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Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 5: Continuity and Differentiability

About this Chapter

Exercise 5.7 introduces **second-order derivatives**: the derivative of $f'(x)$ is $f''(x)$, the rate of change of the slope. Q1–Q10 compute d^2y/dx^2 for standard functions; Q11–Q17 verify that y satisfies a given differential equation.

Topics covered: d^2y/dx^2 • Product/chain rule twice • Verifying ODEs • Implicit second derivatives

Quick Formula Sheet

Notation: $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$, also written $f''(x)$, y'' or y_2.

Linearity: $(\alpha f + \beta g)'' = \alpha f'' + \beta g''$.

Product rule iterates: $(uv)'' = u''v + 2u'v' + uv''$ (Leibnitz with $n = 2$).

Exercise 5.7

Q 5.1 **Find the second-order derivative of $x^2 + 3x + 2$.**

SOLUTION

Concept used. Power-rule derivatives applied twice.

Step 1. Let $y = x^2 + 3x + 2$.

Step 2. First derivative:

$$\frac{dy}{dx} = 2x + 3.$$

Step 3. Second derivative:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(2x + 3) = 2.$$

Final Answer: $\dfrac{d^2y}{dx^2} = 2.$

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Bombay*

Quick reading. Quadratic; second derivative is the constant leading coefficient times $2 \cdot 1$.

Step 1. $y'' = 2$.

Final Answer: 2.

Q 5.2 **Find the second-order derivative of x^{20}.**

SOLUTION

Concept used. Power rule: $\dfrac{d}{dx}x^n = nx^{n-1}$.

Step 1. $\dfrac{dy}{dx} = 20x^{19}$.

Step 2. $\dfrac{d^2y}{dx^2} = 20 \cdot 19 \cdot x^{18} = 380x^{18}$.

Final Answer: $\dfrac{d^2y}{dx^2} = 380x^{18}$.

EXPERT'S SOLUTION *: Sneha Iyer, Ph.D Mathematics, IIT Delhi*

Quick reading. $(x^n)'' = n(n-1)x^{n-2}$, here $20 \times 19 = 380$.

Step 1. $20 \cdot 19 = 380$.

Final Answer: $380x^{18}$.

Q 5.3 **Find the second-order derivative of $x \cdot \cos x$.**

SOLUTION

Concept used. Product rule.

Step 1. Let $y = x\cos x$. Product rule:

$$\frac{dy}{dx} = \cos x + x(-\sin x) = \cos x - x\sin x.$$

Step 2. Differentiate again. Product rule on $x\sin x$:

$$\frac{d^2y}{dx^2} = -\sin x - \big(\sin x + x\cos x\big) = -\sin x - \sin x - x\cos x = -2\sin x - x\cos x.$$

Final Answer: $\dfrac{d^2y}{dx^2} = -(x\cos x + 2\sin x)$.

EXPERT'S SOLUTION *: Vivaan Gupta, M.Tech CS, IIT Madras*

Quick reading. Leibnitz's rule with $n = 2$: $(uv)'' = u''v + 2u'v' + uv''$ with $u = x$, $v = \cos x$:

Step 1. $u''v = 0$; $2u'v' = 2\cdot 1\cdot(-\sin x) = -2\sin x$; $uv'' = x(-\cos x)$.

Step 2. Sum: $-2\sin x - x\cos x$.

Final Answer: $-(2\sin x + x\cos x)$.

Q 5.4 Find the second-order derivative of $\log x$.

SOLUTION

Concept used. $\dfrac{d}{dx}\log x = 1/x$; $\dfrac{d}{dx}(1/x) = -1/x^2$.

Step 1. $\dfrac{dy}{dx} = \dfrac{1}{x}$.

Step 2. $\dfrac{d^2y}{dx^2} = -\dfrac{1}{x^2}$.

Final Answer: $\dfrac{d^2y}{dx^2} = -\dfrac{1}{x^2}$ $(x > 0)$.

EXPERT'S SOLUTION : *Aanya Mehta, Ph.D Pure Mathematics, IISc Bangalore*

Quick reading. Two power-rule steps.

Step 1. $(\log x)' = 1/x = x^{-1}$.

Step 2. $(x^{-1})' = -x^{-2}$.

Final Answer: $-1/x^2$.

Q 5.5 Find the second-order derivative of $x^3 \log x$.

SOLUTION

Concept used. Product rule applied twice.

Step 1. First derivative: $u = x^3$, $v = \log x$, $u' = 3x^2$, $v' = 1/x$.

$$\frac{dy}{dx} = 3x^2 \log x + x^3 \cdot \frac{1}{x} = 3x^2 \log x + x^2.$$

Step 2. Second derivative: differentiate $3x^2 \log x + x^2$.

$$\frac{d}{dx}(3x^2 \log x) = 6x \log x + 3x^2 \cdot \frac{1}{x} = 6x \log x + 3x.$$

$$\frac{d}{dx}(x^2) = 2x.$$

Step 3. Sum:

$$\frac{d^2y}{dx^2} = 6x \log x + 3x + 2x = 6x \log x + 5x = x(6 \log x + 5).$$

Final Answer: $\frac{d^2y}{dx^2} = x(6 \log x + 5)$ $(x > 0)$.

EXPERT'S SOLUTION : *Priya Singh, M.Sc Applied Mathematics, IIT Kanpur*

Quick reading. Differentiate twice with product rule each time.

Step 1. $y' = 3x^2 \log x + x^2$.

Step 2. $y'' = 6x \log x + 3x + 2x = x(6 \log x + 5)$.

Final Answer: $x(6 \log x + 5)$.

Q 5.6 **Find the second-order derivative of $e^x \sin 5x$.**

SOLUTION

Concept used. Product rule twice.

Step 1. First derivative:

$$\frac{dy}{dx} = e^x \sin 5x + e^x \cdot 5\cos 5x = e^x(\sin 5x + 5\cos 5x).$$

Step 2. Second derivative. Let $g(x) = \sin 5x + 5\cos 5x$. Then $g'(x) = 5\cos 5x + 5(-5\sin 5x) = 5\cos 5x - 25\sin 5x$. Apply product rule:

$$\frac{d^2y}{dx^2} = e^x g(x) + e^x g'(x) = e^x\big(g(x) + g'(x)\big).$$

Now

$$g(x) + g'(x) = (\sin 5x + 5\cos 5x) + (5\cos 5x - 25\sin 5x) = -24\sin 5x + 10\cos 5x.$$

Step 3. Therefore

$$\frac{d^2y}{dx^2} = e^x(10\cos 5x - 24\sin 5x) = 2e^x(5\cos 5x - 12\sin 5x).$$

Final Answer: $\frac{d^2y}{dx^2} = 2e^x(5\cos 5x - 12\sin 5x)$.

EXPERT'S SOLUTION *: Karan Joshi, M.Sc Mathematics, IIT Bombay*

Structural observation. For $y = e^{ax}\sin(bx)$, the operator $(D-a)^2 + b^2$ annihilates: $y'' = 2ay' + (b^2 - a^2)(-1)y$ etc. Direct product-rule is cleaner here.

Step 1. $y' = e^x(\sin 5x + 5\cos 5x)$.

Step 2. $y'' = e^x(\sin 5x + 5\cos 5x + 5\cos 5x - 25\sin 5x) = e^x(10\cos 5x - 24\sin 5x)$.

Final Answer: $2e^x(5\cos 5x - 12\sin 5x)$.

Q 5.7 **Find the second-order derivative of $e^{6x}\cos 3x$.**

SOLUTION

Concept used. Product + chain rules.

Step 1. First derivative:

$$\frac{dy}{dx} = 6e^{6x}\cos 3x + e^{6x}(-3\sin 3x) = e^{6x}(6\cos 3x - 3\sin 3x).$$

Step 2. Second derivative. Let $g = 6\cos 3x - 3\sin 3x$. Then $g' = -18\sin 3x - 9\cos 3x$.

$$\frac{d^2y}{dx^2} = 6e^{6x}g + e^{6x}g' = e^{6x}(6g + g').$$

Compute:

$$\begin{aligned} 6g + g' &= 6(6\cos 3x - 3\sin 3x) + (-18\sin 3x - 9\cos 3x) \\ &= 36\cos 3x - 18\sin 3x - 18\sin 3x - 9\cos 3x \\ &= 27\cos 3x - 36\sin 3x. \end{aligned}$$

Step 3. Hence

$$\frac{d^2y}{dx^2} = e^{6x}(27\cos 3x - 36\sin 3x) = 9e^{6x}(3\cos 3x - 4\sin 3x).$$

Final Answer: $\frac{d^2y}{dx^2} = 9e^{6x}(3\cos 3x - 4\sin 3x).$

EXPERT'S SOLUTION *: Diya Bhat, Ph.D Mathematics, IIT Delhi*

Quick reading. Same pattern as Q6 with new coefficients.

Step 1. $y' = e^{6x}(6\cos 3x - 3\sin 3x).$

Step 2. $y'' = e^{6x}(27\cos 3x - 36\sin 3x) = 9e^{6x}(3\cos 3x - 4\sin 3x).$

Final Answer: $9e^{6x}(3\cos 3x - 4\sin 3x).$

Q 5.8 **Find the second-order derivative of** $\tan^{-1} x$.

SOLUTION

Concept used. $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$; differentiate again via quotient rule.

Step 1. $\frac{dy}{dx} = \frac{1}{1+x^2}$.

Step 2. Differentiate:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(1+x^2)^{-1} = -(1+x^2)^{-2} \cdot 2x = -\frac{2x}{(1+x^2)^2}.$$

Final Answer: $\frac{d^2y}{dx^2} = -\frac{2x}{(1+x^2)^2}$.

EXPERT'S SOLUTION : *Yash Nair, M.Tech CS, IIT Madras*

Quick reading. Chain rule on $(1+x^2)^{-1}$.

Step 1. $y' = (1+x^2)^{-1}$.

Step 2. $y'' = -2x/(1+x^2)^2$.

Final Answer: $-2x/(1+x^2)^2$.

Q 5.9 **Find the second-order derivative of** $\log(\log x)$.

SOLUTION

Concept used. Chain rule on first derivative; quotient rule on second.

Step 1. First derivative (Ex 5.4 Q8):

$$\frac{dy}{dx} = \frac{1}{x \log x}.$$

Step 2. Differentiate using quotient rule with numerator 1 and denominator $x \log x$:

$$\frac{d^2y}{dx^2} = -\frac{(x \log x)'}{(x \log x)^2} = -\frac{\log x + 1}{(x \log x)^2}.$$

Here we used $(x \log x)' = \log x + x \cdot 1/x = \log x + 1$.

Final Answer: $\frac{d^2y}{dx^2} = -\frac{1+\log x}{(x \log x)^2}$ $(x > 1)$.

EXPERT'S SOLUTION : *Aditi Banerjee, M.Sc Mathematics, ISI Kolkata*

Quick reading. Two derivatives, each a chain rule.

Step 1. $y' = 1/(x \log x)$.

Step 2. $y'' = -(\log x + 1)/(x \log x)^2$.

> **Final Answer:** $-(1 + \log x)/(x \log x)^2$.

Q 5.10 **Find the second-order derivative of** $\sin(\log x)$**.**

SOLUTION

Concept used. Chain rule and quotient rule.

Step 1. $\dfrac{dy}{dx} = \cos(\log x) \cdot \dfrac{1}{x} = \dfrac{\cos(\log x)}{x}$.

Step 2. Differentiate using quotient rule with $u = \cos(\log x)$, $v = x$: $u' = -\sin(\log x)/x$, $v' = 1$.

$$\frac{d^2y}{dx^2} = \frac{u'v - uv'}{v^2} = \frac{(-\sin(\log x)/x) \cdot x - \cos(\log x) \cdot 1}{x^2} = \frac{-\sin(\log x) - \cos(\log x)}{x^2}.$$

> **Final Answer:** $\dfrac{d^2y}{dx^2} = -\dfrac{\sin(\log x) + \cos(\log x)}{x^2}$ $(x > 0)$.

EXPERT'S SOLUTION : *Tara Pillai, Ph.D Mathematics, IIT Delhi*

Quick reading. Chain, then quotient.

Step 1. $y' = \cos(\log x)/x$.

Step 2. Quotient rule with simplification.

> **Final Answer:** $-(\sin(\log x) + \cos(\log x))/x^2$.

Q 5.11 **If** $y = 5\cos x - 3\sin x$**, prove that** $\dfrac{d^2y}{dx^2} + y = 0$**.**

SOLUTION

Concept used. Differentiate twice and substitute.

Step 1. $\dfrac{dy}{dx} = -5\sin x - 3\cos x.$

Step 2. $\dfrac{d^2y}{dx^2} = -5\cos x + 3\sin x = -(5\cos x - 3\sin x) = -y.$

Step 3. Therefore $\dfrac{d^2y}{dx^2} + y = -y + y = 0.$

> **Final Answer:** $\dfrac{d^2y}{dx^2} + y = 0.$

EXPERT'S SOLUTION *: Krishna Rao, M.Sc Mathematics, IIT Bombay*

Structural observation. Every linear combination of $\sin x$ and $\cos x$ satisfies $y'' + y = 0$ (the simple harmonic equation).

Step 1. Linear combinations of $\sin, \cos$ are eigenfunctions of d^2/dx^2 with eigenvalue -1.

> **Final Answer:** Verified.

Q 5.12 **If $y = \cos^{-1} x$, find $\dfrac{d^2y}{dx^2}$ in terms of y alone.**

SOLUTION

Concept used. Express $\sin y, \cos y$ in terms of x and back-substitute.

Step 1. $y = \cos^{-1} x \Rightarrow x = \cos y$. First derivative:

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} = -\frac{1}{\sin y}.$$

(Used $\sqrt{1-\cos^2 y} = \sin y$, taking the positive root for $y \in (0, \pi)$.)

Step 2. Differentiate $\dfrac{dy}{dx} = -\dfrac{1}{\sin y} = -\csc y$ w.r.t. x:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-\csc y) = \csc y \cot y \cdot \frac{dy}{dx} = \csc y \cot y \cdot \left(-\frac{1}{\sin y}\right).$$

Step 3. Simplify:

$$= -\frac{\csc y \cot y}{\sin y} = -\frac{\cot y}{\sin^2 y} = -\frac{\cos y/\sin y}{\sin^2 y} = -\frac{\cos y}{\sin^3 y}.$$

Final Answer: $\dfrac{d^2y}{dx^2} = -\dfrac{\cos y}{\sin^3 y} \ \big(y \in (0,\pi)\big)$.

EXPERT'S SOLUTION *: Meera Chatterjee, M.Sc Mathematics, ISI Kolkata*

Strategic angle. Convert x-expression to y-expression once dy/dx is known.

Step 1. $y' = -1/\sin y$.

Step 2. Differentiate w.r.t. x via chain rule on $-\csc y$.

Final Answer: $-\cos y/\sin^3 y$.

Q 5.13 If $y = 3\cos(\log x) + 4\sin(\log x)$, **show that** $x^2y_2 + xy_1 + y = 0$.

SOLUTION

Concept used. Compute y_1, y_2 and substitute. Notation: $y_1 = dy/dx$, $y_2 = d^2y/dx^2$.

Step 1. $\dfrac{dy}{dx} = -3\sin(\log x)/x + 4\cos(\log x)/x = \dfrac{-3\sin(\log x) + 4\cos(\log x)}{x}$. So $xy_1 = -3\sin(\log x) + 4\cos(\log x)$.

Step 2. Differentiate $xy_1 = -3\sin(\log x) + 4\cos(\log x)$ w.r.t. x. Left side product rule:

$$\frac{d}{dx}(xy_1) = y_1 + xy_2.$$

Right side chain rule:

$$\frac{d}{dx}\big[-3\sin(\log x) + 4\cos(\log x)\big] = \frac{-3\cos(\log x) - 4\sin(\log x)}{x}.$$

Step 3. Equate:

$$y_1 + xy_2 = \frac{-3\cos(\log x) - 4\sin(\log x)}{x} = -\frac{y}{x}.$$

(Since $y = 3\cos(\log x) + 4\sin(\log x)$.)

Step 4. Multiply both sides by x:

$$xy_1 + x^2y_2 = -y \implies x^2y_2 + xy_1 + y = 0.$$

Final Answer: $x^2y_2 + xy_1 + y = 0$.

EXPERT'S SOLUTION : *Rohit Verma, M.Sc Mathematics, ISI Kolkata*

Strategic angle. Multiply xy_1 before second differentiation to keep the algebra clean.

Step 1. $xy_1 = -3\sin(\log x) + 4\cos(\log x)$.

Step 2. Differentiate; combine with $y_1 + xy_2 = -y/x$.

Step 3. Multiply by x to obtain ODE.

> **Final Answer:** ODE verified.

Q 5.14 **If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$.**

SOLUTION

Concept used. Differentiate twice, substitute, simplify.

Step 1. $y_1 = Ame^{mx} + Bne^{nx}$.

Step 2. $y_2 = Am^2e^{mx} + Bn^2e^{nx}$.

Step 3. Compute the LHS:

$$\begin{aligned} y_2 - (m+n)y_1 + mny &= (Am^2e^{mx} + Bn^2e^{nx}) \\ &\quad - (m+n)(Ame^{mx} + Bne^{nx}) \\ &\quad + mn(Ae^{mx} + Be^{nx}) \\ &= Ae^{mx}\left[m^2 - (m+n)m + mn\right] \\ &\quad + Be^{nx}\left[n^2 - (m+n)n + mn\right] \\ &= Ae^{mx}\left[m^2 - m^2 - mn + mn\right] \\ &\quad + Be^{nx}\left[n^2 - mn - n^2 + mn\right] \\ &= Ae^{mx} \cdot 0 + Be^{nx} \cdot 0 = 0. \end{aligned}$$

> **Final Answer:** $y_2 - (m+n)y_1 + mny = 0$.

EXPERT'S SOLUTION : *Pranav Kapoor, M.Sc Mathematics, IIT Bombay*

Structural observation. The characteristic equation is $r^2 - (m+n)r + mn = (r-m)(r-n) = 0$, with roots m, n. The general solution is exactly $Ae^{mx} + Be^{nx}$.

Step 1. Solutions e^{mx} and e^{nx} each satisfy the ODE.

Step 2. Linear combination also satisfies.

Final Answer: Verified.

Q 5.15 **If $y = 500e^{7x} + 600e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$.**

SOLUTION

Concept used. Differentiate twice; each derivative pulls a factor of ± 7.

Step 1. $y_1 = 500 \cdot 7e^{7x} + 600 \cdot (-7)e^{-7x} = 3500e^{7x} - 4200e^{-7x}$.

Step 2. $y_2 = 3500 \cdot 7e^{7x} - 4200 \cdot (-7)e^{-7x} = 24500e^{7x} + 29400e^{-7x}$.

Step 3. Factor 49:

$$y_2 = 49(500e^{7x} + 600e^{-7x}) = 49y.$$

Final Answer: $\frac{d^2y}{dx^2} = 49y$.

EXPERT'S SOLUTION *: Aditya Patel, M.Sc Mathematics, IIT Bombay*

Quick reading. Each $e^{\pm 7x}$ has $y_2 = 49y$; linear combinations inherit.

Step 1. $y_2 = 49y$.

Final Answer: Verified.

Q 5.16 **If $e^y(x+1) = 1$, show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.**

SOLUTION

Concept used. Take $\log$ to simplify, then differentiate twice.

Step 1. From $e^y(x+1) = 1$: $e^y = 1/(x+1)$, hence $y = -\log(x+1)$.

Step 2. First derivative:

$$\frac{dy}{dx} = -\frac{1}{x+1}.$$

Step 3. Second derivative:

$$\frac{d^2y}{dx^2} = \frac{1}{(x+1)^2}.$$

Step 4. Compute $(dy/dx)^2 = 1/(x+1)^2$, which equals d^2y/dx^2.

Final Answer: $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 = \frac{1}{(x+1)^2}$.

EXPERT'S SOLUTION *: Ishita Reddy, Ph.D Mathematics, IIT Delhi*

Strategic angle. Convert to explicit form $y = -\log(x+1)$ first.

Step 1. $y' = -1/(x+1)$, $y'' = 1/(x+1)^2$.

Step 2. $(y')^2 = 1/(x+1)^2 = y''$.

Final Answer: Verified.

Q 5.17 **If $y = (\tan^{-1} x)^2$, show that $(x^2+1)^2 y_2 + 2x(x^2+1)y_1 = 2$.**

SOLUTION

Concept used. Differentiate twice, simplify using $\tan^{-1} x$ relations.

Step 1. $y_1 = 2\tan^{-1} x \cdot \frac{1}{1+x^2}$, so

$$(1+x^2)y_1 = 2\tan^{-1} x.$$

Step 2. Differentiate both sides of $(1+x^2)y_1 = 2\tan^{-1} x$ w.r.t. x:

$$2xy_1 + (1+x^2)y_2 = \frac{2}{1+x^2}.$$

Step 3. Multiply both sides by $(1+x^2)$:

$$2x(1+x^2)y_1 + (1+x^2)^2 y_2 = 2.$$

Rearrange:

$$(x^2+1)^2 y_2 + 2x(x^2+1)y_1 = 2.$$

Final Answer: $(x^2+1)^2 y_2 + 2x(x^2+1)y_1 = 2$.

EXPERT'S SOLUTION : *Kavya Singh, M.Sc Mathematics, IIT Bombay*

Strategic angle. Multiply through by $(1 + x^2)$ early to keep the algebra clean.

Step 1. $(1 + x^2)y_1 = 2\tan^{-1} x$.

Step 2. Differentiate, multiply by $(1 + x^2)$ again, the right side becomes 2.

Final Answer: Verified.

Key Takeaways

- d^2y/dx^2 is the derivative of dy/dx; apply differentiation rules twice in succession.
- For $y = Ae^{mx} + Be^{nx}$ the second derivative satisfies the characteristic ODE $y'' - (m + n)y' + mny = 0$.
- For implicit relations or proofs of an ODE, multiply by useful factors of x or $(1 + x^2)$ before differentiating again; the algebra usually telescopes neatly.

End of Exercise 5.7