\cos^2 t)}{(\cos 2t)^{3/2}}.$$

(Sign positive, no overall minus.) Therefore

$$\frac{dy}{dx} = \frac{\cos^2 t\sin t(3-4\cos^2 t)}{\sin^2 t\cos t(3-4\sin^2 t)} = \frac{\cos t(3-4\cos^2 t)}{\sin t(3-4\sin^2 t)} = \frac{-\cos 3t}{\sin 3t} = -\cot 3t.$$

Final Answer: $\frac{dy}{dx} = -\cot 3t.$

EXPERT'S SOLUTION : *Tara Pillai, Ph.D Mathematics, IIT Delhi*

Strategic angle. Both x and y are cubics in $\sin t$ or $\cos t$ divided by $\sqrt{\cos 2t}$. Triple-angle identities collapse the answer.

Step 1. Differentiate by quotient rule; the $\sqrt{\cos 2t}$ factors cancel in the ratio.

Step 2. Use $\cos t(3 - 4\cos^2 t) = -\cos 3t$, $\sin t(3 - 4\sin^2 t) = \sin 3t$.

Step 3. Ratio simplifies to $-\cot 3t$.

Final Answer: $-\cot 3t$.

Q 5.8 **Find** $\frac{dy}{dx}$ **if** $x = a\left(\cos t + \log\tan\frac{t}{2}\right),\ y = a\sin t$.

SOLUTION

Concept used. Parametric differentiation with $\log\tan(t/2)$. Its derivative is $\frac{1}{\tan(t/2)} \cdot \sec^2(t/2) \cdot \frac{1}{2} = \frac{1}{2\sin(t/2)\cos(t/2)} = \frac{1}{\sin t}$.

Step 1. Compute derivatives:

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\sin t}\right) = a \cdot \frac{1 - \sin^2 t}{\sin t} = a \cdot \frac{\cos^2 t}{\sin t}.$$

$$\frac{dy}{dt} = a\cos t.$$

Step 2. Ratio:

$$\frac{dy}{dx} = \frac{a\cos t}{a\cos^2 t/\sin t} = \frac{\cos t\sin t}{\cos^2 t} = \frac{\sin t}{\cos t} = \tan t.$$

Final Answer: $\frac{dy}{dx} = \tan t$.

EXPERT'S SOLUTION : *Krishna Rao, M.Sc Mathematics, IIT Bombay*

Quick reading. The derivative of $\log\tan(t/2)$ is $1/\sin t$, which combines with $-\sin t$ to give $\cos^2 t/\sin t$.

Step 1. $dx/dt = a\cos^2 t/\sin t$.

Step 2. $dy/dt = a\cos t$.

Step 3. Ratio $= \tan t$.

Final Answer: $\tan t$.

Q 5.9 **Find $\frac{dy}{dx}$ if $x = a\sec\theta,\ y = b\tan\theta$.**

SOLUTION

Concept used. Parametric differentiation; recall $\frac{d}{d\theta}\sec\theta = \sec\theta\tan\theta$, $\frac{d}{d\theta}\tan\theta = \sec^2\theta$.

Step 1. Differentiate:

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta, \qquad \frac{dy}{d\theta} = b\sec^2\theta.$$

Step 2. Ratio:

$$\frac{dy}{dx} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}\cdot\frac{\sec\theta}{\tan\theta} = \frac{b}{a}\cdot\frac{1/\cos\theta}{\sin\theta/\cos\theta} = \frac{b}{a\sin\theta} = \frac{b}{a}\csc\theta.$$

Final Answer: $\frac{dy}{dx} = \frac{b}{a}\csc\theta = \frac{b}{a\sin\theta}$.

EXPERT'S SOLUTION *: Meera Chatterjee, M.Sc Mathematics, ISI Kolkata*

Picture-first. Eliminating θ: $\sec^2 - \tan^2 = 1 \Rightarrow x^2/a^2 - y^2/b^2 = 1$, a hyperbola.

Step 1. $dx/d\theta = a\sec\theta\tan\theta$, $dy/d\theta = b\sec^2\theta$.

Step 2. Ratio $= (b/a)\csc\theta$.

Final Answer: $(b/a)\csc\theta$.

Q 5.10 **Find $\frac{dy}{dx}$ if $x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta)$.**

SOLUTION

Concept used. Parametric differentiation with product rule on the $\theta\cdot\sin\theta$ and $\theta\cdot\cos\theta$ terms.

Step 1. Differentiate:

$$\frac{dx}{d\theta} = a\big(-\sin\theta + \sin\theta + \theta\cos\theta\big) = a\theta\cos\theta,$$
$$\frac{dy}{d\theta} = a\big(\cos\theta - \cos\theta + \theta\sin\theta\big) = a\theta\sin\theta.$$

Step 2. Ratio:

$$\frac{dy}{dx} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta.$$

Final Answer: $\frac{dy}{dx} = \tan\theta$.

EXPERT'S SOLUTION : *Rohit Verma, M.Sc Mathematics, ISI Kolkata*

Structural observation. The $-\sin\theta$ and $\sin\theta$ cancel in $dx/d\theta$; same with $\cos\theta$ in $dy/d\theta$. Clean cancellation.

Step 1. $dx/d\theta = a\theta\cos\theta$.

Step 2. $dy/d\theta = a\theta\sin\theta$.

Step 3. Ratio $= \tan\theta$.

Final Answer: $\tan\theta$.

Q 5.11 **If $x = a^{\sin^{-1}t}$, $y = a^{\cos^{-1}t}$, show that $\frac{dy}{dx} = -\frac{y}{x}$.**

SOLUTION

Concept used. Parametric differentiation of $a^{f(t)}$: $\frac{d}{dt}a^{f(t)} = a^{f(t)}\log a \cdot f'(t)$.

Step 1. Compute derivatives:

$$\frac{dx}{dt} = a^{\sin^{-1}t}\log a \cdot \frac{1}{\sqrt{1-t^2}} = \frac{x\log a}{\sqrt{1-t^2}}.$$
$$\frac{dy}{dt} = a^{\cos^{-1}t}\log a \cdot \left(-\frac{1}{\sqrt{1-t^2}}\right) = -\frac{y\log a}{\sqrt{1-t^2}}.$$

Step 2. Ratio:

$$\frac{dy}{dx} = \frac{-y\log a/\sqrt{1-t^2}}{x\log a/\sqrt{1-t^2}} = -\frac{y}{x}.$$

Step 3. Hence proved.

Final Answer: $\dfrac{dy}{dx} = -\dfrac{y}{x}$.

♥ Why a constant product

Note $\sin^{-1} t + \cos^{-1} t = \pi/2$, so $xy = a^{\sin^{-1} t} \cdot a^{\cos^{-1} t} = a^{\pi/2}$, a constant. Differentiating $xy = c$ implicitly gives $y + xy' = 0$, i.e. $y' = -y/x$, matching the parametric calculation.

EXPERT'S SOLUTION *: Aditya Patel, M.Sc Mathematics, IIT Bombay*

Strategic angle. Use the hidden relation $xy = a^{\pi/2}$.

Step 1. $\sin^{-1} t + \cos^{-1} t = \pi/2 \Rightarrow xy = a^{\pi/2}$ (const).

Step 2. Differentiate: $y + xy' = 0 \Rightarrow y' = -y/x$.

Final Answer: $dy/dx = -y/x$.

Key Takeaways

- For parametric curves: $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ where $dx/dt \neq 0$. Express answers in terms of the parameter.
- Half-angle and double-angle identities are the workhorses; spot $1 \pm \cos\theta$ patterns immediately.
- Sometimes a hidden algebraic relation between x and y (Q11: $xy = a^{\pi/2}$) gives the derivative more cleanly than the parametric route.

End of Exercise 5.6